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Growth Order of Eigenfunctions of Schrödinger Operators with Potentials Admitting Some Integral Conditions II —Applications—

By

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Abstract

In this paper we give applications of M. Arai and J. Uchiyama [2, Theorems 1.1 and 1.2], which treat the sharp estimates of the growth orders of the eigenfunctions of the Schrödinger operators with potentials oscillating violently at infinity. We can generalize the results of G.B. Khosrovshahi, H.A. Levine and L.E. Payne (Trans. Amer. Math. Soc., **253** (1979), 211-228) and H. Kalf and V.K. Kumar (Trans. Amer. Math. Soc., **275** (1983), 215-229). Also we generalize the result of S. Agmon (J. Analyse Math., **23** (1970), 1-25).

§0. Introduction

Let us consider a not identically vanishing solution $u(x)\in H^2_{loc}(\Omega)$ of a second order elliptic equation

$$(0.1) \qquad -\sum_{j=1}^{n}(\partial_j+\sqrt{-1}b_j(x))^2u(x)+(q_1(x)+q_2(x))u(x)=0$$

for $x\in\Omega:=\{x|x\in\boldsymbol{R}^n,\ |x|>R_0\}$, where $\partial_j=\partial/(\partial x_j)$, $b_j(x)$ and $q_1(x)$ are real-valued functions, $q_2(x)$ is a complex-valued function.

In this paper we give applications of Arai-Uchiyama [2, Theorems 1.1 and 1.2]. So we rewrite their assumptions and results for completeness of this paper. We use the same notation as in Arai-Uchiyama [2].

Assumptions. Let us consider the equation (0.1). We assume that there exist some real-valued functions $\psi_i(r)\in C^0[R_0,\infty)$, $\sigma_i(r)$, $\eta_i(r)\in C^1[R_0,\infty)$ ($i=1, 2$), $\tau(r)\in C^0[R_0,\infty)$ and some constants a_i ($i=1, 2$) satisfying the follow-

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ing conditions (B.1)-(F.2):

(B.1) each $b_j(x)$ is a real-valued function;

(B.2) for any $w(x)\in H^1_{loc}(\Omega)$ we have
$b_j(x)w(x)$, $(\partial_j b_k(x))w(x)\in L^2_{loc}(\Omega)$;

(C.1) $q_1(x)$ is a real-valued function;

(C.2) for any $w(x)\in H^1_{loc}(\Omega)$ we have $q_1(x)|w(x)|^2\in L^1_{loc}(\Omega)$;

(C.3) for any $w(x)\in H^1_{loc}(\Omega)$ we have $\partial_r q_1(x)\cdot|w(x)|^2\in L^1_{loc}(\Omega)$;

(D.1) $q_2(x)$ may be a complex-valued function;

(D.2) for any $w(x)\in H^1_{loc}(\Omega)$ we have $q_2(x)|w(x)|^2\in L^1_{loc}(\Omega)$;

(E.1) $\sigma_i(r)$ is bounded in $[R_0, \infty)$ and $\sigma_i(r)>0$ $(i=1, 2)$ for any $r\geq R_0$;

(E.2) $\eta_i(r)$ is bounded in $[R_0, \infty)$ and we have for any $r\geq R_0$

$$\eta_i(r)<2 \quad (i=1, 2);$$

(E.3) $\psi_i(r)>0$ $(i=1, 2)$ for any $r\geq R_0$;

(E.4) $\lim_{r\to\infty} r^{-1}\psi_i(r)\sigma_i(r)^{-1}=0$ $(i=1, 2)$;

(E.5) $r\{\sigma_1'(r)-\sigma_2'(r)-(\eta_1'(r)-\eta_2'(r))\}$ is bounded in $[R_0, \infty)$;

(E.6) there exists some constant $C_1\geq 1$ such that for any $r\geq R_0$

$$C_1^{-1}\sigma_2(r)\leq \tau(r)\leq C_1;$$

(E.7) $\lim_{r\to\infty} \psi_2(r)^2\sigma_2(r)^{-1}\exp\left(\int_{R_0}^r \frac{r(t)-\eta_2(t)}{t}dt\right)=0$;

(F.1) $a_1>1$, $a_2>0$;

(F.2) $\limsup_{r\to\infty} \psi_i(r)^2\sigma_i(r)^{-1}[r\partial_r q_1+\eta_i(r)q_1+a_i\sigma_i(r)^{-1}|rq_2-Q_i'(r)|^2$
$+(2-\eta_i(r))^{-1}|Bx|^2]<0$ $(i=1, 2)$,

where we put

$Q_i(r)=4^{-1}(\eta_i(r)-\sigma_i(r))$,
$B(x)=(\partial_j b_k(x)-\partial_k b_j(x))$ is an $n\times n$ matrix.

