Spherical Functions of the Principal Series Representations of $Sp(2, \boldsymbol{R})$ as Hypergeometric Functions of C₂-Type

By

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§0. Introduction

In this article we determine explicitly the systems of differential equations satisfied by spherical functions with non-trivial K-types of the principal series and the generalized principal series representations of $Sp(2, \boldsymbol{R})$. Then we obtain series expansions and integral formulas of spherical functions of the generalized principal series representation.

We shall define spherical functions. Let G be a real reductive Lie group and K be maximal compact subgroup of G, $P_0=M_0A_0N_0$ be a parabolic subgroup of G. Let H_π be an admissible representation of G and (τ, V_τ), (η, V_η) be irreducible representations of K which is contained in H_π. We call elements of $\mathrm{Hom}_K(V_\tau, C_\eta^\infty(K\backslash G)) \simeq C_\eta^\infty(K\backslash G) \otimes_K V_\tau^* = C_{\eta,\tau}^\infty(K\backslash G/K)$ spherical functions of type-(η, τ), where $C_\eta^\infty(K\backslash G)$ is the space of smooth sections of the homogeneous vector bundle over $K\backslash G$ associated to V_η and V_τ^* is the contragredient representation of V_τ. Let $\phi \in \mathrm{Hom}_{(\mathfrak{g}, K)}(H_\pi, C_\eta^\infty(K\backslash G))$ and $i \in \mathrm{Hom}_K(V_\tau, H_\pi)$, then $\phi \circ i$ is a spherical function attached to H_π.

There are many studies on the system of differential equations satisfied by spherical functions for 1-dimensional K-types. Moreover they are generalized as the Weyl group invariant commuting differential operators with continuous parameters, which are introduced by generalizing root multiplicities (cf. [DG1], [DG2], [H1], [HO], [Ko], [OO], [Op1], [Op2], [Os], [OS], [Sh]). On the other hand, there are few studies for vector-valued spherical functions. Besides, spherical functions are rarely calculated in explicit forms except for rank one cases. Therefore it is interesting to study vector-valued spherical functions of higher rank Lie group explicitly.

In this article we treat $Sp(2, \mathbf{R})$ as G and the principal series and the generalized principal series representation as H_π. We call $H_{\pi_0} = \operatorname{Ind}_P^G$ (a character of P) the principal series representation and $H_{\pi_1} = \operatorname{Ind}_{P_J}^G(\sigma)$ the generalized principal series representation. Here $P = MAN$ is a minimal parabolic subgroup of G, $P_J = M_J A_J N_J$ is the Jacobi parabolic subgroup of G and σ is a tensor product of a discrete series representation of M_J and a character of $A_J N_J$.

We give explicit formulas of the systems of differential equations satisfied by spherical functions of H_{π_0} and H_{π_1}. If H_π has the infinitesimal character, its spherical function is the eigenfunction of elements of $Z(\mathfrak{g})$, the center of the universal enveloping algebra $U(\mathfrak{g})$. $Z(\mathfrak{g})$ for $G = Sp(2, \mathbf{R})$ is generated by two elements. One is the Casimir element of order 2, the other is of order 4. It is difficult to calculate the radial part of the latter operator with respect to KAK-decomposition. We avoid the difficulty by using shift operators, which are defined by means of the Schmid operator. Its name comes from the property of shifting the parameter of K-types. Moreover, this method is useful for studying the reducibility of the differential equations for H_{π_1}. We choose τ, η from K-types of minimal dimension in H_π, which is 1- or 2-dimensional. We can obtain spherical functions for higher dimensional K-types from those for minimal dimensional K-types and shift operators in principle.

We shall give series expansions and integral representations for the solutions of the system of the differential equations of H_{π_1}.

The main results of this article are the following.

Theorem 0.1 (Theorem 7.3).   *For* $\eta = (k, k)$, $\tau = (l, l) \in \widehat{K}$ *with* $k \equiv l$ mod 2, *the system of differential equations satisfied by spherical functions* $\phi \in C_{\eta,\tau}^\infty$ $(K \backslash G / K)$ *of* $H_{\pi_0} = \operatorname{Ind}_P^G(\sigma \otimes a^{\mu+\rho} \otimes 1_N)$ *is the following.*

$$(0.1) \quad R(L)\phi(x_1, x_2) = (\mu_1^2 + \mu_2^2 - 5)\phi(x_1, x_2)$$

$$(0.2) \quad R(D_{l-2}^+) \circ R(D_l^-)\phi(x_1, x_2) = 4\{\mu_1^2 - (l-1)^2\}\{\mu_2^2 - (l-1)^2\}\phi(x_1, x_2)$$

Theorem 0.2 (Theorem 7.4).   *For* $\eta = (k, k-1)$, $\tau = (l, l-1) \in \widehat{K}$ *with* $k \equiv l$ mod 2, *the system of differential equations satisfied by spherical functions* $\phi \in C_{\eta,\tau}^\infty(K \backslash G / K)$ *of* $H_{\pi_0} = \operatorname{Ind}_P^G(\sigma \otimes a^{\mu+\rho} \otimes 1_N)$ *is the following.*

$$(0.3) \quad R(L)\phi(x_1, x_2) = (\mu_1^2 + \mu_2^2 - 5)\phi(x_1, x_2)$$

$$(0.4) \quad R(E_{l-1}^+) \circ R(E_l^-)\phi(x_1, x_2) = \begin{cases} -\{\mu_1^2 - (l-1)^2\}\phi(x_1, x_2) & \text{if } l : \text{odd} \\ -\{\mu_2^2 - (l-1)^2\}\phi(x_1, x_2) & \text{if } l : \text{even} \end{cases}$$

Here, $(l_1, l_2) \in \{(m, n) \in \mathbf{Z}^2 \mid m - n \geq 0\} \simeq \widehat{K}$ represents the irreducible representation of K with the highest weight (l_1, l_2), whose dimension is $l_1 - l_2 + 1$. L

is the Casimir operator of $U(\mathfrak{g})$ and $D_l^{\pm}$ and $E_l^{\pm}$ are shift operators, which shift the parameter of τ. The radial parts $R(L)$, $R(D_l^{\pm})$ and $R(E_l^{\pm})$ are calculated in §5 and 6.

