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Canonical Representations Generated by Translationally Quasi-invariant Measures

By

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§1. Introduction

Let G be a transformation group acting on some measurable space $(X, \mathfrak{B})$, and μ be a G-quasi-invariant probability measure. That is, μ_g defined by $\mu_g := \mu \circ g^{-1}$ is equivalent to μ $(\mu_g \cong \mu)$ for all $g \in G$. Then a natural representation of G in L^2_μ arises such that,

$$V_{\mu,\theta}(g) : f(x) \in L^2_\mu \longmapsto \sqrt{\frac{d\mu_g}{d\mu}}\theta(x, g) f(g^{-1}x) \in L^2_\mu,$$

where $\theta(x, g)$ is a 1-cocycle.

We call it a canonical representation generated by μ.

In this paper, X is always a locally convex Hausdorff linear space over $\boldsymbol{R}$ and $\mathfrak{B}$ is the cylindrical σ-algebra on X. And G consists of parallel displacement on X, $x \longmapsto x + \varphi$, where φ runs through a linear subspace Φ of X. Up to the present time the representation of this type is considered together with the representation such type as

$$U_\mu(x) : f(x) \in L^2_\mu \longmapsto \exp(i\langle x, x^* \rangle) f(x) \in L^2_\mu,$$

where $x^* \in X^*$ (topological dual space of X) in view of the field theory in quantum mechanics. *However we shall treat here* $V_{\mu,\theta}$ *alone* and discuss their various properties. The first important problem is a decomposition of these non irreducible representations. We shall carry out it using a direct integral of Hilbert spaces. This is one of main results of our subject and discussed in Section 2. If Φ-quasi-invariant measure is also Ψ-quasi-invariant, then it becomes an interesting problem to discuss relations of the two representations of

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Φ and Ψ. These considerations not only clarify the theoretical structure, but also offer a technical tool assuring that some kinds of reasoning will be carried out smoothly. Those will be done in Section 3. Section 4 is a study of a Φ-ergodic measure μ, a spectral measure σ and the multiplicity of the representation $V_{\mu,\theta}$. It will be shown that the ergodicity of μ derives that the spectral measure σ is also X^*-ergodic and the representation $V_{\mu,\theta}$ has uniform multiplicity, and that the multiplicity 1 together with the X^*-ergodicity of σ implies that μ is Φ-ergodic. However we don't have a definite relation with these three relations, though it seems that there exists some interesting connection between multiplicity and ergodicity. Next we shall consider countable direct product of such representations V_{μ_n,θ_n} $(n=1, 2, \cdots)$ in Section 5 and obtain a result like the theorem of Kakutani type. This is the second one of our main results. The third one discussed in Section 6 is a decision of maximal spectral type of Gaussian measures γ on locally convex spaces. Moreover it will be turned out that under assumptions that θ is a trivial 1-cocycle or some special one, $V_{\gamma,\theta}$ is equivalent to $V_{\gamma',\theta'}$, if and only if γ is equivalent to some translation of γ'.

§2. Irreducible Decomposition of Canonical Representations by Direct Integrals

2.1 Basic notation. Let X be a locally convex Hausdorff linear space over $\mathbb{R}$ and $\mathfrak{B}$ be the cylindrical σ-algebra on X. That is, $\mathfrak{B}$ is the minimal σ-field with which all the continuous linear functionals $x^*\in X^*$ are measurable. Now we shall take G as a translation group defined by some linear subspace Φ of X. So the probability measure which we are confronted with is Φ-quasi-invariant one which is defined by a relation,

$$\mu_\varphi \cong \mu$$

for all $\varphi\in\Phi$, where $\mu_\varphi(\cdot)=\mu(\cdot-\varphi)$. In what follows we always assume that L^2_μ is separable. Let $V_{\mu,\theta}$ be the canonical representation of Φ defined by

$$V_{\mu,\theta}(\varphi): h(x)\in L^2_\mu \longmapsto \sqrt{\frac{d\mu_\varphi}{d\mu}}(x)\theta(x, \varphi)h(x-\varphi)\in L^2_\mu, \tag{2.1}$$

where θ is a 1-cocycle. That is, $\theta(x, \varphi)$ is a complex valued measurable function of x for each fixed $\varphi\in\Phi$ whose absolute value is always 1 and

$$\theta(x, \varphi_2)\theta(x-\varphi_2, \varphi_1)=\theta(x, \varphi_1+\varphi_2) \tag{2.2}$$

for μ-a.e.x, and

$$\theta(x, 0)=1. \tag{2.3}$$

Now we shall demand that a one parameter group of operators $V_{\mu,\theta}(t\varphi)$ for each fixed $\varphi\in\Phi$ is continuous. Since

$$\left\|\sqrt{\frac{d\mu_{t\varphi}}{d\mu}}(x)-1\right\|_2 \longrightarrow 0 \ (t\rightarrow 0)$$

(For example, see [9].), for the above requirement it is necessary and sufficient that

$$\theta(x, t\varphi) \longrightarrow 0 \text{ in } \mu, \text{ as } t \rightarrow 0 \tag{2.4}$$

for each fixed $\varphi \in \Phi$. From now on we shall impose the above condition (2.4) upon θ. We remark that (2.4) is immediately extended to an n-variable form,

$$\theta(x, t_1\varphi_1+\cdots+t_n\varphi_n) \longrightarrow \theta(x, t_1^{\circ}\varphi_1+\cdots+t_n^{\circ}\varphi_n) \text{ in } \mu, \quad \text{as } (t_1, \cdots, t_n) \longrightarrow (t_1^{\circ}, \cdots, t_n^{\circ}) \tag{2.5}$$

for each fixed $\varphi_1, \cdots, \varphi_n \in \Phi$. Now $\langle V_{\mu,\theta}(\varphi)h, h\rangle_2$ ($\langle\cdot, \cdot\rangle_2$ is the inner product of L^2_μ.) is positive definite and continuous on any finite dimensional subspace of Φ. Therefore the infinite dimensional Bochner's theorem assures an existence of $\sigma_{\mu,\theta,h} \equiv \sigma_h$ called an h-adjoint measure on a measurable space $(\Phi^a, \mathfrak{C}_\Phi)$ such that

$$\widehat{\sigma_h}(\varphi) := \int_{\Phi^a} \exp(i\langle\varphi, \varphi^a\rangle)\sigma_h(d\varphi^a) = \langle V_{\mu,\theta}(\varphi)h, h\rangle_2, \tag{2.6}$$

where Φ^a is the algebraic dual space of Φ, $\mathfrak{C}_\Phi$ is the minimal σ-algebra on Φ^a with which all the linear functionals, $\varphi^a \in \Phi^a \longmapsto \langle\varphi, \varphi^a\rangle \in \boldsymbol{R}$ are measurable.

Theorem 2.1. *Let d be the Hellinger distance defined on probability measures on $(\Phi^a, \mathfrak{C}_\Phi)$. Then we have*

$$d(\sigma_h, \sigma_g) \leq \|h-g\|_2 \tag{2.7}$$

for all $h, g \in L^2_\mu$.

(The Hellinger distance is defined as follows;

$$d^2(\sigma_h, \sigma_g) := \int_{\Phi^a} \left|\sqrt{\frac{d\sigma_h}{dp}}(\varphi^a) - \sqrt{\frac{d\sigma_g}{dp}}(\varphi^a)\right|^2 p(d\varphi^a),$$

where p is an arbitrary σ-finite measure, as far as σ_h and σ_g is absolutely continuous with p.

Proof. We shall devide the proof into four steps.

(I) Put $L_n : \varphi^a \in \Phi^a \longmapsto (\langle\varphi_1, \varphi^a\rangle, \cdots, \langle\varphi_n, \varphi^a\rangle) \in \boldsymbol{R}^n$, where $\varphi_1, \cdots, \varphi_n$ is a linearly independent set of Φ. We shall observe the explicit form of the image measure $L_n\sigma_{\mu,\theta,h}$. So let us decompose X into a direct product using a dual system $\{\varphi_j^*\}$ ($\varphi_j^* \in X^*$ *s.t.*, $\langle\varphi_k, \varphi_j^*\rangle = \delta_{k,j}$) as follows;

$$x \in X \longleftrightarrow (x_1, \cdots, x_n, \xi) \in \boldsymbol{R}^n \times (\Phi_n^*)^\perp,$$

where $x_j=\langle x, \varphi_j^*\rangle$ and $\xi=x-\sum_{j=1}^n\langle x, \varphi_j^*\rangle\varphi_j$ which will be denoted by $q_n(x)$. Since the natural measurable space $(\boldsymbol{R}^n, \mathfrak{B}(\boldsymbol{R}^n))$ is standard, so any μ is decomposed into the regular conditional probabilities $\mu^\xi(\cdot)$ given $q_n=\xi$. Namely there exists a family of probability measures $\{\mu^\xi\}_{\xi\in(\Phi_n^*)^\perp}$ on $(\boldsymbol{R}^n, \mathfrak{B}(\boldsymbol{R}^n))$ which satisfies,

$\mu^\xi(A)$ is a measurable function of $\xi\in(\Phi_n^*)^\perp$ for each fixed $A\in\mathfrak{B}(\boldsymbol{R}^n)$, (The measurable structure on $(\Phi_n^*)^\perp$ is an induced one from $(X, \mathfrak{B})$.) and

$$\mu(A\times B)=\int_B\mu^\xi(A)q_n\mu(d\xi) \tag{2.8}$$

for all $A\in\mathfrak{B}(\boldsymbol{R}^n)$ and all measurable sets $B\subseteq(\Phi_n^*)^\perp$.

As μ is Φ-quasi-invariant, so μ^ξ is equivalent to the Lebesgue measure λ_n on $\boldsymbol{R}^n$ for $q_n\mu$-a.e.ξ. Thus we have

$$\mu\cong\lambda_n\times q_n\mu. \tag{2.9}$$

Here we shall take a unitary map,

$$T_n: h(x)\in L^2_\mu \longmapsto \sqrt{\frac{d\mu}{d(\lambda_n\times q_n\mu)}}(x_1, \cdots, x_n, \xi)h(x_1, \cdots, x_n, \xi)\in L^2_{\lambda_n\times q_n\mu}.$$

Then $V_{\mu,\theta}(t_1\varphi_1+\cdots+t_n\varphi_n)$ is converted as follows.

$$\begin{aligned}T_nV_{\mu,\theta}(t_1\varphi_1+\cdots+t_n\varphi_n)T_n^{-1}: h(x_1, \cdots, x_n, \xi) &\longmapsto \\ \theta(x_1\varphi_1+\cdots+x_n\varphi_n+\xi,\ t_1\varphi_1+\cdots+t_n\varphi_n)&h(x_1-t_1, \cdots, x_n-t_n, \xi).\end{aligned} \tag{2.10}$$

As θ is continuous in probability, so using the Slutsky's theorem (For example, see [4].) we deduce that there exists a jointly measurable function θ^* of $((x_1, \cdots, x_n), \xi, (t_1, \cdots, t_n))$ such that for each fixed $(t_1, \cdots, t_n)\in\boldsymbol{R}^n$,

$$\theta^*(x_1, \cdots, x_n, \xi, t_1, \cdots, t_n)=\theta(x_1\varphi_1+\cdots+x_n\varphi_n+\xi,\ t_1\varphi_1+\cdots+t_n\varphi_n)$$

for $\lambda_n\times q_n\mu$-a.e.$(x_1, \cdots, x_n, \xi)$.

(II) For a while we shall fix ξ and shall write x and t instead of $(x_1, \cdots, x_n)$ and $(t_1, \cdots, t_n)$, respectively. Further we shall write $\theta_\xi^*(x, t)$ instead of $\theta^*(x, \xi, t)$. Then it follows from (2.2) that

$$\theta_\xi^*(x, t)=\theta_\xi^*(x-t, s)^{-1}\theta_\xi^*(x, s+t) \tag{2.11}$$

for $\lambda_n\times\lambda_n\times\lambda_n$-a.e.$(x, s, t)$. Here we shall change the variable s to $\sigma=t+s-x$. Then (2.11) becomes

$$\theta_\xi^*(x, t)=\theta_\xi^*(x-t, x+\sigma-t)^{-1}\theta_\xi^*(x, x+\sigma) \tag{2.12}$$

for $\lambda_n\times\lambda_n\times\lambda_n$-a.e.$(x, t, \sigma)$, and therefore there exists a $\sigma_0\in\boldsymbol{R}^n$ such that

$$\theta_\xi^*(x, t)=\theta_\xi^*(x-t, x-t+\sigma_0)^{-1}\theta_\xi^*(x, x+\sigma_0) \tag{2.13}$$

for $\lambda_n\times\lambda_n$-a.e.(x, t). We put

$$q_n(x,\ \xi) := \theta_\xi^*(x,\ x+\sigma_0)^{-1}.$$

Then it follows from (2.13) that

$$\theta_\xi^*(x,\ \xi,\ t) = q_n(x-t,\ \xi) q_n(x,\ \xi)^{-1}, \tag{2.14}$$

for $\lambda_n \times q_n\mu \times \lambda_n$-a.e.$(x,\ \xi,\ t)$.

(III) We shall calculate the characteristic function of $L_n\sigma_h := L_n\sigma_{\mu,\theta,h}$.

$$\begin{aligned}
&\widehat{L_n\sigma_h}(t_1,\ \cdots,\ t_n) \\
&= \int_{R^n} \exp(i\sum_{j=1}^n t_j y_j) L_n\sigma_h(dy) \\
&= \int_{\Phi^a} \exp(i\sum_{j=1}^n t_j \langle \varphi_j,\ \varphi^a\rangle) \sigma_h(d\varphi^a) \\
&= \langle V_{\mu,\theta}(\sum_{j=1}^n t_j\varphi_j) h,\ h\rangle_2 = \langle T_n V_{\mu,\theta}(\sum_{j=1}^n t_j\varphi_j) T_n^{-1} T_n h,\ T_n h\rangle_2 \\
&= \int_{R^n}\int_{(\Phi_n^*)^\perp} \theta(x,\ \xi,\ t) T_n h(x-t,\ \xi) \overline{T_n h}(x,\ \xi) q_n\mu(d\xi) dx \\
&= \int_{R^n}\int_{(\Phi_n^*)^\perp} \exp(i\sum_{j=1}^n t_j y_j) |\mathcal{F}(q_n(\cdot,\ \xi) T_n h(\cdot,\ \xi))|^2 (y_1, \cdots,\ y_n) q_n\mu(d\xi) dy,
\end{aligned}$$

where $\mathcal{F}$ is the usual Fourier transform,

$$(\mathcal{F} f)(y_1,\ \cdots,\ y_n) = (2\pi)^{-n/2} \int_{R^n} \exp(i\sum_{j=1}^n y_j x_j) f(x_1,\ \cdots,\ x_n) dx.$$

Note that the last equality holds for almost all $t=(t_1,\ \cdots,\ t_n)$. However each term appeared in this series of equalities are all continuous function of t, so it holds for every point. It follows that

$$\frac{dL_n\sigma_h}{dy} = \int_{(\Phi_n^*)^\perp} |\mathcal{F}(q_n(\cdot,\ \xi) T_n h(\cdot,\ \xi))|^2 (y) q_n\mu(d\xi). \tag{2.15}$$

(IV) Put $p = 2^{-1}(\sigma_h + \sigma_g)$. Then there exists a linearly independent at most countable set $\{\varphi_1,\ \cdots,\ \varphi_n,\ \cdots\} \subset \Phi$ such that both $\frac{d\sigma_h}{dp}(\varphi^a)$ and $\frac{d\sigma_g}{dp}(\varphi^a)$ are measurable with respect to $\mathfrak{C}$, where $\mathfrak{C}$ is the minimal σ-field with which all the functions $\langle \varphi_n,\ \varphi^a\rangle$ are measurable. If the suffix n runs through $\{1,\ \cdots,\ N\}$, then a new sub-σ-field of $\mathfrak{C}$ appears. We shall denote it by $\mathfrak{C}_N$. Note that the conditional expectation $\frac{d\sigma_h}{dp}$ with respect to the sub-σ-field $\mathfrak{C}_N$ is $\frac{dL_N\sigma_h}{dL_N p}$. So using the martingale convergence theorem we have from (2.15)

$$\begin{aligned} d^2(\sigma_h, \sigma_g) &= \int_{\Phi^a}\left|\sqrt{\frac{d\sigma_h}{dp}(\varphi^a)}-\sqrt{\frac{d\sigma_g}{dp}(\varphi^a)}\right|^2 p(d\varphi^a) \\ &= \lim_{n\to\infty}\int_{R^n}\left|\sqrt{\frac{dL_n\sigma_h}{dL_np}(y)}-\sqrt{\frac{dL_n\sigma_g}{dL_np}(y)}\right|^2 dL_np(dy) \\ &= \lim_{n\to\infty}\int_{R^n}\left|\sqrt{\frac{dL_n\sigma_h}{dy}(y)}-\sqrt{\frac{dL_n\sigma_g}{dy}(y)}\right|^2 dy \\ &\leq \lim_{n\to\infty}\int_{R^n}\int_{(\Phi_n^*)^\perp}|\mathcal{F}(q_n(\cdot, \xi)T_n(h-g)(\cdot, \xi))|^2 q_n\mu(d\xi)dy \\ &= \|h-g\|_2^2. \qquad \square \end{aligned}$$

Theorem 2.2. *There exists a probability measure $\sigma_{\mu,\theta}$ on $(\Phi^a, \mathfrak{C}_\Phi)$ which have the following property.*
(P) $\sigma_{\mu,\theta}(E)=0$ *if and only if* $\sigma_{\mu,\theta,h}(E)=0$ *for all* $h\in L^2_\mu$.