Theorems 1.1 and 1.2 of Arai-Uchiyama [2] are as follows:

Theorem 0.1. *Let $u(x)\in H^2_{loc}(\Omega)$ satisfy the equation* (0.1). *Assume that* (B)-(F) *hold. If* supp$[u]$ *is not a compact set in $\overline{\Omega}$, then we have*

$$\liminf_{R\to\infty} \Phi_1(R)\int_{|x|=R}[|D_r u|^2+\{r^{-2}+(q_1)_-\}|u|^2]dS>0,$$

where

$$(f)_-(x)=\max\{0,\ -f(x)\}\geq 0 \text{ for a real-valued function } f(x),$$

$$\Phi_1(r)=\exp\Big(\int_{R_0}^{r}\frac{\sigma_1(t)+\eta_1(t)}{2t}dt\Big).$$

Theorem 0.2. *In addition to the assumptions of Theorem* 0.1, *assume that*

(G) *there exist some real-valued bounded function* $Q_0(x)\in C^1(\bar{\Omega})$, *some positive and non-decreasing function* $p(r)\in C^1[R_0, \infty)$ *and some constant* $0<d_1<1$ *satisfying the following* (G.1)-(G.3):

(G.1) *for any* $w(x)\in C_0^\infty(\Omega)$ *we have*

$$\int_\Omega\{(q_1)_-(x)+(\mathrm{Re}[q_2]-r^{-1}\partial_r Q_0)_-(x)\}|w(x)|^2dx$$
$$\leq\int_\Omega\{d_1|\nabla w(x)|^2+p(r)|w(x)|^2\}dx\ ;$$

(G.2) $\liminf_{r\to\infty} p(r)p(r+1)^{-1}>0$;

(G.3) $p'(r)^2p(r)^{-3}$ *is bounded in* $[R_0, \infty)$.

Then we have

(1) $\liminf_{R\to\infty} p(R)\Phi_1(R)\int_{R<|x|<R+1}|u(x)|^2dx>0$;

(2) *moreover if* $\int_{R_0}^{\infty}p(R)^{-1}\Phi_1(R)^{-1}dR=\infty$ *then* $u(x)\notin L^2(\Omega)$.

The aims of this paper are the following:

(1) to give a generalization of the results of Khosrovshahi-Levine-Payne [7] and Kalf-Kummar [5] (Theorem 1.1);

(2) to give a generalization of the result of Agmon [1] (Theorem 1.4);

(3) to check that our results cover all the ones given by Uchiyama-Yamada [9] (Application 1.6).

We will apply our Theorem 0.2 to von Neumann-Wigner [8] example in other article.

§1. Applications of Theorems 0.1 and 0.2

Theorem 1.1. *Let* $b_j(x)\in C^1(\bar{\Omega})$ $(j=1, \cdots, n)$, $V_1(x)\in C^1(\bar{\Omega})$ *and* $V_3(r)\in C^0[R_0, \infty)$ *be real-valued functions, and* $V_2(x)\in C^0(\bar{\Omega})$ *be a complex-valued function. Let* $u(x)\in H^2_{loc}(\Omega)\cap L^2(\Omega)$ *satisfy*

$$(1.1)\qquad -\sum_{j=1}^{n}(\partial_j+\sqrt{-1}b_j(x))^2u(x)+\{V_1(x)+V_2(x)+V_3(r)\}u(x)=\lambda u(x)$$

for $x\in\Omega:=\{x||x|>R_0\}$. *Let*

$$Q(r)=\int_{R_0}^{r}tV_3(t)dt.$$

We assume:

(1) $V_1(x)$ *is bounded on* $\bar{\Omega}$,

(2) $\limsup_{r\to\infty} V_1(x)=0$,

(3) $0\leq\limsup_{r\to\infty} r\partial_r V_1(x)=L<\infty$,

(4) $\limsup_{r\to\infty}\{|rV_2(x)|^2+|B(x)x|^2\}=K^2<\infty$,

(5) $\limsup_{r\to\infty} Q(r)-\liminf_{r\to\infty} Q(r)=M<1$,

(6) $\lambda>\Lambda:=\frac{1}{2}\cdot\frac{1}{1-M^2}[K^2+L+\sqrt{K^2(K^2+2L)+L^2M^2}]$.

Then we have $u(x)\equiv 0$ *on* Ω. (*The condition* $L\geq 0$ *in* (3) *is automatically satisfied by the condition* (1).)

Remark 1.2. Noting the results given in Arai-Uchiyama [2, §8], we can treat the case $V_3=V_3(x)$. In this case we can replace (4) and (5) with the following (4)′ and (5)′ respectively.

(4)′ $\limsup_{r\to\infty}\{|rV_2(x)|^2+|\nabla Q(x)|^2-|\partial_r Q(x)|^2+|B(x)x|^2\}=K^2<\infty$,

where

$$Q(x)=\int_{R_0}^{|x|}tV_3(t\hat{x})dt\ \ (\hat{x}=x/|x|),$$

(5)′ $\limsup_{r\to\infty} Q(x)-\liminf_{r\to\infty} Q(x)=M<1$.