Spherical functions are considered as functions of two variables and we take certain special coordinates (x_1, x_2) for variables.

We denote $\delta(x_1, x_2; k, l) = (\mathrm{ch}\, x_1 \cdot \mathrm{ch}\, x_2)^{\frac{k+l}{2}} (\mathrm{sh}\, x_1 \cdot \mathrm{sh}\, x_2)^{-\frac{k-l}{2}}$ and $y_i = -\mathrm{sh}^2 x_i$.

Let $H_{\pi_1} = \mathrm{Ind}_{P_J}^G(\sigma_J \otimes a_J^{\nu_1 + \rho_J} \otimes 1_{N_J})$, where $\sigma_J = (\varepsilon, D_l)$, $\varepsilon \in \widehat{\{\pm 1\}}$ and D_l is the discrete series representation of $SL(2, \boldsymbol{R})$ with the Blattner parameter l. Note that $M_J \simeq \{\pm 1\} \times SL(2, \boldsymbol{R})$.

Then spherical functions of H_{π_1} satisfy the reducible system of (0.1) and (0.2) or (0.3) and (0.4). This follows from the minimality of dim η and dim τ among K-types in H_{π_1}. Then we have the following.

Theorem 0.3 (Theorem 8.7). *If $\phi = \delta(x_1, x_2; k, l)^{-1}\psi$ is a spherical function of H_{π_1} with 1-dimensional K-types (l, l), (k, k), then ψ has the following series expansion and integral representation up to constant.*

(i) $\quad \psi(y_1, y_2) = \sum_{m_i \geq 0} \dfrac{\left(\dfrac{1}{2}\right)_{m_1} \left(\dfrac{1}{2}\right)_{m_2} (\mu_+)_{m_1+m_2}(\mu_-)_{m_1+m_2}}{m_1!\, m_2!\, (1)_{m_1+m_2} \left(\dfrac{3+k-l}{2}\right)_{m_1+m_2}} y_1^{m_1} y_2^{m_2}$

(ii) $\quad \psi(y_1, y_2) = \displaystyle\int_0^1 {}_2F_1\left(\mu_+, \mu_-; \dfrac{3+k-l}{2}; ty_1 + (1-t)y_2\right) t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt$

Here we set $\mu_{\pm} = -(l - 2 \pm \nu_1)/2$, $(\lambda)_k = \dfrac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ and ${}_2F_1$ is the classical Gaussian hypergeometric function which is analytic around the origin.

Theorem 0.4 (Theorem 8.9). *If $\phi = \phi_{01} v_0^{\eta} \otimes v_1^{\tau *} + \phi_{10} v_1^{\eta} \otimes v_0^{\tau *}$ is a spherical function of H_{π_1} with 2-dimensional K-types $(l, l-1)$, $(k, k-1)$, then $\psi_{01}(x_1, x_2) = \delta(x_1, x_2; k, l) \mathrm{ch}^{-1} x_1 \phi_{01}(x_1, x_2)$ has the following series expansion and integral representation up to constant.*

(i) $\quad \psi_{01}(y_1, y_2) = \sum_{m_i \geq 0} \dfrac{\left(\dfrac{3}{2}\right)_{m_1} \left(\dfrac{1}{2}\right)_{m_2} (\mu_+)_{m_1+m_2}(\mu_-)_{m_1+m_2}}{m_1!\, m_2!\, (2)_{m_1+m_2} \left(\dfrac{3+k-l}{2}\right)_{m_1+m_2}} y_1^{m_1} y_2^{m_2}$

(ii) $\quad \psi_{01}(y_1, y_2) = \displaystyle\int_0^1 {}_2F_1\left(\mu_+, \mu_-; \dfrac{3+k-1}{2}; ty_1 + (1-t)y_2\right) t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}} dt$

Here we set $\mu_{\pm} = -(l - 2 \pm \sqrt{\nu_1^2 - 2l + 1})/2$ and ${}_2F_1$ is the classical Gaussian hypergeometric function which is analytic around the origin.

In order to get Theorem 0.1 and Theorem 0.2, we use the method in [MO1], where Miyazaki and Oda constructed explicitly the system of differential equations satisfied by Whittaker functions for H_{π_0} where $G = Sp(2, \mathbf{R})$. Whittaker functions are elements of $\mathrm{Hom}_K(V_\tau, C_\chi^\infty(N\backslash G))$, where (χ, C_χ) is a unitary character of N and $C_\chi^\infty(N\backslash G)$ is a smooth sections of homogeneous line bundle over $N\backslash G$ associated to C_χ. They calculated the radial parts of the Casimir operator and shift operators for the double coset decomposition $N\backslash G/K$. In our case, we shall calculate the radial parts of the Casimir operator and shift operators for the double coset decomposition $K\backslash G/K$.

In order to obtain Theorems 0.3 and Theorem 0.4, we use the similar method as in [DG1] and [DG2]. Spherical function of H_{π_1} are a generalization of the Appell's hypergeometric functions F_1.

The organization of this article is as follows.

In § 1, 2 and 3, we give a brief review of the structure of $G = Sp(2, \mathbf{R})$ and the representations of G and its maximal compact subgroup K. All of lemmas and propositions in § 1, 2 and 3 are found in [MO1]. In § 4, we see the symmetric properties of spherical functions. In § 5, we calculate the radial part of the Casimir operator. In §6, we define shift operators by using Schmid operators. In § 7, we get the system of differential equations satisfied by spherical functions and in § 8, we have series expansions and integral formulas of spherical functions. In §9, we obtain a relation between Appell's hypergeometric functions F_1 and F_2.

