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Geometric Construction of $*$-Representations of the Weyl Algebra with Degree 2

By

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Abstract

Let $\mathscr{W}_2$ denote the Weyl algebra generated by self-adjoint elements $\{p_j, q_j\}_{j=1,2}$ satisfying the canonical commutation relations. In this paper we discuss $*$-representations $\{\pi\}$ of $\mathscr{W}_2$ such that $\pi(p_j)$ and $\pi(q_j)$ $(j=1, 2)$ are essentially self-adjoint operators but π is not exponentiable to a representation of the associated Weyl system. We first construct a class of such $*$-representations of $\mathscr{W}_2$ by considering a non-simply connected space $\Omega=\boldsymbol{R}^2\backslash\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N\}$ and a one-dimensional representations of the fundamental group $\pi_1(\Omega)$. Non-exponentiability of those $*$-representations comes from the geometry of the universal covering space $\widetilde{\Omega}$ of Ω. Then we show that our $*$-representations of $\mathscr{W}_2$ are related, by unitary equivalence, with Reeh-Arai's ones, which are based on a quantum system on the plane under a perpendicular magnetic field with singularities at $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N$, and, by doing that , we classify the Reeh-Arai's $*$-representations up to unitary equivalence. We further discuss extension and irreducibility of those $*$-representations. Finally, for the $*$-representations of $\mathscr{W}_2$, we calculate the defect numbers which measure the distance to the exponentiability.

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§1. Introduction

As an algebraic object for quantum mechanics with n degree of freedom, we consider the $*$-algebra $\mathcal{W}_n$ with the self-adjoint generators $\{p_j, q_j\}_{j=1,\cdots,n}$ satisfying the canonical commutation relations (CCR);

$$\begin{aligned}&[p_j, q_k] = -i\delta_{jk}I \text{ and}\\&[p_j, p_k] = [q_j, q_k] = 0\\&\quad \text{for } j, k = 1, \cdots, n.\end{aligned} \tag{1.1}$$

The $*$-algebra $\mathcal{W}_n$ is called the Weyl algebra or CCR algebra. Recall that we do not have bounded $*$-representations of $\mathcal{W}_n$, thus we need to study unbounded ones.

In general, $(\pi, \mathcal{D})$, simply π, is called a $*$-representation of a $*$-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ if

$$\begin{aligned}&\mathcal{D} \text{ is a dense subspace of } \mathcal{H},\\&\pi \text{ is an algebraic homomorphism of } \mathcal{A} \text{ into End } \mathcal{D},\\&\text{and } \pi(a)^* \supset \pi(a^*) \text{ for } a \in \mathcal{A}.\end{aligned} \tag{1.2}$$

Since the Weyl algebra $\mathcal{W}_n$ has self-adjoint generators $\{p_j, q_j\}$, we can define a $*$-representation $(\pi, \mathcal{D})$ of $\mathcal{W}_n$ by giving symmetric operators $\{\pi(p_j), \pi(q_j)\}$ with the common and invariant domain $\mathcal{D}$ such that those satisfy the CCR.

To avoid a difficulty to analyze directly unbounded $*$-representations of $\mathcal{W}_n$, we often consider the Weyl system $\{u_j(s), v_j(s); s \in \boldsymbol{R}, j=1, \cdots, n\}$ satisfying the Weyl relations (WR):

$$\begin{aligned}&u_j(s)v_k(t) = e^{i\delta_{jk}st}v_k(t)u_j(s),\\&[u_j(s), u_k(t)] = [v_j(s), v_k(t)] = 0\\&\quad \text{for } s, t \in \boldsymbol{R} \text{ and } j, k = 1, \cdots, n.\end{aligned} \tag{1.3}$$

The WR is the integrated form of the CCR, that is, we formally get the WR from the CCR for the formal series $u_j(s) = \exp(isp_j)$ and $v_j(s) = \exp(isq_j)$. And then we consider a (strongly continuous) unitary representation of the WR instead of a $*$-representation of $\mathcal{W}_n$.

It is a well-known fact due to von Neumann that any irreducible unitary representation of the WR is unitarily equivalent to the Schrödinger representation ρ_S in $L^2(\boldsymbol{R}^n)$, which is defined by

$$(\rho_S(u_j(s))f)(x_1, \cdots, x_n)=f(x_1, \cdots, x_j+s, \cdots, x_n) \text{ and}$$
$$(\rho_S(v_j(s))f)(x_1, \cdots, x_n)=e^{isx_j}f(x_1, \cdots, x_n) \tag{1.4}$$
$$\text{for } f\in L^2(\boldsymbol{R}^n) \text{ and } j=1, \cdots, n.$$

For a given unitary representation of the WR, we can easily get a $*$-representation of the Weyl algebra $\mathcal{W}_n$ as its differential. In fact, as for the Schrödinger representation of the WR, we obtain the following $*$-representation π_S of $\mathcal{W}_n$, which is also called the Schrödinger representation :

$$\begin{aligned} &\mathcal{D}=\mathcal{D}(\pi_S(p_j))=\mathcal{D}(\pi_S(q_j))=\mathcal{S}(\boldsymbol{R}^n)\\ &(\pi_S(p_j)f)(x_1, \cdots, x_n)=-i\frac{\partial f}{\partial x_j}(x_1, \cdots, x_n),\\ &(\pi_S(q_j)f)(x_1, \cdots, x_n)=x_jf(x_1, \cdots, x_n),\\ &\qquad\qquad\text{for } f\in\mathcal{D}. \end{aligned} \tag{1.5}$$

From the viewpoint of representation theory of Lie groups and their Lie algebras, we can restate that a unitary representation of the WR gives that of the Heisenberg group, and the associated Lie algebra representation gives a $*$-representation of the Weyl algebra. But the converse does not hold in general, that is, for a $*$-representation π of $\mathcal{W}_n$, even if $\pi(p_j)$ and $\pi(q_j)$ are essentially (ess.) self-adjoint, the unitary operators $\{\exp is\overline{\pi(p_j)}, \exp is\overline{\pi(q_j)}\}$ do not necessarily satisfy the WR.

In this paper we shall say that a $*$-representation of $\mathcal{W}_n$ is quasi-exponentiable if the generators p_j and q_j $(j=1, \cdots, n)$ are represented as ess. self-adjoint operators. And a quasi-exponentiable $*$-representation π of $\mathcal{W}_n$ is said to be exponentiable if $\{\exp is\overline{\pi(p_j)}, \exp is\overline{\pi(q_j)}\}$ satisfy the WR (cf. [S4] §10.5). We also use the same notions for $*$-representations of the polynominal algebras $\mathcal{P}_n=\mathcal{P}(x_1, \cdots, x_n)$ generated by the self-adjoint elements $\{x_j\}$. In this case, a $*$-representation π of $\mathcal{P}_n$ is said to be exponentiable if the self-adjoint operators $\{\overline{\pi(x_j)}\}$ are strongly commuting with each other. The first example of quasi-exponentiable but non-exponentiable $*$-representations of the polynomial algebras was got by Nelson [N]. In other words, he constructed the two ess. self-adjoint operators A and B in a Hilbert space such that they have a common invariant domain $\mathcal{D}$ and satisfy $[\mathrm{A}, \mathrm{B}]=0$ on $\mathcal{D}$, but they do not strongly commute, that is, $[e^{is\overline{\mathrm{A}}}, e^{it\overline{\mathrm{B}}}]\neq 0$ for some $s\neq 0$ and $t\neq 0$. Stimulated by the Nelson's example, some authers could get quasi-exponentiable but non-exponentiable $*$-representations of $\mathcal{W}_1$ (c.f. [F1], [RS]). It is notable that non-exponentiability of the $*$-representations of $\mathcal{W}_1$ and $\mathcal{P}_2$ cited in [RS] pp. 273–275 easily follows from the geometry of the Riemann surface associated with $\sqrt{z}$. After the Nelson's work, many authors have studied $*$-representations of the Weyl algebras and the polynomial algebras and constructed many examples of non-exponentiable ones (e.g. [F1], [J], [JM], [P1-2], [Pou], [Pu], [S1-4], [W]). But these examples, except some of those, are of $\mathscr{W}_1$ or $\mathscr{P}_2$.

In the recent papers [R] and [A1], Reeh and Arai found quasi-exponentiable but non-exponentiable $*$-representations of the Weyl Algebra $\mathscr{W}_2$ by considering quantum systems on the plane $\boldsymbol{R}^2$ with perpendicular magnetic fields concentrated at finite points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N \in \boldsymbol{R}^2$. The Reeh-Arai's $*$-representations of $\mathscr{W}_2$ are natural in the sense that they come from the quantum systems as above and non-exponentiability of those corresponds to the Aharonov-Bohm effect (c.f. [A1], [A2], [R])

It is suggested by Reeh [R] that his $*$-representations of $\mathscr{W}_2$ might be related with the Nelson's observation. The first purpose of this paper is to clarify this point. In §2.2, following the spirit of Nelson, we construct a class of quasi-exponentiable $*$-representations of $\mathscr{W}_2$ by considering the universal covering space and the fundamental group of the non-simply connected space;

$$\Omega = \boldsymbol{R}^2 \backslash \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N\}. \tag{1.6}$$

Then, we show in Theorem 3.2 that they are related by unitary equivalence with the $*$-representations given by Reeh and Arai. And the second purpose of this paper is to classify the Reeh-Arai's $*$-representations of $\mathscr{W}_2$ up to unitary equivalence (c.f. Corollaries 3.3 and 3.7).

Each $*$-representation π of $\mathscr{W}_2$ given in §2 is quasi-exponentiable and it is exponentiable if and only if $\overline{\pi(p_1)}$ and $\overline{\pi(p_2)}$ are strongly commuting (c.f. Theorem 2.2 and Theorem 2.4). In this paper, by restricting π, we often consider π as a $*$-representation of the polynomial subalgebra $\mathscr{P}_2 = \mathscr{P}(p_1, p_2)$ of $\mathscr{W}_2$ generated by p_1, p_2, and I. The restriction has its own interest. The third purpose of this paper is to show fundamental properties of the $*$-representations of $\mathscr{W}_2$ and $\mathscr{P}_2$ given in §2. In §3.2 we show that every $*$-representation π of $\mathscr{W}_2$ in §2 is irreducible and, more strongly, that the associated $*$-representation of $\mathscr{P}_2 = \mathscr{P}(p_1, p_2)$ is irreducible if π is not exponentiable (c.f. Theorem 3.8 and 3.9). In §3.3 we consider extending $*$-representations of $\mathscr{W}_2$ and $\mathscr{P}_2 = \mathscr{P}(p_1, p_2)$ in §2 by taking larger domains in the same Hilbert space or in a larger Hilbert space. In particular, we show that any extension of a non-exponentiable $*$-representation of $\mathscr{W}_2$ or $\mathscr{P}_2$ in §2 is also non-exponentiable (c.f. Theorem 3.11).