Proof. Take a countable dense set $\{h_1, \cdots, h_n, \cdots\}\subset L^2_\mu$ and choose positive sequence α_n $(n=1, \cdots)$ such that $\sum_{n=1}^\infty \alpha_n\|h_n\|_2^2=1$. Then a measure $\sum_{n=1}^\infty \alpha_n\sigma_{\mu,\theta,h_n}$ is a desired one in virtue of the above theorem. $\square$

$\sigma_{\mu,\theta}$ is called the spectral measure of the representation $V_{\mu,\theta}$, and sometimes it is simply written as σ, if no confusion arises.

Theorem 2.3. *Let h, $g\in L^2_\mu$. Then there exists a complex valued measure $\sigma_{\mu,\theta,h,g}\equiv\sigma_{h,g}$ on $(\Phi^a, \mathfrak{C}_\Phi)$ such that*

$$\langle V_{\mu,\theta}(\varphi)h, g\rangle_2=\int_{\Phi^a}\exp(i\langle\varphi, \varphi^a\rangle)\sigma_{h,g}(d\varphi^a),$$

for all $\varphi\in\Phi$ which satisfies

(2.16) $\sigma_{h,g}$ *is absolutely continuous with* $\sigma_{\mu,\theta}$

(2.17) $$\left|\int_{\Phi^a}F(\varphi^a)\sigma_{h,g}(d\varphi^a)\right|\leq\|F\|_{\infty,\sigma}\|h\|_2\|g\|_2$$

for all $F\in L^\infty_\sigma$.

Proof. The existence of such $\sigma_{h,g}$ and the absolute continuity is obvious. Next using the same technique in the proof of Theorem 2.1, we have

(2.18)
$$\frac{dL_n\sigma_{h,g}}{dy}=\int_{(\Phi_n^*)^\perp}\mathcal{F}(q_n(\cdot, \xi)T_nh(\cdot, \xi))(y, \xi)\overline{\mathcal{F}}(q_n(\cdot, \xi)T_ng(\cdot, \xi))(y)q_n\mu(d\xi).$$

It follows that

$$\int_{\Phi^a}\left|\frac{d\sigma_{h,g}}{d\sigma_{\mu,\theta}}(\varphi^a)\right|\sigma_{\mu,\theta}(d\varphi^a)=\lim_{n\to\infty}\int_{R^n}\left|\frac{dL_n\sigma_{h,g}}{dL_n\sigma_{\mu,\theta}}(y)\right|L_n\sigma_{\mu,\theta}(dy)$$
$$=\lim_{n\to\infty}\int_{R^n}\left|\frac{dL_n\sigma_{h,g}}{dy}\right|dy$$
$$\leq\underline{\lim}_{n\to\infty}\int_{R^n}\left\{\int_{(\Phi_n^*)^\perp}|\mathcal{F}(q_n(\cdot,\ \xi)T_nh(\cdot,\ \xi))|^2q_n\mu(d\xi)\right\}^{1/2}\cdot$$
$$\left\{\int_{(\Phi_n^*)^\perp}|\mathcal{F}(q_n(\cdot,\ \xi)T_ng(\cdot,\ \xi))|^2q_n\mu(d\xi)\right\}^{1/2}dy\leq\|h\|_2\|g\|_2.$$

(2.17) is a direct consequence of the above inequality. □

By virtue of the above theorem, we can define a bounded operator $T_{\mu,\theta}(F)\equiv T(F)$ on L^2_μ for each $F\in L^\infty_\sigma$ such that

$$\langle T(F)h,\ g\rangle_2:=\int_{\Phi^a}F(\varphi^a)\sigma_{\mu,\theta,h,g}(d\varphi^a), \tag{2.19}$$

for all $h,\ g\in L^2_\mu$.

Proposition 2.1. *$T(F)$ has the following properties.*

$$T(F)^*=T(\bar{F}) \tag{2.20}$$

$$T(F\cdot G)=T(F)\cdot T(G) \tag{2.21}$$

$$T \text{ is injective and } \|T(F)\|_{op}=\|F\|_\infty. \tag{2.22}$$

Proof. As for (2.20),

$$\widehat{\sigma_{g,h}}(\varphi)=\langle V_{\mu,\theta}(\varphi)g,\ h\rangle_2=\overline{\langle V_{\mu,\theta}(-\varphi)h,\ g\rangle_2}$$
$$=\int_{\Phi^a}\exp(i\langle\varphi,\ \varphi^a\rangle)\overline{\sigma_{h,g}}(d\varphi^a)=\widehat{\overline{\sigma_{h,g}}}(\varphi).$$

So we have $\sigma_{g,h}=\overline{\sigma_{h,g}}$. It follows that

$$\langle T(F)^*h,\ g\rangle_2=\overline{\int_{\Phi^a}F(\varphi^a)\sigma_{g,h}(d\varphi^a)}$$
$$=\int_{\Phi^a}\bar{F}(\varphi^a)\sigma_{h,g}(d\varphi^a)=\langle T(\bar{F})h,\ g\rangle_2.$$

We shall prove (2.21). It is easy to see that the equality holds in the case $F(\varphi^a)=\exp(i\langle\varphi_1,\ \varphi^a\rangle)$ and $G(\varphi^a)=\exp(i\langle\varphi_2,\ \varphi^a\rangle)$. Next we shall substitute the indicator function χ_E of a measurable set $E\in\mathfrak{C}$ for $\exp(i\langle\varphi_1,\ \varphi^a\rangle)$. Then $\langle T(\chi_E\cdot G)h,\ g\rangle_2$ and $\langle T(\chi_E)T(G)h,\ g\rangle_2$ are both regarded as complex valued measures on $\mathfrak{C}$ and their characteristic functions coincide by the first step. Thus they are equal to each other. Lastly we substitute χ_F, $F\in\mathfrak{C}$ for $G=\exp(i\langle\varphi_2,\ \varphi^a\rangle)$ and repeat the same argument as above. So (2.21) certainly holds in the case that F and G are step functions. The general case easily follows from (2.17).

As for (2.22), if we have $T(F)=0$ for some $F\in L_{\sigma}^{\infty}$, then we may assume that $F(\varphi^{a})\geq 0$ for σ-a.e.φ^{a}, because $T(|F|^{2})=T(F)T(\bar{F})=0$. Since $\int_{\Phi^{a}}F(\varphi^{a})\sigma_{h}(d\varphi^{a})=0$ for all $h\in L_{\mu}^{2}$, $F(\varphi^{a})=0$ for σ_{h}-a.e.φ^{a}, which is equivalent to $F(\varphi^{a})=0$ for $\sigma_{\mu,\theta}$-a.e.φ^{a}.

The norm equality is a direct consequence of $*$-isomorphism $T_{\mu,\theta}$. (For example see p5 in [8] or refer the later discussions.) □

2.2 Decomposition of $V_{\mu,\theta}$ by a direct integral. We shall start at the following equality,

$$\langle T(F)h,\ g\rangle_{2}=\int_{\Phi^{a}}F(\varphi^{a})\frac{d\sigma_{h,g}}{d\sigma}(\varphi^{a})\sigma(d\varphi^{a}), \tag{2.23}$$

and follow after J. Diximier [1]. First we shall take and fix a dense $\boldsymbol{Q}$-linear subspace $V\subset L_{\mu}^{2}$ whose cardinal is countable. Then there exists a σ-negligible set N such that for every $\varphi^{a}\in N^{c}$, the function $(v,\ v')\longmapsto\frac{d\sigma_{v,v'}}{d\sigma}(\varphi^{a})$ is a positive definite sesquilinear form on V. Let $H_{\mu,\theta}(\varphi^{a})\equiv H(\varphi^{a})$ be the Hilbert space obtained from V by passing to the quotient and completing with this sesquilinear form. We put $H(\varphi^{a})=\{0\}$ for $\varphi^{a}\in N$ and denote the image of v_{n} by the canonical map $V\longmapsto H(\varphi^{a})$ by $v_{n}(\varphi^{a})$. Then

$$\frac{d\sigma_{v_{n},v_{m}}}{d\sigma}(\varphi^{a})=\langle v_{n}(\varphi^{a})|v_{m}(\varphi^{a})\rangle_{H(\varphi^{a})},\ \text{if}\ \varphi^{a}\in N^{c}\ \text{and}\ =0\ \text{otherwise}.$$

So they are $\mathfrak{C}_{\Phi}$-measurable. Further $\{v_{n}(\varphi^{a})\}_{n}$ forms a total sequence ($\Longleftrightarrow$ spans a dense linear subspace) in each $H(\varphi^{a})$. Consequently there exists exactly one measurable structure $R_{\mu,\theta}\equiv R$ on the $H(\varphi^{a})$'s with which every field, $\varphi^{a}\longmapsto v_{n}(\varphi^{a})$ ($n=1,\ \cdots,$) is measurable. (See, p167 in [1].) Hence a direct integral $H:=\int^{\oplus}H(\varphi^{a})\sigma(d\varphi^{a})$ has been constructed with the spectral measure σ. We note that

$$\dim H(\varphi^{a})\geq 1 \tag{2.24}$$

for σ-a.e.φ^{a}. Because $K:=\{\varphi^{a}|\dim H(\varphi^{a})=0\}\ni\varphi^{a}$ is equivalent to $\frac{d\sigma_{v_{n},v_{m}}}{d\sigma}(\varphi^{a})=0$ for all n, m from which it follows that $\sigma_{v_{n},v_{m}}(K)=0$. As $\{v_{n}\}_{n}$ is dense, so we have $\sigma(K)=0$ by virtue of (2.7).

Theorem 2.4. (1) *L_{μ}^{2} is isomorphic to H by a unitary map $S_{\mu,\theta}\equiv S$ such that*

$$ST(F)=T^H(F)S \tag{2.25}$$

for all $F\in L^\infty_\sigma$, *where* $T^H(F): \eta(\varphi^a)\in H \longmapsto F(\varphi^a)\eta(\varphi^a)\in H$ *is the diagonalisable operator.*

(2) *If* L^2_μ *is isomorphic to another direct integral* $H'=\int^\oplus H'(\varphi^a)\sigma'(d\varphi^a)$ *with another spectral measure* σ' *by a map* S' *which have the same property as* (2.25), *then* $\dim H(\varphi^a)=\dim H'(\varphi^a)$ *for* σ*-a.e.*φ^a.

Proof. For (1), let F_n $(n=1, \cdots, N)\in L^\infty_\sigma$. Then

$$\begin{aligned}\left\|\sum_{n=1}^N T(F_n)v_n\right\|_2^2 &= \sum_{n,m=1}^N \int_{\Phi^a} F_n(\varphi^a)\bar{F}_m(\varphi^a)\frac{d\sigma_{v_n,v_m}}{d\sigma}(\varphi^a)\sigma(d\varphi^a)\\ &= \sum_{n,m=1}^N \int_{\Phi^a} F_n(\varphi^a)\bar{F}_m(\varphi^a)\langle v_n(\varphi^a)|v_m(\varphi^a)\rangle_{H(\varphi^a)}\sigma(d\varphi^a)\\ &= \int_{\Phi^a}\left\|\sum_{n=1}^N F_n(\varphi^a)v_n(\varphi^a)\right\|^2_{H(\varphi^a)}\sigma(d\varphi^a).\end{aligned}$$

Therefore a map $S: \sum_{n=1}^N T(F_n)v_n \longmapsto \sum_{n=1}^N F_n(\varphi^a)v_n(\varphi^a)$ is well defined and is extended to a unitary map from L^2_μ to H. That the image of S is dense follows from the totality of $\{v_n\}_n$.
For (2), from the assumption, $S'S^{-1}$ is a decomposable operator. That is, $S'S^{-1}$ induces an isomorphic operator $\Sigma(\varphi^a): H(\varphi^a) \longmapsto H'(\varphi^a)$ for λ-a.e.φ^a. (See, p187 in [1].) Thus the corresponding dimensions are the same one. □

Example 1. $\boldsymbol{R}^n$-quasi-invariant measure μ on $(\boldsymbol{R}^n, \mathfrak{B}(\boldsymbol{R}^n))$. That is, $\mu(dx)=\rho(x)dx$, $\rho(x)>0$ for λ_n-a.e.x.
In this case, as we have seen in (2.14), for any 1-cocycle $\theta(x, \varphi)$ there exists some measurable function $q(x)$ with $|q(x)|=1$ such that for each $\varphi\in\boldsymbol{R}^n$, $\theta(x, \varphi)=q(x-\varphi)q(x)^{-1}$ for a.e.$x\in\boldsymbol{R}^n$.

$$\begin{aligned}\langle V_{\mu,q}(\varphi)h, g\rangle_2 &= \int_{\boldsymbol{R}^n}\sqrt{\frac{d\mu_\varphi}{d\mu}}(x)q(x-\varphi)h(x-\varphi)\bar{q}(x)\bar{g}(x)\mu(dx)\\ &= \int_{\boldsymbol{R}^n}\sqrt{\rho(x-\varphi)}q(x-\varphi)h(x-\varphi)\sqrt{\rho(x)}\,\bar{q}(x)\bar{g}(x)dx\\ &= \int_{\boldsymbol{R}^n}\exp(i\langle\varphi, y\rangle)\mathcal{F}(q\sqrt{\rho}h)(y)\bar{\mathcal{F}}(q\sqrt{\rho}g)(y)dy.\end{aligned}$$

Thus the unitary map S_q is $\rho(y)^{-1/2}\,\mathcal{F}(q\sqrt{\rho}h)(y)$, the spectral measure is μ, and $H_q(y)=\boldsymbol{R}$.

Theorem 2.5. *Let* μ *and* μ' *be* Φ*-quasi-invariant probability measures on* $(X, \mathfrak{B})$, *and* θ *and* θ' *be* 1*-cocycles. Then* $(V_{\mu,\theta}, \Phi)$ *is equivalent to* $(V_{\mu',\theta'}, \Phi)$, *if and only if* $\sigma_{\mu,\theta}\cong\sigma_{\mu',\theta'}$ *and* $\dim H_{\mu,\theta}(\varphi^a)=\dim H_{\mu',\theta'}(\varphi^a)$ *for* $\sigma_{\mu,\theta}-$*a.e.*φ^a.

Proof. For the proof, we only have to note that an operator which commutes with every diagonalisable operator is decomposable. □

Theorem 2.6. *Let λ be any finite measure on $(\Phi^a, \mathfrak{C}_\Phi)$ which is absolutely continuous with $\sigma_{\mu,\theta}$. Then there exists some $h \in L^2_\mu$ such that $\sigma_{\mu,\theta,h} = \lambda$.*

Proof. We shall take $\eta \in \int^{\oplus} H_{\mu,\theta}(\varphi^a)\sigma_{\mu,\theta}(d\varphi^a)$ such that $\|\eta(\varphi^a)\| = 1$ for λ-a.e. φ^a. The existence of such η is assured by (2.24). (Also see, p162 in [1].) Then an element h defined by $h(\varphi^a) = S^{-1}\left(\sqrt{\frac{d\lambda}{d\sigma}}\eta\right)$ is a desired one. □

§3. Lifting and Restriction of Canonical Representations

3.1 Dense case. As before let μ be a probability measure on $(X, \mathfrak{B})$ and put

$$A^{\circ}_\mu := \{\varphi \in X \mid \mu_{t\varphi} \cong \mu \text{ for all } t \in \boldsymbol{R}\},$$

which is the maximal linear space consists of admissible shifts of μ. We give a metric d_μ on A°_μ such that

$$d_\mu(\varphi_1, \varphi_2) := \sup_{|t| \leq 1} \left\| \sqrt{\frac{d\mu_{t\varphi_1}}{d\mu}} - \sqrt{\frac{d\mu_{t\varphi_2}}{d\mu}} \right\|_2.$$

It is well known that d_μ has the following properties. (See, [9].)

(P.1) d_μ is translationally invariant and (A°_μ, d_μ) is separable. (The separability is a consequence of the assumption that L^2_μ is separable.)
(P.2) $\varphi \to 0$ in d_μ if and only if $\|\mu_{t\varphi} - \mu\|_{\text{tot}} \to 0$ for any fixed $t \in \boldsymbol{R}$.
(P.3) The topology derived from d_μ is stronger than the weak topology $\sigma(X, X^*)$.
(P.4) If $\{\varphi_n\}_n$ is a Cauchy sequence in d_μ, and $\varphi_n \to \varphi$ in $\sigma(X, X^*)$, then $\varphi_n \to \varphi$ in d_μ.
(P.5) $\varphi \to 0$ in d_μ, if and only if $V(t\varphi) \to I$ in the strong operator topology for any fixed $t \in \boldsymbol{R}$.

Proposition 3.1. *Let Φ be a linear subspace of A°_μ and $\theta(x, \varphi)$ be a 1-cocycle defined for $\varphi \in \Phi$. Then $\varphi \to 0$ in d_μ if and only if $\langle \varphi, \cdot \rangle \to 0$ in $\sigma_{\mu,\theta}$.*

Proof. ($\Rightarrow$) From the assumption and (P.5), we have $\langle V_{\mu,\theta}(t\varphi)h, h\rangle_2 \to \|h\|_2^2$ for each fixed $h \in L^2_\mu$. Especially, $\int_{\Phi^a} \exp(it\langle \varphi, \varphi^a\rangle)\sigma_{\mu,\theta}(d\varphi^a) \to 1$ for each fixed $t \in \boldsymbol{R}$, which is equivalent to

$\langle \varphi, \cdot \rangle \to 0$ in $\sigma_{\mu,\theta}$.
($\Leftarrow$) Since $\langle \varphi, \cdot \rangle$ converges to 0 in $\sigma_{\mu,\theta,1}$, so

$$\langle V_{\mu,\theta}(t\varphi)1, 1\rangle_2 = \int_{\Phi^a} \exp(i\langle \varphi, \varphi^a\rangle)\sigma_{\mu,\theta,1}(d\varphi^a) \to 1$$

for each fixed $t \in \boldsymbol{R}$. Thus we have

$$\int_X \left\{1 - \sqrt{\frac{d\mu_{t\varphi_n}}{d\mu}(x)}\right\}^2 \mu(dx) = 2\left\{1 - \int_X \sqrt{\frac{d\mu_{t\varphi_n}}{d\mu}(x)}\mu(dx)\right\}$$
$$\leq 2\{1 - \mathrm{Re}\langle V_{\mu,\theta}(t\varphi)1, 1\rangle\} \to 0. \qquad \square$$

Now let Φ and Ψ be two linear subspaces of A_μ° such that $\Phi \subseteq \Psi \subseteq A_\mu^\circ$ and ι be the imbedding map from Φ to Ψ. We shall denote the corresponding operator $T_{\mu,\theta}(\cdot)$ to Φ and Ψ by $T^{\Phi}_{\mu,\theta}(\cdot)$ and $T^{\Psi}_{\mu,\theta}(\cdot)$, respectively. Then it is easily checked that

$$\langle T^{\Phi}_{\mu,\theta}(E)h, g\rangle_2 = \langle T^{\Psi}_{\mu,\theta}(({}^t\iota)^{-1}(E))h, g\rangle_2 \tag{3.1}$$

for all $E \in \mathfrak{C}_\Phi$. Hence the spectral measure can be chosen as

$$\sigma^{\Phi}_{\mu,\theta} = {}^t\iota\sigma^{\Psi}_{\mu,\theta}. \tag{3.2}$$

Lemma 3.1. *If Φ is dense in Ψ with the derived topology from d_μ, then* $\mathfrak{C}_\Psi = ({}^t\iota)^{-1}(\mathfrak{C}_\Phi)$ mod $\sigma^{\Psi}_{\mu,\theta}$.