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§ 1.   The Structure of Lie Groups and Lie Algebras

1.1   The Structure of $Sp(2, \mathbf{R})$

We will introduce basic notations about $G = Sp(2, \mathbf{R})$ and its Lie algebra $\mathfrak{g} = \mathfrak{sp}(2, \mathbf{R})$. Put

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \in M(4, \mathbf{R}),$$

where I_2 is the identity matrix of degree 2. The symplectic group $Sp(2, \mathbf{R})$ is given by

$$Sp(2, \mathbf{R}) = \{g \in GL(4, \mathbf{R}) | {}^t g J g = J\}.$$

A maximal compact subgroup of $G = Sp(2, \boldsymbol{R})$ is given by

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(2, \boldsymbol{R}) \,\middle|\, A, B \in M(2, \boldsymbol{R}) \right\},$$

which is isomorphic to the unitary group

$$U(2) = \{ g \in GL(2; \boldsymbol{C}) \,|\, {}^t \overline{g} \cdot g = I_2 \}$$

via a homomorphism

$$u : K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + \sqrt{-1} B \in U(2).$$

1.2 The Structure of $\mathfrak{sp}(2, \boldsymbol{R})$

The Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \mathfrak{sp}(2, \boldsymbol{R}) = \{ X \in M(4, \boldsymbol{R}) \,|\, JX + {}^t XJ = 0 \},$$

and that of K is given by

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A, B \in M(2, \boldsymbol{R}), \; {}^t A = -A, \; {}^t B = B \right\}.$$

We define a Cartan involution θ on $\mathfrak{g}$ by

$$\theta(X) = -{}^t X \quad \text{for } X \in \mathfrak{g}.$$

Then $\mathfrak{k}$ is the 1-eigenspace of θ and the -1-eigenspace is

$$\mathfrak{p} = \{ X \in \mathfrak{g} \,|\, \theta(X) = -X \} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \,\middle|\, A, B \in M(2, \boldsymbol{R}), \; {}^t A = A, \; {}^t B = B \right\},$$

which gives the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The linear map

$$\mathfrak{k} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + \sqrt{-1} B \in \mathfrak{u}(2)$$

defines an isomorphism of Lie algebras from $\mathfrak{k}$ to the Lie algebra

$$\mathfrak{u}(2) = \{ C \in M(2, \boldsymbol{C}) \,|\, {}^t \overline{C} + C = 0 \}.$$

We again denote this map by u.

Let $\mathfrak{a}$ be

$$\mathfrak{a} = \left\{ \left(\begin{array}{cc|cc} t_1 & & & \\ & t_2 & & \\ \hline & & -t_1 & \\ & & & -t_2 \end{array} \right) \in \mathfrak{p} \right\},$$

which is a maximal abelian subspace. And its $\boldsymbol{R}$-basis is given by

$$H_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & -1 & \\ & & & 0 \end{array} \right) \text{ and } H_2 = \left(\begin{array}{cc|cc} 0 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & -1 \end{array} \right).$$

We denote $e_1, e_2 \in \mathfrak{a}^*$ the dual basis of H_1 and H_2. That is, $e_i(H_j) = \delta_{ij}$. Then the restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$ is given by

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{ \pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2 \}.$$

We fix a positive root system Δ^+ as

$$\Delta^+ = \{ 2e_1, 2e_2, e_1 + e_2, e_1 - e_2 \}.$$

The root spaces $\mathfrak{g}_\alpha (\alpha \in \Delta^+)$ are one dimensional and a basis E_α of them are given by

$$E_{2e_1} = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline & & & \\ & & & \end{array} \right), \quad E_{e_1+e_2} = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline & & & \\ & & & \end{array} \right),$$

$$E_{2e_2} = \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 1 \\ \hline & & & \\ & & & \end{array} \right), \quad E_{e_1-e_2} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array} \right).$$

We choose the basis $E_{-\alpha}$ of $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta^+$ by $E_{-\alpha} = {}^t E_\alpha$.

We have a nilpotent subalgebra $\mathfrak{n} = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$.

1.3   A Minimal Parabolic Subgroup of $Sp(2, \boldsymbol{R})$

Let A, N be the closed subgroups of G corresponding to $\mathfrak{a}, \mathfrak{n}$. And we define a closed subgroup M of G by $M = Z_K(\mathfrak{a})$ (the centralizer of $\mathfrak{a}$ in K). Then

$$M = \left\{ \left(\begin{array}{cccc} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \epsilon_1 & \\ & & & \epsilon_2 \end{array} \right) \middle| \epsilon_1, \epsilon_2 \in \{ \pm 1 \} \right\}$$

and $P = MAN$ is a minimal parabolic subgroup of G. We define two generators of M by

$$d_1=\begin{pmatrix} -1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & 1 \end{pmatrix}, \quad d_2=\begin{pmatrix} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{pmatrix}.$$

1.4 The Jacobi Parabolic Subgroup

We will define the maximal parabolic subgroup called the Jacobi parabolic subgroup. Let

$$M_J=\left\{\begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ \hline 0 & 0 & \epsilon & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, \boldsymbol{R}),\ \epsilon\in\{\pm1\}\right\},$$

$$A_J=\left\{\begin{pmatrix} t & & \\ & 1 & \\ \hline & & t^{-1} \\ & & & 1 \end{pmatrix} \middle| t\in\boldsymbol{R}_{>0}\right\}$$

and

$$N_J=\left\{\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix}\right\}.$$

Then,

$$P_J=M_JA_JN_J=\left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix}\right\}$$

is called the Jacobi parabolic subgroup of $Sp(2, \boldsymbol{R})$.