Schmüdgen [S4] introduced the defect number for two self-adjoint operators to measure the distance to the strong commutativity. The last purpose of this paper is to calculate the Schmüdegen's defect number for $\overline{\pi(p_1)}$ and $\overline{\pi(p_2)}$, where π is a $*$-representation of $\mathscr{W}_2$ in §2. As a result, we show in Theorem 4.1 that the defect number is equal to the number of essentially singular points $\boldsymbol{a}_j$ (c.f.

Definition 3.6). In our case the defect number is considered as a distance of π to the exponentiability.

§2. Construction of Non-exponentiable $*$-Representations

In this section we state examples of quasi-exponentiable $*$-representations of the Weyl algebra $\mathscr{W}_2$ which are generally non-exponentiable. After reviewing the recent Reeh-Arai's examples, in subsection 2.2 we construct examples by a different method, that is, by generalizing the way to construct the Nelson's example mentioned in §1.

2.1. Reeh-Arai's $*$-Representations

Let A_1 and A_2 be real valued C^∞-functions on Ω (c.f. (1.6)) such that

$$D_x A_2 = D_y A_1, \tag{2.1}$$

where D_x and D_y denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively. Then, in this paper, the pair $\boldsymbol{A}=(A_1, A_2)$ is called a vector potential on Ω with singular points $\boldsymbol{a}_j$ $(j=1, \cdots, N)$. For a vector potential $\boldsymbol{A}=(A_1, A_2)$, we define the four operators P_j and Q_j $(j=1, 2)$ in $L^2(\Omega)\simeq L^2(\boldsymbol{R}^2)$ as follows;

$$\begin{gathered} P_1=-iD_x-A_1,\ P_2=-iD_y-A_2, \\ Q_1=x,\ Q_2=y, \\ \mathscr{D}(P_j)=\mathscr{D}(Q_j)=C_0^\infty(\Omega)\ (j=1, 2), \end{gathered} \tag{2.2}$$

where $\mathscr{D}(\cdot)$ means the domain of the associated operator, $C_0^\infty(\Omega)$ is the subspace of all $\boldsymbol{C}$-valued C^∞-functions on Ω with compact supports in Ω, and A_1, A_2, x, and y in the definition (2.2) denote the multiplication operators by themselves. It is easily seen that P_j and Q_j $(j=1, 2)$ are symmetric operators with the invariant domain $C_0^\infty(\Omega)$ and satisfy the CCR;

$$\begin{aligned} &[p_j, Q_k]=-i\delta_{jk}I\ (j, k=1, 2), \\ &[P_1, P_2]=[Q_1, Q_2]=0. \end{aligned} \tag{2.3}$$

Among these relations, only the relation $[P_1, P_2]=0$ is due to the condition (2.1) for $\boldsymbol{A}$. Thus, by setting

$$\pi_A(p_j)=P_j \text{ and } \pi_A(q_j)=Q_j\ (j=1, 2) \tag{2.4}$$

we obtain a $*$-representation π_A of $\mathscr{W}_2$. We often use the same notation π_A for the $*$-representation of $\mathscr{P}_2$ given by restriction of π_A to the polynomial subalgebra $\mathscr{P}_2=\mathscr{P}(p_1, p_2)$ of $\mathscr{W}_2$.

It is easily shown that Q_j $(j=1, 2)$ are ess. self-adjoint and $\exp(is\overline{Q_j})$ $(j=$

1, 2) are the multiplication operators by e^{isx} and e^{isy}, respectively. As for P_j, the following lemma is proved in [A1].

Lemma 2.1. ([A1], Theorem 2.1) *The operators P_j $(j=1, 2)$ are ess. self-adjoint and satisfy*

$$(e^{is\overline{P_1}}f)(x, y)=\exp\left(-i\int_0^s A_1(x+x', y)dx'\right)f(x+s, y)$$
$$(e^{is\overline{P_2}}f)(x, y)=\exp\left(-i\int_0^s A_2(x, y+y')dy'\right)f(x, y+s)$$
for $f\in L^2(\Omega)$ and a.e. $(x, y)\in\Omega$.

Thus the $*$-representation π_A is quasi-exponentiable.

For a vector potential $\boldsymbol{A}=(A_1, A_2)$ with singular points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N$ and sufficiently small $\varepsilon>0$ (e.g. $0<\varepsilon<\min_{i\neq j}|\boldsymbol{a}_i-\boldsymbol{a}_j|$), set

$$\gamma_j^\varepsilon(t)=(a_{j1}+\varepsilon\cos 2\pi t,\ a_{j2}+\varepsilon\sin 2\pi t)\ (0\leq t\leq 1), \tag{2.5}$$

where

$$\boldsymbol{a}_j=(a_{j1}, a_{j2}) \qquad (j=1, 2, \cdots, N). \tag{2.6}$$

And further set

$$c_j=\oint_{\gamma_j^\varepsilon}\boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{r} \qquad (j=1, \cdots, N), \tag{2.7}$$

where γ_j^ε denotes not only the function γ_j^ε; $[0, 1]\rightarrow\Omega$ but also the anticlockwise oriented continuous loop in Ω given by the range of the function. Note that, by the condition (2.1) and the Green's theorem, we have

$$\oint_{\gamma_j}\boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{r}=c_j \qquad (j=1, 2,\cdots, N) \tag{2.8}$$

for any other loop γ_j in Ω which is sufficiently smooth (e.g. piecewise C^1-class) and homotopic (base point free) to γ_j^ε in Ω. Arai [A1] showed

Theorem 2.2. ([A1], Theorem 4.2) *For a vector potential $\boldsymbol{A}=(A_1, A_2)$ on Ω, the following conditions are equivalent.*

(i) *The $*$-representation π_A of $\mathscr{W}_2$ are exponentiable.*

(ii) *The self-adjoint operators $\overline{\pi_A(p_j)}=\overline{P_j}$ $(j=1, 2)$ are strongly commuting, or equivalently, the $*$-representation π_A of $\mathscr{P}_2=\mathscr{P}(p_1, p_2)$ are exponentiable.*

(iii) $c_j\in 2\pi\boldsymbol{Z}$ $(j=1, \cdots, N)$.

Remark 2.3. Some of the quasi-exponentiable $*$-representations of $\mathcal{W}_2$ in this subsection naturally appear in quantum systems on the plane with perpendicular magnetic fields concentrated at $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N$. Then, non-exponentiability of those is due to that the magnetic fields are not locally quantized, and it corresponds to the Aharonov-Bohm effect. The readers can be reffered to [A1] and [A2] for the details.

2.2. Construction of $*$-Representations of Nelson's Type

In this subsection we begin by briefly introducing the Nelson's example mentioned in the introduction.

Let $\mathcal{R}_2$ be the Riemann surface associated with $\sqrt{z}$, equip it with the local coordinate (x, y) and the (local) Lebesgue measure $\boldsymbol{\mu}$ and set

$$\begin{gathered} P_1=-iD_x,\ P_2=-iD_y, \\ Q_1=x,\ Q_2=y, \\ \mathcal{D}=\mathcal{D}(P_j)=\mathcal{D}(Q_j)=C_0^\infty(\mathcal{R}_2\backslash(0,\ 0)). \end{gathered} \tag{2.9}$$

Then the operators P_j and Q_j $(j=1, 2)$ clearly satisfy the CCR. Furthermore, it easily follows that P_j and Q_j are ess. self-adjoint operators in $L^2(\mathcal{R}_2, \boldsymbol{\mu})$ with the common invariant domain $\mathcal{D}$ and that $\overline{P_1}$ and $\overline{P_2}$ generate translation groups on the sheet of $\mathcal{R}_2$ along the x- and y-axis, respectively, so that

$$[\exp is\overline{P_1},\ \exp it\overline{P_2}]\neq 0 \tag{2.10}$$

for some $s\neq 0$ and $t\neq 0$. Thus, by setting

$$\pi(p_j)=P_j \text{ and } \pi(q_j)=Q_j \qquad (j=1,\ 2) \tag{2.11}$$

we get a non-exponentiable $*$-representation of $\mathcal{W}_2$ and $\mathcal{P}_2=\mathcal{P}(p_1, p_2)$ in $L^2(\mathcal{R}_2, \boldsymbol{\mu})$. The $*$-representation of $\mathcal{P}_2$ is due to Nelson.

Here we note that the $*$-representation π is a direct sum of $*$-subrepresentations $\pi_\pm$. In fact, by setting

$$\begin{aligned} \mathcal{H}_\pm=\{&f\in L^2(\mathcal{R}_2,\ \boldsymbol{\mu});\ f(\boldsymbol{r}_1)=\pm f(\boldsymbol{r}_2) \text{ for almost all} \\ &\text{pairs of distinct points } \boldsymbol{r}_1 \text{ and } \boldsymbol{r}_2 \text{ in } \mathcal{R}_2 \text{ with the} \\ &\text{same coordinate}\}, \\ \mathcal{D}_\pm=\mathcal{D}&\cap\mathcal{H}_\pm, \end{aligned}$$

and by denoting $\pi_\pm$ the restriction of π to the domain $\mathcal{D}_\pm$, we get

$$\begin{gathered} L^2(\mathcal{R}_2,\ \boldsymbol{\mu})=\mathcal{H}_+\oplus\mathcal{H}_-,\ \mathcal{D}=\mathcal{D}_+\oplus\mathcal{D}_-, \text{ and} \\ \pi=\pi_+\oplus\pi_-. \end{gathered}$$

Non-exponentiability of π is only due to that of π_-.

In this subsection, we generalize the way to construct the $*$-representation π_- of $\mathcal{W}_2$ by following the Nelson's spirit. As in the subsection 2.1, for finite fixed points $\boldsymbol{a}_j=(a_{j1}, a_{j2})$ $(j=1, \cdots, N)$ in $\boldsymbol{R}^2$, we set $\Omega=\boldsymbol{R}^2\backslash\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N\}$. To describe the universal covering space $\tilde{\Omega}$ and the fundamental group $\pi_1(\Omega)$ of Ω, we will fix some notations, which are used throughout this paper.