Remark 1.3. We state several results obtained before from which we have the conclusion that λ ($>\Lambda$) is not an eigenvalue. (Λ defined above will be denoted as Λ_{AU}.) Kato [6] considered the case $V_1(x)\equiv V_3(r)\equiv b_j(x)\equiv 0$ and gave

$$\Lambda_K = K^2.$$

Agmon [1] considered the case $V_3(r)\equiv b_j(x)\equiv 0$, $K=0$ and gave

$$\Lambda_A = \frac{L}{2}.$$

Eastham-Kalf [3, p. 187] considered the case $V_3(r)\equiv 0$ and gave

$$\Lambda_{EK} = \frac{1}{2}\{\tilde{K}^2 + L + \sqrt{\tilde{K}^2(\tilde{K}^2+2L)}\},$$

where $\tilde{K} = \limsup_{r\to\infty}\{r|V_2(x)| + |B(x)x|\}$ ($\geq K$). Khosrovshahi-Levine-Payne [7] considered the case $b_j(x)\equiv 0$ and gave under the condition $M<4^{-1}$

$$\Lambda_{KLP} = \max\left\{\left[\frac{K+\sqrt{K^2+2L(1-2M)}}{2(1-2M)}\right]^2, \frac{2K^2+L(1-4M)}{2(1-4M)^2}\right\}.$$

Kalf-Kumar [5] treated the case $V_3 = V_3(x)$ and $\lim_{r\to\infty}\{|\nabla Q(x)|^2 - |\partial_r Q(x)|^2\} = 0$, and gave under the condition $M<2^{-1}$

$$\Lambda_{KK} = \left[\frac{\tilde{K}+\sqrt{\tilde{K}^2+2L(1-2M)}}{2(1-2M)}\right]^2.$$

(We remark that we correct the error in the representation of Λ_{KLP} given in Khosrovshahi-Levine-Payne [7] and Kalf-Kumar [5].) It is obvious that $\Lambda_{KK} = \Lambda_{EK}$ under the assumption of Eastham-Kalf [3] and that Λ_{EK}, Λ_{KLP}, Λ_{KK} and Λ_{AU} reduce to Λ_K or Λ_A under the Kato's or Agmon's assumptions, respectively. After careful consideration, noting Remark 1.2, we can see $\Lambda_{KLP}\geq\Lambda_{KK}\geq\Lambda_{AU}$.

Theorem 1.4 (Generalization of Agmon's Theorem.) *Let* $u(x)\in H^2_{loc}(\Omega)$ *satisfy*

$$(1.2) \qquad -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\,b_j(x)\right)^2 u(x) + \{V_1(x) + V_2(x) + V_3(r)\}u(x) = \lambda u(x) \text{ in } \Omega$$

and $u(x)\not\equiv 0$ *in* $\Omega := \{x \mid |x| > R_0\}$. *Assume*

(AG.1)

λ is a real constant,

$\lambda - V_1(x)\in C^0(\bar{\Omega})$ is a positive function possessing a continuous radial

derivative,
$V_2(x)\in C^0(\bar{\Omega})$ *is a complex-valued function,*
$V_3(r)\in C^1[R_0, \infty)$ *is a real-valued function.*
each $b_j(x)\in C^1(\bar{\Omega})$ *is a real-valued function,*

(AG.2) *There exist a number* $\alpha>0$, *continuous non-negative functions* $\zeta_1(r)$, $\zeta_2(r)$ *and* $\mu(r)$ *defined on* $r\geq R_0$, *and a real-valued function* $Q(r)\in C^1[R_0, \infty)$, *satisfying*

(i) $\int_{R_0}^{\infty}\zeta_i(r)dr<\infty \qquad (i=1,2),$

(ii) $\limsup_{r\to\infty} r\{\zeta_1(r)+\zeta_2(r)\}<2\alpha,$

(iii) $-\partial_r V_1(x)+\left(\frac{2(1-\alpha)}{r}+\zeta_1(r)\right)(\lambda-V_1(x))\geq 0$ *for* $r\geq R_0$,

(iv) $|V_2(x)|\leq\zeta_2(r)(\lambda-V_1(x))^{1/2}$ *for* $r\geq R_0$,

(v) $Q'(r)=rV_3(r)$ *for* $r\geq R_0$,

(vi) $|Q(r)|\leq r\mu(r)\{\zeta_1(r)+\zeta_2(r)\}$ *for* $r\geq R_0$,

(vii) $|B(x)x|^2\leq r\mu(r)\{\zeta_1(r)+\zeta_2(r)\}(\lambda-V_1(x))$ *for* $r\geq R_0$,

(viii) $\lim_{r\to\infty}\mu(r)=0.$

Then we have

$$\liminf_{R\to\infty} R^{-\alpha}\int_{R_0<|x|<R}(\lambda-V_1(x))|u(x)|^2dx>0. \tag{1.3}$$

Remark 1.5. Agmon[1] showed the above result for $b_j(x)\equiv 0$ and $V_3(r)\equiv 0$ (, which means $Q(r)\equiv 0$), under the above conditions (i), (iii) and the following more restricted conditions than ours

(ii)′ $\limsup_{r\to\infty} r\{\zeta_1(r)+\zeta_2(r)\}<\alpha,$

(iv)′ $\max_{|x|=r}|V_2(x)|\leq\zeta_2(r)\min_{|x|=r}(\lambda-V_1(x))^{1/2}$ for $r\geq R_0$.

(v)-(viii) did not relate to his problem.

In the similar fashion as Remark 1.2, we can treat the case $V_3=V_3(x)$. In this case we can replace (v) and (vii) with the following (v)′ and (vii)′.

(v)′ $\partial_r Q(x)=rV_3(x)$ for $r\geq R_0$,

(vii)′ $|\nabla Q(x)|^2-|\partial_r Q(x)|^2+|B(x)x|^2$
$\leq r\mu(r)\{\zeta_1(r)+\zeta_2(r)\}(\lambda-V_1(x))$ for $r\geq R_0$.