1.5 A Compact Cartan Subalgebra

We fix a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by $\mathfrak{h}=\boldsymbol{R}T_1+\boldsymbol{R}T_2$ with

$$T_1=\begin{pmatrix} & & 1 & \\ & & & 0 \\ \hline -1 & & & \\ & 0 & & \end{pmatrix}, \quad T_2=\begin{pmatrix} & & 0 & \\ & & & 1 \\ \hline 0 & & & \\ & -1 & & \end{pmatrix}.$$

Let $\beta_1', \beta_2'\in\mathfrak{h}^*$ be the dual basis of T_1, T_2. Then the root spaces are defined by

$$\mathfrak{g}_{\mathbb{C}}^{\beta}=\{X\in\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}|[H, X]=\beta(H)X,\ {}^{\vee}H\in\mathfrak{h}\},$$

for $\beta \in \mathfrak{h}_C^*$ and the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$ is

$$\Sigma = \Sigma(\mathfrak{g}_C, \mathfrak{h}_C) = \{\beta \in \mathfrak{h}_C^* \setminus \{0\} | \mathfrak{g}_C^\beta \neq 0\}$$
$$= \{\pm 2\beta_1, \pm 2\beta_2, \pm(\beta_1 \pm \beta_2)\},$$

where $\beta_i = \sqrt{-1}\beta_i'$. We take a basis X_β of the root space $\mathfrak{g}_C^\beta (\beta \in \Sigma)$ as follows.

$$X_{2\beta_1} = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{\beta_1+\beta_2} = \begin{pmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{pmatrix},$$

$$X_{\beta_1-\beta_2} = \begin{pmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{pmatrix}, \quad X_{2\beta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{pmatrix},$$

and $X_{-\beta} = \overline{X}_\beta$. Then we have

$$\mathfrak{k}_C = \mathfrak{h}_C \oplus C X_{\beta_1-\beta_2} \oplus C X_{-\beta_1+\beta_2},$$

and set

$$\mathfrak{p}_\pm = C X_{\pm 2\beta_1} \oplus C X_{\pm(\beta_1+\beta_2)} \oplus C X_{\pm 2\beta_2}.$$

Then $\mathfrak{p}_C = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, and we call $\Sigma_n = \{\pm 2\beta_1, \pm(\beta_1+\beta_2), \pm 2\beta_2\}$ the set of non-compact roots and its subset $\Sigma_n^+ = \{2\beta_1, \beta_1+\beta_2, 2\beta_2\}$ the set of positive non-compact roots.

We define a norm on $\Sigma(\mathfrak{g}, \mathfrak{h})$ with $|\beta| = \sqrt{|c_1|^2 + |c_2|^2}$ for $\beta = c_1\beta_1 + c_2\beta_2 \in \Sigma$.

We have decompositions of X_β corresponding to the Iwasawa decomposition.

Lemma 1.1.

(1.1)
$$\begin{cases} X_{2\beta_1} = -\sqrt{-1}\,T_1 + H_1 + 2\sqrt{-1}\,E_{2e_1} \\ X_{\beta_1+\beta_2} = 2\overline{X} + 2E_{e_1-e_2} + 2\sqrt{-1}\,E_{e_1+e_2} \\ X_{2\beta_2} = -\sqrt{-1}\,T_2 + H_2 + 2\sqrt{-1}\,E_{2e_2} \\ X_{-2\beta_1} = \sqrt{-1}\,T_1 + H_1 - 2\sqrt{-1}\,E_{2e_1} \\ X_{-\beta_1-\beta_2} = -2X + 2E_{e_1-e_2} - 2\sqrt{-1}\,E_{e_1+e_2} \\ X_{-2\beta_2} = \sqrt{-1}\,T_2 + H_2 - 2\sqrt{-1}\,E_{2e_2} \end{cases}$$

Here,

$$\begin{cases} X = \dfrac{1}{2}(E_{e_1-e_2} - E_{-e_1+e_2}) - \dfrac{\sqrt{-1}}{2}(E_{e_1+e_2} - E_{-e_1-e_2}) \in \mathfrak{k}_C, \\ \overline{X} = -\dfrac{1}{2}(E_{e_1-e_2} - E_{-e_1+e_2}) - \dfrac{\sqrt{-1}}{2}(E_{e_1+e_2} - E_{-e_1-e_2}) \in \mathfrak{k}_C. \end{cases}$$

Proof. These are the consequences of direct calculations.

§2. Representations of K

2.1 Irreducible Representations of K

We will recall some basic facts about the representations of K and its complexification K_C in this section. We have seen K is isomorphic to $U(2)$ in Section 1, hence we consider the irreducible representations of $\mathfrak{u}(2)$, which correspond to the irreducible finite dimensional holomorphic representations of $\mathfrak{u}(2)_C = \mathfrak{gl}(2, \mathbf{C})$.

We take a basis of $\mathfrak{gl}(2, \mathbf{C})$ as

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \overline{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that above X and $\overline{X}$ are the images of X and $\overline{X}$ defined in Lemma 1.1 under the map u. The irreducible finite dimensional holomorphic representations of $\mathfrak{gl}(2, \mathbf{C})$ which determine the representations of $GL(2, \mathbf{C})$ are parametrized by the set of dominant weights

$$\{\lambda = (l_1, l_2) \in \mathbf{Z} \oplus \mathbf{Z} \mid l_1 \geq l_2\}.$$

For each dominant weight $\lambda = (l_1, l_2)$, we set $d = l_1 - l_2 \geq 0$. Then the dimension of the representation space V_λ associated to λ is $d+1$. We can choose a basis $\{v_k \mid 0 \leq k \leq d\}$ in V_λ so that the representation τ_λ is given by

$$(2.1) \qquad \begin{cases} \tau_\lambda(Z)v_k = (l_1 + l_2)v_k, \\ \tau_\lambda(H')v_k = (2k - d)v_k, \\ \tau_\lambda(X)v_k = (k+1)v_{k+1}, \\ \tau_\lambda(\overline{X})v_k = (d+1-k)v_{k-1}. \end{cases}$$

When we put $H'_1 = (Z + H')/2$ and $H'_2 = (Z - H')/2$, we have

$$\tau_\lambda(H'_1)v_k = (k + l_2)v_k \text{ and } \tau_\lambda(H'_2)v_k = (-k + l_1)v_k.$$

Note that $H'_i = -\sqrt{-1}\, T_i$.