Proof. Put

$$\mathfrak{C} := \{E \in \mathfrak{C}_\Psi \mid {}^\exists E_1 \in \mathfrak{C}_\Phi \text{ s.t., } E = ({}^t\iota)^{-1}(E_1) \bmod \sigma^{\Psi}_{\mu,\theta}\}.$$

Then $\mathfrak{C}$ is a σ-field. Moreover for any fixed $\psi \in \Psi$, $\langle \psi, \psi^a\rangle$ is equal to a $\mathfrak{C}$-measurable function for $\sigma^{\Psi}_{\mu,\theta}$-a.e.$\psi^a$, because $\langle \varphi_n, \psi^a\rangle$ converges to $\langle \psi, \psi^a\rangle$ in $\sigma^{\Psi}_{\mu,\theta}$, where $\{\varphi_n\}_n$ is any sequence of Φ such that $d_\mu(\varphi_n, \psi) \to 0$ ($n \to \infty$). $\square$

Under the same assumption of Lemma 3.1 we also see that for any $F \in \mathrm{L}^\infty_{\sigma^{\Psi}_{\mu,\theta}}$, there exists $f \in \mathrm{L}^\infty_{\sigma^{\Phi}_{\mu,\theta}}$ such that $F(\psi^a) = f({}^t\iota\psi^a)$ for $\sigma^{\Psi}_{\mu,\theta}$-a.e.$\psi^a$.

Theorem 3.1. *Let $\Phi \subseteq \Psi \subseteq A_\mu^\circ$ and Φ be dense in Ψ. If an expression of the direct integral for $(V_{\mu,\theta}, \Phi)$ is $\int^\oplus H^{\Phi}_{\mu,\theta}(\varphi^a)\sigma^{\Phi}_{\mu,\theta}(d\varphi^a)$, then $\int^\oplus H^{\Phi}_{\mu,\theta}({}^t\iota\psi^a)$ $\sigma^{\Psi}_{\mu,\theta}(d\psi^a)$ is an expression of the direct integral for $(V_{\mu,\theta}, \Psi)$.*

Proof. For the proof it is enough to show that $\dim H^{\Phi}_{\mu,\theta}({}^t\iota\psi^a) = \dim H^{\Psi}_{\mu,\theta}(\psi^a)$ for $\sigma^{\Psi}_{\mu,\theta}$-a.e.$\psi^a$.
Now take $F \in \mathrm{L}^\infty_{\sigma^{\Psi}_{\mu,\theta}}$ and $h, g \in \mathrm{L}^2_\mu$. Then

$$\langle T^{\Psi}_{\mu,\theta}(F)h,\ g\rangle_2=\langle T^{\Psi}_{\mu,\theta}(f\circ{}^t\iota)h,\ g\rangle_2=\langle T^{\Phi}_{\mu,\theta}(f)h,\ g\rangle_2.$$

Thus,

$$(3.3)\quad \int_{\Psi^a}F(\psi^a)\langle S^{\Psi}_{\mu,\theta}h|S^{\Psi}_{\mu,\theta}g\rangle(\psi^a)\sigma^{\Psi}_{\mu,\theta}(d\psi^a) = \int_{\Psi^a}F(\psi^a)\langle S^{\Phi}_{\mu,\theta}h|S^{\Phi}_{\mu,\theta}g\rangle({}^t\iota\psi^a)\sigma^{\Psi}_{\mu,\theta}(d\psi^a).$$

Here we shall choose a sequence $\{h_n\}_n\subset L^2_\mu$ which satisfies $\{S^{\Psi}_{\mu,\theta}h_n(\psi^a)\}_n$ and $\{S^{\Phi}_{\mu,\theta}h_n(\varphi^a)\}_n$ form total sequence at every $H^{\Psi}_{\mu,\theta}(\psi^a)$ and $H^{\Phi}_{\mu,\theta}(\varphi^a)$, respectively. Then it follows from (3.3) that for all N and for all $q_n\in\boldsymbol{Q}$, $\|\sum_{n=1}^N q_nS^{\Psi}_{\mu,\theta}h_n(\psi^a)\| =\|\sum_{n=1}^N q_nS^{\Phi}_{\mu,\theta}h_n({}^t\iota\psi^a)\|$ for $\sigma^{\Psi}_{\mu,\theta}$-a.e.$\psi^a$. Therefore a map $S^{\Psi}_{\mu,\theta}h_n(\psi^a)\longmapsto S^{\Phi}_{\mu,\theta}h_n({}^t\iota\psi^a)$ is extended to a unitary map from $H^{\Psi}_{\mu,\theta}(\psi^a)$ to $H^{\Phi}_{\mu,\theta}({}^t\iota\psi^a)$. □

Theorem 3.2. *Let μ and μ' be probability measures on $(X,\mathfrak{B})$, Φ, Ψ be linear subspaces of X such that $\Phi\subseteq\Psi\subseteq A^\circ_\mu\cap A^\circ_{\mu'}$, and θ, θ' be 1-cocycles defined on $X\times\Psi$.*

(1) *If $\sigma^{\Phi}_{\mu,\theta}\cong\sigma^{\Phi}_{\mu',\theta'}$ and Φ is dense in Ψ with either d_μ or $d_{\mu'}$, then it also holds with another one and $\sigma^{\Psi}_{\mu,\theta}\cong\sigma^{\Psi}_{\mu',\theta'}$.*

(2) *If $(V_{\mu,\theta},\ \Phi)\cong(V_{\mu',\theta'},\ \Phi)$ and Φ is dense in Ψ with either d_μ or $d_{\mu'}$, then it also holds that $(V_{\mu,\theta},\ \Psi)\cong(V_{\mu',\theta'},\ \Psi)$.*

Proof. (1). Assume that Φ is dense in Ψ. Then for any $\psi\in\Psi$, there exists a sequence $\{\varphi_n\}_n\subset\Phi$ such that $d_\mu(\varphi_n,\ \psi)\to 0$ and thus $\varphi_n\to\psi$ in $\sigma(X,\ X^*)$. It follows from the assumption and from Proposition 3.1 that $\{\varphi_n\}_n$ is also a Cauchy sequence in $d_{\mu'}$. Therefore they converge to ψ by virtue of (P.4).
As for the second half of (a), repeating the same work in the proof of Lemma 3.1 we see that for any $E\in\mathfrak{C}_\Psi$ there exists $E_1\in\mathfrak{C}_\Phi$ such that $\sigma^{\Psi}_{\mu,\theta}(E\ominus({}^t\iota)^{-1}(E_1)) =\sigma^{\Psi}_{\mu',\theta'}(E\ominus({}^t\iota)^{-1}(E_1))=0$. The equivalence of measures immediately follows from this matter.
For (2), it is a direct consequence of (1) in the above theorem and Theorem 3.1. □

3.2 General case. Let Φ and Ψ be two linear subspaces of A°_μ such that $\Phi\subseteq\Psi$. We shall examine how the expression of the direct integral for $(V_{\mu,\theta},\ \Psi)$ derives it for $(V_{\mu,\theta},\ \Phi)$. So let us take linearly independent sets $\{\varphi_n\}_n$ and $\{\varphi'_n\}_n$ such that φ_n $(n=1,\cdots)$ span a dense linear subspace Φ_0 of Φ, and both φ_n and φ'_n $(n=1,\cdots)$ span a dense linear subspace Ψ_0 of Ψ. From what we have seen in 3.1, it is essential to observe the direct integrals for $(V_{\mu,\theta},\ \Phi_0)$ and $(V_{\mu,\theta},\ \Psi_0)$ and to discuss their mutual relation for this problem. An advantage of changing the spaces into new ones is that $(\Psi_0,\ \mathfrak{C}_{\Psi_0})$ is Borel isomorphic to the natural measurable space $(\boldsymbol{R}^\infty,\ \mathfrak{B}(\boldsymbol{R}^\infty))$ of the countable direct product of $\boldsymbol{R}$ and therefore $(\Psi_0^a, \mathfrak{C}_{\Psi_0})$ is a standard space. In particular, $\sigma_{\mu,\theta}^{\Psi_0}$ is decomposed to regular conditional probabilities $Q_{\mu,\theta}^{\varphi_0^a}$ given ${}^t\iota_0=\varphi_0^a$ as follows, where ι_0 is the imbedding map from Φ_0 to Ψ_0.

$$\sigma_{\mu,\theta}^{\Psi_0}(({}^t\iota_0)^{-1}(E)\cap F)=\int_E Q_{\mu,\theta}^{\varphi_0^a}(F)\sigma_{\mu,\theta}^{\Phi_0}(d\varphi_0^a) \tag{3.4}$$

for all $E\in\mathfrak{C}_{\Phi_0}$, where $\sigma_{\mu,\theta}^{\Phi_0}={}^t\iota_0\sigma_{\mu,\theta}^{\Psi_0}$. Let R^{Ψ_0} be the measurable field used to define the direct integral $\int^{\oplus} H_{\mu,\theta}^{\Psi_0}(\psi_0^a)\sigma_{\mu,\theta}^{\Psi_0}(d\psi_0^a)$ and put

$$H_{\mu,\theta}^{\Phi_0}(\varphi_0^a):=\left\{\eta\in R^{\Psi_0}\middle|\int_{\Psi_0^a}\|\eta(\psi_0^a)\|^2 Q_{\mu,\theta}^{\varphi_0^a}(d\psi_0^a)<\infty\right\}.$$

In order to assign a measurable field with respect to $\{H_{\mu,\theta}^{\Phi_0}(\varphi_0^a)\}$, we shall take $\{\eta_n\}_n$ which satisfies

(P.6) if $\dim H_{\mu,\theta}^{\Psi_0}(\psi_0^a)=\infty$, $\eta_1(\psi_0^a), \cdots, \eta_n(\psi_0^a), \cdots$ is a c.o.n.s.in $H_{\mu,\theta}^{\Psi_0}(\psi_0^a)$,
(P.7) if $\dim H_{\mu,\theta}^{\Psi_0}(\psi_0^a)=d<\infty$, $\eta_1(\psi_0^a), \cdots, \eta_d(\psi_0^a)$ is a c.o.n.s.in $H_{\mu,\theta}^{\Psi_0}(\psi_0^a)$ and $\eta_n(\psi_0^a)=0$ for $n>d$.
(For the existence of such $\{\eta_n\}_n$ see p166 in [1].)

Further we shall take a countable algebra $\mathcal{F}_0=\{F_n\}_n$ which generates $\mathfrak{C}_{\Psi_0}$. Then $\chi_{F_p}\cdot\eta_n$ $(n, p=1, \cdots)$ belongs to $H_{\mu,\theta}^{\Phi_0}(\varphi_0^a)$ and forms a total sequence in $H_{\mu,\theta}^{\Phi_0}(\varphi_0^a)$, as is easily seen. Besides,

$$\langle\chi_{F_p}\cdot\eta_n|\chi_{F_q}\cdot\eta_m\rangle(\varphi_0^a)=\delta_{n,m}Q_{\mu,\theta}^{\varphi_0^a}(F_p\cap F_q)$$

is $\mathfrak{C}_{\Phi_0}$-measurable. Thus there exists a unique measurable field R^{Φ_0} with which all the $\chi_{F_p}\cdot\eta_n$ are measurable, (See, p167 in [1].) and a direct integral $\int^{\oplus} H_{\mu,\theta}^{\Phi_0}(\varphi_0^a)\sigma_{\mu,\theta}^{\Phi_0}(d\varphi_0^a)$ is defined with the measure $\sigma_{\mu,\theta}^{\Phi_0}$.

Theorem 3.3. *An expression of the direct integral* $(V_{\mu,\theta}, \Phi)$ *is* $\int^{\oplus} H_{\mu,\theta}^{\Phi_0}({}^tJ\varphi^a)\sigma_{\mu,\theta}^{\Phi}(d\varphi^a)$, *where* J *is the imbedding map from* Φ_0 *to* Φ.

Proof. By Theorem 3.1, it is enough to show that L_μ^2 is canonically isomorphic to $\int^{\oplus} H_{\mu,\theta}^{\Phi_0}(\varphi_0^a)\sigma_{\mu,\theta}^{\Phi_0}(d\varphi_0^a)$. For this we put $h_n:=(S_{\mu,\theta}^{\Psi_0})^{-1}\eta_n$ and take $f\in L^\infty_{\sigma_{\mu,\theta}^{\Phi_0}}$ and $F_n, F_n'\in L^\infty_{\sigma_{\mu,\theta}^{\Psi_0}}$ $(n, n'=1, \cdots, N)$. Then

$$\langle T_{\mu,\theta}^{\Phi_0}(f)(\sum_{n=1}^N T_{\mu,\theta}^{\Psi_0}(F_n)h_n), \sum_{n=1}^N T_{\mu,\theta}^{\Psi_0}(F_n')h_n\rangle_2$$
$$=\langle T_{\mu,\theta}^{\Psi_0}(f\circ{}^t\iota_0)(\sum_{n=1}^N T_{\mu,\theta}^{\Psi_0}(F_n)h_n), \sum_{n=1}^N T_{\mu,\theta}^{\Psi_0}(F_n')h_n\rangle_2$$

$$
\begin{aligned}
&= \sum_{n,m=1}^{N} \int_{\Psi_0^a} f({}^t\iota_0 \psi_0^a) F_n(\psi_0^a) \overline{F'_m}(\psi_0^a) \langle \eta_n | \eta_m \rangle (\psi_0^a) \sigma_{\mu,\theta}^{\Psi_0}(d\psi_0^a) \\
&= \sum_{n,m=1}^{N} \int_{\Phi_0^a} f(\varphi_0^a) \sigma_{\mu,\theta}^{\Phi_0}(d\varphi_0^a) \int_{\Psi_0^a} F_n(\psi_0^a) \overline{F'_m}(\psi_0^a) \langle \eta_n | \eta_m \rangle (\psi_0^a) Q_{\mu,\theta}^{\varphi_0^a}(d\psi_0^a) \\
&= \int_{\Phi_0^a} f(\varphi_0^a) \langle \sum_{n=1}^{N} F_n \cdot \eta_n \, \Big| \, \sum_{n=1}^{N} F'_n \cdot \eta_n \rangle (\varphi_0^a) \sigma_{\mu,\theta}^{\Phi_0}(d\varphi_0^a).
\end{aligned}
$$

So a map $S_{\mu,\theta}^{\Phi_0} : \sum_{n=1}^{N} T_{\mu,\theta}^{\Psi_0}(F_n) h_n \longmapsto \sum_{n=1}^{N} F_n \eta_n$ has a unitary extension from L_μ^2 to $\int^{\oplus} H_{\mu,\theta}^{\Phi_0}(\varphi_0^a) \sigma_{\mu,\theta}^{\Phi_0}(d\varphi_0^a)$ possessing the Property (2.25). □

Corollary 3.4. *If the representation* $(V_{\mu,\theta}, \Psi)$ *has uniform multiplicity, say* n, *(That is,* $\dim H_{\mu,\theta}^{\Psi}(\psi^a) = n$ *for* $\sigma_{\mu,\theta}^{\Psi}$*-a.e.*$\psi^a$*) then we have* $\dim H_{\mu,\theta}^{\Phi}(\varphi^a) = n \cdot \dim(L^2_{Q_{\mu,\theta}^{{}^t J\varphi^a}})$ *for* $\sigma_{\mu,\theta}^{\Phi}$*-a.e.*$\varphi^a$.

§4. Ergodicity and Multiplicity

4.1 Ergodicity and uniform multiplicity. As was stated in the Introduction, $U_\mu(x^*)$ defined by

$$U_\mu(x^*) : h(x) \in L_\mu^2 \longrightarrow \exp(i\langle x, x^* \rangle) h(x) \in L_\mu^2$$

satisfies the following commutation relations with $T(F)$.

$$V_{\mu,\theta}(\varphi) U_\mu(x^*) = \exp(-i\langle \varphi, x^* \rangle) U_\mu(x^*) V_{\mu,\theta}(\varphi) \tag{4.1}$$

$$T_{\mu,\theta}(F_{x^*}) U_\mu(x^*) = U_\mu(x^*) T_{\mu,\theta}(F), \tag{4.2}$$

where $F_{x^*}(\cdot) = F(\cdot + x^*)$.