Application 1.6. Here we refer to the results of Uchiyama-Yamada [9]. Though they treated more general second-order elliptic equations and we can show that our theory in [2, §8] recovers their results completely, we restrict ourselves to the case of the equation (0, 1) for simplicity. In this case their assumptions become as follows.

(UY.1) our Assumptions (B)-(D) are assumed.

(UY.2) there exist some real constants $0<\alpha<\beta$, $\tilde{\gamma}_i\in\boldsymbol{R}$, $\tilde{a}_1>1$, $\tilde{a}_2>0$, $\tilde{b}_i>1$, $0<\tilde{d}_1<1$, $\delta_1\in\boldsymbol{R}$, $\delta_2\leq\beta-2$, $\tilde{C}_i>0$ $(i=1,\ 2)$ and some real-valued functions $\sigma(r)\in C^1(\bar{\Omega})$, $\eta(r)\in C^1(\bar{\Omega})$ such that the following (UY.3)-(UY.13) hold;

(UY.3) $\sigma(r)$ is bounded and $\sigma(r)>0$ for any $x\in\bar{\Omega}$;

(UY.4) $\eta(r)$ is bounded in $\bar{\Omega}$ and $\tilde{\gamma}_1+\limsup_{r\to\infty}\eta(r)<2$;

(UY.5) $(2-2\beta<)2-2\alpha<\tilde{\gamma}_2+\liminf_{r\to\infty}\eta(r)\leq\tilde{\gamma}_2+\limsup_{r\to\infty}\eta(r)<2$;

(UY.6) $\lim_{r\to\infty} r^{\beta-\alpha}\sigma(r)=\infty$;

(UY.7) $\lim_{r\to\infty} r^{1-\beta}\sigma(r)^{-1}\sigma'(r)=0$ and $\lim_{r\to\infty} r^{1-\beta}\sigma(r)^{-1}\eta'(r)=0$;

(UY.8) $\limsup_{r\to\infty} r^{2-2\beta}\sigma(r)^{-1}[r\partial_r q_1+(\tilde{\gamma}_i+\eta(r))q_1$
$+\tilde{a}_i\sigma(r)^{-1}|rq_2|^2+\tilde{b}_i(2-\tilde{\gamma}_i-\eta(r))^{-1}|Bx|^2]$
<0 $(i=1,\ 2)$;

(UY.9) for any $w(x)\in C_0^\infty(\Omega)$ we have

$$\int_\Omega (q_1)_-|w|^2dx\leq\int_\Omega\{\tilde{d}_1|\nabla w|^2+\tilde{C}_1 r^{\delta_1}|w|^2\}dx\,;$$

(UY.10) for any $x\in\bar{\Omega}$ we have $(\mathrm{Re}[q_2])_-(x)\leq\tilde{C}_2\min\{r^{\delta_2},\ r^{\beta-2}\sigma(r)\}$.

Then they gave the following results for the solution $u(x)\in H^2_{loc}(\Omega)$ of (0.1) whose support is not a compact set in $\bar{\Omega}$:

$$\liminf_{r\to\infty} R^{\tilde{\gamma}_1/2}\Phi(R)\int_{|x|=R}[|D_r u|^2+\{r^{-2}+(q_1)_-\}|u|^2]dS>0,$$
$$\liminf_{R\to\infty} R^{(\tilde{\gamma}_1/2)+\max\{0,\ \delta_1,\ \delta_2\}}\Phi(R)\int_{R<|x|<R+1}|u|^2dx>0,$$

where

$$\Phi(R)=\exp\Big(\int_{R_0}^R\frac{\sigma(t)+\eta(t)}{2t}dt\Big).$$

Apply our Theorems 0.1 and 0.2 to the above circumstances, and we can show their results. So our Assumptions treat wider classes of potentials than those of Uchiyama-Yamada [9]. In fact let for $i=1, 2$

$$\psi_i(r)=r^{1-\beta},$$
$$\sigma_i(r)=\sigma(r),$$
$$\eta_i(r)=\tilde{\gamma}_i+\eta(r).$$

By (UY.3)-(UY.5), (E.1) and (E.2) hold. It is obvious that (E.3) and (E.5) hold. By (UY.6) and $\alpha>0$, we can see that (E.4) holds. Noting (UY.5), we choose $\tau_0>0$ satisfying $2-2\alpha+\tau_0<\tilde{\gamma}_2+\liminf_{r\to\infty}\eta(r)$, and put $\tau(t)\equiv\tau_0$, by which and (UY.3) we have (E.6). Then for sufficiently large r we have $2-2\alpha+\tau_0-(\tilde{\gamma}_2+\eta(r))\leq 0$. Noting (UY.3), (UY.6) and

$$\psi_2^2\sigma_2^{-1}\exp\left(\int_{R_0}^r\frac{\tau(t)-\eta_2(t)}{t}dt\right)$$
$$=R_0^{2-2\alpha}r^{2(\alpha-\beta)}\sigma(r)^{-2}\cdot\sigma(r)\cdot\exp\left(\int_{R_0}^r\frac{2-2\alpha+\tau_0-(\tilde{\gamma}_2+\eta(t))}{t}dt\right),$$

we have (E.7). Let a_i satisfy $\tilde{a}_1>a_1>1$ and $\tilde{a}_2>a_2>0$. Then we have (F.2) from (UY.8) and the fact $\psi_i\sigma_i^{-1}Q_i'=o(1)$, which follows from (UY.7). Let $d_1=\tilde{d}_1$, $Q_0(x)\equiv 0$ and $p(r)=(\tilde{C}_1+\tilde{C}_2)r^{\max\{0,\delta_1,\delta_2\}}$. Then by (UY.9) and (UY.10) we have (G). Therefore, noting $\Phi_1(r)=(r/R_0)^{\tilde{\gamma}_1/2}\Phi(r)$, by our Theorems 0.1 and 0.2 we have their results.