If it is necessary to refer explicitly to the dominant weight λ, we denote v_k by v_k^λ.

Lemma 2.1. *When $\mathfrak{p}_+$ is considered as a K_C-module by the adjoint*

action, we have an isomorphism $\mathfrak{p}_+ \simeq V_{(2,0)}$ *given by*

$$(X_{2\beta_2},\ X_{\beta_1+\beta_2},\ X_{2\beta_1}) \longmapsto (v_0,\ v_1,\ v_2).$$

Similarly for $\mathfrak{p}_-$, *we have* $\mathfrak{p}_- \simeq V_{(0,-2)}$ *by*

$$(X_{-2\beta_1},\ X_{-\beta_1-\beta_2},\ X_{-2\beta_2}) \longmapsto (v_0,\ -v_1,\ v_2).$$

Proof. It is proved by direct calculations. $\qquad\square$

We shall give the following realization of one and two dimensional irreducible representations of K as the subspaces of the function space $C^\infty(K)$ with right regular action.

Lemma 2.2 (1-dimensional representation). *Define* $\sigma_l \in \widehat{M}$ *by* $\sigma_l(d_1) = \sigma_l(d_2) = (\pm 1)^l$ *and let* $f_l \in C^\infty(K)$ *be* $f_l(k) = \det u(k)^l$ *for* $k \in K$. *Here* u *is defined in section 1.1. Then,*

$$f_l \in C^\infty_{\sigma_l}(M\backslash K)$$

and with the right regular action, $\mathbb{C}f_l \subset C^\infty_{\sigma_l}(M\backslash K)$ *is a realization of* $(l,\ l) \in \widehat{K}$. *Here we set*

$$C^\infty_{\sigma_l}(M\backslash K) = \{f \in C^\infty(K) | f(mk) = \sigma_l(m)f(k),\ for\ {}^\forall m \in M,\ {}^\forall k \in K\}.$$

Proof. For $m \in M$,

$$f_l(mk) = \det u(m)^l \cdot \det u(k)^l = \begin{cases} (-1)^l f_l(k) & m = d_1,\ d_2, \\ f_l(k) & m = \pm I_2, \end{cases}$$

holds. Therefore, $f_l \in C^\infty_{\sigma_l}(K)$. Since $f_l(xk) = f_l(x) \cdot \det u(k)^l$, the left action of $\mathfrak{k}_C$ is $Xf = \overline{X}f = 0$ ($\because \det u(\exp tX) = \det u(\exp t\overline{X}) = 1$ for ${}^\forall t \in \mathbf{R}$) and $H'_1 f_l = H'_2 f_l = lf_l$ ($\because \det u(\exp tH'_1) = \det u(\exp tH'_2) = e^t$ for ${}^\forall t \in \mathbf{R}$). This completes the proof. $\qquad\square$

Lemma 2.3 (2-dimensional representation). *If* l *is even, let* $(f_{l,0}(k),\ f_{l,1}(k))$ *be the first row of the* 2×2 *matrix* $\det(u(k))^{l-1}u(k)$ *and if* l *is odd, let* $(f_{l,0}(k),\ f_{l,1}(k))$ *be the second row of the* 2×2 *matrix* $\det(u(k))^{l-1}u(k)$. *For* $\sigma^l \in \widehat{M}$ *with* $\sigma^l(d_1) = -\sigma^l(d_2) = (-1)^{l+1}$, *let*

$$C^\infty_{\sigma^l}(M\backslash K) = \{f \in C^\infty(K) | f(mk) = \sigma^l(m)f(k),\ {}^\forall m \in M,\ {}^\forall k \in K\}.$$

Then, for each $l \in \mathbf{Z}$, $f_{l,0}$ *and* $f_{l,1}$ *above belong to* $C^\infty_{\sigma^l}(M\backslash K)$, *and* $\mathbb{C}f_{l,0} \oplus \mathbb{C}f_{l,1}$ *realizes* $(l,\ l-1) \in \widehat{K}$ *with the right regular action. In particular,*

$$Xf_{l,0} = 0,\quad Xf_{l,1} = f_{l,0},$$

$$\overline{X}f_{l,0}=f_{l,1}, \quad \overline{X}f_{l,1}=0,$$

and

$$H_1'f_{l,0}=lf_{l,0}, \quad H_2'f_{l,0}=(l-1)f_{l,0},$$
$$H_1'f_{l,1}=(l-1)f_{l,1}, \quad H_2'f_{l,1}=lf_{l,1},$$

hold.

Proof. The former part follows immediately from the fact

$$\det(u(xk))^{l-1}u(xk)=\det(u(x))^{l-1}\det(u(k))^{l-1}u(x)u(k),$$

for each x, $k \in K$. For the latter part, use the fact

$$(f_{l,0}(xk), f_{l,1}(xk))=(f_{l,0}(x), f_{l,1}(x))\det(u(k))^{l-1}u(k),$$

for any x, $k \in K_C$. The lemma is an immediate consequence of this. $\square$

Lemma 2.4. *Let $w_S \in K$ be the element such that $u(w_S)=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