Theorem 4.1. *Let* μ *be a* Φ*-quasi-invariant measure on* $(X, \mathfrak{B})$. *Then the spectral measure* σ *for* $(V_{\mu,\theta}, \Phi)$ *is* X^**-quasi-invariant. (More exactly,* $X^* | \Phi$*-quasi-invariant.)*
Moreover if μ *is* Φ*-ergodic, (That is,* $\mu(B) = 1$ *or* 0 *provided that* $\mu((B - \varphi) \ominus B) = 0$ *for all* $\varphi \in \Phi$*) then* σ *is also* X^**-ergodic.*

Proof. Let $h \in L_\mu^2$ and $E \in \mathfrak{C}_\Phi$. Then

$$
\begin{aligned}
\sigma_{\mu,\theta,h}(E - x^*) &= \langle T((\chi_E)_{x^*}) h, h \rangle_2 = \langle T(\chi_E) U_\mu(-x^*) h, U_\mu(-x^*) h \rangle_2 \\
&= \sigma_{\mu,\theta,U(-x^*)h}(E).
\end{aligned}
$$

Thus,

$$\sigma(E) = 0 \Longleftrightarrow \forall h, \sigma_{\mu,\theta,h}(E) = 0 \Longleftrightarrow \forall h, \sigma_{\mu,\theta,U(-x^*)h}(E) = 0 \Longleftrightarrow \sigma(E - x^*) = 0.$$

If $\sigma((E - x^*) \ominus E) = 0$ for all $x^* \in X^*$, then a projection $T(\chi_E)$ commutes with every $U_\mu(x^*)$ and $V_{\mu,\theta}(\varphi)$. As the ergodicity of μ is equivalent to the irreducibility of $\{U_\mu(x^*),\ V_{\mu,\theta}(\varphi)\}$, so $T(\chi_E)=\mathrm{I}$ or 0 and the conclusion follows from the injectivity of T. □

Theorem 4.2. *Put* $D_{p,\mu,\theta}:=\{\varphi^a\in\Phi^a|\dim H^{\Phi}_{\mu,\theta}(\varphi^a)=p\}$. *Then*

(1) $\sigma^{\Phi}_{\mu,\theta}((D_{p,\mu,\theta}-x^*)\ominus D_{p,\mu,\theta})=0$ *for all* $x^*\in X^*$.

(2) *There exist at most countable* Φ*-quasi-invariant measures* μ_p *such that* μ_p *are mutually singular with each other*, μ *is a convex sum of* μ_p's *and* $(V_{\mu_p,\theta},\ \Phi)$ *has uniform multiplicity* p.

(3) *In particular*, *if* μ *is* Φ*-ergodic*, *then the corresponding representation has a uniform multiplicity*.

Proof. Let $h_1,\ h_2\in\mathrm{L}^2_\mu$ and $F\in\mathrm{L}^\infty_\sigma$. Then

$$\begin{aligned}&\int_{\Phi^a}F(\varphi^a)\langle S_{\mu,\theta}U_\mu(-x^*)h_1|S_{\mu,\theta}U_\mu(-x^*)h_2\rangle(\varphi^a)\sigma(d\varphi^a)\\&=\int_{\Phi^a}F(\varphi^a+x^*)\langle S_{\mu,\theta}h_1|S_{\mu,\theta}h_2\rangle(\varphi^a)\sigma(d\varphi^a)\\&=\int_{\Phi^a}F(\varphi^a)\langle S_{\mu,\theta}h_1|S_{\mu,\theta}h_2\rangle(\varphi^a-x^*)\frac{d\sigma_{x^*}}{d\sigma}(\varphi^a)\sigma(d\varphi^a).\end{aligned}$$

So we have

$$\begin{aligned}(4.3)\quad&\langle S_{\mu,\theta}U_\mu(-x^*)h_1|S_{\mu,\theta}U_\mu(-x^*)h_2\rangle(\varphi^a+x^*)\\&=\langle S_{\mu,\theta}h_1|S_{\mu,\theta}h_2\rangle(\varphi^a)\frac{d\sigma_{x^*}}{d\sigma}(\varphi^a+x^*)\end{aligned}$$

for σ-a.e.φ^a. This leads us the following conclusion that $S_{\mu,\theta}U_\mu(-x^*)S^{-1}_{\mu,\theta}$ induces a similar transformation from $H^{\Phi}_{\mu,\theta}(\varphi^a)$ to $H^{\Phi}_{\mu,\theta}(\varphi^a+x^*)$ for σ-a.e.φ^a. Thus dim $H(\varphi^a)$ is almost all invariant under the actions of $x^*\in X^*$.
For the second half, note that a projection $T(\chi_{D_{p,\mu,\theta}})$ commutes with all $U_\mu(x^*)$ and $V_{\mu,\theta}(\varphi)$. Hence there exists $\Omega_{p,\mu,\theta}\in\mathfrak{B}$ such that $T(\chi_{D_{p,\mu,\theta}})h=\chi_{\Omega_{p,\mu,\theta}}\cdot h$ for all $h\in\mathrm{L}^2_\mu$ and $\mu((\Omega_{p,\mu,\theta}-x^*)\ominus\Omega_{p,\mu,\theta})=0$ for all $\varphi\in\Phi$. Put

$$\mu_{p,\theta}(B):=\mu(\Omega_{p,\mu,\theta})^{-1}\mu(B\cap\Omega_{p,\mu,\theta}),$$

for all $B\in\mathfrak{B}$. Then $\mu_{p,\theta}$ $(p=1,\ \cdots)$ have desired properties. □

From Theorem 4.1 and Theorem 4.2 we have a conclusion that the ergodicity of the original measure μ derives the ergodicity of the spectral measure σ and uniform multiplicity p. The converse does not hold in general. In fact for the case $p=2,\ \cdots,\ \infty$ we have the following example.

Example 2. $X=\boldsymbol{R}^\infty$, $\Phi=\boldsymbol{R}^\infty_0$ ($\boldsymbol{R}^\infty_0$ is the countable direct sum of $\boldsymbol{R}$.)

$$\mu=\sum_{n=1}^{p}\alpha_n\nu_{(n-1)\tau} \text{ with } \alpha_n>0,\ \sum_{n=1}^{p}\alpha_n=1,$$

where ν is a product type of 1-dimensional measures, which is $\boldsymbol{R}_0^\infty$-quasi-invariant and $\tau\in\boldsymbol{R}^\infty$ is choosen such as $\nu_{n\tau}$ is singular with ν for every n. θ is a trivial 1-cocycle, so we shall omit the suffix θ. It is obvious that μ is not $\boldsymbol{R}_0^\infty$-ergodic for $p\geq 2$. Let h, $g\in \mathrm{L}_\mu^2$ and F be a bounded measurable function on $\boldsymbol{R}^\infty$. Then

$$\langle T_\mu(F)h,\ g\rangle_2=\sum_{n=1}^{p}\alpha_n\langle T_{\nu_{(n-1)\tau}}(F)\chi_{X_n}\cdot h,\ \chi_{X_n}\cdot g\rangle_2, \tag{4.4}$$

where $\{X_n\}_n$ is a Borel partition of $\boldsymbol{R}^\infty$ such that $\nu_{(n-1)\tau}(X_m)=\delta_{n,m}$. Note that $\sigma_{\nu_{(n-1)\tau}}$ are all equivalent to σ_ν. So it follows from (4.4) that we have $\sigma_\mu=\sigma_\nu$, which is $\boldsymbol{R}_0^\infty$-ergodic, because ν is $\boldsymbol{R}_0^\infty$-ergodic.
In order to see the multiplicity, let the expression of the direct integral ($V_{\nu_{(n-1)\tau}}$, $\boldsymbol{R}_0^\infty$) be $H^n=\int^\oplus H^n(x)\sigma_\nu(dx)$. Then we have dim $H^n(x)=1$ for σ_ν-a.e.x which will be seen in Section 5. We form a direct sum of Hilbert spaces $H^n(x)$, $H(x) :=\sum_{n=1}^{p\oplus}H^n(x)$.
Take a sequence $\{v_i^n\}_i\subset H^n$ such that $\{v_i^n(x)\}_i$ is total in each $H^n(x)$. Then $\{v_{i_1}, \cdots, v_{i_m}, \cdots(x)=(v_{i_1}^1(x)\cdots, v_{i_m}^m(x), \cdots)|v_{i_m}^m=0$ except finite numbers of $m\}$ forms a total sequence at each $H(x)$ and $\langle v_{i_1}, \cdots, v_{i_m},\cdots|v_{j_1}, \cdots, v_{j_m}, \cdots\rangle(x)$ are measurable. Therefore a measurable field structure is induced in such a manner as before and a direct integral $H:=\int^\oplus H(x)\sigma_\nu(dx)$ is defined. Let us define a map S from L_μ^2 to H such that

$$S(h)=(\sqrt{\alpha_n}\,S_{\nu_{(n-1)\tau}}(\chi_{X_n}\cdot h)),$$

where $S_{\nu_{(n-1)\tau}}$ is the canonical map from $\mathrm{L}_{\nu_{(n-1)\tau}}^2$ to H^n. Then

$$\begin{aligned}\langle T(F)h,\ g\rangle_2&=\sum_{n=1}^{p}\alpha_n\int_{\boldsymbol{R}^\infty}F(x)\langle S_{\nu_{(n-1)\tau}}\chi_{X_n}\cdot h|S_{\nu_{(n-1)\tau}}\chi_{X_n}\cdot g\rangle(x)\sigma_\nu(dx)\\&=\int_{\boldsymbol{R}^\infty}F(x)\langle Sh|Sg\rangle(x)\sigma_\nu(dx).\end{aligned}$$

Further it is easy to see that S is onto, so S is a unitary map with the desired property. Consequently (V_μ, $\boldsymbol{R}_0^\infty$) has the $\boldsymbol{R}_0^\infty$-ergodic spectral measure and has uniform multiplicity p.

4.2 Multiplicity 1.

Proposition 4.1. *($V_{\mu,\theta},\Phi$) is cyclic, if and only if it has uniform multiplicity* 1.

Proof. ($\Leftarrow$) The sufficiency is obvious.
($\Rightarrow$) Let h_0 be a cyclic element of the representation and put $\eta_0 := S_{\mu,\theta}h_0$, and $A := \{\varphi^a \in \Phi^a | \eta_0(\varphi^a) = 0\}$. Then

$$\langle T(\chi_A)h,\ T(F)h_0\rangle_2 = \int_A \bar{F}(\varphi^a)\langle h|\eta_0\rangle(\varphi^a)\sigma_{\mu,\theta}(d\varphi^a) = 0$$

for all $h \in L^2_\mu$ and $F \in L^\infty_\lambda$, which shows that $\sigma_{\mu,\theta}(A) = 0$. New let $\{h_n\}_n$ be a sequence from L^2_μ such that $\eta_n := S_{\mu,\theta}h_n$ has the properties like (P.6) and (P.7). Set $k_n := S^{-1}_{\mu,\theta}(\langle \eta_n|\eta_0\rangle \eta_1 - \langle \eta_1|\eta_0\rangle \eta_n)$ for $n > 1$. Then $\langle k_n,\ T(F)\eta_0\rangle_2 = 0$ and therefore

$$\langle \eta_0|\eta_n\rangle(\varphi^a) = 0 \text{ and } \langle \eta_0|\eta_1\rangle(\varphi^a)\eta_n(\varphi^a) = 0 \tag{4.5}$$

for $\sigma_{\mu,\theta}$-a.e. φ^a. If $\langle \eta_0|\eta_1\rangle(\varphi^a) = 0$ on some set $B \in \mathfrak{C}_\Phi$ with positive measure, then it follows from (4.5) that $B \subseteq A$ which contradicts to $\sigma(A) = 0$. So $\eta_n = 0$ for $n \geq 2$. □

Theorem 4.3. *So far as cyclic representations are concerned,*
(1) μ *is Φ-ergodic if and only if $\sigma_{\mu,\theta}$ is X^*-ergodic.*
(2) $(V_{\mu,\theta}, \Phi) \cong (V_{\mu',\theta'}, \Phi)$, *if and only if* $\sigma_{\mu,\theta} \cong \sigma_{\mu',\theta'}$.

Proof. Suppose that $\sigma_{\mu,\theta}$ is X^*-ergodic and $\mu((B - \varphi) \ominus B) = 0$ for all $\varphi \in \Phi$. We put $Ph := \chi_B \cdot h$ for $h \in L^2_\mu$, and $Q := S_{\mu,\theta}PS^{-1}_{\mu,\theta}$. Since P commutes with all $T(F)$, $F \in L^\infty_\sigma$, so Q commutes with all $T^H(F)$. Thus there exists $E \in \mathfrak{C}_\Phi$ such that $Q = T^H(\chi_E)$. Now

$$\langle T(\chi_E)h,\ g\rangle_2 = \langle QSh,\ Sg\rangle_2 = \langle Ph,\ g\rangle_2$$

for all h, $g \in L^2_\mu$, which implies $T(\chi_E) = P$. Besides,

$$T((\chi_E)_{x^*}) = U_\mu(x^*)T(\chi_E)U_\mu(-x^*) = P = T(\chi_E).$$

It follows that $\sigma((E - x^*) \ominus E) = 0$ and therefore $\sigma(E) = 1$ or 0, which is equivalent to $\mu(B) = 1$ or 0. The rest of the proof is immediate. □

Of course there is an example of non ergodic measure with cyclic representation.

Example 3. $X = \boldsymbol{R}^\infty$, $\Phi = \boldsymbol{R}^\infty_0$, $\mu = \sum_{n=1}^\infty a_n g_{c_n}$, with $a_n > 0$, $\sum_{n=1}^\infty a_n = 1$, where $\{c_n\}_n$ is a mutually different positive sequence and g_c is a standard Gaussian measure with mean 0 and variance c. θ is a trivial 1-cocycle, so we shall omit the suffix θ. Later in Section 6, it will be shown that the representation $(V_{g_c}, \boldsymbol{R}^\infty_0)$ has a spectral measure $g_{(4c)^{-1}}$ and has uniform multiplicity 1. For the Gaussian case the spectral measure is attained at constant function. As before,

$$(4.6)\qquad \langle T_\mu(F)h,\ g\rangle_2=\sum_{n=1}^{\infty}\alpha_n\langle T_{g_{c_n}}(F)\chi_{X_n}F,\ \chi_{X_n}g\rangle_2$$

for all $h,\ g\in L^2_\mu$ and $F\in L^\infty_{\sigma_\mu}$, where $\{X_n\}_n$ is a Borel partion of $\boldsymbol{R}^\infty$ such that $g_{c_n}(X_m)=\delta_{n,m}$. It follows that

$$\sigma_\mu=\sum_{n=1}^{\infty}\alpha_n\sigma_{g_{c_n}}=\sum_{n=1}^{\infty}\alpha_n g_{(4c_n)^{-1}},$$

so σ_μ is not $\boldsymbol{R}_0^\infty$-ergodic.

We take another Borel partition $\{Y_n\}_n$ such that $g_{(4c_n)^{-1}}(Y_m)=\delta_{n,m}$. Let $H^n=\int_{Y_n}^{\oplus}H^n(y)\alpha_n^{-1}\sigma_\mu(dy)$ be a direct integral for $(V_{g_{c_n}},\ \boldsymbol{R}_0^\infty)$ and S_n be the canonical map from $L^2_{g_{c_n}}$ to H^n, and put $H(y):=H^n(y)$, if $y\in Y_n$. As for the measurable field structure on $H(y)$'s, we shall consider induced one from each measurable field structure on $H(y)$'s. Now we shall show that L^2_μ is isomorphic to a direct integral $H:=\int_{\boldsymbol{R}^\infty}^{\oplus}H(y)\sigma_\mu(dy)$ by a map S defined by $(Sh)(y)=S_n(\chi_{X_n}h)(y)$ for $y\in Y_n$. For,

$$\begin{aligned}\langle T_\mu(F)h,\ g\rangle_2&=\sum_{n=1}^{\infty}\alpha_n\int_{Y_n}F(y)\langle S_n(\chi_{X_n}h)|S_n(\chi_{X_n}g)\rangle(y)\alpha_n^{-1}\sigma_\mu(dy)\\&=\int_{\boldsymbol{R}^\infty}F(y)\langle Sh|Sg\rangle(y)\sigma_\mu(dy).\end{aligned}$$

Thus $(V_\mu,\ \boldsymbol{R}_0^\infty)$ is cyclic, however μ is not $\boldsymbol{R}_0^\infty$-ergodic.

4.3 Unitary cocycles. We have seen that the ergodicity of the spectral measure and the uniform multiplicity 1 implies that the original measure is ergodic and that the ergodicity of the original measure implies that the ergodicity of the spectral measure and a uniform multiplicity.

However we don't yet know whether the uniform multiplicity can be taken the place of uniform multiplicity 1.

Let us make the following device in order to approach to this problem.