Remark 1.7.

(1) Uchiyama-Yamada [9, Example 1.7, Cases 2 and 3] gave the examples such that $\tilde{\gamma}_1+\lim_{r\to\infty}\eta(r)=2-2\beta$. Noting (UY.5), we cannot, in general, put $\tilde{\gamma}_1=\tilde{\gamma}_2$.

(2) The above condition (UY.7) can be replaced with

(UY.7)′ $\lim_{r\to\infty} r^{1-\beta}\sigma(r)^{-1}(\eta(r)-\sigma(r))'=0.$

In fact in the above argument we use (UY.7) only to show $\psi_i\sigma_i^{-1}Q_i'=o(1)$, which also comes out from (UY.7)′.

§2. Proof of Theorem 1.1

Lemma 2.1. *Let $K\geq 0$, $L\geq 0$, $0\leq M<1$ be constants and*

$$f(t)=\frac{1}{2}\left\{\frac{K^2+2L}{t}+\frac{K^2}{2-t}\right\}\ (0<t<2).$$

Then we have

$$\inf_{0<\beta<2-2M}(\sup\{f(t)|\beta\le t\le\beta+2M\})=\Lambda,$$

where

$$\Lambda=\frac{1}{2}\cdot\frac{1}{1-M^2}\{K^2+L+\sqrt{K^2(K^2+2L)+L^2M^2}\},$$

which is the one given in the statement of Theorem 1.1.

Proof. First let us consider the case $K>0$. Put

$$t_0=\frac{2\sqrt{K^2+2L}}{K+\sqrt{K^2+2L}}.$$

Then we have

$$\begin{cases}1\le t_0<2,\\ \min\limits_{0<t<2} f(t)=f(t_0)=2^{-1}\{K^2+L+K\sqrt{K^2+2L}\},\\ f(t) \text{ is continuous in } (0,\ 2),\\ f(t) \text{ is decreasing in } (0,\ t_0) \text{ and increasing in } (t_0,\ 2).\end{cases}$$

Let us consider the case $0<M<1$. Noting $\lim_{t\downarrow 0} f(t)=\lim_{t\uparrow 2} f(t)=\infty$, we apply the intermediate-value theorem. Then for any $s\in(f(t_0),\ \infty)$ there uniquely exist some $t_1=t_1(s)\in(0,\ t_0)$ and $t_2=t_2(s)\in(t_0,\ 2)$ such that $s=f(t_1(s))=f(t_2(s))$. We can see that $t_2(s)-t_1(s)$ is a positive continuous increasing function of $s\in(f(t_0),\ \infty)$, $\lim_{s\downarrow f(t_0)}(t_2(s)-t_1(s))=0$, $\lim_{s\to\infty}(t_2(s)-t_1(s))=2$. Noting $0<M<1$, we apply the intermediate-value theorem again. Then there uniquely exists some s_0 $\in(f(t_0),\ \infty)$ such that $t_2(s_0)-t_1(s_0)=2M$. We put $\beta_1=t_1(s_0)$. Since $t_1(s_0)+2M$ $=t_2(s_0)<2$ we have $\beta_1\in(0,\ 2-2M)$ and $f(\beta_1)=f(\beta_1+2M)$.

If $M=0$, then we put $\beta_1=t_0$.

So for $M\ge 0$, we have $0<\beta_1<2-2M$ and

$$\begin{aligned}&\min_{0<\beta<2-2M}(\sup\{f(t)|\beta\le t\le\beta+2M\})\\ &\quad=\min_{0<\beta<2-2M}(\max\{f(\beta),\ f(\beta+2M)\})\\ &\quad=f(\beta_1).\end{aligned}$$

By $f(\beta_1)=f(\beta_1+2M)$ we have

$$L\beta_1^2-2\beta_1\{K^2+(2-M)L\}+2(1-M)(K^2+2L)=0.$$

Thus

$$\beta_1=\begin{cases}L^{-1}\{K^2+(2-M)L-\sqrt{K^2(K^2+2L)+L^2M^2}\} & \text{if } L>0,\\ 1-M & \text{if } L=0.\end{cases}$$

Since for $L\geq 0$

$$\begin{aligned}&2^{-1}\beta_1^{-1}(K^2+2L)\\&\quad=4^{-1}(1-M)^{-1}\{\sqrt{K^2(K^2+2L)+L^2M^2}+K^2+(2-M)L\},\\&2^{-1}(2-\beta_1)^{-1}K^2\\&\quad=4^{-1}(1+M)^{-1}\{\sqrt{K^2(K^2+2L)+L^2M^2}+K^2-LM\},\end{aligned}$$

we have $f(\beta_1)=\Lambda$.