(i) *If l is even, then $f_{l,0}$ and $f_{l,1}$ are evaluated at the 4×4 identity matrix I_4 and w_S as*

$$\begin{pmatrix} f_{l,0}(I_4) \\ f_{l,1}(I_4) \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} f_{l,0}(w_S) \\ f_{l,1}(w_S) \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(ii) *If l is odd, then*

$$\begin{pmatrix} f_{l,0}(I_4) \\ f_{l,1}(I_4) \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} f_{l,0}(w_S) \\ f_{l,1}(w_S) \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Proof. Let $r_\theta \in K$ be the element such that $\mu(r_\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Then from the definition of $f_{l,i}$'s,

$$\begin{pmatrix} f_{l,0}(r_\theta) \\ f_{l,1}(r_\theta) \end{pmatrix}=\begin{cases} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} & \text{if } l \text{ is even,} \\ \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} & \text{if } l \text{ is odd.} \end{cases}$$

Taking $\theta=0$ or $\theta=\dfrac{\pi}{2}$, we have the above lemma. $\square$

2.2 Tensor products of the representations of K

Lemma 2.5. (i) *The tensor product $V_\lambda \otimes \mathfrak{p}_+$ has the decomposition into irreducible factors as*

$$V_\lambda \otimes \mathfrak{p}_+ \simeq \begin{cases} V_{(l_1+2, l_2)} \oplus V_{(l_1+1, l_2+1)} \oplus V_{(l_1, l_2+2)} & \text{if } l_1 > l_2, \\ V_{(l_1+2, l_2)} & \text{if } l_1 = l_2. \end{cases}$$

Here we set $V_{(l_1, l_2)} = \{0\}$ for $l_1 < l_2$.

(ii) *The tensor product $V_\lambda \otimes \mathfrak{p}_-$ has the decomposition into irreducible factors as*

$$V_\lambda \otimes \mathfrak{p}_- \simeq \begin{cases} V_{(l_1, l_2-2)} \oplus V_{(l_1-1, l_2-1)} \oplus V_{(l_1-2, l_2)} & \text{if } l_1 > l_2, \\ V_{(l_1, l_2-2)} & \text{if } l_1 = l_2. \end{cases}$$

Here we set $V_{(l_1, l_2)} = \{0\}$ for $l_1 < l_2$.

Let P_+^{up}, P_-^{even}, and P_+^{down} be projectors from $V_\lambda \otimes \mathfrak{p}_+$ to the factors $V_{(l_1+2, l_2)}$, $V_{(l_1+1, l_2+1)}$, and $V_{(l_1, l_2+2)}$, respectively. And let P_-^{up}, P_-^{even}, and P_-^{down} be projectors from $V_\lambda \otimes \mathfrak{p}_-$ to the factors $V_{(l_1, l_2-2)}$, $V_{(l_1-1, l_2-1)}$, and $V_{(l_1-2, l_2)}$, respectively. We will write explicit formulas of these projectors below.

We denote $v_k^{(2,0)}$ and $v_k^{(0,-2)}$ ($k = 0, 1, 2$) by w_k^+ and w_k^- ($k = 0, 1, 2$) respectively.

Lemma 2.6. *Set $\mu_+ = (l_1+2, l_2)$, $\mu_- = (l_1, l_2-2)$. Then up to scalar, the projector $P_\pm^{up}$ is given by*

(i) $P_\pm^{up}(v_k^\lambda \otimes w_2^\pm) = \dfrac{(k+1)(k+2)}{2} v_{k+2}^{\mu_\pm}$,

(ii) $P_\pm^{up}(v_k^\lambda \otimes w_1^\pm) = (k+1)(d+1-k) v_{k+1}^{\mu_\pm}$,

(iii) $P_\pm^{up}(v_k^\lambda \otimes w_0^\pm) = \dfrac{(d+1-k)(d+2-k)}{2} v_k^{\mu_\pm}$,

for $0 \le k \le d$.

Lemma 2.7. *Set $\nu_\pm = (l_1 \pm 1, l_2 \pm 1)$. Then up to scalar, the projector $P_\pm^{even}$ is given by*

(i) $P_\pm^{even}(v_k^\lambda \otimes w_2^\pm) = (k+1) v_{k+1}^{\nu_\pm}$,

(ii) $P_\pm^{even}(v_k^\lambda \otimes w_1^\pm) = (d-2k) v_k^{\nu_\pm}$,

(iii) $P_\pm^{even}(v_k^\lambda \otimes w_0^\pm) = -(d+1-k) v_{k-1}^{\nu_\pm}$,

for $0 \le k \le d$. We set $v_k^{\nu_\pm} = 0$ for $k \le -1$ or $k \ge d+1$.

Lemma 2.8. *Set $\pi_+ = (l_1, l_2+2)$, $\pi_- = (l_1-2, l_2)$. Then up to scalar, the*

projector $P_{\pm}^{down}$ *is given by*

$$\text{(i)} \quad P_{\pm}^{down}(v_k^\lambda \otimes w_2^\pm) = v_k^{\pi_\pm},$$

$$\text{(ii)} \quad P_{\pm}^{down}(v_k^\lambda \otimes w_1^\pm) = -2v_{k-1}^{\pi_\pm},$$

$$\text{(iii)} \quad P_{\pm}^{down}(v_k^\lambda \otimes w_0^\pm) = v_{k-2}^{\pi_\pm},$$

for $0 \le k \le d$. *We set* $v_k^{\pi_\pm} = 0$ *for* $k \le -1$ *or* $k \ge d-1$.

The proofs of the above lemmas are easy. It is enough to find the highest weight vectors in $V_\lambda \otimes \mathfrak{p}^\pm$ corresponding to the factors $V_{\mu_\pm}$, $V_{\nu_\pm}$, and $V_{\pi_\pm}$ respectively. The other steps of the proofs are settled by induction.

§3. Principal Series Representations of $Sp(2, \boldsymbol{R})$

3.1 The Definition of the Principal Series Representations

We will start with the definition of the principal series representation of $Sp(2, \boldsymbol{R})$.