Let μ be Φ-quasi-invariant, $(V_{\mu,\theta},\ \Phi)$ have uniform multiplicity p, and K be a Hilbert space of dimension p. Then L^2_μ is canonically isomorphic to $L^2_\sigma(K)$ of all square summable K-valued functions by a map $S_{\mu,\theta}$. Put

$$\widetilde{V_{\mu,\theta}}(\varphi)=S_{\mu,\theta}V_{\mu,\theta}(\varphi)S_{\mu,\theta}^{-1}\text{ and }\widetilde{U_{\mu,\theta}}(\varphi^*)=S_{\mu,\theta}U_\mu(\varphi^*)S_{\mu,\theta}^{-1}$$

for all $\varphi\in\Phi$ and $\varphi^*\in X^*$. Then it follows from (4.1) that

$$(4.7)\quad (\widetilde{U_{\mu,\theta}}(\varphi^*)k)(\varphi^a)=\exp(-i\langle\varphi,\ \varphi^a+\varphi^*\rangle)\widetilde{U_{\mu,\theta}}(\varphi^*)(\exp(i\langle\varphi,\ \cdot\rangle)k)(\varphi^a)$$

for all $k\in K$ and

$$(4.8)\quad \int_{\Phi^a}\exp(i\langle\varphi,\ \varphi^a\rangle)\|\widetilde{U_{\mu,\theta}}(\varphi^*)k\|_K^2\sigma(d\varphi^a)$$

$$=\|k\|_K^2 \exp(-i\langle\varphi, \varphi^*\rangle)\int_{\Phi^a}\exp(i\langle\varphi, \varphi^a\rangle)\sigma(d\varphi^a).$$

Thus, we have

$$\|\widetilde{U_{\mu,\theta}}(\varphi^*)k\|_K=\sqrt{\frac{d\sigma_{-\varphi^*}}{d\sigma}(\varphi^a)}\|k\|_K \tag{4.9}$$

for σ-a.e.φ^a. Therefore $U_{\mu,\theta}(\varphi^a, \varphi^*)$ defined by

$$U_{\mu,\theta}(\varphi^a, \varphi^*)k=\left(\frac{d\sigma_{-\varphi^*}}{d\sigma}(\varphi^a)\right)^{-1/2}(\widetilde{U_{\mu,\theta}}(\varphi^*)k)(\varphi^a) \tag{4.10}$$

is unitary for σ-a.e.φ^a. Consequently,

$$(\widetilde{U_{\mu,\theta}}(\varphi^*)f)(\varphi^a)=\sqrt{\frac{d\sigma_{-\varphi^*}}{d\sigma}(\varphi^a)}\,U_{\mu,\theta}(\varphi^a, \varphi^*)f(\varphi^a+\varphi^*) \tag{4.11}$$

for all $f\in L^2_\sigma(K)$. It is easily checked that

$$U_{\mu,\theta}(\varphi^a, \varphi_2^*)U_{\mu,\theta}(\varphi^a+\varphi_2^*, \varphi_1^*)=U_{\mu,\theta}(\varphi^a, \varphi_1^*+\varphi_2^*) \tag{4.12}$$

$$U_{\mu,\theta}(\varphi^a, 0)=I \tag{4.13}$$

for σ-a.e.φ^a. We call a system of unitary operators possessing the properties (4.12) and (4.13) a unitary cocycle.

Conversely, suppose that a Φ^*-quasi-invariant probability measure σ on $(\Phi^a, \mathfrak{C}_\Phi)$, complex separable Hilbert space K and a unitary cocycle $\{U(\varphi^a, \varphi^*)\}$ are given. We define unitary operators $\widetilde{V}(\varphi)$, $\widetilde{U}(\varphi^*)$ on $L^2_\sigma(K)$ such that

$$\widetilde{V}(\varphi): f(\varphi^a)\longmapsto \exp(i\langle\varphi, \varphi^a\rangle)f(\varphi^a)$$

$$\widetilde{U}(\varphi^*): f(\varphi^a)\longmapsto\sqrt{\frac{d\sigma_{-\varphi^*}}{d\sigma}(\varphi^a)}\,U(\varphi^a, \varphi^*)f(\varphi^a+\varphi^*).$$

If $\widetilde{U}(\varphi^*)$ is a cyclic representation of Φ^* and the representations $\{\widetilde{V}(\varphi), \widetilde{U}(\varphi^*)\}$ is irreducible, then it follows that there correspondes a Φ-ergodic measure μ on $(\Phi^*)^a$ such that

$$\langle\widetilde{U}(\varphi^*)h_0, h_0\rangle_2=\int_{(\Phi^*)^a}\exp(i\langle\varphi^*, x\rangle)\mu(dx), \tag{4.14}$$

where $h_0\in L^2_\sigma(K)$ is a cyclic vector with $\|h_0\|_2=1$. So an operator S^{-1} defined by

$$S^{-1}: \sum_{n=1}^N a_n\widetilde{U}(\varphi_n^*)h_0\in L^2_\sigma(K)\longmapsto\sum_{n=1}^N a_n\exp(i\langle\varphi_n^*, \cdot\rangle)\in L^2_\mu$$

has a unitary extension denoted by the same letter, and $U(\varphi^*):=S^{-1}\widetilde{U}(\varphi^*)S$ and $V(\varphi):=S^{-1}\widetilde{V}(\varphi)S$ have the following explicit form.

$$U(\varphi^*): h(x) \longmapsto \exp(i\langle\varphi^*, x\rangle)h(x) \tag{4.15}$$

$$V(\varphi): h(x) \longmapsto \sqrt{\frac{d\mu_\varphi}{d\mu}}(x)\theta(x, \varphi)h(x-\varphi), \tag{4.16}$$

where θ is some 1-cocycle. (For these discussions, see Chapter IV in [3].) Therefore an original situation is realized as $X=(\Phi^*)^a$ with weak topology. Consequently in order to give a counter example for this problem it is enough to construct a measure σ and a unitary cocycle $\{U(\varphi^a, \varphi^*)\}$ on $L^2_\sigma(K)$ with $\dim(K)\geq 2$ such that $\tilde{U}(\varphi^*)$ is cyclic and $\{\tilde{U}(\varphi^*), \tilde{V}(\varphi)\}$ is irreducible.

A typical example of a unitary cocycle is given by

$$U(\varphi^a, \varphi^*)=U(\varphi^a)^{-1}U(\varphi^a+\varphi^*), \tag{4.17}$$

where U is a measureable map from Φ^a to the unitary group on K equipped with a Borel field generated by the strong operator topology. However the representation $\{\tilde{U}(\varphi^*), \tilde{V}(\varphi)\}$ derived from this cocycle is equivalent to a representation $\{\tilde{U}_0(\varphi^*), \tilde{V}_0(\varphi)\}$ derived from the trivial unitary cocycle by a map, $f(\varphi^a) \longmapsto U(\varphi^a)^{-1}f(\varphi^a)$. So it derives a non irreducible representation, if $\dim(K)\geq 2$. Similarly if $\{U(\varphi^a, \varphi^*)\}$ consists of commutative operators, then the corresponding representation is reducible.

We finally remark that in the case $X=\boldsymbol{R}^\infty$, $\Phi=\boldsymbol{R}_0^\infty$, an explicit form of unitary cocycle is decided as follows.

$$U(x, s)=U_1(x^1)\cdots U_n(x^n)U_n((x+s)^n)^{-1}\cdots U_1((x+s)^1)^{-1}, \tag{4.18}$$

where $x=(x_1, \cdots, x_n, \cdots)$, $x^n=(x_n, x_{n+1}, \cdots)\in\boldsymbol{R}^\infty$, $s=(s_1, \cdots, s_n, 0, 0, \cdots)\in\boldsymbol{R}_0^\infty$ and U_n is a measurable map from $\boldsymbol{R}^\infty$ to the unitary group on K.

Anyway, it seems to the author that this problem will be solved negatively. And if so, it is quite interesting to construct a multiplicity formula for ergodic measures.

§5. Product Representation

5.1 Finite product. Let X_n $(n=1, \cdots, N<\infty)$ be a locally convex Hausdorff space over $\boldsymbol{R}$, $\mathfrak{B}_n$ be the cylindrical σ-algebra on X_n, μ_n be a Φ_n-quasi-invariant probability measure on $(X_n, \mathfrak{B}_n)$ and θ_n be a 1-cocycle with property (2.4). Put

$$X:=X_1\times\cdots\times X_N,\ \mathfrak{B}:=\mathfrak{B}_1\times\cdots\times\mathfrak{B}_N \text{ and } \mu:=\mu_1\times\cdots\times\mu_N.$$

It is easily checked that $\mathfrak{B}$ coincides with the cylindrical σ-algebra on X and that $A_\mu^0 \supseteq \Phi := \Phi_1 \times \cdots \times \Phi_N$. So a unitary representation of Φ is defined as

$$(5.1)\quad V_{\mu,\theta}((\varphi_1, \cdots, \varphi_N)) : f(x_1, \cdots, x_N) \in \mathrm{L}_\mu^2 \longmapsto \prod_{n=1}^{N} \sqrt{\frac{d(\mu_n)_{\varphi_n}}{d\mu_n}}(x_n)\theta_n(x_n, \varphi_n) f(x_1 - \varphi_1, \cdots, x_N - \varphi_N) \in \mathrm{L}_\mu^2.$$

In this subsection first we shall find an expression of the direct integral for $(V_{\mu,\theta}, \Phi)$ using factor expressions.
So, choose $h_n^0 \in \mathrm{L}_{\mu_n}^2$ such that $\sigma_{\mu_n,\theta_n,h_n} = \sigma_{\mu_n,\theta_n}$ $(=:\sigma_n)$ for each n, and set $\sigma := \sigma_1 \times \cdots \times \sigma_N$. Since we have

$$(5.2)\quad \langle T_{\mu,\theta}(F_1 \otimes \cdots \otimes F_N)(h_1 \otimes \cdots \otimes h_N), g_1 \otimes \cdots \otimes g_N \rangle_2 = \prod_{n=1}^{N} \langle T_{\mu_n}(F_n) h_n, g_n \rangle_2 = \int_{\Phi^a} \prod_{n=1}^{N} F_n(\varphi_n^a) \langle S_{\mu_n,\theta_n}^{\Phi_n} h_n | S_{\mu_n,\theta_n}^{\Phi_n} g_n \rangle (\varphi_n^a) \sigma(d\varphi^a)$$

for all $h_n, g_n \in \mathrm{L}_{\mu_n}^2$ and for all $F_n \in \mathrm{L}_{\sigma_n}^\infty$, it follows that $\sigma_{\mu,\theta} = \sigma$, and it is natural to define

$$H_{\mu,\theta}(\varphi^a) = \otimes_{n=1}^{N} H_{\mu_n,\theta_n}(\varphi_n^a)$$

for $\varphi^a = (\varphi_1^a, \cdots, \varphi_N^a)$. Next a measurable field structure R is defined such that $\eta(\varphi^a) \in R$ if and only if $\langle \eta(\varphi^a) | \eta_1(\varphi_1^a) \otimes \cdots \otimes \eta_N(\varphi_N^a) \rangle$ is measurable for each $\eta_n \in R^{\Phi_n}$ $(n=1, \cdots, N)$. Thus a direct integral $H = \int^{\oplus} H_{\mu,\theta}(\varphi^a) \sigma(d\varphi^a)$ is constructed and it is easily checked that a map, $h_1 \otimes \cdots \otimes h_N \longmapsto \otimes_{n=1}^{N} (S_{\mu_n,\theta_n} h_n)(\varphi_n^a)$ has a unitary extension from L_μ^2 to H. Settling these arguments,

Theorem 5.1. *An expression of the direct integral for* $(V_{\mu,\theta}, \Phi)$ *is* $\int^{\oplus} \otimes_{n=1}^{N} H_{\mu_n,\theta_n}(\varphi_n^a) \sigma(d\varphi^a)$. *Thus the spectral measure* σ *is the product of each spectral measure of the factor and the multiplicity is the product of each multiplicity.*

Here we shall make addition to the ergodicity of μ for a little while.

Theorem 5.2. *Under the same notation as in this paragraph, if* μ_n *is* Φ_n*-ergodic for each* n, *then* μ *is* Φ*-ergodic.*

Proof. It is enough to show it in the case $N=2$. The general case follows from the mathematical induction.
So let $A \in \mathfrak{B}$ and $\mu((A - (\varphi_1, \varphi_2)) \ominus A) = 0$ for all $\varphi_j \in \Phi_j$ $(j=1, 2)$. Then there exists a countable set $\{x_{j,m}^*\}_m \subset X_j^*$ $(j=1, 2)$ such that $A \in \mathfrak{C}_1 \times \mathfrak{C}_2$, where $\mathfrak{C}_j$ is the minimal σ-algebra with which all the $x_{j,m}^*$ $(m=1, \cdots)$ are measurable.
Now let us define a metric on Φ_j such that

$$d^j(\varphi,\ \psi):=\|(\widehat{\mu_j})_\varphi-(\widehat{\mu_j})_\psi\|_{\text{tot}},$$

where $\widehat{\mu_j}=\mu_j|\mathfrak{C}_j$. Since $L^2_{\widehat{\mu_j}}$ is separable, so is $(\Phi_j,\ d^j)$. We take a countable dense set $\{\varphi_{j,m}\}_m$ from $(\Phi_j,\ d^j)$. Then using Fubini's theorem, we see that there exists some $N_1\in\mathfrak{C}_2$ such that $\mu_2(N_1)=0$ and for all $y\in N_1^c$

$$\mu_1((A^y-\varphi_{1,m})\ominus A^y)=0 \tag{5.3}$$

for all m. Since $d^1(\psi_n,\ \psi)\longrightarrow 0$ implies $\mu_1((E-\psi_n)\ominus(E-\psi))\longrightarrow 0$ for any $E\in\mathfrak{C}_1$, which will be shown later soon, so for all $y\in N_1^c$

$$\mu_1((A^y-\varphi)\ominus A^y)=0 \tag{5.4}$$

for all $\varphi\in\Phi_1$. It follows from the ergodicity of μ_1 that

$$\mu_1(A^y)=0 \text{ or } 1. \tag{5.5}$$

Notice that $x\in\{A-(0,\ \varphi)\}^y\Longleftrightarrow(x,\ y+\varphi)\in A\Longleftrightarrow x\in A^{y+\varphi}$.
Hence for any $\varphi\in\Phi_2$

$$\mu_1(A^{y+\varphi}\ominus A^y)=0 \tag{5.6}$$

for μ_2-a.e.y in virtue of Fubini's theorem. Thus we have

$$\mu_1(A^{y\pm\varphi_{2,m}}\ominus A^y)=0\ (m=1,\ \cdots) \tag{5.7}$$

for all $y\in N_2^c$, where N_2^c is some μ_2-negligible set. Now put $F:=\{y\in\Phi_2|\mu_1(A^y)=1\}$. It follows from (5.7) that $F\cap N_2^c\subseteq(F\pm\varphi_{2,m})\cap N_2^c$. Hence discussing in the same way as above, we see that $\mu_2(F)=1$ or 0. Consequently,

$$\mu(A)=\int_{X_2}\mu_1(A^y)\mu_2(dy)=\int_{F\cap N_1^c}\mu_1(A^y)\mu_2(dy)=\mu_2(F)=1 \text{ or } 0. \qquad \square$$

Lemma 5.1. *If* $d^1(\psi_n,\ 0)\longrightarrow 0\ (n\longrightarrow\infty)$, *then* $\widehat{\mu_1}((E-\psi_n)\ominus E)\longrightarrow 0$ $(n\longrightarrow\infty)$ *for each fixed* $E\in\mathfrak{C}_1$.

Proof. It is clear that for any $\epsilon>0$, there exists a function $F(x)$ of the form, $F(x)=f(\langle x,\ x^*_{1,1}\rangle,\ \cdots,\ \langle x,\ x^*_{1,k}\rangle)$, where f is a continuous bounded function on $\boldsymbol{R}^k$ with $\|f\|_\infty=1$ such that $\|F-\chi_E\|_1<\epsilon$. It follows that

$$\begin{aligned}\widehat{\mu_1}((E-\psi_n)\ominus E)&=\int_{X_1}|\chi_E(x+\psi_n)-\chi_E(x)|\widehat{\mu_1}(dx)\\&\leq\int_{X_1}|\chi_E(x+\psi_n)-F(x+\psi_n)|\widehat{\mu_1}(dx)+\int_{X_1}|F(x+\psi_n)\\&\quad-F(x)|\widehat{\mu_1}(dx)+\int_{X_1}|F(x)-\chi_E(x)|\widehat{\mu_1}(dx)\\&<2\epsilon+\int_{X_1}|F(x+\psi_n)-F(x)|\widehat{\mu_1}(dx)+2\int_{X_1}\left|\frac{d(\widehat{\mu_1})_{\psi_n}}{d\widehat{\mu_1}}(x)-1\right|\widehat{\mu_1}(dx).\end{aligned}$$

The last term in the right hand of the last inequality converges to 0 due to the definition of d^1, so we only have to check that $\langle \psi_n, x^* \rangle \longrightarrow 0$ $(n \longrightarrow \infty)$ for any $x^* \in \{x^*_{1,1}, \cdots, x^*_{1,k}\}$. Now let $B \in \mathfrak{B}(\boldsymbol{R})$ and put $p = x^* \mu_1$. Then

$$p(B - \langle \psi_n, x^* \rangle) = \mu_1((x^*)^{-1}(B) - \psi_n) \longrightarrow p(B) \ (n \longrightarrow \infty).$$

Thus $\langle \psi_n, x^* \rangle \longrightarrow 0$ $(n \longrightarrow \infty)$, as is easily seen. □

5.2 Countably infinite product. As before, let X_n be a locally convex Hausdorff space over $\boldsymbol{R}$, $\mathfrak{B}_n$ be the cylindrical σ-algebra on X_n, μ_n be a probability measure on $(X_n, \mathfrak{B}_n)$ such that $\Phi_n \subset A^\circ_{\mu_n}$ and θ_n be a 1-cocycle. We put $X := \prod_{n=1}^\infty X_n$, $\mathfrak{B} := \prod_{n=1}^\infty \mathfrak{B}_n$ and $\mu := \prod_{n=1}^\infty \mu_n$. It is easily checked that $\mathfrak{B}$ coincides with the cylindrical σ-algebra on X, and $A^\circ_\mu \supset \Phi := \{(\varphi_n) \in \prod_{n=1}^\infty \Phi_n \mid \varphi_n = 0$ except finite numbers of $n\}$. Further if $\Phi_n = A^\circ_{\mu_n}$ holds for all n, then Φ is dense in A^0_μ with respect to the Kakutani's metric d_μ. We put

$$\theta(x, \varphi) := \prod_{n=1}^\infty \theta_n(x_n, \varphi_n) \tag{5.8}$$

for $x = (x_1, \cdots, x_n, \cdots) \in X$ and $\varphi = (\varphi_1, \cdots, \varphi_n, \cdots) \in \Phi$.