Next let us consider the case $K=0$. In this case we have $f(t)=Lt^{-1}$ and

$$\begin{aligned}&\inf_{0<\beta<2-2M}(\sup\{f(t)|\beta\leq t\leq\beta+2M\})\\&\quad=\inf_{0<\beta<2-2M}f(\beta)=\frac{1}{2}\cdot\frac{L}{1-M}=\Lambda.\end{aligned}$$ □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Noting $\lambda>\Lambda$ by the condition (6) in the statement of Theorem 1.1, we choose $\varepsilon_0\in(0,3^{-1}(\lambda-\Lambda))$. Since Λ is continuous in $M\in[0,1)$, by Lemma 2.1 there exists $M'\in(M,1)$ such that

$$\begin{aligned}&\inf_{0<\beta<2-2M'}\left(\sup\left\{\frac{1}{2}\left\{\frac{K^2+2L}{t}+\frac{K^2}{2-t}\right\}\middle|\beta\leq t\leq\beta+2M'\right\}\right)\\&\quad<\Lambda+\varepsilon_0.\end{aligned}$$

Thus there exists some $\beta_0\in(0,2-2M')$ such that for any $t\in[\beta_0,\beta_0+2M']$ we have

$$\frac{1}{2}\left\{\frac{K^2+2L}{t}+\frac{K^2}{2-t}\right\}<\Lambda+\varepsilon_0. \tag{2.1}$$

Let

$$\mu_1=\liminf_{r\to\infty}Q(r),\ \mu_2=\limsup_{r\to\infty}Q(r),$$

and we have $M=\mu_2-\mu_1$. Let

$$\alpha_0=\beta_0+M'-\mu_1-\mu_2,$$

and we have

$$\begin{aligned}&\alpha_0+2\liminf_{r\to\infty}Q(r)=\alpha_0+2\mu_1=\beta_0+M'-M>\beta_0,\\&\alpha_0+2\limsup_{r\to\infty}Q(r)=\alpha_0+2\mu_2=\beta_0+M'+M<\beta_0+2M'.\end{aligned}$$

Therefore there exists $R_1>R_0$ such that for any $r\geq R_1$ we have

$$0<\beta_0<\alpha_0+2Q(r)<\beta_0+2M'<2. \tag{2.2}$$

By (2.1), for any $r\geq R_1$ we have

$$\Lambda+\varepsilon_0>\frac{1}{2}\left\{\frac{K^2+2L}{\alpha_0+2Q(r)}+\frac{K^2}{2-\alpha_0-2Q(r)}\right\}$$
$$=\frac{L}{\alpha_0+2Q(r)}+\frac{K^2}{\{\alpha_0+2Q(r)\}\{2-\alpha_0-2Q(r)\}}.$$

By the conditions (2), (3), (4) and (2.2), there exists $R_2\geq R_1$ such that for any $r\geq R_2$ we have

$$\Lambda+2\varepsilon_0$$
$$>\frac{r\partial_r V_1(x)}{\alpha_0+2Q(r)}+V_1(x)+\frac{|rV_2(x)|^2+|B(x)x|^2}{\{\alpha_0+2Q(r)\}\{2-\alpha_0-2Q(r)\}}.$$

And by (2.2), there exists some constant $C_3>0$ such that for any $r\geq R_2$ we have

$$\varepsilon_0>\frac{C_3\{2-\alpha_0-2Q(r)\}}{\{\alpha_0+2Q(r)\}}.$$

Adding above two inequalities and noting $\lambda>\Lambda+3\varepsilon_0$ and (2.2), for any $r\geq R_2$ we have

$$\frac{1}{2-\alpha_0-2Q(r)}[r\partial_r V_1(x)+\{\alpha_0+2Q(r)\}\{V_1(x)-\lambda\} \\ +\frac{1}{2-\alpha_0-2Q(r)}\{|rV_2(x)|^2+|B(x)x|^2\}] \\ <-C_3. \tag{2.3}$$

Let

$$q_1(x)=V_1(x)-\lambda,$$
$$q_2(x)=V_2(x)+V_3(r).$$

Then the equation (1.1) coincides with (0.1). Our Assumptions (B)-(D) are satisfied. Now we shall check over all the conditions given in Assumptions (E)-(G).

Let for $i=1, 2$

$$\sigma_i(r)=2-\alpha_0-2Q(r),$$
$$\eta_i(r)=\alpha_0+2Q(r),$$
$$\tau(r)=2^{-1}\beta_0,$$
$$\psi_i(r)\equiv 1,$$
$$Q_0(x)=Q(r),$$
$$p(r)\equiv C_4,$$

where $C_4>0$ is a sufficiently large constant. Then we have

$$Q_i(r)=2^{-1}(\alpha_0-1)+Q(r),$$
$$\sigma_i(r)+\eta_i(r)=2,$$
$$\Phi_1(r)=rR_0^{-1}.$$

Noting (2.2) it can be easily seen that (E) and (G) except (E.7) are satisfied. By (2.2), for any $t \geq R_2$ we have $\tau(t) - \eta_2(t) \leq -2^{-1}\beta_0$, which leads us to (E.7). By (2.3), (F.2) holds with $a_i = 1$. By Arai-Uchiyama [2, Remark 1.6] we can see that (F.2) with (F.1) holds.