The simple symbols $\gamma, \nu, \cdots$ are used for general (oriented) continuous paths in Ω which are the ranges of continuous mappings of $[0, 1]$ into Ω. The product $\gamma\circ\nu$ and the inverse γ^{-1} are defined as usual. And $[\gamma], [\nu], \cdots$ denote the homotopy equivalence classes (end points fixed) of the paths $\gamma, \nu, \cdots$. For an arbitrarily fixed point $\boldsymbol{r}_0$ in Ω, the symbols $\gamma^r, \nu^r, \cdots$ denote continuous paths in Ω with the initial point $\boldsymbol{r}_0$ and the final point $\boldsymbol{r}\in\Omega$. In particular, continuous loops starting from $\boldsymbol{r}_0$ are denoted by $\gamma^0, \nu^0, \cdots$. Then the universal covering space $\tilde{\Omega}$ and the fundamental group $\pi_1(\Omega)$ are given by

$$\tilde{\Omega}=\{[\gamma^r]\,;\ \boldsymbol{r}\in\Omega,\ \gamma^r\subset\Omega\}, \tag{2.12}$$

$$\pi_1(\Omega)=\{[\gamma^0]\,;\ \gamma^0\subset\Omega\}, \tag{2.13}$$

where the product and the inverse in $\pi_1(\Omega)$ are induced by those for continuous paths and $\tilde{\Omega}$ is equipped with the local coordinate (x, y) and the local Lebesgue measure $\boldsymbol{\mu}$ coming from those of Ω. Denote by γ_j^0 $(j=1, \cdots, N)$ a continuous loop which is homotopic (base point free) to the continuous loop γ_j^ε for sufficiently small $\varepsilon>0$ (c.f. (2.5)), then $\pi_1(\Omega)$ is the free group with the N-generators $[\gamma_j^0]$ $(j=1, \cdots, N)$. We fix the loops γ_j^0 and use those throughout this paper.

Let φ be any one-dimensional representation of $\pi_1(\Omega)$, that is, group homomorphism of $\pi_1(\Omega)$ into $S^1=\{z\in\boldsymbol{C}\,;\ |z|=1\}$. In what follows, we define a quasi-exponentiable $*$-representation of $\mathscr{W}_2$ induced by φ.

A function f on the universal convering space $\tilde{\Omega}$ is said to be φ-invariant if f satisfies

$$f[\gamma^r\circ\gamma^0]=\varphi[\gamma^0]f[\gamma^r] \text{ for all } \gamma^0 \text{ and } \gamma^r. \tag{2.14}$$

Note that, for φ-invariant functions f and g on $\tilde{\Omega}$, the value $f[\gamma^r]\overline{g[\gamma^r]}$ at each $[\gamma^r]$ depends only on $\boldsymbol{r}=(x, y)$. Here we set

$$L^2(\tilde{\Omega}, \varphi)=\Big\{f : f \text{ is a } \varphi\text{-invariant and } \boldsymbol{\mu}\text{-measurable function on } \tilde{\Omega} \text{ such that } \int|f[\gamma^r]|^2 d\mu(\boldsymbol{r})<+\infty\Big\}, \tag{2.15}$$

$$(f, g)=\int_\Omega f[\gamma^r]\overline{g[\gamma^r]}d\mu(\boldsymbol{r}) \text{ for } f, g\in L^2(\tilde{\Omega}, \varphi),$$

$$C_0^\infty(\tilde{\Omega}, \varphi)=\{f\in C^\infty(\tilde{\Omega})\,;\ f \text{ is } \varphi\text{-invariant and } \rho(\operatorname{supp} f) \text{ is compact in } \Omega\}, \tag{2.16}$$

where μ is the Lebesgue measure on $\boldsymbol{R}^2$ and $\rho : \widetilde{\Omega} \longrightarrow \Omega$ denotes the projection with $\rho[\gamma^r]=\boldsymbol{r}$. Then, $L^2(\widetilde{\Omega}, \varphi)$ is a Hilbert space with the inner product $(\ , \ )$ and $C_0^\infty(\widetilde{\Omega}, \varphi)$ is a dense subspace of $L^2(\widetilde{\Omega}, \varphi)$. Set

$$\begin{gathered} P_1=-iD_x,\ P_2=-iD_y, \\ Q_1=x,\ Q_2=y, \\ \mathscr{D}=\mathscr{D}(P_j)=\mathscr{D}(Q_j)=C_0^\infty(\widetilde{\Omega}, \varphi) \qquad (j=1, 2). \end{gathered} \tag{2.17}$$

Then, we easily observe that P_j and Q_j $(j=1, 2)$ are symmetric operators with the common invariant domain $\mathscr{D}$ satisfying the CCR, so that we have a $*$-representation π_φ of $\mathscr{W}_2$ by

$$\pi_\varphi(p_j)=P_j \text{ and } \pi_\varphi(q_j)=Q_j \qquad (j=1, 2). \tag{2.18}$$

Remark that, if φ is trivial, $L^2(\widetilde{\Omega}, \varphi)$ (resp. $C_0^\infty(\widetilde{\Omega}, \varphi)$) is identified with $L^2(\Omega)$ (resp. $C_0^\infty(\Omega)$), and then π_φ is equivalent to the restriction of the Schrödinger representation to $C_0^\infty(\Omega)$. As for general φ, we also remark that $L^2(\widetilde{\Omega}, \varphi)$ and $C_0^\infty(\widetilde{\Omega}, \varphi)$ are non-trivial. In fact, for any $\boldsymbol{s}\in\Omega$ and $[\gamma^{\boldsymbol{s}}]\in\widetilde{\Omega}$, take connected open neighborhoods U and $\widetilde{U}$ of $\boldsymbol{s}$ and $[\gamma^{\boldsymbol{s}}]$, respectively, such that the projection ρ maps $\widetilde{U}$ onto U homeomorphically. Then, for $h\in C_0^\infty(\Omega)$ with supp $h\subset U$, we get a function $f\in C_0^\infty(\widetilde{\Omega}, \varphi)$ with $f[\gamma^r]=h(\boldsymbol{r})$ for $[\gamma^r]\in\widetilde{U}$ by setting

$$f[\gamma^r]=\begin{cases} \varphi[\gamma^0]^{-1}h(\boldsymbol{r}) & \text{if } [\gamma^r\circ\gamma^0]\in\widetilde{U} \text{ for some } \gamma^0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.4. *Let φ be a one-dimensional representation of the fundamental group $\pi_1(\Omega)$ of Ω and π_φ the $*$-representation of $\mathscr{W}_2$ defined by* (2.18), *then*

(1) *π_φ is quasi-exponentiable, that is, $\pi_\varphi(p_j)=P_j$ and $\pi_\varphi(q_j)=Q_j$ are ess. self-adjoint operators in $L^2(\widetilde{\Omega}, \varphi)$ and*

(2) *the following conditions are equivalent* :
- (i) *The $*$-representation π_φ of $\mathscr{W}_2$ is exponentiable.*
- (ii) *The $*$-representation π_φ of $\mathscr{P}_2=\mathscr{P}(p_1, p_2)$ is exponentiable, that is, $\overline{\pi_\varphi(p_1)}=\overline{P_1}$ and $\overline{\pi_\varphi(p_2)}=\overline{P_2}$ are strongly commuting.*
- (iii) *φ is trivial.*

Remark 2.5. In the next section, we will show that π_φ is unitarily equivalent to π_A defined in §2.1 under a correspondence between φ and A. By the unitary equivalence, the above theorem follows from Lemma 2.1 and Theorem 2.2 for π_A. But Theorem 2.4 has its own interest. Here we give the proof without using the unitary equivalence.

Proof. The first assertion follows from the same discussion as in [RS] Example 1 (p. 273). We note that the self-adjoint operators $\overline{\pi(p_j)}=\overline{P_j}$ $(j=1, 2)$ generate the 1-parameter unitary groups on $L^2(\tilde{\Omega}, \varphi)$ of the translations on the sheet of $\tilde{\Omega}$ along the x-axis (for $j=1$) and y-axis (for $j=2$), while $\overline{\pi(q_j)}=\overline{Q_j}$ $(j=1, 2)$ generate those of the multiplications by e^{isx} (for $j=1$) and e^{isy} (for $j=2$).

We will show the second part of the theorem. The equivalence of (i) and (ii) easily follows from the above statement. To show the equivalence of (ii) and (iii), we use the following graphic notations:

$$\underset{s}{\lrcorner}\!\!\uparrow t\circ\gamma^r,\quad t\overset{s}{\ulcorner\!\!\rightarrow}\circ\gamma^r,\quad \underset{s}{\downarrow\!\square} t\circ\gamma^r, \text{ etc.}$$

For example, the first notation denotes the continuous path;

$$\boldsymbol{r}_0\overset{\gamma^r}{\rightsquigarrow}\boldsymbol{r}=(x, y)\rightarrow(x+s, y)\rightarrow(x+s, y+t).$$

Under these notations, we have, for $f\in L^2(\tilde{\Omega}, \varphi)$ and $s, t\in\boldsymbol{R}$,

$$\begin{aligned}(e^{is\overline{P_1}}e^{it\overline{P_2}}f)[\gamma^r]&=f[\underset{s}{\lrcorner}\!\!\uparrow t\circ\gamma^r]\\ &=f[(t\overset{s}{\ulcorner\!\!\rightarrow}\circ\gamma^r)\circ((\gamma^r)^{-1}\circ\underset{s}{\downarrow\!\square} t\circ\gamma^r)]\\ &=\varphi[(\gamma\ \ )^{-1}\circ\underset{s}{\downarrow\!\square} t\circ\gamma^r]f[t\overset{s}{\ulcorner\!\!\rightarrow}\circ\gamma^r]\\ &=\varphi[(\gamma^r)^{-1}\circ\underset{s}{\downarrow\!\square} t\circ\gamma^r]e^{it\overline{P_2}}e^{is\overline{P_1}}f)[\gamma^r]\\ &\quad\text{for almost all } [\gamma^r]\in\tilde{\Omega},\end{aligned}$$

and

$$\begin{aligned}&(e^{is\overline{P_1}}e^{it\overline{P_2}}e^{-is\overline{P_1}}e^{-it\overline{P_2}}f-f)[\gamma^r]\\ &\qquad=(\varphi[(\gamma^r)^{-1}\circ\underset{s}{\downarrow\!\square} t\circ\gamma^r]-1)f[\gamma^r] \qquad (2.19)\\ &\qquad\text{for almost all } [\gamma^r]\in\tilde{\Omega}.\end{aligned}$$

Thus the triviality of φ implies that $e^{is\overline{P_1}}$ and $e^{it\overline{P_2}}$ are commuting.