In this subsection, we shall find a direct integral expression for the representation $(V_{\mu,\theta}, \Phi)$. So let $L^2_{\mu_n}$ be canonically isomorphic to $H_n := \int^\oplus H_{\mu_n,\theta_n}(\varphi^a_n)\sigma_n(d\varphi^a_n)$ by a map S_n. We shall write $\hat{1}_n$ in place of $S_n(1)$. First we shall take a positive measurable function $\rho_n(\varphi^a_n)$ such that

$$\int_{\Phi^a_n} \rho(\varphi^a_n)\sigma_n(d\varphi^a_n) = 1 \text{ and} \tag{5.9}$$

$$\sum_{n=1}^\infty \left\{1 - \int_{\Phi^a_n} \sqrt{\langle \hat{1}_n | \hat{1}_n \rangle(\varphi^a_n)} \sqrt{\rho_n(\varphi^a_n)}\, \sigma_n(d\varphi^a_n)\right\}^{1/2} < \infty. \tag{5.10}$$

Such ρ_n surely exists. For example it may be as well to take $C_\epsilon^{-1}(\langle \hat{1}_n | \hat{1}_n \rangle(\varphi^a_n) + \epsilon)$ for sufficiently small ϵ, where C_ϵ is the normalizing constant.
Next we change each spectral measure $\sigma_n(d\varphi^a_n)$ to $\sigma^0_n(d\varphi^a_n) := \rho_n(\varphi^a_n)\sigma_n(d\varphi^a_n)$, so the inner product of $H_{\mu_n,\theta_n}(\varphi^a_n)$ is altered from $\langle \cdot | \cdot \rangle$ to $\rho_n^{-1/2}\langle \cdot | \cdot \rangle$ which will be denoted by $\langle \cdot | \cdot \rangle_\circ$. Consequently we can rewrite (5.10) in a new form as

$$\sum_{n=1}^\infty \left\{1 - \int_{\Phi^a_n} \sqrt{\langle \hat{1}_n | \hat{1}_n \rangle_\circ(\varphi^a_n)}\, \sigma^\circ_n(d\varphi^a_n)\right\}^{1/2} < \infty. \tag{5.11}$$

It follows from (5.11) that $\{\prod_{n=1}^N \sqrt{\langle \hat{1}_n | \hat{1}_n \rangle_\circ (\varphi^a_n)}\}_N$ forms a Cauchy sequence in L^2_σ, where $\sigma := \prod_{n=1}^\infty \sigma^\circ_n$ is a probability measure on Φ^a. Consequently we have

$$\int_{\Phi^a} \Big| \prod_{n=N+1}^\infty \langle \hat{1}_n | \hat{1}_n \rangle_\circ(\varphi^a_n) - 1 \Big| \sigma(d\varphi^a) \longrightarrow 0 \ (N \longrightarrow \infty). \tag{5.12}$$

Now let $h, g \in L^2_\mu$ be tame functions of the form of separation variables, $h = h_1$

$\otimes\cdots\otimes h_N$ and $g=g_1\otimes\cdots\otimes g_N$, where h_n, $g_n\in L^2_{\mu_n}$ for all $n=1, \cdots, N$. Since we have for $N\leq M$,

$$\begin{aligned}&\langle V_{\mu,\theta}((\varphi_1, \cdots, \varphi_n, 0, 0, \cdots))h, g\rangle_2\\&=\prod_{n=1}^{N}\int_{\Phi_n^a}\exp(i\langle\varphi_n, \varphi^a\rangle)\langle S_n h_n|S_n g_n\rangle(\varphi_n^a)\sigma_n^\circ(d\varphi_n^a)\cdot\\&\quad\prod_{N+1}^{M}\int_{\Phi_n^a}\exp(i\langle\varphi_n, \varphi^a\rangle)\langle\hat{1}_n|\hat{1}_n\rangle_\circ(\varphi_n^a)\sigma_n^\circ(d\varphi_n^a),\end{aligned}$$

so we have for each tame function $F\in L^\infty_\sigma$,

$$\text{(5.13)}\quad\begin{aligned}&\langle T_{\mu,\theta}(F)h, g\rangle_2\\&=\int_{\Phi^a}F(\varphi^a)\prod_{n=1}^{N}\langle S_n h_n|S_n g_n\rangle(\varphi_n^a)\prod_{n=N+1}^{\infty}\langle\hat{1}_n|\hat{1}_n\rangle_0(\varphi_n^a)\sigma(d\varphi^a).\end{aligned}$$

It follows from Theorem 2.1 that we have $\sigma_{\mu,\theta}$ is absolutely continuous with σ. On the other hand, $\sigma_{\mu,\theta}(E)=0$ gives that $\int_E\prod_{n=N+1}^\infty\langle\hat{1}_n|\hat{1}_n\rangle_0(\varphi_n^a)\sigma(d\varphi^a)=0$ for all n, because we only have to substitute the corresponding one to η_1 in (P.6) and (P.7) for h_n and g_n. Hence $\sigma(E)=0$ by virtue of (5.12), and σ is regarded as the spectral measure. Here we shall construct a Hilbert space $H(\varphi^a)$ for each $\varphi^a=(\varphi_n^a)\in\prod_{n=1}^\infty\Phi_n^a=\Phi^a$ as follows.
First we notice that (5.11) is nothing else but that $\{\hat{1}_n(\varphi_n^a)\}_n\in\prod_{n=1}^\infty H_{\mu_n,\theta_n}(\varphi_n^a)$ forms a C_0-sequence for σ-a.e.φ^a. (For C_0-sequence, we refer p21 in [6].) Let us put $H_{\mu,\theta}(\varphi^a):=\otimes_{n=1}^{\infty(\hat{1})}H_{\mu_n,\theta_n}(\varphi_n^a)$, which is the $(\hat{1})$-adic incomplete direct product. A measurable field structure R is induced in such a manner as $\eta(\varphi^a)\in R$ if and only if $\langle\eta(\varphi^a)|\eta_1(\varphi_1^a)\otimes\cdots\otimes\eta_n(\varphi_n^a)\otimes\cdots\rangle$ are measurable for all $\eta_n\in R^{\Phi_n}$, where $\eta_n(\varphi_n^a)=\hat{1}_n(\varphi_n^a)$ except finite numbers of n.

Theorem 5.3. *L^2_μ is isomorphic to $H:=\int^\oplus H_{\mu,\theta}(\varphi^a)\sigma(d\varphi^a)$ with a map S possesing the property* (2.25).

Proof. Put for a tame function $h=h_1\otimes\cdots\otimes h_N\in L^2_\mu$,

$$Sh:=(S_{\mu_1,\theta_1}h_1)(\varphi_1^a)\otimes\cdots\otimes(S_{\mu_N,\theta_N}h_N)(\varphi_N^a)\otimes\hat{1}_{N+1}(\varphi_{N+1}^a)\otimes\cdots.$$

Then (5.13) is rewritten as

$$\text{(5.14)}\qquad\langle T_{\mu,\theta}(F)h, g\rangle_2=\int_{\Phi^a}F(\varphi^a)\langle Sh|Sg\rangle(\varphi^a)\sigma(d\varphi^a)$$

for all $F\in L^\infty_\sigma$ and for all tame functions h, $g\in L^2_\mu$. The rest of the proof is easily checked. □

Corollary 5.4. *If all the $(V_{\mu_n,\theta_n}, \Phi_n)$ are cyclic representations, then so is their product representation $(V_{\mu,\theta}, \Phi)$.*

Theorem 5.5. *Assume that* μ'_n $(n=1, \cdots)$ *be another* Φ_n*-quasi-invariant probability measure on* $(X_n, \mathfrak{B}_n)$, θ'_n *be a* 1*-cocycle, and the representations* $(V_{\mu_n,\theta_n}, \Phi_n)$ *and* $(V_{\mu'_n,\theta'_n}, \Phi_n)$ *be equivalent for all* n. *Put* $\mu' := \prod_{n=1}^{\infty} \mu'_n$ *and* $\theta' := \prod_{n=1}^{\infty} \theta'_n$. *Then in order that* $(V_{\mu,\theta}, \Phi) \cong (V_{\mu',\theta'}, \Phi)$, *it is necessary and sufficient that the spectral measures* $\sigma_{\mu,\theta}$ *and* $\sigma_{\mu',\theta'}$ *are equivalent. And this condition is equivalent to*

$$\sum_{n=1}^{\infty} d^2\{\langle \hat{1}_n | \hat{1}_n \rangle (\varphi_n^a) \sigma_{\mu_n,\theta_n}(d\varphi_n^a),\ \langle \hat{1}'_n | \hat{1}'_n \rangle (\varphi_n^a) \sigma_{\mu'_n,\theta'_n}(d\varphi_n^a)\} < \infty, \tag{5.15}$$

where d *is the Hellinger distance.*

Proof. The necessity is obvious. For the sufficiency, we first notice that the incomplete direct products with different reference vectors are isomorphic to each other. Thus $\sigma_{\mu,\theta} \cong \sigma_{\mu',\theta'}$ implies that $\dim H_{\mu,\theta}(\varphi^a) = \dim H_{\mu',\theta'}(\varphi^a)$ for $\sigma_{\mu,\theta}$-a.e.φ^a and the conclusion follows from Theorem 2.5.
As for the second half, "$\sigma_{\mu,\theta} \cong \sigma_{\mu',\theta'}$" is equivalent to

$$\sum_{n=1}^{\infty} d^2\{\rho_n(\varphi_n^a) \sigma_{\mu_n,\theta_n}(d\varphi_n^a),\ \rho'_n(\varphi_n^a) \sigma_{\mu'_n,\theta'_n}(d\varphi_n^a)\} < \infty \tag{5.16}$$

by virtue of Kakutani's theorem. (See, [5].) On the other hand, by the choice of ρ_n, ρ'_n it holds that

$$\sum_{n=1}^{\infty} d^2\{\langle \hat{1}_n | \hat{1}_n \rangle (\varphi_n^a) \sigma_{\mu_n,\theta_n}(d\varphi_n^a),\ \rho_n(\varphi_n^a) \sigma_{\mu_n,\theta_n}(d\varphi_n^a)\} < \infty \tag{5.17}$$

$$\sum_{n=1}^{\infty} d^2\{\langle \hat{1}'_n | \hat{1}'_n \rangle (\varphi_n^a),\ \sigma_{\mu'_n,\theta'_n}(d\varphi_n^a),\ \rho'_n(\varphi_n^a) \sigma_{\mu'_n,\theta'_n}(d\varphi_n^a)\} < \infty. \tag{5.18}$$

Therefore (5.15) is equivalent to (5.16). □

Example 4. Consider $\boldsymbol{R}^{n_k}$-quasi-invariant measures μ_k on $(\boldsymbol{R}^{n_k}, \mathfrak{B}(\boldsymbol{R}^{n_k}))$ and 1-cocycles θ_k such that $\theta_k(x, t) = q_k(x-t) q_k(x)^{-1}$. Then as we have seen in Example 1, $(V_{\mu_k,\theta_k}, \boldsymbol{R}^{n_k})$ has multiplicity 1 and $\langle \hat{1}_k | \hat{1}_k \rangle (y) \sigma_{\mu_k,\theta_k}(dy) = |\mathcal{F}(q_k \sqrt{\rho_k})|^2(y) dy$, where ρ_k is the density of μ_k with respect to the Lebesgue measure. Thus their product representation $V_{\mu,\theta}$ is cyclic, and $(V_{\mu,\theta}, \boldsymbol{R}_0^{\infty})$ is equivalent to $(V_{\mu',\theta'}, \boldsymbol{R}_0^{\infty})$ if and only if

$$\sum_{k=1}^{\infty} \left\{1 - \int_{\boldsymbol{R}^{n_k}} |\mathcal{F}(q_k \sqrt{\rho_k})|(y) |\mathcal{F}(q'_k \sqrt{\rho'_k})|(y) dy\right\} < \infty. \tag{5.19}$$

Here we shall make addition to the ergodicity of a product measure, as we have done in 5.1.

Theorem 5.6. *Under the notation in this subsection,* μ *is* Φ*-ergodic if and only if* μ_n *is* Φ_n*-ergodic for each* n.

Proof. The necessity is obvious. For the sufficiency, let $A\in\mathfrak{B}$ such that $\mu((A-\varphi)\ominus A)=0$ for all $\varphi\in\Phi$. By the definition of $\mathfrak{B}$, there exist $x^*_{n,j}\in X^*_n$ ($n=1,\cdots,k_j$, $j=1,\cdots$) such that A belongs to the minimal σ-algebra $\mathfrak{C}$ with which all the functions $x\in X\longmapsto\sum_{n=1}^{k_j}\langle x_n, x^*_{n,j}\rangle$ ($j=1,\cdots$) are measurable. Let us denote the minimal σ-algebra on $\prod_{n=1}^{N}X_n$ with which all the functions $(x_1,\cdots,x_N)\longmapsto\sum_{n=1}^{\min(N,k_j)}\langle x_n, x^*_{n,j}\rangle$ are measurable by $\mathfrak{C}_N$. Then we have $A^y\in\mathfrak{C}_N$ for all $y\in\prod_{n=N+1}^{\infty}X_n$. As $\mathfrak{C}_N$ is countably generated, so $L^2_{\mu_1\times\cdots\times\mu_N}(\prod_{n=1}^{N}X_n, \mathfrak{C}_N)$ is separable. Hence $(\Phi_1\times\cdots\times\Phi_N, d_N)$ is separable for the total variation metric d_N derived from $\mu_1\times\cdots\times\mu_N|\mathfrak{C}_N$. Choose a countable dense set $\{\varphi_{N,k}\}_k$ from $(\Phi_1\times\cdots\times\Phi_N, d_N)$, and put $\mu^1:=\mu_1\times\cdots\times\mu_N$ and $\mu^2:=\mu_{N+1}\times\cdots\times\mu_M\times\cdots$. Then it follows from Fubini's theorem that there exists some μ^2-negligible set Ω^c such that for all $y\in\Omega$

$$\mu^1((A^y-\varphi_{N,k})\ominus A^y)=0 \tag{5.20}$$

for all k. Hence proceeding in the same way as before, we have for all $y\in\Omega$

$$\mu^1((A^y-\varphi)\ominus A^y)=0 \tag{5.21}$$

for all $\varphi\in\Phi_1\times\cdots\times\Phi_N$. Since μ^1 is ergodic due to Theorem 5.2, so $\mu^1(A^y)=1$ or 0 for μ^2-a.e.y. Now for any $\epsilon>0$ there exists a tame set $A_\epsilon=\{x\in X|(x_1,\cdots,x_N)\in B_\epsilon\}$ such that $\mu(A\ominus A_\epsilon)<\epsilon$. So

$$\begin{aligned}\mu(A)\mu(A^c_\epsilon)+\mu(A_\epsilon)\mu(A^c)&=\int_{\prod_{N+1}^{\infty}X_n}\{\mu^1(B^c_\epsilon)\mu^1(A^y)+\mu^1((A^y)^c)\mu^1(B_\epsilon)\}\mu_2(dy)\\&=\mu(A\ominus A_\epsilon)<\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\mu(A)\mu(A^c)=0$. □

§6. Gaussian Measure

6.1 Gaussian measure. Let γ be a Gaussian measure on $(X,\mathfrak{B})$. That is, its characteristic function $\hat{\gamma}(x^*)$, $x^*\in X^*$ has the following form.

$$\hat{\gamma}(x^*)=\exp(im(x^*)-2^{-1}v^2(x^*)), \tag{6.1}$$

where

$$m(x^*)=\int_X\langle x, x^*\rangle\gamma(dx) \text{ and } v^2(x^*)=\int_X\{\langle x, x^*\rangle-m(x^*)\}^2\gamma(dx). \tag{6.2}$$

Put $\bar{\gamma}(E):=\gamma(-E)$ for all $E\in\mathfrak{B}$ and let g be a image measure of the convolution of $\gamma*\bar{\gamma}$ by the homothety, $x\longmapsto 2^{-1/2}x$. Then

$$\hat{g}(x^*)=\hat{\gamma}(2^{-1/2}x^*)\,\hat{\bar{\gamma}}(2^{-1/2}x^*)=\exp(-2^{-1}v^2(x^*)) \tag{6.3}$$

and

$$\hat{\gamma}(x^*)=\exp(im(x^*))\,\hat{g}(x^*). \tag{6.4}$$

In this section we shall consider representations such type as

$$V_{\gamma,s}(\varphi):h(x)\in L^2_\gamma \longmapsto \left(\frac{d\gamma_\varphi}{d\gamma}\right)^{1/2+is} h(x-\varphi)\in L^2_\gamma, \tag{6.5}$$

where $\varphi\in A^\circ_\gamma$ and s is a real parameter.
Now consider a map $\mathcal{M}$ defined for the tame bounded functions such that

$\mathcal{M}:f(\langle x, x_1^*\rangle, \cdots, \langle x, x_n^*\rangle)\in L^2_g \longmapsto f(\langle x, x_1^*\rangle - m(x_1^*), \cdots, \langle x, x_n^*\rangle - m(x_n^*))\in L^2_\gamma$.