Now we assume that $\mathrm{supp}[u]$ is not a compact set in $\bar{\Omega}$. Then by Theorem 0.2 we have $u(x) \notin L^2(\Omega)$, which contradicts with the assumptions of Theorem 1.1. Thus $\mathrm{supp}[u]$ is a compact set in $\bar{\Omega}$. Since the unique continuation theorem holds to our problem (, see e.g. Garofalo-Lin [4, Theorem 1.1]), we have $u(x) \equiv 0$ on Ω. □

§3. Proof of Theorem 1.3

Proof of Theorem 1.3. If $\zeta_2(r)$ satisfies the assumptions, then for $\varepsilon_1 > 0$, which will be determined later, $\zeta_2(r)$ replaced with $\zeta_2(r) + r^{-1-\varepsilon_1}$ also satisfies all the conditions given in Theorem 1.3. Thus we can assume, without loss of generality, that for any $r \geq R_0$ we have

$$\zeta_2(r) \geq r^{-1-\varepsilon_1}. \tag{3.1}$$

By (iii) and positivity of $\lambda - V_1(x)$, there exist constants $C_5 > 0$ and $C_6 > 0$ such that for any $r \geq R_0$

$$\lambda - V_1(x) \geq C_5 r^{2(\alpha-1)} \exp\left(-\int_{R_0}^{r} \zeta_1(t)\,dt\right) \geq C_6 r^{2(\alpha-1)}, \tag{3.2}$$

where in the last inequality we used (i).

We put

$$q_1(x) = V_1(x) - \lambda,\ q_2(x) = V_2(x) + V_3(r).$$

Then the equation (1.2) coincides with (0.1). Our Assumptions (B)-(D) are satisfied.

Let $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ be constants determined later and we put

$$\begin{aligned}
&\sigma_i(r) = (1+2\varepsilon_2) r \{\zeta_1(r) + \zeta_2(r)\} - 2Q(r)\ (i=1,\,2),\\
&\eta_1(r) = 2(1-\alpha) + (1+2\varepsilon_2) r \{\zeta_1(r) + \zeta_2(r)\} + 2Q(r),\\
&\eta_2(r) = 2(1-\alpha) + \varepsilon_3 + (1+2\varepsilon_2) r \{\zeta_1(r) + \zeta_2(r)\} + 2Q(r),\\
&\psi_i(r) = r^{1-\alpha}\ (i=1,\,2).
\end{aligned}$$

By (ii), (vi), (viii) and (3.1) there exist some $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $C_7 > 0$ and $R_3 \geq R_0$ such that for any $r \geq R_3$ we have

$$2 - \eta_i(r) \geq C_7\ (i=1,\,2), \tag{3.3}$$

$$|Q(r)| \leq 2^{-1}\varepsilon_2 r \{\zeta_1(r) + \zeta_2(r)\}, \tag{3.4}$$

$$r^{-\varepsilon_1} \leq (1+\varepsilon_2) r \{\zeta_1(r) + \zeta_2(r)\} \leq \sigma_i(r) \leq (1+3\varepsilon_2) r \{\zeta_1(r) + \zeta_2(r)\}. \tag{3.5}$$

So the Assumptions (E.1)-(E.3) are satisfied. Let $0<\varepsilon_1<\min\{\alpha, \varepsilon_3\}$, $0<\tau_0<\varepsilon_3-\varepsilon_1$ and $\tau(t)\equiv\tau_0$. By (3.5), we have (E.4). (E.5) is obvious since $\sigma_1=\sigma_2$ and $\eta_1-\eta_2$ is a constant. (E.6) is also obvious by (ii) and (3.5). By (3.4) and non-negativities of $\zeta_1(r)$ and $\zeta_2(r)$ we have $\tau(t)-\eta_2(t)\leq\tau_0-2(1-\alpha)-\varepsilon_3$ for any $t\geq R_3$. Then by (3.5)

$$\psi_2^2\sigma_2^{-1}\exp\left(\int_{R_0}^{r}\frac{\tau(t)-\eta_2(t)}{t}dt\right)\leq\text{Const } r^{\tau_0+\varepsilon_1-\varepsilon_3},$$

which leads us to (E.7).

Now we shall show (F.2). By (iii), (3.4) and (3.5), we have for any $r\geq R_3$

$$\begin{aligned}
&\sigma_i^{-1}(r\partial_r q_1+\eta_i q_1)\\
&\quad=-r\sigma_i^{-1}\left[\left\{-\partial_r V_1+\left(\frac{2(1-\alpha)}{r}+\zeta_1\right)(\lambda-V_1)\right\}\right.\\
&\qquad\left.+\{\zeta_2+2\varepsilon_2(\zeta_1+\zeta_2)+\varepsilon_3\delta_{i2}r^{-1}+2r^{-1}Q\}(\lambda-V_1)\right]\\
&\quad\leq-r\sigma_i^{-1}\{\zeta_2+\varepsilon_2(\zeta_1+\zeta_2)\}(\lambda-V_1)\\
&\quad\leq-\{r\sigma_i^{-1}\zeta_2+\varepsilon_2(1+3\varepsilon_2)^{-1}\}(\lambda-V_1),
\end{aligned}$$

where δ_{ij} is Kronecker's delta. Let $a_i=1+\varepsilon_2\ (>1)$. By (3.5) and non-negativity of $\zeta_1(r)$ we have $a_i r\zeta_2\leq\sigma_i$ for any $r\geq R_3$. Note that $Q_i=4^{-1}(\eta_i-\sigma_i)=Q(r)+2^{-1}(1-\alpha)+4^{-1}\varepsilon_3\delta_{i2}$. Then by (iv) and (v) we have for any $r\geq R_3$

$$a_i\sigma_i^{-2}|rq_2-Q_i'|^2=a_i r^2\sigma_i^{-2}|V_2|^2\leq a_i r^2\sigma_i^{-2}\zeta_2^2(\lambda-V_1)\leq r\sigma_i^{-1}\zeta_2(\lambda-V_1).$$