To show the converse implication, for each $j=1, \cdots, N$, we take a path γ^r, $s, t\in\boldsymbol{R}$, and $f\in C_0^\infty(\tilde{\Omega}, \varphi)$ such that the loop $(\gamma^r)^{-1}\circ\underset{s}{\downarrow\!\square} t\circ\gamma^r$ is homotopic to the generator γ_j^0 of $\pi_1(\Omega)$ and that $f[\gamma^r]\neq 0$. Then, if we assume the condition (ii), by using (2.19), we have $\varphi[\gamma_j^0]=1$. Thus φ is trivial. □

Remark 2.6. When the space Ω is given by $\Omega=\boldsymbol{R}^2\backslash\{\boldsymbol{a}_1\}$ and the one-dimensional representation φ of $\pi_1(\Omega)$ is given by $\varphi[\gamma_1^0]=-1$ for the single generator $[\gamma_1^0]$ of $\pi_1(\Omega)$, the $*$-representation π_φ of $\mathscr{W}_2$ coinsides with π_- stated in introducing the Nelson's example.

Schmüdgen ([S4], Example 9.4.7) defined a class of $*$-representation of $\mathscr{P}_2$ depending on the parameter c in S^1, which include π_- (for the case $c=-1$). Those are essentially same as our π_φ of $\mathscr{P}_2$, where $\Omega=\boldsymbol{R}^2\backslash\{\boldsymbol{a}_1\}$ and $\varphi[\gamma_1^0]=c$.

§3. Unitary Equivalence and Other Properties of the $*$-Representations

3.1. Unitary Equivalence

In §2, for previously fixed N points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_N$ in $\boldsymbol{R}^2$, we discussed the two classes of quasi-exponentiable $*$-representations of the Weyl algebra $\mathscr{W}_2$, which are constructed by quite different ways. A $*$-representation π_A in the first class is based on a vector potential $\boldsymbol{A}$ on Ω and π_φ in the second class on a one-dimensional representation φ of the fundamental group $\pi_1(\Omega)$. In this subsection we will first show that the two $*$-representations π_A and π_φ are unitarily equivalent under a correspondence between $\boldsymbol{A}$ and φ.

For a vector potential $\boldsymbol{A}$ on Ω and a sufficiently smooth (e.g. piecewise C^1-class) path γ in Ω, we set

$$\Phi_A(\gamma)=\int_\gamma \boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{r}. \tag{3.1}$$

Note that we can take a sufficiently smooth path in each homotopy equivalence class (end points fixed) of continuous paths in Ω and that the real value $\Phi_A(\gamma)$ only depends on the homotopy class of γ by the Green's Theorem. Thus we get a S^1-valued function φ_A on the set of all homotopy equivalence classes (end points fixed) of continuous paths in Ω by

$$\varphi_A[\gamma]=\exp(-i\Phi_A(\gamma)). \tag{3.2}$$

Concerning the product $\gamma\circ\nu$ and the inverse γ^{-1} for continuous paths γ and ν, we easily have

$$\varphi_A[\gamma\circ\nu]=\varphi_A[\gamma]\varphi_A[\nu] \text{ and } \varphi_A[\gamma^{-1}]=\varphi_A[\gamma]^{-1}. \tag{3.3}$$

In what follows, we use the same notation φ_A for the restrictions $\varphi_A|_{\pi_1(\Omega)}$ and $\varphi_A|_{\tilde{\Omega}}$ (c.f. (2.12) and (2.13)).

Lemma 3.1. *For any vector potential $\boldsymbol{A}$ on Ω, the function φ_A gives a one-dimensional representation of $\pi_1(\Omega)$. Conversely, every one-dimensional representation φ of $\pi_1(\Omega)$ is given by such a way for some vector potential $\boldsymbol{A}$*

on Ω.

Proof. For a vector potential A on Ω, it follows from (3.3) that the function φ_A gives a one-dimensional representation of $\pi_1(\Omega)$.

For a one-dimensional representation φ of $\pi_1(\Omega)$ and the generators γ_j^0 ($j=1, \cdots, N$) of $\pi_1(\Omega)$ defined in §2.2, take real numbers c_j ($j=1, \cdots, N$) satisfying

$$\varphi[\gamma_j^0]=e^{-ic_j}, \tag{3.4}$$

and set

$$A_1(\boldsymbol{r})=-\sum_{j=1}^{N}\frac{c_j}{2\pi}\frac{y-a_{j2}}{|\boldsymbol{r}-\boldsymbol{a}_j|^2} \text{ and } A_2(\boldsymbol{r})=\sum_{j=1}^{N}\frac{c_j}{2\pi}\frac{x-a_{j1}}{|\boldsymbol{r}-\boldsymbol{a}_j|^2} \tag{3.5}$$

for $\boldsymbol{r}\in\Omega$. Then, for the vector potential $A=(A_1, A_2)$, we have

$$\Phi_A(\gamma_j^0)=\oint_{\gamma_j^0} A(\boldsymbol{r})\cdot d\boldsymbol{r}=c_j \qquad \text{and} \tag{3.6}$$

$$\varphi[\gamma_j^0]=\varphi_A[\gamma_j^0]=e^{-ic_j} \text{ for } j=1, \cdots, N \tag{3.7}$$

(c.f. [A1], [A2]). Thus φ agrees with the representation φ_A of $\pi_1(\Omega)$ induced by the vector potential A. □

Theorem 3.2. *Let A be a vector potential on Ω and set $\varphi=\varphi_A$. Then the $*$-representations π_A and π_φ of the Weyl algebra $\mathcal{W}_2$ are unitarily equivalent, that is, there exists a unitary operator $V: L^2(\Omega)\to L^2(\tilde{\Omega}, \varphi)$ such that*

$$VC_0^\infty(\Omega)=C_0^\infty(\tilde{\Omega}, \varphi) \quad \textit{and}$$
$$V\pi_A(a)V^*=\pi_\varphi(a) \textit{ for } a\in\mathcal{W}_2.$$

Proof. We first remark that the S^1-valued function φ_A on $\tilde{\Omega}$ is in C^∞-class. For $f\in L^2(\Omega)$, we can define a measurable function Vf on $\tilde{\Omega}$ by

$$(Vf)[\gamma^r]=\varphi_A[\gamma^r]f(\boldsymbol{r}) \text{ for } [\gamma^r]\in\tilde{\Omega}. \tag{3.8}$$

Then we have, for any loop γ^0 starting from $\boldsymbol{r}_0$,

$$\begin{aligned}(Vf)[\gamma^r\circ\gamma^0]&=\varphi_A[\gamma^r\circ\gamma^0]f(\boldsymbol{r})\\&=\varphi_A[\gamma^r]\varphi_A[\gamma^0]f(\boldsymbol{r})\\&=\varphi[\gamma^0](Vf)[\gamma^r],\end{aligned}$$

so that the function Vf on $\tilde{\Omega}$ is φ-invariant. Furthermore we have

$$\int_\Omega |(Vf)[\gamma^r]|^2 d\mu(\boldsymbol{r})=\int_\Omega |f(\boldsymbol{r})|^2 d\mu(\boldsymbol{r})<+\infty.$$

Thus the function Vf is in $L^2(\tilde{\Omega}, \varphi)$ and we can define an isometry

$$V : L^2(\Omega)\ni f\longrightarrow Vf\in L^2(\tilde{\Omega},\ \varphi).$$

We can easily observe that V is a unitary operator. Remark that the discussion also implies $VC_0^\infty(\Omega)=C_0^\infty(\tilde{\Omega},\ \varphi)$. Further we have, for $f\in C_0^\infty(\Omega)$,

$$\begin{aligned}(\pi_\varphi(p_1)Vf)[\gamma^r]&=-i(D_x\varphi_A)[\gamma^r]f(\boldsymbol{r})-i\varphi_A[\gamma^r](D_xf)(\boldsymbol{r})\\&=-A_1(\boldsymbol{r})\varphi_A[\gamma^r]f(\boldsymbol{r})-i\varphi_A[\gamma^r](D_xf)(\boldsymbol{r})\\&=(V\pi_A(p_1)f)[\gamma^r]\end{aligned}$$

and

$$\begin{aligned}(\pi_\varphi(q_1)Vf)[\gamma^r]&=x\varphi[\gamma^r]f(\boldsymbol{r})\\&=\varphi[\gamma^r](\pi_A(q_1)f)(\boldsymbol{r})\\&=(V\pi_A(q_1)f)[\gamma^r]\end{aligned}$$

for $\boldsymbol{r}=(x,\ y)\in\Omega$ and $[\gamma^r]\in\tilde{\Omega}$.

The same equalities also hold for p_2 and q_2. Since p_j, q_j $(j=1,\ 2)$ are the generators of $\mathcal{W}_2$, we obtain the equality

$$V\pi_A(a)V^*=\pi_\varphi(a)\qquad\text{for } a\in\mathcal{W}_2. \tag{3.9}$$

□

Corollary 3.3. (1) *Let A and A' be vector potentials on Ω. Then the $*$-representations π_A and $\pi_{A'}$ of the Weyl algebra $\mathcal{W}_2$ are unitarily equivalent if and only if $\varphi_A=\varphi_{A'}$.*
(2) *Let φ and φ' be one-dimensional representations of $\pi_1(\Omega)$. Then the $*$-representations π_φ and $\pi_{\varphi'}$ of $\mathcal{W}_2$ are unitarily equivalent if and only if $\varphi=\varphi'$.*

Proof. (1) Assume that $\varphi_A=\varphi_{A'}$, then, by Theorem 3.2, we have

$$\pi_A\sim\pi_{\varphi_A}=\pi_{\varphi_A}\sim\pi_{A'},$$

where the notation $\sim$ denotes the unitary equivalence relation.