It is easy to see that $\mathcal{M}$ is well defined and has a unitary extension which will be denoted by the same letter. Put

$$S_\varphi := \mathcal{M}\left(\sqrt{\frac{dg_\varphi}{dg}}\right) \tag{6.6}$$

for $\varphi\in A^\circ_g$. Then $s_\varphi(x)\geq 0$ for γ-a.e.x and

$$\begin{aligned}
&\int_X S^2_\varphi(x)\chi_E(\langle x, x_1^*\rangle, \cdots, \langle x, x_n^*\rangle)\gamma(dx)\\
&=\int_X \chi_E(\langle x, x_1^*\rangle + m_1(x^*), \cdots, \langle x, x_n^*\rangle + m_n(x^*))g_\varphi(dx)\\
&=\int_X \chi_E(\langle x, x_1^*\rangle, \cdots, \langle x, x_n^*\rangle)\gamma_\varphi(dx).
\end{aligned}$$

It derives that

$$\gamma_\varphi\cong\gamma \text{ and } \sqrt{\frac{d\gamma_\varphi}{d\gamma}}=\mathcal{M}\left(\sqrt{\frac{dg_\varphi}{dg}}\right) \tag{6.7}$$

for $\varphi\in A^\circ_g$. In a similar way, we have

$$A^\circ_\gamma = A^\circ_g. \tag{6.8}$$

Note that $\mathcal{M}(h\cdot f)=\mathcal{M}(h)\cdot\mathcal{M}(f)$ for all $h\in L^2_g$ and for all $f\in L^\infty_g$. Thus we have

$$\mathcal{M}\left(\left(\frac{d\gamma_\varphi}{d\gamma}\right)^{1/2+si}\right)=\left(\frac{dg_\varphi}{dg}\right)^{1/2+si}, \tag{6.9}$$

and it follows that

$$\begin{aligned}
&\mathcal{M}\circ V_{g,s}(\varphi)(f)(x)=\mathcal{M}\left(\left(\frac{dg_\varphi}{dg}(\cdot)\right)^{1/2+si} f(\cdot-\varphi)\right)(x)\\
&=\left(\frac{d\gamma_s}{d\gamma}\right)^{1/2+si}(x)(\mathcal{M}f)(x-\varphi)=V_{\gamma,s}(\varphi)\circ\mathcal{M}(f)(x).
\end{aligned}$$

Theorem 6.1. *$(V_{\gamma,s}, A^\circ_\gamma)$ is equivalent to $(V_{g,s}, A^\circ_g)$ by the intertwining* *operator* $\mathcal{M}$. *Moreover* γ *is* Φ*-ergodic if and only if so is* g.

Proof. We only have to show the second assertion. Suppose that γ is Φ-ergodic and $g((B-\varphi)\ominus B)=0$ for all $\varphi\in\Phi$. Then as $\mathcal{M}(\chi_E(\cdot-\varphi))(x)=(\mathcal{M}\chi_E)(x-\varphi)$, which is first valid for the tame set E and generally holds by limiting procedure, so $\mathcal{M}\chi_B=const$ mod γ and we have $\chi_B=1$ or 0 mod g. The converse will be proved similarly. □

By the above theorem it is sufficient to consider only centered Gaussian measures g for our subject. Now set

$$\|x^*\|_g:=\left\{\int_X\langle x, x^*\rangle^2 g(dx)\right\}^{1/2} \tag{6.10}$$

for $x^*\in X^*$, and put

$$H_g:=\{\varphi\in X\,|\,|\langle\varphi, x^*\rangle|\leq{}^{\exists}C_\varphi\|x^*\|_g\}, \tag{6.11}$$

which is called the reproducing kernel Hilbert space. Then for any $\varphi\in H_g$ there exists a unique $W_\varphi(x)$ belonging to the L^2-closure W of $\{\langle x, x^*\rangle|x^*\in X^*\}$ such that

$$\langle\varphi, x^*\rangle=\int_X\langle x, x^*\rangle W_\varphi(x)g(dx). \tag{6.12}$$

Since a map $\varphi\longmapsto W_\varphi$ is one to one, so an inner product structure is naturally induced from L^2_g. Moreover it is well known that

$$A_g^\circ=H_g \tag{6.13}$$

$$\frac{dg_\varphi}{dg}(x)=\exp(W_\varphi(x)-2^{-1}\|\varphi\|_g^2)\text{ and} \tag{6.14}$$

$$W_\varphi(x+h)=W_\varphi(x)+\langle\varphi, h\rangle_{H_g} \tag{6.15}$$

for all φ, $h\in H_g$. As for the Kakutani's metric d_g on A_μ°, we have

$$d_g(\varphi, 0)^2=2\{1-\exp(-8^{-1}\|\varphi\|_{H_g}^2)\}. \tag{6.16}$$

Thus the topologies on H_g derived from d_g and the norm $\|\cdot\|_{H_g}$ coincide and it is stronger than $\sigma(X, X^*)$. As we assume that L^2_g is separable, so is H_g.

Proposition 6.1. *The followings are all equivalent.*

(1) g *is* H_g*-ergodic.*

(2) *The* L^2*-closure* W' *of* $\{W_\varphi\}_{\varphi\in H_g}$ *coincides with* W.

(3) *There exists an orthonormal set* $\{h_n\}_n\subset H_g$ *such that*

$$\langle x, x^*\rangle = \sum_{n=1}^{\infty} \langle h_n, x^*\rangle W_{h_n}(x) \tag{6.17}$$

for all $x^* \in X^*$, *where the equality holds in* L^2_g*-sense.*

Proof. (1)$\Rightarrow$(2). Take any $S(x) \in W \cap (W')^{\perp}$. Then there exists a sequence $\{x_n^*\}_n \subset X^*$ such that $\langle x, x_n^*\rangle \longrightarrow S(x)$ $(n \longrightarrow \infty)$ in L^2_g. Since

$$\lim_{n\to\infty}\langle \varphi, x_n^*\rangle = \lim_{n\to\infty}\int_X \langle x, x_n^*\rangle W_\varphi(x) g(dx) = \int_X S(x) W_\varphi(x) g(dx) = 0,$$

so, if necessary, taking a subsequence $\{x_n^*\}_n$ we have

$$S(x+\varphi) = \lim_{n\to\infty}\langle x+\varphi, x_n^*\rangle = S(x)$$

for g-a.e.x. Thus $S(x) = const \mod g$ and the constant is equal to 0 because $\int_X S(x) g(dx) = 0$.

(2)$\Rightarrow$(3). Let $\{h_n\}_n \subset H_g$ be a complete orthonormal set in the completion on H_g. By the assumption $\langle x, x^*\rangle$ is adherent to the linear span of W'_{h_n}s. Therefore (6.17) exactly holds by virtue of (6.12).

(3)$\Rightarrow$(1). Suppose that $g((B-\varphi) \ominus B) = 0$ for all $\varphi \in H_g$. By the assumption there exists a measurable set $\hat{B} \subseteq \boldsymbol{R}^\infty$ such that $\chi_B(x) = \chi_{\hat{B}}((W_{h_1}(x), \cdots, W_{h_n}(x), \cdots))$. Now take any $(\alpha_1, \cdots, \alpha_n) \in \boldsymbol{R}^n$ and set $\varphi := \alpha_1 h_1 + \cdots + \alpha_n h_n$. As the image of g by a map $x \in X \longrightarrow (W_{h_1}(x), \cdots, W_{h_n}(x), \cdots) \in \boldsymbol{R}^\infty$ is the standard Gaussian measure G on $\boldsymbol{R}^\infty$, so

$$\begin{aligned}
&\int_{\boldsymbol{R}^\infty} |\chi_{\hat{B}}(y + (\alpha_1, \cdots, \alpha_n, 0, 0, \cdots)) - \chi_{\hat{B}}(y)| G(dy) \\
&= \int_X |\chi_{\hat{B}}((W_{h_n}(x+\varphi)) - \chi_{\hat{B}}((W_{h_n}(x))| g(dx) \\
&= \int_X |\chi_B(x+\varphi) - \chi_B(x)| g(dx) = 0.
\end{aligned}$$

It follows that $G(\hat{B}) = 1$ or 0, because G is $\boldsymbol{R}_0^\infty$-ergodic. So, $g(B) = 1$ or 0. □

Remark. We don't yet know whether the conditions of Proposition 6.1 always hold or not, and further H_g is always complete or not. However if γ is extended to a weak Radon measure on X, (the extension is unique,) then the above questions are all solved affirmatively. Besides, it holds certainly that $m \in (X^*)^a$ in (6.2) belongs to X. Moreover provided that X is $\sigma(X, X^*)$-complete, then H_g is complete with or without Radon extensibility, which is easily seen by virtue of (P.4).

Proposition 6.2. *Let* $\tilde{X}$ *be the completion of* $(X, \sigma(X, X^*))$ *and put* $\tilde{g}$ *be the image measure of* g *by the imbedding map* $\iota : X \longmapsto \tilde{X}$. *Then if* g

is H_g-ergodic, then so is $\widetilde{g}$, and H_g is dense in $H_{\widetilde{g}}$.

Proof. It is easy to see that $H_g \subseteq H_{\widetilde{g}}$ and $\widetilde{g}$ is H_g-ergodic, and that the norm on $H_{\widetilde{g}}$ is an extension of the norm on H_g. We shall prove that H_g is dense in $H_{\widetilde{g}}$. Suppose that it would be false. Then there exists some $\widetilde{S}(\widetilde{x})$ $(\neq 0)$ belonging to the L^2-closure of $\{\langle \widetilde{x}, x^*\rangle\}_{x^* \in X^*}$ such that $\int_{\widetilde{X}} \widetilde{S}(\widetilde{x}) \widetilde{W}_h(\widetilde{x}) \widetilde{g}(d\widetilde{x}) = 0$ for all $h \in H_g$. Thus proceeding in the same way as before, we have $\widetilde{S}(\widetilde{x} + h) = \widetilde{S}(\widetilde{x})$ for $\widetilde{g}$-a.e. $\widetilde{x}$. It follows that $\widetilde{S}(\widetilde{x}) = 0$ for $\widetilde{g}$-a.e. $\widetilde{x}$, which contradicts to the assumption. □

In a little while, we shall take and fix a complete orthonormal set $\{h_n\}_n \subset H_g$ in the completion of H_g and $\{\omega_q\}_{q \in Q}$ which is a c.o.n.s. in $W \cap (W')^{\perp}$. As we have seen,

$$\langle x, x^*\rangle = \sum_{n=1}^{\infty} \langle h_n, x^*\rangle W_{h_n}(x) + \sum_{q \in Q} \alpha_q \omega_q(x), \tag{6.18}$$

where $\alpha_q = \int_X \langle x, x^*\rangle \omega_q(x) g(dx)$, and it holds that for all $\varphi \in H_g$

$$\omega_q(x + \varphi) = \omega_q(x) \tag{6.19}$$

for g-a.e.x. Now we shall find a direct integral for the representation $(V_{g,s}, H_g)$. So let us take tame functions h and g such that $h(x) = H(W_{h_1}(x), \cdots, W_{h_N}(x), \omega_1(x), \cdots, \omega_M(x))$, $g(x) = G(W_{h_1}(x), \cdots, W_{h_N}(x), \omega_1(x), \cdots, \omega_M(x))$.
We calculate $\langle V_{g,s}(\varphi)h, g\rangle_2$ for $\varphi = \sum_{n=1}^{\infty} \varphi_n h_n \in H_g$.

$$\begin{aligned}
&\langle V_{g,s}(\varphi)h, g\rangle_2 \\
&= \int_X \left(\frac{dg_\varphi}{dg}\right)^{1/2+si} h(x-\varphi)\, \overline{g}(x) g(dx) \\
&= \exp(-2^{-1}(2^{-1}+si)\|\varphi\|_{H_g}^2) \int_X \exp((2^{-1}+si) W_\varphi(x)) H(W_{h_1}(x) - \varphi_1, \cdots, W_{h_N}(x) \\
&\quad - \varphi_N, \omega_1(x), \cdots, \omega_M(x)) \overline{G}(W_{h_1}(x), \cdots, W_{h_N}(x), \omega_1(x), \cdots, \omega_M(x)) g(dx) \\
&= \exp(-2^{-1}(2^{-1}+si)\|\varphi\|_{H_g}^2) \exp(2^{-1}(2^{-1}+si)^2 \sum_{n=N+1}^{\infty} \varphi_n^2) \cdot \\
&\quad \int_{R^N} \int_{R^M} \exp((2^{-1}+si) \sum_{n=1}^{N} \varphi_n x_n) \cdot \\
&\quad H(x_1 - \varphi_1, \cdots, x_N - \varphi_N, y_1, \cdots, y_M) \overline{G}(x_1, \cdots, x_N, y_1, \cdots, y_M) g_N(dx) g_M(dy),
\end{aligned}$$

where g_N is the standard Gaussian measure on $\boldsymbol{R}^N$. And

$$\begin{aligned}
&\langle V_{g,s}(\varphi)h, g\rangle_2 \\
&= \exp(-8^{-1}(1+4s^2) \sum_{n=N+1}^{\infty} \varphi_n^2) \int_{R^N} \int_{R^M} (2\pi)^{-N/2} \exp(-2^{-1}(2^{-1}+si) \\
&\quad \sum_{n=1}^{N} (x_n - \varphi_n)^2) H(x_1 - \varphi_1, \cdots, x_N - \varphi_N, y_1, \cdots, y_M) \exp(-2^{-1}(2^{-1}-si) \sum_{n=1}^{N} x_n^2) \cdot
\end{aligned}$$

$\bar{G}(x_1, \cdots, x_N, y_1, \cdots, y_M)g_M(dy)dx.$

Consequently,

$$(6.20)\quad \langle V_{g,s}(\varphi)h, g\rangle_2 = \exp(-8^{-1}(1+4s^2)\sum_{n=N+1}^{\infty}\varphi_n^2)\int_{R^N}\int_{R^M}\exp(i\sum_{n=1}^{N}\varphi_n x_n) \mathcal{F}\{(2\pi)^{-N/4}\exp(-2^{-1}(2^{-1}+si)\sum_{n=1}^{N}\xi_n^2)\cdot H(\xi, y)\}(x)\overline{\mathcal{F}}\{(2\pi)^{-N/4}\exp(-2^{-1}(2^{-1}+si)\sum_{n=1}^{N}\xi_n^2)K(\xi, y)\}(x)dxg_M(dy).$$

Now consider a measure m_H whose density is given by

$$(6.21)\quad \frac{m_H(dx)}{dx}=\int_{R^M}|\mathcal{F}\{(2\pi)^{-N/4}\exp(-2^{-1}(2^{-1}+si)\sum_{n=1}^{N}\xi_n^2)H(\xi, y)\}|^2(x)\gamma_M(dy).$$

Then (6.20) is rewritten as

$$\langle V_{g,s}(\varphi)h, h\rangle_2=\exp(-8^{-1}(1+4s^2)\sum_{n=N+1}^{\infty}\varphi_n^2)\hat{m}_H((\varphi_1, \cdots, \varphi_N)).$$

Here let us take a Gaussian measure G_s on $(H^a, \mathfrak{C}_H)$ such that $\hat{G}_s(\varphi)=\exp(-8^{-1}(1+4s^2)\|\varphi\|_{H_g}^2)$ and put $P_N(\varphi):=\sum_{n=1}^{N}\varphi_n h_n$, $Q_N:=\mathrm{I}-P_N$ and $h_n^*(\varphi):=\langle\varphi, h_n\rangle_{H_g}$. Using an isomorphism η from H^a to $\boldsymbol{R}^N\times\{h_1, \cdots, h_N\}^{\perp}$ such that

$$\eta : \varphi^a \longmapsto ((\langle h_1, \varphi^a\rangle, \cdots, \langle h_N, \varphi^a\rangle), \varphi^a-\sum_{n=1}^{N}\langle h_n, \varphi^a\rangle h_n^*),$$

we obtain an image measure $\eta^{-1}(m_H\times {}^tQ_NG_s)$ on H^a. It is quite easy to see that its characteristic function coincides with $\langle V_{g,s}(\varphi)h, h\rangle_2$. Further m_1 (corresponding to $H=1$) is equivalent to the Lebesgue measure λ_N on $\boldsymbol{R}^N$, so $m_H\times {}^tQ_NG_s$ is absolutely continuous with $m_1\times {}^tQ_NG_s$ and $m_1\times {}^tQ_NG_s$ correspondes to G_s by the above map η. It gives that G_s is a spectral measure for $(V_{g,s}, H_g)$. (It is realized as $H=1$.)

Now let us consider the multiplicity. We shall rewrite (6.20) as the following form using the spectral measure G_s and the standard Gaussian measure g_Q on $\boldsymbol{R}^Q$.

$$(6.22)\quad \langle V_{g,s}(\varphi)h, g\rangle_2 =\int_{H^a}\int_{R^Q}\exp(i\langle\varphi, \varphi^a\rangle)\left\{(2\pi)^{-1/4}\sqrt{\frac{2}{1+2si}}\right\}^{-N}\exp\left(\frac{1}{1+2si}\sum_{n=1}^{N}\langle h_n, \varphi^a\rangle^2\right)\cdot \mathcal{F}\{(2\pi)^{-N/4}\exp(-2^{-1}(2^{-1}+si)\sum_{n=1}^{N}\xi_n^2)H(\xi, y)\}(\langle h_1, \varphi^a\rangle, \cdots, \langle h_N, \varphi^a\rangle)\cdot \left\{(2\pi)^{-1/4}\sqrt{\frac{2}{1-2si}}\right\}^{-N}\exp\left(\frac{1}{1-2si}\sum_{n=1}^{N}\langle h_n, \varphi^a\rangle^2\right)\cdot$$

$$\mathcal{F}\{(2\pi)^{-N/4}\exp(-2^{-1}(2^{-1}+si)\sum_{n=1}^{N}\xi_n^2)\cdot$$
$$G(\xi, y)\}(\langle h_1, \varphi^a\rangle, \cdots, \langle h_N, \varphi^a\rangle)G_s(d\varphi^a)g_Q(dy).$$

Hence a map

$$(6.23)\quad S_s: H(W_{h_1}(x), \cdots, W_{h_N}(x), \omega_1(x), \cdots, \omega_M(x))$$
$$\longmapsto\left\{(2\pi)^{-1/4}\sqrt{\frac{2}{1+2si}}\right\}^{-N}\exp\left(\frac{1}{1+2si}\sum_{n=1}^{N}\langle h_n, \varphi^a\rangle^2\right)\mathcal{F}\{(2\pi)^{-N/4}\cdot$$
$$\exp(-2^{-1}(2^{-1}+si)\sum_{n=1}^{N}\xi_n^2)H(\xi_1, \cdots, \xi_N, y_1, \cdots, y_M)\}$$

is well defined for the tame functions h and it has a unitary extension from L_g^2 to $L_{Gs}^2(H^a, L_{gQ}^2(\boldsymbol{R}^Q))$ of all $L_{gQ}^2(\boldsymbol{R}^Q)$-valued square summable functions.
(The onto property follows from

$$S_s\left(\left(\frac{dg_\varphi}{dg}\right)^{1/2+si}h(x-\varphi)\right)(\varphi^a)=\exp(i\langle\varphi, \varphi^a\rangle)(S_sh)(\varphi^a).)$$

And we have

$$(6.24)$$
$$\int_X\left(\frac{dg_\varphi}{dg}\right)^{1/2+si}h(x-\varphi)g(x)g(dx)=\int_{H^a}\exp(i\langle\varphi, \varphi^a\rangle)\langle S_sh|S_sg\rangle_{L_{gQ}^2}(\varphi^a)G_s(d\varphi^a).$$

We settle these arguments as the following theorem.