Definition 3.1. *Let* σ *be an irreducible unitary representation of* M *and* $\mu \in \mathfrak{a}_C^*$, $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha = 2e_1 + e_2$. *We define the action of* G *on the space*

$$C^\infty\text{-Ind}_P^G(\sigma \otimes a^{\mu+\rho} \otimes 1_N) = \{f \in C^\infty(G) | f(mang) = \sigma(m)a^{\mu+\rho}f(g),$$

$$^\forall g \in G, \ ^\forall m \in M, \ ^\forall a \in A, \ ^\forall n \in N\}$$

by $\pi_0(g)f(x) = f(xg)$ *for any* $f \in C^\infty\text{-Ind}_P^G(\sigma \otimes a^{\mu+\rho} \otimes 1_N)$ *and any* $g \in G$.

We call this representation of G *a principal series representation of* G. *Hereafter we denote this representation by* (π_0, H_{π_0}) *in short.*

There is the K-type theorem on the principal series. If we put $H_{\pi_0, K}$ for K-finite vectors in $C^\infty\text{-Ind}_P^G(\sigma \otimes a^{\mu+\rho} \otimes 1_N)$, then we have the following.

Proposition 3.2. *For* $\sigma \in \widehat{M}$, *write* $\varepsilon_1 = \sigma(d_1)$ *and* $\varepsilon_2 = \sigma(d_2)$. *The multiplicity of* $\tau_{(l_1, l_2)}$ *in* $H_{\pi_0, K}$ *is given by*

$$\#\{m \in \boldsymbol{Z} | l_2 \le m \le l_1, (-1)^m = \varepsilon_1, (-1)^{l_1+l_2-m} = \varepsilon_2\}.$$

In particular,
(i) *If* $\varepsilon_1 = \varepsilon_2$, *then the representation* $\tau_{(l,l)}$ *occurs in* $H_{\pi_0, K}$ *with multiplicity one for any* $l \in \boldsymbol{Z}$ *such that* $(-1)^l = \varepsilon_1 = \varepsilon_2$.

(ii) *If* $(\varepsilon_1, \varepsilon_2) = (1, -1)$ *or* $(-1, 1)$, *then for any integer* l *the* K-*type* $\tau_{(l,l-1)}$

occurs in $H_{\pi_0, K}$ *with multiplicity one.*

Proof. Because $H_{\pi_0, K}|_K \simeq \mathrm{Ind}_M^K(\sigma)$ and the Frobenius reciprocity,

$$[H_{\pi_0, K}|_K : \tau_{(l_1, l_2)}] = [\mathrm{Ind}_M^K(\sigma) : \tau_{(l_1, l_2)}] = [\tau_{(l_1, l_2)}|_M : \sigma].$$

Since $d_i = \exp \pi \sqrt{-1} H_i'$ for $i = 1, 2$, then $\tau_{(l_1, l_2)}(d_1)v_k = \exp \pi \sqrt{-1}(l_2 + k)v_k$ and $\tau_{(l_1, l_2)}(d_2)v_k = \exp \pi \sqrt{-1}(l_1 - k)v_k$. Here, v_k is a weight vector in $V_{(l_1, l_2)}$ defined in section 2. That is, for $m = l_2 + k$, $\tau_{(l_1, l_2)}(d_1)|_{Cv_k} = (-1)^m$ and $\tau_{(l_1, l_2)}(d_2)|_{Cv_k} = (-1)^{l_1 + l_2 - m}$. Therefore $\tau_{(l_1, l_2)}(M)|_{Cv_k} \simeq \sigma$ holds if and only if

$$k \in \{m \in \mathbf{Z} \mid l_2 \leq m \leq l_1, \ (-1)^m = \varepsilon_1, \ (-1)^{l_1 + l_2 - m} = \varepsilon_2\}.$$

The latter parts are easy consequence of the former. $\square$

3.2   The Generalized Principal Series Representation of $Sp(2, \mathbf{R})$

We define the generalized principal series representation of $Sp(2, \mathbf{R})$.

Definition 3.3.   *Let* (σ_J, V_{σ_J}) *be a discrete series representation of* $M_J \simeq \{\pm 1\} \times SL(2, \mathbf{R})$ *given by a pair* (ε, ξ), *where* $\varepsilon \in \widehat{\{\pm 1\}}$ *and* ξ *is a discrete series representation of* $SL(2, \mathbf{R})$. *And* $\nu_1 \in \mathfrak{a}_{J,c}^*$, $\rho_J = \frac{1}{2}\{(e_1 - e_2) + 2e_1 + (e_1 + e_2)\} = 2e_1$. *We define the action of* G *on the space*

$$C^\infty\text{-}\mathrm{Ind}_{P_J}^G(\sigma_J \otimes a_J^{\nu_1 + \rho_J} \otimes 1_{N_J})$$

$$= \{f : G \to V_{\sigma_J} \mid f \text{ is of } C^\infty\text{-class}, \ f(m_J a_J n_J g) = \sigma_J(m_J)a_J^{\nu_1 + \rho_J}f(g),$$
$$\text{for } {}^\forall g \in G, \ {}^\forall m_J \in M_J, \ {}^\forall a_J \in A_J, \ {}^\forall n_J \in N_J\},$$

by $\pi_1(g)f(x) = f(xg)$ *for any* $f \in C^\infty\text{-}\mathrm{ind}_{P_J}^G(\sigma_J \otimes a_J^{\nu_1 + \rho_J} \otimes 1_{N_J})$ *and any* $g \in G$. *We call this representation of* G *a generalized principal series representation of* G. *Hereafter we denote this representation by* (π_1, H_{π_1}) *in short.*

The discrete series of $SL(2, \mathbf{R})$ are parametrized by the Blattner parameter $l \in \{n \in \mathbf{Z} \mid n \geq 2 \text{ or } n \leq -2\}$. We denote by D_l the discrete series representation with the Blattner parameter l if $l \geq 2$ and D_l $(l \leq -2)$ is the contragredient representation of D_{-l}.