Conversely, assume that π_A and $\pi_{A'}$ are unitarily equivalent and denote by W the unitary operator on $L^2(\Omega)\simeq L^2(\boldsymbol{R}^2)$ such that

$$W\pi_A(p_j)W^*=\pi_{A'}(p_j),\ W\pi_A(q_j)W^*=\pi_{A'}(q_j)$$

for $j=1$, 2. Since π_A and $\pi_{A'}$ are quasi-exponentiable, we have

$$W\exp(is\overline{\pi_A(p_j)})W^*=\exp(is\overline{\pi_{A'}(p_j)}), \tag{3.10}$$

$$W\exp(is\overline{\pi_A(q_j)})W^*=\exp(is\overline{\pi_{A'}(q_j)}), \tag{3.11}$$

for $j=1$, 2 and $s\in\boldsymbol{R}$. Remark that $\exp(is\overline{\pi_A(q_j)})=\exp(is\overline{\pi_{A'}(q_j)})(j=1,\ 2)$ are the multiplication operators of e^{isx} (for $j=1$) and e^{isy} (for $j=2$) and that those generate the maximal abelian von Neumann algebra of all multiplication operators of functions in $L^\infty(\boldsymbol{R}^2)$. The equality (3.11) implies that the unitary operator W is a multiplication of a function $w \in L^\infty(\boldsymbol{R}^2)$. For $s, t \in \boldsymbol{R}$, define a function $\varphi_{s,t}$ (resp. $\varphi'_{s,t}$) in $L^\infty(\boldsymbol{R}^2)$ by

$$\varphi_{s,t}(x, y) = \varphi_A[\gamma(x, y; s, t)] \tag{3.12}$$

$$(\text{resp. } \varphi'_{s,t}(x, y) = \varphi_{A'}[\gamma(x, y; s, t)]),$$

where $\gamma(x, y; s, t)$ denotes the rectangular loop;

$$(x, y) \longrightarrow (x+s, y) \longrightarrow (x+s, y+t) \longrightarrow (x, y+t) \longrightarrow (x, y). \tag{3.13}$$

Then, by using (3.10), (3.12), and Lemma 2.1 we have

$$\begin{aligned}\varphi'_{s,t} &= e^{is\overline{\pi_{A'}(p_1)}} e^{it\overline{\pi_{A'}(p_2)}} e^{-is\overline{\pi_{A'}(p_1)}} e^{-it\overline{\pi_{A'}(p_2)}} \\ &= W e^{is\overline{\pi_A(p_1)}} e^{it\overline{\pi_A(p_2)}} e^{-is\overline{\pi_A(p_1)}} e^{-it\overline{\pi_A(p_2)}} W^* \\ &= w\varphi_{s,t}\overline{w} = \varphi_{s,t}\end{aligned} \tag{3.14}$$

for all $s, t \in \boldsymbol{R}$. For each singular point $\boldsymbol{a}_j$, take $(x, y) \in \Omega$ and $s, t > 0$ such that the only singular point $\boldsymbol{a}_j$ is surrounded by the rectangular loop $\gamma(x, y; s, t)$ in Ω, then we have

$$\varphi_A[\gamma_j^0] = \varphi_{s,t}(x, y) = \varphi'_{s,t}(x, y) = \varphi_{A'}[\gamma_j^0].$$

Hence we get $\varphi_A = \varphi_{A'}$ (as one-dimensional representations of $\pi_1(\Omega)$).
(2) For φ and φ', take vector potentials $\boldsymbol{A}$ and $\boldsymbol{A}'$ on Ω such that $\varphi = \varphi_A$ and $\varphi' = \varphi_{A'}$ (c.f. Lemma 3.1). Then the assertion (2) follows from Theorem 3.2 and (1) of this corollary. □

Corollary 3.4. (1) *Let $\boldsymbol{A} = (A_1, A_2)$ be a vector potential on Ω satisfying $\varphi_A[\gamma_N^0] = 1$. Then we can take a vector potential $\boldsymbol{A}' = (A_1', A_2')$ on $\Omega' = \boldsymbol{R}^2 \setminus \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{N-1}\}$ such that $\pi_A \sim \pi_{A'}|_{C_0^\infty(\Omega)}$.*
(2) *Let Ω and Ω' be as in (1), and φ and φ' be one-dimensional representations of $\pi_1(\Omega)$ and $\pi_1(\Omega')$, respectively, satisfying $\varphi[\gamma_N^0] = 1$ and $\varphi' = \varphi|_{\pi_1(\Omega')}$. Then there exists a subspace $\mathcal{M}$ of $C_0^\infty(\widetilde{\Omega}', \varphi')$ such that $\mathcal{M}$ is dense in $L^2(\widetilde{\Omega}', \varphi')$ and $\pi_\varphi \sim \pi_{\varphi'}|_{\mathcal{M}}$.*

Proof. (1) We set

$$c_j = \Phi_A(\gamma_j^0) = \oint_{\gamma_j^0} \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r} \qquad (j = 1, \cdots, N-1),$$

$$A_1'(\boldsymbol{r}) = -\sum_{j=1}^{N-1} \frac{c_j}{2\pi} \frac{y - a_{j2}}{|\boldsymbol{r} - \boldsymbol{a}_j|^2}, \quad A_2'(\boldsymbol{r}) = \sum_{j=1}^{N-1} \frac{c_j}{2\pi} \frac{x - a_{j1}}{|\boldsymbol{r} - \boldsymbol{a}_j|^2},$$

$$(\boldsymbol{r} = (x, y) \in \Omega'),$$

$$\boldsymbol{A}' = (A_1', A_2'), \text{ and } \boldsymbol{A}'' = \boldsymbol{A}'|_\Omega.$$

Then $\boldsymbol{A}'$ and $\boldsymbol{A}''$ are vector potentials on Ω' and Ω, respectively. It follows from the definition of $\pi_{A'}$ and $\pi_{A''}$ that $\pi_{A''}=\pi_{A'}|_{C_0^\infty(\Omega)}$. On the other hand, since the equalities

$$\varphi_{A''}[\gamma_j^0]=e^{-ic_j}=\varphi_A[\gamma_j^0]\ (j=1,\cdots,N-1) \text{ and}$$
$$\varphi_{A''}[\gamma_N^0]=1=\varphi_A[\gamma_N^0]$$

hold, the assertion (1) of Corollary 3.3 implies $\pi_A\sim\pi_{A''}$. Combining these, we get

$$\pi_A\sim\pi_{A''}=\pi_{A'}|_{C_0^\infty(\Omega)}.$$

(2) For a one-dimensional representation φ of $\pi_1(\Omega)$ with $\varphi[\gamma_N^0]=1$, we take a vector potential $\boldsymbol{A}$ with $\varphi_A=\varphi$ by Lemma 3.1. Further take a vector potential $\boldsymbol{A}'$ on Ω' as in the proof of (1), then we have $\varphi_{A'}=\varphi'$ and $\pi_A\sim\pi_{A'}|_{C_0^\infty(\Omega)}$. Thus we get the assertion (2) by Theorem 3.2. □

Remark 3.5. In Corollary 3.4 (1), we denote W the unitary operator satisfying

$$W\pi_A(a)W^*=\pi_{A'}(a)|_{C_0^\infty(\Omega)}\quad a\in\mathscr{W}_2.$$

It follows that $\pi_{A'}(p_j)|_{C_0^\infty(\Omega)}$ and $\pi_{A'}(q_j)|_{C_0^\infty(\Omega)}$ are ess. self-adjoint operators and

$$W\exp(is\overline{\pi_A(p_j)})W^*=\exp(is\overline{\pi_{A'}(p_j)}),$$
$$W\exp(is\overline{\pi_A(q_j)})W^*=\exp(is\overline{\pi_{A'}(q_j)})$$
$$\text{for } j=1,2 \text{ and } s\in\boldsymbol{R}.$$

Thus we can say that π_A and $\pi_{A'}$ in Corollary 3.4 (1) are unitarily equivalent at the level of unitary operators generated by $\{\pi_A(p_j), \pi_A(q_j)\}$ and $\{\pi_{A'}(p_j), \pi_{A'}(q_j)\}$.

Definition 3.6. For a vector potential $\boldsymbol{A}$ on Ω, if $\varphi_A[\gamma_j^0]=1$ for some j, we can remove the singular point $\boldsymbol{a}_j$ of $\boldsymbol{A}$ in the sence of Corollary 3.4 (1) and Remark 3.5. Thus we will say that a singular point $\mathbf{a}_j$ is removable if $\varphi_A[\gamma_j^0]=1$ and essentially singular otherwise. We denote the set of essentially singular points of $\boldsymbol{A}$ by S_A.

For a subset S of $\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_N\}$, set $\Omega'=\boldsymbol{R}^2\backslash S$. Then $\{\gamma_j^0;\ \boldsymbol{a}_j\in S\}$ is a set of generators of $\pi_1(\Omega')$, where γ_j^0 $(j=1,\cdots,N)$ are the generators of $\pi_1(\Omega)$ defined in §2.2. Thus we can naturally consider $\pi_1(\Omega')$ as a subgroup of $\pi_1(\Omega)$. For a vector potential $\boldsymbol{A}$ on Ω, the restriction $\varphi_A|_{\pi_1(\Omega')}$ plays an important role as in the following corollary.

Corollary 3.7. *Let* $\boldsymbol{A}_1$ *(resp.* $\boldsymbol{A}_2$*) be a vector potential on* $\Omega_1\subset\boldsymbol{R}^2$ *(resp.* Ω_2

$\subset \mathbb{R}^2$) with finite singular points. Then the $*$-representations π_{A_1} and π_{A_2} of the Weyl algebra $\mathcal{W}_2$ are unitarily equivalent at the level of unitaries generated by $\{\overline{\pi_{A_1}(p_j)}, \overline{\pi_{A_1}(q_j)}\}$ and $\{\overline{\pi_{A_2}(p_j)}, \overline{\pi_{A_2}(q_j)}\}$ if and only if the following conditions are satisfied:

$$\text{(i)} \quad S_{A_1} = S_{A_2} \ (\equiv S)$$
$$\text{(ii)} \quad \varphi_{A_1}|_{\pi_1(\mathbb{R}^2 \setminus S)} = \varphi_{A_2}|_{\pi_1(\mathbb{R}^2 \setminus S)}$$

Proof. Assume that the conditions (i) and (ii) are satisfied. It follows from Corollary 3.4 and Remark 3.5 that, for $k=1, 2$, there exists a vector potential A'_k on $\mathbb{R}^2 \setminus S$ such that π_{A_k} and $\pi_{A'_k}$ are unitarily equivalent at the level of unitaries and $\varphi_{A'_1} = \varphi_{A_k}|_{\pi_1(\mathbb{R}^2 \setminus S)}$. Since $\varphi_{A'_1} = \varphi_{A'_2}$, $\pi_{A'_1}$ and $\pi_{A'_2}$ are unitarily equivalent (c.f. Corollary 3.3), and, hence, π_{A_1} and π_{A_2} are unitarily equivalent at the level of unitaries.