Theorem 6.2. *Let g be a centered Gaussian measure on $(X, \mathfrak{B})$ and H_g be the reproducing kernel space. Then the spectral measure G_s for $(V_{g,s}, H_g)$ is given by*

$$\hat{G}_s(\varphi)=\exp(-8^{-1}(1+4c^2)\|\varphi\|_{H_g}^2)=\int_X\left(\frac{dg_\varphi}{dg}\right)^{1/2+si}(x)g(dx).$$

Further the direct integral is realized as $L_{Gs}^2\otimes L_{R^Q}^2$ and the corresponding map from L_g^2 to $L_{Gs}^2\otimes L_{R^Q}^2$ is given by (6.23). *Therefore it has uniform mutiplicity* 1 *or* ∞ *according to whether g is H-ergodic or not.*

Corollary 6.3. *Let g_s be the image measure of g by the homothety, $x \longmapsto (1+4s^2)^{-1/2}x$. Then the representation $(V_{g,s}, H_g)$ is equivalent to $(V_{gs,0}, H_{gs})$.*

Proof. It is obvious that $A_g^\circ=A_{gs}^\circ$ and the ergodic notion is invariant for these measures. Thus two representations have the same multiplicity. Let $\lambda_{gs,0}$ be the spectral measure of $(V_{gs,0}, H_{gs})$. Then $\hat{\lambda}_{gs,0}(\varphi)=\exp(-8^{-1}\|\varphi\|_{H_{gs}}^2)$ and

$$\|\varphi\|_{H_{g_s}}=\sup_{\|x^*\|_{g_s}=1}\langle\varphi,\ x^*\rangle=\sup_{\|x^*\|_g=\sqrt{1+4s^2}}\langle\varphi,\ x^*\rangle=\sqrt{1+4s^2}\,\|\varphi\|_{Hg}.$$

Thus they have the same spectral measure. □

Remark. For the case g is H_g-ergodic, S_s is regarded as a kind of Fourie transform on infinite dimensional spaces. Without proofs we shall give an interesting formula such that

$$(6.25)\qquad \begin{cases} S_s\circ V_{g,s}(\varphi)\circ S_s^{-1}\tilde{J}(\varphi^a)=\exp(i\langle\varphi,\ \varphi^a\rangle)\,\tilde{J}(\varphi^a) \\ S_s\circ U_g(x^*)\circ S_s^{-1}\tilde{J}(\varphi^a)\Big(\dfrac{d(G_s)_{-x^*}}{dG_s}\Big)^{1/2-si}\tilde{J}(\varphi^a+x^*) \end{cases}$$

for all $\tilde{J}\in L^2_{\lambda g,s}$. That is s is changed to $-s$ under the dual operation. Besides, g and G_s correspondes to the norms of the same type, if $s=\sqrt{3}/2$.

Theorem 6.4. *Let s' be a real number and g' be another centered Gaussian measure on $(X,\ \mathfrak{B})$ such that $A_g^\circ=A_{g'}^\circ=:E$.*
(1) *If the representations $(V_{g,s},\ E)$ and $(V_{g',s'},\ E)$ are equivalent, then both g and g' are E-ergodic or so is neither g nor g'.*
(2) *If both g and g' are E-ergodic or so is neither g nor g', then $(V_{g,s},\ E)$ and $(V_{g',s'},\ E)$ are equivalent, if and only if the spectral measure $\lambda_{g,s}$ and $\lambda_{g',s'}$ are equivalent.*
(3) *Under an assumption that both g and g' are E-ergodic, the representations $(V_{g,s},\ E)$ and $(V_{g',s'},\ E)$ are equivalent, if and only if g_s is equivalent to $g'_{s'}$, where g_s is the image measure of g by the homothety, $x\longmapsto(1+4s^2)^{-1/2}x$.*

Proof. There is nothing to prove (1) and (2) and the sufficiency of (3). In order to prove the necessity of (3), it is sufficient to consider the case $s=s'=0$ due to Corollary 6.3. Let $\tilde{X}$ be the completion of $(X,\ \sigma(X,\ X^*))$, and $\tilde{g},\ \tilde{g}'$ be the image measures of g and g' by the imbedding map $\iota: X\longmapsto\tilde{X}$, respectively. As it holds that $\mathfrak{B}=({}^t\iota)^{-1}(\tilde{\mathfrak{B}})$, where $\tilde{\mathfrak{B}}$ is the cylindrical σ-algebra on $\tilde{X}$, so we only have to check that $\tilde{g}\cong\tilde{g}'$ for the proof. By the assumption spectral measures are equivalent. Thus the norm $\|\cdot\|_{Hg}$ on E is equivalent to $\|\cdot\|_{Hg'}$ and we have

$$(6.26)\qquad A_{\tilde{g}}^\circ=A_{\tilde{g}'}^\circ=:\tilde{E},$$

which follows from Proposition 6.2. Now let G and G' be the spectral measure on E^a for the original representations such that $\hat{G}(\varphi)=\exp(-8^{-1}\|\varphi\|_{Hg}^2)$ and $\hat{G}'(\varphi)=\exp(-8^{-1}\|\varphi\|_{Hg'}^2)$, respectively. Moreover let $\tilde{G}$ and $\tilde{G}'$ be probability measures on $\tilde{E}^a$ such that $\hat{\tilde{G}}(\varphi)=\exp(-8^{-1}\|\varphi\|_{H_{\tilde{g}}}^2)$ and $\hat{\tilde{G}}'(\varphi)=\exp(-8^{-1}\|\varphi\|_{H\tilde{g}'}^2)$, respectively. Then we have ${}^t\eta\tilde{G}=G$ and ${}^t\eta\tilde{G}'=G'$, where $\eta: E$

$\longmapsto \tilde{E}$ is the imbedding map. Further it is easily checked that $\mathfrak{C}_{\tilde{E}}=(t\eta)^{-1}(\mathfrak{C}_E)$ mod $\tilde{G}+\tilde{G}'$. Thus $\tilde{G}\cong\tilde{G}'$ follows from our assumption $G\cong G'$. Here we shall use the following well known fact. (For example, see [2] or [11].)

Theorem 6.5. *In order that $\tilde{G}\cong\tilde{G}'$ holds, it is necessary and sufficient that there exists a positive definite isomorphic operator P on $\tilde{E}$ such that $\mathrm{I}-P$ is a Hilbert-Schmidt type and $\|P\varphi\|_{H_{\tilde{g}}}=\|\varphi\|_{H_{\tilde{g}}}$ for all $\varphi\in\tilde{E}$.*

Let $\tilde{h}_1, \cdots, \tilde{h}_n, \cdots$, be the complete system of eigen vectors of P, and $(1+\lambda_n)$ be the corresponding eigen value. And put

$$\rho(\tilde{x})=\prod_{n=1}^{\infty}(1+\lambda_n)\exp(-2^{-1}(2\lambda_n+\lambda_n^2)W_{\tilde{h}_n}^2(x)). \tag{6.27}$$

The infinite product in (6.27) exactly converges in $\mathrm{L}_{\tilde{g}}^1$. We shall calculate the characteristic function of the measure $\rho(\tilde{x})\tilde{g}(d\tilde{x})$.

$$A(x^*):=\int_{\tilde{X}}\exp(i\langle\tilde{x}, x^*\rangle)\rho(\tilde{x})\tilde{g}(d\tilde{x}). \tag{6.28}$$

Since $\tilde{g}$ is $\tilde{E}$-ergodic by virtue of Proposition 6.2, so we have

$$\langle\tilde{x}, x^*\rangle=\sum_{n=1}^{\infty}\langle\tilde{h}_n, x^*\rangle W_{\tilde{h}_n}(\tilde{x}) \text{ in } \mathrm{L}_{\tilde{g}}^2. \tag{6.29}$$

Consequently,

$$\begin{aligned}
A(x^*)&=\lim_{N\to\infty}\int_{\tilde{X}}\exp(i\sum_{n=1}^{N}\langle\tilde{h}_n, x^*\rangle W_{\tilde{h}_n}(\tilde{x}))\rho(\tilde{x})\tilde{g}(d\tilde{x})\\
&=\lim_{N\to\infty}\lim_{M\to\infty}\int_{\tilde{X}}\exp(i\sum_{n=1}^{N}\langle\tilde{h}_n, x^*\rangle W_{\tilde{h}_n}(\tilde{x}))\cdot\\
&\quad\prod_{n=1}^{M}(1+\lambda_n)\exp(2^{-1}(2\lambda_n+\lambda_n^2)W_{\tilde{h}_n}^2(\tilde{x}))\tilde{g}(d\tilde{x})\\
&=\lim_{N\to\infty}\lim_{M\to\infty}\int_{R^M}\exp(i\sum_{n=1}^{N}\langle\tilde{h}_n, x^*\rangle x_n)(2\pi)^{-M/2}\prod_{n=1}^{M}(1+\lambda_n)\exp(2^{-1}(1+\lambda_n)^2x_n^2)dx\\
&=\lim_{N\to\infty}\exp\left(-2^{-1}\sum_{n=1}^{N}\left(\frac{\langle\tilde{h}_n, x^*\rangle}{1+\lambda_n}\right)^2\right).
\end{aligned}$$

Since $\|(1+\lambda_n)^{-1}\tilde{h}_n\|_{H_{\tilde{g}}}=1$ and $\tilde{g}'$ is $\tilde{E}$-ergodic, so it follows from the corresponding formula to (6.29) to $\tilde{g}'$ that $A(x^*)=\exp(-2^{-1}\|x^*\|_{\tilde{g}'}^2)=\widehat{\tilde{g}}'(x^*)$. Therefore $\tilde{g}'(d\tilde{x})=\rho(\tilde{x})g(d\tilde{x})$ and $\tilde{g}'\cong\tilde{g}$. □

Lastly we shall give an example which is not equivalent to any representation with a real 1-cocycle.

Example 5. Under the same notation in this section, let g be a Gaussian measure on $(X, \mathfrak{B})$, H_g be the reproducing kernel space, and $h_1, \cdots, h_n, \cdots$ be a c.o.n.s. in H_g. Then for any $\varphi \in H_g$

$$\sum_{n=1}^{\infty}\{W_{h_n}(x-\varphi)^3-3W_{h_n}(x-\varphi)-W_{h_n}^3(x)+3W_{h_n}(x)\}$$

converges for g-a.e.x, because it is equal to

$$\sum_{n=1}^{\infty}\{-3(W_{h_n}^2(x)-1)\langle\varphi, h_n\rangle_H+3W_{h_n}(x)\langle\varphi, h_n\rangle_H^2-\langle\varphi, h_n\rangle_H^3\},$$

and it is well known that $\sum_{n=1}^{\infty}a_n(X_n^2(\omega)-1)$, $\sum_{n=1}^{\infty}a_nX_n(\omega)$ converges if and only if $\{a_n\}_n\in l^2$, respectively, where $\{X_n(\omega)\}_n$ is i.i.d. random variables which obey to the normal law $N(0, 1)$. Put for $c\in\boldsymbol{R}$,

$$\zeta_c(x, \varphi):=\exp(ic\sum_{n=1}^{\infty}(W_{h_n}^3(x-\varphi)-3W_{h_n}(x-\varphi)-W_{h_n}^3(x)+3W_{h_n}(x)).$$

Then ζ_c is a 1-cocycle. Now consider a representation of H_g such that

$$V_{g,c}(\varphi): h(x)\in L_g^2 \longmapsto \sqrt{\frac{dg_\varphi}{dg}}(x)\zeta_c(x, \varphi)h(x-\varphi)\in L_g^2.$$

Let us calculate the value $\langle V_{g,c}(\varphi)h, g\rangle_2$.

$$\langle V_{g,c}(\varphi)h, k\rangle_2=\int_X \exp(-4^{-1}\|\varphi\|_{H_g}^2+2^{-1}W_\varphi(x))\cdot$$
$$\exp(ic\sum_{n=1}^{\infty}(W_{h_n}^3(x-\varphi)-3W_{h_n}(x-\varphi)-W_{h_n}^3(x)+3W_{h_n}(x)))h(x-\varphi)\overline{g}(x)g(dx)$$
$$=\prod_{n=N+1}^{\infty}\exp(-4^{-1}\varphi_n^2)\int_{-\infty}^{\infty}\exp(2^{-1}\varphi_n t)\exp(ic(-3t^2\varphi_n+3t\varphi_n^2-\varphi_n^3+3\varphi_n))\cdot$$
$$(2\pi)^{-1/2}\exp(-2^{-1}t^2)dt\cdot\int_{R^N}\int_{R^M}\exp(i\sum_{n=1}^{N}\varphi_n x_n)\mathcal{F}\{(2\pi)^{-N/4}\exp(\sum_{n=1}^{N}(-4^{-1}\xi_n^2$$
$$+ic(\xi_n^3-3\xi_n)))\cdot$$
$$H(\xi_1, \cdots, \xi_N, y_1, \cdots, y_M)\}(x)\overline{\mathcal{F}}\{(2\pi)^{-N/4}\exp(\sum_{n=1}^{N}(4^{-1}\xi_n^2+ic(\xi_n^3-3\xi_n)))\cdot$$
$$G(\xi_1, \cdots, \xi_N, y_1, \cdots, y_M)\}(x)dxg_M(dy),$$

where $\varphi_n=\langle\varphi, h_n\rangle_{H_g}$. Hereafter we shall proceed a similar manner as before and obtain a spectral measure σ_c on $(H_g^a, \mathfrak{C}_{H_g})$ whose characteristic function is

$$\text{(6.30)}\quad \hat{\sigma}_c(\varphi)=\prod_{n=1}^{\infty}\int_{-\infty}^{\infty}\exp(i\varphi_n t)(2\pi)^{-1/2}|\mathcal{F}\{\exp(-4^{-1}\xi^2+ic(\xi^3-3\xi))\}|^2(t)dt.$$

(The spectral measure is attained at $h=1$.)
By the way, if $(V_{g,c}, H_g)$ would be equivalent to some $(V_{\mu,\theta}, H_g)$ with a real 1-cocycle θ, then σ_c must be quasi-invariant under the map $-I: \varphi^a \longmapsto -\varphi^a$. Because we have $\langle V_{\mu,\theta}(\varphi)h, h\rangle_2=\langle V_{\mu,\theta}(-\varphi)h, h\rangle_2$ for all real valued functions $h\in L_\mu^2$ from which the invariance of the spectral measure $\sigma_{\mu,\theta}$ follows. Since σ_c is regarded as a product measure of the countable copies of a 1-dimensional

measure, so by the theorem of Kakutani (See, [5]),

$$\rho_c(t):=|\mathcal{F}\{\exp(-4^{-1}\xi^2)\exp(ic(\xi^3-3\xi))\}|^2(t)$$

must be an even function of t. Consequently,

$$\psi_c(u):=\int_{-\infty}^{\infty}\exp(itu)\rho_c(t)dt$$

is a real valued function. We shall calculate it exactly.

$$\begin{aligned}\psi_c(u)&=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(-2^{-1}(t-2^{-1}u)^2-8^{-1}u^2\cdot\\&\quad+ic[(t-u)^3-3(t-u)-t^3+3t])dt\\&=(2\pi)^{-1/2}\exp(-8^{-1}u^2+ic[3u-4^{-1}u^3])\int_{-\infty}^{\infty}\exp(-2^{-1}t^2-3icut^2)dt\\&=\exp(-8^{-1}u^2+ic[3u-4^{-1}u^3]-2^{-1}\,\mathrm{Log}(1+6ciu))\\&=\exp(-8^{-1}u^2-4^{-1}\log(1+36c^2u^2)+i[c(3u-4^{-1}u^3)-2^{-1}\tan^{-1}6cu]).\end{aligned}$$

Thus $\psi_c(u)$ has a non-zero imaginary part for each c, which is a contradiction. Since ψ_c is different for each c, so σ_c is singular with each other, and $(V_{g,c}, H)$ is a different representation for each c.

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Added in proof : After writing this paper, the author was informed that there is a counterexample, which is stated below, for the ergodicity of Gaussian measures on linear spaces which is due to Bogachev.

Example: By a result of Fremlin (See the following reference A.1), there exist a centered Gaussian measure g on a Banach space X and $x_0^* \in X^*$ such that $R_\gamma x_0^*$ is not in X, where $R_\gamma x_0^*$ is defined as a functional on X^* by

$$R_\gamma x_0^*(x^*) = \int_X \langle x, x^* \rangle \langle x, x_0^* \rangle g(dx).$$

Since X is a Banach space, the set $E := \{w \in W | R_\gamma w \in H\}$ is a closed linear space of the Hilbert space W. Let x_E^* be the orthogonal projection of x_0^* to E and put $w_0 := x_0^* - x_E^*$. Then for any $h \in H$, $w_0(x+h) = w_0(x)$ holds for g-a.e. x, which is easily verified. Hence there exists a non-trivial H-invariant function, so g is not H-ergodic.
Thus the first question of Remark often Proposition 6.1 is solved negatively.

A.1 Fremlin, D.H. and Talagrand, M., A Gaussian measure on l^∞, *Ann. Probab.*, **8** (1980), 1192–1193.
A.2 Bogachev., V.I., Gaussian measures on linear spaces, *preprint*.