By (vii), (3.3) and $\sigma_i\geq r(\zeta_1+\zeta_2)$ we have for any $r\geq R_3$

$$\sigma_i^{-1}(2-\eta_i)^{-1}|Bx|^2\leq C_7^{-1}\mu(r)(\lambda-V_1).$$

Thus, noting (3.2), for any $r\geq R_3$ we have

$$\begin{aligned}
&\psi_i^2\sigma_i^{-1}[r\partial_r q_1+\eta_i q_1+a_i\sigma_i^{-1}|rq_2-Q_i'|^2+(2-\eta_i)^{-1}|Bx|^2]\\
&\quad\leq-r^{2-2\alpha}\{\varepsilon_2(1+3\varepsilon_2)^{-1}-C_7^{-1}\mu(r)\}(\lambda-V_1)\\
&\quad\leq-C_6\{\varepsilon_2(1+3\varepsilon_2)^{-1}-C_7^{-1}\mu(r)\},
\end{aligned}$$

which and (viii) lead us to (F.2).

Since unique continuation theorem can be applied to our problem (1.2) and $u\not\equiv0$, we can admit that $\operatorname{supp}[u]$ is not a compact set. Thus we can apply Theorem 0.1 and, using positivity of $\lambda-V_1(x)$, we obtain

$$\liminf_{R\to\infty}\int_{|x|=R}\Phi_1(r)[|Du|^2+\{r^{-2}+(\lambda-V_1(x))\}|u|^2]dS>0,$$

where $\Phi_1(r)=\exp\left(\int_{R_0}^{r}\frac{\sigma_1(t)+\eta_1(t)}{2t}dt\right)\leq C_8 r^{1-\alpha}$ by (i) with some positive constant C_8. Now by (ii) and (3.2) there exist some constants $R_4\geq R_3$ and $C_9>0$ such that for any $r\geq R_4$ we have

$$r^2(\lambda - V_1(x)) \geq \text{Const} > 0, \tag{3.6}$$

$$\int_{|x|=r} [|Du|^2 + (\lambda - V_1(x))|u|^2] dS \geq C_9 r^{\alpha-1}, \tag{3.7}$$

$$\zeta_2(r) < 2\alpha r^{-1}. \tag{3.8}$$

By (iv), (v), (3.6) and (3.8), we have

$$|q_2 - r^{-1}Q'| = |V_2| \leq 2^{-1}\{\zeta_2^2 + (\lambda - V_1)\} \leq \text{Const}(\lambda - V_1).$$

Thus $\tilde{p}(x) = \text{Const}(\lambda - V_1(x))$ and $Q_0(x) = Q(r)$ satisfy the assumptions of Arai-Uchiyama [2, Lemma 6.2] and there exist some $R_5 \geq R_4$ and $C_{10} > 0$ such that for any $\varphi(r) \in C_0^1(R_5, \infty)$ we have

$$\int_\Omega \varphi(r)^2 [|Du|^2 + (\lambda - V_1(x))|u|^2] dx$$
$$\leq C_{10} \int_\Omega \{(\lambda - V_1(x))\varphi(r)^2 + \varphi'(r)^2\}|u|^2 dx.$$

Let $\varphi_1(t) \in C_0^\infty(\boldsymbol{R})$ be a function satisfying $0 \leq \varphi_1(t) \leq 1$ in $(-\infty, \infty)$, $\varphi_1(t) = 0$ for $t \leq 1/4$ and $t \geq 1$, and $\varphi_1(t) = 1$ for $1/2 \leq t \leq 3/4$. Let $R > 4R_5$ and put $\varphi_R(r) = \varphi_1(r/R)$. Then for any $R > 4R_5$, $\varphi_R(r) \in C_0^1(R_5, \infty)$ and (3.7) holds for any $r > R/2$. Integrate both sides of (3.7) in $R/2 < r < (3R)/4$. Then there exists some $C_{11} > 0$ such that for any $R > 4R_5$ we have

$$C_{11} R^\alpha \leq \int_{\frac{R}{2} < |x| < \frac{3}{4}R} [|Du|^2 + (\lambda - V_1(x))|u|^2] dx$$
$$\leq \int_\Omega \varphi_R(|x|)^2 [|Du|^2 + (\lambda - V_1(x))|u|^2] dx$$
$$\leq C_{10} \int_\Omega \{(\lambda - V_1(x))\varphi_R(r)^2 + \varphi_R'(r)^2\}|u|^2] dx$$
$$\leq \text{Const} \int_{R_0 < |x| < R} (\lambda - V_1(x))|u|^2 dx,$$

where in the last inequality, noting (3.6), we have used $\varphi_R'(r)^2 \leq \text{Const}\ r^{-2} \leq \text{Const}(\lambda - V_1(x))$. Thus we have (1.3). □

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