Conversely, assume that there exists a unitary operator W on $L^2(\mathbb{R}^2) \simeq L^2(\Omega_1) \simeq L^2(\Omega_2)$ such that

$$W \exp(is\overline{\pi_{A_1}(p_j)}) W^* = \exp(is\overline{\pi_{A_2}(p_j)}),$$
$$W \exp(is\overline{\pi_{A_1}(q_j)}) W^* = \exp(is\overline{\pi_{A_2}(q_j)}),$$

for $j=1, 2$ and $s \in \mathbb{R}$. And set, for $k=1, 2$,

$$\varphi_{s,t}^{(k)}(x, y) = \exp(-i \oint_{\gamma(x, y;\, s, t)} A_k(r) \cdot dr).$$

Then, by the same calculation as (3.14), we have $\varphi_{s,t}^{(1)} = \varphi_{s,t}^{(2)}$ for all $s, t \in \mathbb{R}$. Here we remark that, for each $k=1, 2$, the family of functions $\{\varphi_{s,t}^{(k)}\}$ determines the positions of ess. singular points for A_k and the restriction φ_{A_k} to $\pi_1(\mathbb{R}^2 \setminus S_{A_k})$. This fact and the above equalities lead us to the assertions (i) and (ii). □

3.2. Irreducibility

To consider irreducibility of the $*$-representations of $\mathcal{W}_2$ or $\mathcal{P}_2$ given in §2, we first discuss their commutants.

For a $*$-representation $(\pi, \mathcal{D})$ of a $*$-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ (c. f. (1.2)), we often use the two commutants of $\pi(\mathcal{A})$ as follows:

$$\pi(\mathcal{A})'_{\mathrm{s}} = \left\{ T \in \mathcal{B}(\mathcal{H})\,;\ \begin{array}{l} T\mathcal{D} \subset \mathcal{D} \text{ and } \pi(a)T = T\pi(a) \\ \text{for all } a \in \mathcal{A} \end{array} \right\}, \tag{3.15}$$

$$\pi(\mathcal{A})'_{\mathrm{w}} = \left\{ T \in \mathcal{B}(\mathcal{H})\,;\ \begin{array}{l} (T\pi(a)\xi, \eta) = (T\xi, \pi(a^*)\eta) \\ \text{for all } \xi, \eta \in \mathcal{D} \text{ and all } a \in \mathcal{A} \end{array} \right\}, \tag{3.16}$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. Those are respectively called strong commutant and weak commutant of $\pi(\mathcal{A})$ and satisfy

$$\pi(\mathscr{A})'_{\mathrm{s}} \subset \pi(\mathscr{A})'_{\mathrm{w}}. \tag{3.17}$$

For the case that π is a $*$-representation of $\mathscr{A}=\mathscr{W}_2$ given in §2, since $\pi(p_j)$ and $\pi(q_j)$ are ess. self-adjoint, we can define another commutant C_π by

$$C_\pi=\{\exp(is\overline{\pi(p_j)}),\ \exp(is\overline{\pi(q_j)})\ ;\ s\in\boldsymbol{R},\ j=1,\ 2\}'. \tag{3.18}$$

Then we easily have

$$C_\pi \subset \pi(\mathscr{W}_2)'_{\mathrm{w}}. \tag{3.19}$$

By the following theorem, for each $*$-representation π of $\mathscr{W}_2$ given in §2, we will show the strong irreducibility $\pi(\mathscr{W}_2)'_{\mathrm{w}}=\boldsymbol{C}1$ (c.f. (3.17) and (3.19)).

Theorem 3.8. *For each $*$-representation π of $\mathscr{W}_2$ constructed in* §2, *we have $\pi(\mathscr{W}_2)'_{\mathrm{w}}=\boldsymbol{C}1$.*

Proof. By Theorem 3.2, we may set $\pi=\pi_{\boldsymbol{A}}$ for a vector potential $\boldsymbol{A}$ on Ω.

Let T be any operator in $\pi(\mathscr{W}_2)'_{\mathrm{w}}$, then T is weakly commuting with $\pi(q_j)$, that is,

$$(T\pi(q_j)f,\ g)=(Tf,\ \pi(q_j)g) \text{ for } f,\ g\in C_0^\infty(\Omega) \text{ and } j=1,\ 2.$$

Since $\pi(q_j)$ is ess. self-adjoint and T is bounded, we get

$$T\overline{\pi(q_j)}=\overline{\pi(q_j)}\,T \text{ on } \mathscr{D}(\overline{\pi(q_j)}) \text{ for } j=1,\ 2,$$

and so,

$$Te^{is\overline{\pi(q_j)}}=e^{is\overline{\pi(q_j)}}\,T \text{ for } j=1,\ 2, \text{ and } s\in\boldsymbol{R}.$$

Recall that $e^{is\overline{\pi(q_j)}}$ $(j=1,\ 2)$ are the multiplication operators by e^{isx} for $j=1$ and e^{isy} for $j=2$, and that they generate the maximal abelian von Neumann algebra which consists of all multiplication operators by functions in $L^\infty(\Omega)$. Thus there exists a function $\varphi\in L^\infty(\Omega)$ such that

$$Tf=\varphi f \text{ for } f\in L^2(\Omega).$$

Since $T=\varphi$ weakly commutes with $\pi(p_j)$ $(j=1,\ 2)$ on $\mathscr{D}$, it also weakly commutes with $-iD_x$ and $-iD_y$ on $\mathscr{D}$. Noting that $-iD_x$ and $-iD_y$ are ess. self-adjoint and that $-i\overline{D_x}$ and $-i\overline{D_y}$ respectively generate the translations along the x-axis and y-axis, we can conclude that $T=\varphi$ commutes with those translations, so that φ is a constant almost everywhere. Thus we have $T=\lambda I$ for some $\lambda\in\boldsymbol{C}$. □

When the representation π of $\mathscr{W}_2$ given in §2 is not exponentiable, we can get a stronger result for its irreducibility.

Theorem 3.9. *Assume that a $*$-representation π of $\mathscr{W}_2$ in §2 is not exponentiable, then the restriction of π to $\mathscr{P}_2=\mathscr{P}(p_1, p_2)$ is irreducible, that is,* $\pi(\mathscr{P}_2)'_{\mathrm{w}}=\boldsymbol{C}I$.

Proof. By the same discussion as in Theorem 3.8, we have

$$\begin{aligned}\pi(\mathscr{P}_2)'_{\mathrm{w}}&\subset\{\pi(p_1),\ \pi(p_2)\}'_{\mathrm{w}}\subset\{\overline{\pi(p_1)},\ \overline{\pi(p_2)}\}'_{\mathrm{s}}\\&\subset\{\exp(is\overline{\pi(p_1)}),\ \exp(is\overline{\pi(p_2)}\ ;\ s\in\boldsymbol{R}\}',\end{aligned}$$

where $\{\cdot\}'_{\mathrm{s}}$ (resp. $\{\cdot\}'_{\mathrm{w}}$) denotes the strong (resp. weak) commutant defined by the same way as (3.15) (resp. (3.16)). We will show that

$$\{\exp(is\overline{\pi(p_1)}),\ \exp(is\overline{\pi(p_2)})\ ;\ s\in\boldsymbol{R}\}'=\boldsymbol{C}I.$$

By Theorem 3.2 and Corollary 3.3, we may set $\pi=\pi_{\boldsymbol{A}}$, where the vector potential $\boldsymbol{A}=(A_1, A_2)$ on Ω is of the form (3.5). To show the above equality, by Remark 3.5, we may assume that each $\boldsymbol{a}_j$ is ess. singular, that is, $c_j\notin 2\pi\boldsymbol{Z}$ ($j=1, \cdots, N$). Here we set

$$A_1^{(j)}(\boldsymbol{r})=-\frac{c_j}{2\pi}\frac{y-a_{j2}}{|\boldsymbol{r}-\boldsymbol{a}_j|^2},\quad A_2^{(j)}(\boldsymbol{r})=\frac{c_j}{2\pi}\frac{x-a_{j1}}{|\boldsymbol{r}-\boldsymbol{a}_j|^2},\quad \boldsymbol{A}^{(j)}=(A_1^{(j)},\ A_2^{(j)}).$$

Then, we have

$$\varphi_{s,t}=\exp\left(-i\oint_{\gamma(x,\ y,\ s,\ t)}\boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{r}\right)=\prod_{j=1}^{N}\varphi_{s,t}^{(j)},$$

where we put

$$\varphi_{s,t}^{(j)}(x,\ y)=\exp\left(-i\oint_{\gamma(x,\ y,\ s,\ t)}\boldsymbol{A}^{(j)}(\boldsymbol{r})\cdot d\boldsymbol{r}\right)\ (j=1,\ \cdots,\ N)$$

(c.f. (3.12) and (3.13)). Remark that

$$\varphi_{s,t}^{(j)}(x,\ y)=\begin{cases}e^{ic_j}\text{ or }e^{-ic_j} & \text{if }(x,\ y)\text{ is surrounded by }\gamma(a_{j1},\ a_{j2}\ ;\ -s,\ -t)\\ 1 & \text{otherwise.}\end{cases}$$

By $e^{\pm ic_j}\neq 1$ ($j=1, \cdots, N$) and a slightly tedious consideration, we observe that the family of functions $\{\varphi_{s,t}\}_{s,\ t\in\boldsymbol{R}}$ on Ω separates sufficiently small neighborhoods of any two points in Ω, so that, we have

$$\{\varphi_{s,t}\ ;\ s,\ t\in\boldsymbol{R}\}''=L^\infty(\Omega)=L^\infty(\Omega)'.$$

Thus we get

$$\begin{aligned}\{e^{is\overline{\pi(p_1)}},\ e^{is\overline{\pi(p_2)}}\ ;\ s\in\boldsymbol{R}\}'&=\{e^{is\overline{\pi(p_1)}},\ e^{is\overline{\pi(p_2)}}\ ;\ s\in\boldsymbol{R}\}'\cap\{\varphi_{s,t}\ ;\ s,\ t\in\boldsymbol{R}\}'\\&=\{e^{is\overline{\pi(p_1)}},\ e^{is\overline{\pi(p_2)}}\ ;\ s\in\boldsymbol{R}\}'\cap L^\infty(\Omega)\end{aligned}$$

$$=CI,$$

where the first equality is due to the argument for (3.14) and the last equality follows from the proof of Theorem 3.8. □

3.3. Extensions

For a representation $(\pi, \mathcal{D})$ of an algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, another representation $(\tilde{\pi}, \tilde{\mathcal{D}})$ of $\mathcal{A}$ in a possibly larger Hilbert space $\mathcal{K}$ is called an extension of $(\pi, \mathcal{D})$, if $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and we have

$$\tilde{\mathcal{D}} \supset \mathcal{D}, \text{ and } \tilde{\pi}(a)|_{\mathcal{D}} = \pi(a) \text{ for } a \in \mathcal{A}, \tag{3.20}$$

and then we denote $\tilde{\pi} \supset \pi$.

In constructing the $*$-representations π_A and π_φ of the Weyl-algebra $\mathcal{W}_2$ in §2, to simplify the discussion we took $C_0^\infty(\Omega)$ and $C_0^\infty(\tilde{\Omega}, \varphi)$ as the domains of π_A and π_φ, respectively. However we have other possibilities to choose the domains of them. So it is an interesting problem how large domains of them we can take in the same Hibert spaces.

Let $(\tilde{\pi}, \tilde{\mathcal{D}})$ is a $*$-representation of $\mathcal{W}_2$ which is an extension of the $*$-representation $(\pi_\varphi, C_0^\infty(\tilde{\Omega}, \varphi))$ in the same Hilbert space, then we can easily show that $(\tilde{\pi}, \tilde{\mathcal{D}})$ is non-exponentiable iff $(\pi, \mathcal{D})$ is non-exponentiable. But, if $(\tilde{\pi}, \tilde{\mathcal{D}})$ is an extension in a larger Hilbert space, the exponentiability of $(\tilde{\pi}, \tilde{\mathcal{D}})$ is not so clear. It is our second problem in this section.

To solve the first problem we will first recall the fundamentals of the $*$-representation theory of $*$-algebras. For a $*$-representation $(\pi, \mathcal{D})$ of a $*$-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, its adjoint representation $(\pi^*, \mathcal{D}^*)$ of $\mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{D}^* &= \bigcap_{a \in \mathcal{A}} \mathcal{D}(\pi(a)^*) \text{ and} \\ \pi^*(a) &= \pi(a^*)^*|_{\mathcal{D}^*} \text{ for } a \in \mathcal{A}. \end{aligned} \tag{3.21}$$

Then π^* is an extension of π as an algebra representation but, in general, it does not preserve the involution. For any $*$-representation $\tilde{\pi}$ extending π in $\mathcal{H}$, we easily have

$$\pi \subset \tilde{\pi} \subset \tilde{\pi}^* \subset \pi^*. \tag{3.22}$$

If a $*$-representation π satisfies $\pi = \pi^*$, π is said to be self-adjoint, and then, it follows from (3.22) that π is a maximal $*$-representation of $\mathcal{A}$ in $\mathcal{H}$.

For our purpose, the following proposition is useful.

Proposition ([S4] Corollary 8.1.13) *Let $\mathcal{G}$ be a Lie algebra with the basis $\{x_1, \cdots, x_d\}$ and $\mathcal{E}(\mathcal{G})$ its enveloping algebra. If $(\pi, \mathcal{D})$ is a $*$-repre-*

sentation of $\mathscr{E}(\mathscr{G})$ in a Hilbert space $\mathscr{H}$ such that all $\pi(x_j)$'s are ess. self-adjoint, then its adjoint $(\pi^*, \mathscr{D}^*)$ is a self-adjoint $*$-representation.

For a $*$-representation π_φ of $\mathscr{W}_2$ or $\mathscr{P}_2$ in §2.2, by taking its adjoint and using the Sobolev lemma as in the argument in [S4] Example 9.4.7, we get the following proposition.

Proposition 3.10. *Let φ be a one-dimensional representation of the fundamental group $\pi_1(\Omega)$ of Ω.*
(1) *Set*

$$\widetilde{\mathscr{D}} = \left\{ f \in C^\infty(\widetilde{\Omega}, \varphi) ; \begin{array}{l} x^n y^m D_x^\alpha D_y^\beta f \in L^2(\widetilde{\Omega}, \varphi), \\ \text{for all } \alpha, \beta, m, n \in \mathbf{Z}_{\geq 0} \end{array} \right\},$$
$$\tilde{\pi}(p_1) = -iD_x, \quad \tilde{\pi}(p_2) = -iD_y$$
$$\tilde{\pi}(q_1) = x, \qquad \tilde{\pi}(q_2) = y,$$

then $(\tilde{\pi}, \widetilde{\mathscr{D}})$ is a self-adjoint $*$-representation and, hence, the maximum $*$-representation of $\mathscr{W}_2$ which extends $(\pi_\varphi, C_0^\infty(\widetilde{\Omega}, \varphi))$. In particular, $(\tilde{\pi}, \widetilde{\mathscr{D}})$ is unitarily equivalent to the Schrödinger representation if and only if φ is trivial.
(2) *Set*

$$\widetilde{\mathscr{D}} = \{ f \in C^\infty(\widetilde{\Omega}, \varphi) ; D_x^\alpha D_y^\beta f \in L^2(\widetilde{\Omega}, \varphi), \text{ for } \alpha, \beta \in \mathbf{Z}_{\geq 0} \},$$
$$\tilde{\pi}(p_1) = -iD_x, \quad \tilde{\pi}(p_2) = -iD_y,$$

then $(\tilde{\pi}, \widetilde{\mathscr{D}})$ is a self-adjoint $*$-representation and, hence, the maximum $*$-representation of $\mathscr{P}_2 = \mathscr{P}(p_1, p_2)$ which extends $(\pi_\varphi, C_0^\infty(\widetilde{\Omega}, \varphi))$.

The next theorem gives an answer to the second problem.

Theorem 3.11. *If φ is a non-trivial representation of $\pi_1(\Omega)$, then any extension $(\tilde{\pi}, \widetilde{\mathscr{D}})$ of $(\pi_\varphi, C_0^\infty(\widetilde{\Omega}, \varphi))$ as a $*$-representation in a possibly larger Hilbert space $\mathscr{K}$ is non-exponentiable.*

Proof. We assume that φ is nontrivial and $(\tilde{\pi}, \widetilde{\mathscr{D}})$ is an exponentiable extension of $\pi = \pi_\varphi$. Then, set

$$\widetilde{U}_j(s) = exp(is\,\overline{\tilde{\pi}(p_j)}) \text{ and}$$
$$U_j(s) = exp(is\,\overline{\pi(p_j)})$$

for $s \in \mathbf{R}$, $j = 1, 2$. By the assumption, for the self-adjoint operators $\overline{\tilde{\pi}(p_j)}$ in $\mathscr{K}$ and $\overline{\pi(p_j)}$ in $L^2(\widetilde{\Omega}, \varphi)$, we have

$$\overline{\tilde{\pi}(p_j)} \supset \overline{\pi(p_j)}, \qquad \text{and so,}$$
$$(1+i\lambda\overline{\tilde{\pi}(p_j)})^{-1} \supset (1+i\lambda\overline{\pi(p_j)})^{-1} \quad \text{for all } \lambda \in \boldsymbol{R}.$$

Since the unitary operators $\tilde{U}_j(s)$ and $U_j(s)$ are described by the resolvents of $\overline{\tilde{\pi}(p_j)}$ and $\overline{\pi(p_j)}$, we get

$$\tilde{U}_j(s) \supset U_j(s).$$

Since U_1 and U_2 do not commute (c.f. Theorem 2.4 (2)), thus $\tilde{U}_1$ and $\tilde{U}_2$ also do not. This contradiction completes the proof. □

§4. Defect Numbers of the $*$-Representations

For two self-adjoint operators A and B in a Hilbert space, Schmüdgen [S4] studied the non-negative integer given by

$$\dim \operatorname{Range}[(A-\alpha)^{-1}, (B-\beta)^{-1}], \tag{4.1}$$

where α and β are complex numbers in the resolvent sets $\boldsymbol{C}\backslash\sigma(A)$ and $\boldsymbol{C}\backslash\sigma(B)$, respectively. And he showed in [S4] Lemma 9.3.11 that the integer does not depend on $\alpha \in \boldsymbol{C}\backslash\sigma(A)$ and $\beta \in \boldsymbol{C}\backslash\sigma(B)$. The integer is called the defect number of the couple $\{A, B\}$ and denoted by $d(A, B)$. Since the self-adjoint operators A and B strongly commute if and only if $[(A-\alpha)^{-1}, (B-\beta)^{-1}]=0$ for all $\alpha \in \boldsymbol{C}\backslash\sigma(A)$ and $\beta \in \boldsymbol{C}\backslash\sigma(B)$, we may say that the defect number $d(A, B)$ measures the distance to the strong commutativity.

In §2 we constructed a class of quasi-exponentiable $*$-representations $\{\pi\}$ of the Weyl algebra $\mathcal{W}_2$ and we showed that the $*$-representation π is exponentiable if and only if the self-adjoint operators $\overline{\pi(p_1)}$ and $\overline{\pi(p_2)}$ strongly commutes. In this section we compute the defect number $d(\overline{\pi(p_1)}, \overline{\pi(p_2)})$ which also measures the distance to the exponentiability of π.

Theorem 4.1. *Let $\pi=\pi_A$ be a $*$-representation of the Weyl algebra $\mathcal{W}_2$ induced by a vector potential $\boldsymbol{A}$ on Ω in §2.1. Then the defect number $d(\overline{\pi(p_1)}, \overline{\pi(p_2)})$ is equal to the number of the essentially singular points of the vector potential $\boldsymbol{A}$.*

Proof. Let $\varphi=\varphi_A$ denote the S^1-valued function induced by $\boldsymbol{A}$ on the set of homotopy equivalence classes (end points fixed) of all continuous paths in Ω (c.f. §3.1). We use the same notation φ for the one-dimensional representation of $\pi_1(\Omega)$ which is given by the restriction of φ to $\pi_1(\Omega)$ (c.f. Lemma 3.1). By using Theorem 3.2, we may identify π with the $*$-representation π_φ (c.f. §2.2), for which we compute the defect number $d(\overline{\pi(p_1)}, \overline{\pi(p_2)})$.

To state the proof, we first introduce some notations. Define the two orders

$<$ and $<<$ for two points $\boldsymbol{a}=(a_1, a_2)$ and $\boldsymbol{b}=(b_1, b_2)$ in $\boldsymbol{R}^2$ by

$$\boldsymbol{a}<\boldsymbol{b} \Longleftrightarrow \begin{cases} a_2<b_2 \\ \text{or} \\ a_2=b_2,\ a_1<b_1 \end{cases} \tag{4.2}$$

$$\boldsymbol{a}<<\boldsymbol{b} \Longleftrightarrow a_1<b_1 \text{ and } a_2<b_2. \tag{4.3}$$

For the singular points $\boldsymbol{a}_j$ $(j=1, \cdots, N)$, we may assume

$$\boldsymbol{a}_1<\boldsymbol{a}_2<\cdots<\boldsymbol{a}_N \tag{4.4}$$

To express an element of $\pi_1(\Omega)$ and $\widetilde{\Omega}$ we take a base point $\boldsymbol{r}_0\in\Omega$ such that

$$\boldsymbol{r}_0<<\boldsymbol{a}_j \ (j=1, \cdots, N) \tag{4.5}$$

For each $j=0, 1, \cdots, N$, define a simply connected subspace Ω_j of Ω by

$$\Omega_j=\boldsymbol{R}^2 \Big\backslash \left(\left(\bigcup_{k=1}^{j}\{x=a_{k1},\ y\leq a_{k2}\}\right)\cup\left(\bigcup_{k=j+1}^{N}\{x=a_{k1},\ y\geq a_{k2}\}\right)\right), \tag{4.6}$$

and then, for each point $\boldsymbol{r}=(x, y)\in\Omega$ with $x\neq a_{k1}$ $(k=1, \cdots, N)$, take a continuous path $\gamma_j^{\boldsymbol{r}}$ in Ω_j with the initial point $\boldsymbol{r}_0$ and the final point $\boldsymbol{r}$. Note that the homotopy equivalence class $[\gamma_j^{\boldsymbol{r}}]$ (end points fixed) does not depend on the choice of $\gamma_j^{\boldsymbol{r}}$.

Since the resolvent $(\overline{\pi(p_j)}+i)^{-1}$ is given by

$$(\overline{\pi(p_j)}+i)^{-1}=-i\int_0^\infty e^{-s}\exp(is\,\overline{\pi(p_j)})ds, \tag{4.7}$$

for $j=1,2$, we have

$$\begin{aligned} L&\equiv[(\overline{\pi(p_1)}+i)^{-1}, (\overline{\pi(p_2)}+i)^{-1}] \\ &=-\int_0^\infty\int_0^\infty e^{-s-t}[\exp(is\,\overline{\pi(p_1)}), \exp(it\,\overline{\pi(p_2)})]dsdt, \end{aligned} \tag{4.8}$$

hence, for $f\in L^2(\widetilde{\Omega}, \varphi)$,

$$\begin{aligned} (Lf)[\gamma^{\boldsymbol{r}}]&=-\int_0^\infty\int_0^\infty e^{-s-t}(f[\underset{s}{\llcorner\!\!\uparrow} t\circ\gamma^{\boldsymbol{r}}]-f[t\,\overset{s}{\ulcorner\!\!\rightarrow}\circ\gamma^{\boldsymbol{r}}])dsdt \\ &=-e^{x+y}\int_x^\infty\int_y^\infty e^{-u-v}(f[\overset{(u,\,v)}{\llcorner\!\!\uparrow}\circ\gamma^{\boldsymbol{r}}]-f[\ulcorner\!\!\rightarrow^{(u,\,v)}\circ\gamma^{\boldsymbol{r}}])dudv, \end{aligned}$$

where we set $\boldsymbol{r}=(x, y)$ and $\boldsymbol{r}'=(u, v)=(x+s, y+t)$.

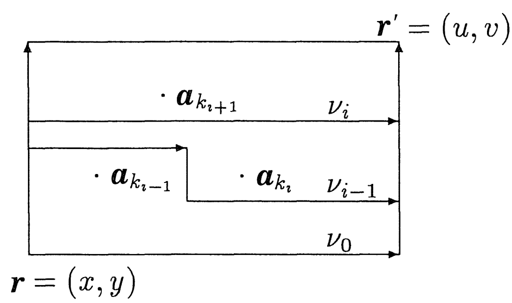
To see the range of the operator L and to calculate the above integral, we may consider the points $\boldsymbol{r}=(x, y)$ and $\boldsymbol{r}'=(u, v)$ such that $\boldsymbol{r}<<\boldsymbol{r}'$, $x\neq a_{j1}$, $u\neq a_{j1}$ $(j=1, \cdots, N)$, and the rectangular loop $\gamma(x, y\,;\, s, t)$ is in Ω. For those points, we will show the equality

$$f[\overset{(u,v)}{\lrcorner\!\!\uparrow}\circ\gamma^{r}]-f[\ulcorner\!\!\rightarrow^{(u,v)}\circ\gamma^{r}]$$

$$=\sum_{j=1}^{N}\chi_{r<<a_j}\chi_{r'>>a_j}(\varphi[\gamma_j^0]-1)\varphi[(\gamma_j^r)^{-1}\circ\gamma^r]f[\gamma_j^{r'}],$$

where $\chi_{r<<a_j}$ and $\chi_{r'>>a_j}$ denote the characteristic functions of $\boldsymbol{r}$ and $\boldsymbol{r}'$ with respect to the sets $\{\boldsymbol{r}\,;\ \boldsymbol{r}<<\boldsymbol{a}_j\}$ and $\{\boldsymbol{r}'\,;\ \boldsymbol{r}'>>\boldsymbol{a}_j\}$, respectively. When any singlar point of $\boldsymbol{A}$ is not surrounded by $\gamma(x, y\,;\ s, t)$, we have $[\overset{(u,v)}{\lrcorner\!\!\uparrow}\circ\gamma^{r}]=[\ulcorner\!\!\rightarrow^{(u,v)}\circ\gamma^{r}]$ and $\chi_{r<<a_j}\chi_{r'>>a_j}=0$ $(j=1, \cdots, N)$ and hence, the equality holds. Now we assume the singular points $\boldsymbol{a}_{k_1}, \cdots, \boldsymbol{a}_{k_l}$ $(1\leq k_1\leq\cdots\leq k_l\leq N)$ are surrounded by $\gamma(x, y\,;\ s, t)$, then, by taking account of the order of $\{\boldsymbol{a}_j\}$, we can take continuous paths ν_i $(i=1, \cdots, l)$ inside the rectangular $\gamma(x, y;s, t)$ such that

$$\nu_i\subset\Omega_{k_i},\ \nu_i(0)=\boldsymbol{r},\ \nu_i(1)=\boldsymbol{r}',\ (i=1, \cdots, l).$$



We further set $\nu_0={}_{r}\overset{r'}{\lrcorner\!\!\uparrow}$. It follows from the definition of ν_i $(i=0, 1, \cdots, l)$ that each loop $(\nu_i)^{-1}\circ\nu_{i-1}$ is homotopic (base point free) to $\gamma_{k_i}^0$ and that each path $(\nu_i)\circ\gamma_{k_i}^r$ is homotopic (end points fixed) to $\gamma_{k_i}^{r'}$. Thus we have

$$f[\overset{(u,v)}{\lrcorner\!\!\uparrow}\circ\gamma^{r}]-f[\ulcorner\!\!\rightarrow^{(u,v)}\circ\gamma^{r}]$$

$$=\sum_{i=1}^{l}(f[\nu_{i-1}\circ\gamma^r]-f[\nu_i\circ\gamma^r])$$

$$=\sum_{i=1}^{l}(f[(\nu_i\circ\gamma^r)\circ(\nu_i\circ\gamma^r)^{-1}\circ(\nu_{i-1}\circ\gamma^r)]-f[\nu_i\circ\gamma^r])$$

$$=\sum_{i=1}^{l}(\varphi[(\nu_i\circ\gamma^r)^{-1}\circ(\nu_{i-1}\circ\gamma^r)]-1)f[\nu_i\circ\gamma^r]$$

$$=\sum_{i=1}^{l}(\varphi[(\nu_i)^{-1}\circ\nu_{i-1}]-1)f[\nu_i\circ\gamma_{k_i}^r\circ(\gamma_{k_i}^r)^{-1}\circ\gamma^r]$$

$$=\sum_{i=1}^{l}(\varphi[\gamma_{k_i}^0]-1)\varphi[(\gamma_{k_i}^r)^{-1}\circ\gamma^r]f[\gamma_{k_i}^{r'}]$$

$$=\sum_{j=1}^{N}\chi_{r<<a_j}\chi_{r'>>a_j}(\varphi[\gamma_j^0]-1)\varphi[(\gamma_j^r)^{-1}\circ\gamma^r]f[\gamma_j^{r'}].$$

This completes the proof of equality.

By using this equality, we can calculate the integral as follows:

$$(Lf)[\gamma^r]=-\sum_{j=1}^{N}\alpha_j I_j(f)h_j[\gamma^r]$$

for almost all $[\gamma^r]$, where we set

$$\begin{aligned}&\alpha_j=\varphi[\gamma_j^0]-1,\\&I_j(f)=\int_{a_{j1}}^{\infty}\int_{a_{j2}}^{\infty}e^{-u-v}f[\gamma_j^{r'}]dudv\quad(\boldsymbol{r}'=(u,v)),\\&h_j[\gamma^r]=e^{x+y}\chi_{r<<a_j}\varphi[(\gamma_j^r)^{-1}\circ\gamma^r]\ (\boldsymbol{r}=(x,y))\end{aligned}\tag{4.9}$$

for $j=1,\cdots,N$. Note that the functions h_j $(j=1,\cdots,N)$ are in $L^2(\tilde{\Omega},\varphi)$ and that those are linearly independent since they have the distinct supports $\{[\gamma^r];\ \boldsymbol{r}<<\boldsymbol{a}_j\}$ in $\tilde{\Omega}$. Furthermore, each linear functional I_j on $L^2(\tilde{\Omega},\varphi)$ is continuous and, hence, given by $I_j(f)=(f,g_j)$ for some $g_j\in L^2(\tilde{\Omega},\varphi)$. By the definition of I_j, the support of g_j is $\{[\gamma^r];\ \boldsymbol{a}_j<<\boldsymbol{r}\}$ and the family $\{g_j\}$ are also linearly independent. Taking account of those facts, we can conclude that the dimension of the range of the operator

$$L(\cdot)=-\sum_{j=1}^{N}\alpha_j(\cdot,g_j)h_j$$

is equal to the number of $\{j;\ \alpha_j\neq 0\}$. This completes the proof of the theorem. □

Remark 4.2. (1) Let π be a $*$-representation of $\mathscr{W}_2$ given in §2 and $\tilde{\pi}$ be any extension of π as a $*$-representation in the same Hilbert space. Since $\pi(p_j)\subset\tilde{\pi}(p_j)$ and $\pi(p_j)$ is ess. self-adjoint, we have $\overline{\pi(p_j)}=\overline{\tilde{\pi}(p_j)}$ for $j=1,2$. Hence the defect number of $\{\overline{\tilde{\pi}(p_1)},\ \overline{\tilde{\pi}(p_2)}\}$ is the same as that for π.

For a $*$-representation π of $\mathscr{W}_2$ in §2, in Proposition 3.10 we showed that there exists the maximum extension $\tilde{\pi}$ of π in the same Hilbert space and that $\tilde{\pi}$ is unitary equivalent to the Schrödinger representation π_S if only if π is exponentiable. Thus we might say that the defect number $d(\overline{\pi(p_1)},\ \overline{\pi(p_2)})$ measures the distance between $\tilde{\pi}$ and π_S.
(2) Schmüdgen has calculated the defect number for his $*$-representation of $\mathscr{P}_2$ (c.f. Remark 2.6 (1) and [S4] pp257–258). His result is a special case of Theorem 4.1.

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