For K-finite vectors $H_{\pi_1, K}$, we have the following K-type theorem.

Proposition 3.4.   *For* $\sigma_J = (\varepsilon, D_l) \in \hat{M}_J$ $(l \geq 2)$ *and* $\nu_1 \in \mathfrak{a}_{1,c}^*$, *the multiplicity of* $\tau_{(l_1, l_2)}$ *in* $H_{\pi_1, K}$ *is given by*

$$\#\{m \in \mathbf{Z} \mid m \equiv l \bmod 2, \ (-1)^{l_1 + l_2 - m} = \varepsilon(d_1), \ \max(l_2, l) \leq m \leq l_1\}.$$

In particular,
(i)   *If* $\varepsilon(d_1) = (-1)^l$, *the representation* $\tau_{(k, k)}$ $(k \in \mathbf{Z}, \ k \equiv l \bmod 2, \ k \geq l)$ *and*

$\tau_{(l,k)}(k \in \boldsymbol{Z}, \ k \equiv l \bmod 2, \ k \leq l)$ occurs in $H_{\pi_1, K}$ with multiplicity one.

(ii) If $\varepsilon(d_1) = (-1)^{l+1}$, the representation $\tau_{(k,k-1)}$ $(k \in \boldsymbol{Z}, \ k \geq l)$ and $\tau_{(l,k-1)}$ $(k \in \boldsymbol{Z}, \ k \equiv l, \ k \leq l)$ occurs in $H_{\pi_1, K}$ with multiplicity one.

Proof. By the Frobenius reciprocity,

$$
\begin{aligned}
& [H_{\pi_1, K}|_K : \tau_{(l_1, l_2)}] \\
&= [\mathrm{Ind}_{K \cap M_J}^K (\sigma_J|_{K \cap M_J}) : \tau_{(l_1, l_2)}] \\
&= \sum_{\omega \in (K \cap M_J)^\wedge} [\sigma_J|_{K \cap M_J} : \omega] \cdot [\mathrm{Ind}_{K \cap M_J}^K(\omega) : \tau_{(l_1, l_2)}] \\
&= \sum_{\omega \in (K \cap M_J)^\wedge} [\sigma_J|_{K \cap M_J} : \omega] \cdot [\tau_{(l_1, l_2)}|_{K \cap M_J} : \omega].
\end{aligned}
$$

Since $K \cap M_J \simeq \{\pm 1\} \times SO(2)$, $\omega \in (K \cap M_J)^\wedge$ is specified by $\omega(d_1)$ and $\omega|_{SO(2)}$. $\omega(d_1)$ is in $\{\pm 1\}$ and $\widehat{SO(2)} = \left\{ \chi_m \middle| \chi_m \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) = e^{\sqrt{-1}\,m\theta} \right\}$. Then the K-type theorem for D_l gives

$$
\begin{aligned}
[\sigma_J|_{K \cap M_J} : \omega] &= [(\varepsilon, D_l) : (\omega(d_1), \chi_m)] \\
&= \begin{cases} 1 & \text{if } m \equiv l \bmod 2, \ m \geq l, \ w(d_1) = \varepsilon(d_1), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

On the other hand,

$$
\tau_{(l_1, l_2)|K \cap M_J} = \bigoplus_{l_2 \leq m \leq l_1} ((-1)^{l_1 + l_2 - m}, \chi_m)
$$

holds and we get the former part of the proposition. The latter parts are easy consequences of the former. $\square$

§4. The Space of Spherical Functions

4.1 The Definition of Spherical Functions

Let (η, V_η), (τ, V_τ) be in $\widehat{K}$. Then we define the space

$$
\begin{aligned}
C_\eta^\infty(K \backslash G) = \{ f : G \to V_\eta | f \text{ is a } C^\infty\text{-function,} \\
f(kg) = \eta(k) f(g), \ {}^\forall k \in K, \ {}^\forall g \in G \},
\end{aligned}
$$

which is a G-module under the right regular action. We put

$$
\begin{aligned}
C_{\eta, \tau}^\infty(K \backslash G/K) = \{ f : G \to V_\eta \otimes V_{\tau^*} | f \text{ is a } C^\infty\text{-function,} \\
f(k_1 g k_2) = \eta(k_1) \otimes \tau^*(k_2)^{-1} f(g), \ {}^\forall k_1, \ {}^\forall k_2 \in K, \ {}^\forall g \in G \},
\end{aligned}
$$

where τ^* is the contragredient representation of τ.

Let (π, H_π) be an admissible $(\mathfrak{g}, K)$-module such that the multiplicity of τ in π is one. Consider a $(\mathfrak{g}, K)$-module homomorphism

$$\phi_\pi : H_\pi \longrightarrow C_\eta^\infty(K\backslash G),$$

and $i : V_\tau \hookrightarrow H_\pi$ be the unique K-homomorphism up to scalar. Then,

$$\phi_{\pi,\tau,i} = \phi_\pi \circ i \in \mathrm{Hom}_K(V_\tau, C_\eta^\infty(K\backslash G)) \simeq C_\eta^\infty(K\backslash G) \otimes_K V_{\tau*} \simeq C_{\eta,\tau}^\infty(K\backslash G/K).$$

We can write $\phi_{\pi,\tau,i}$ explicitly as follows.

Let H_π^* be the contragredient representation of H_π and we assume there is the unique non-trivial K-homomorphism $j^* : V_{\eta*} \hookrightarrow H_\pi^*$. Choose bases $\{v_n^\tau | 0 \le n \le d_\tau\}$, $\{v_n^\eta | 0 \le n \le d_\eta\}$ of V_τ and V_η and let $\{w_n^{\tau*} | 0 \le n \le d_{\tau*}\}$, $\{w_n^{\eta*} | 0 \le n \le d_{\eta*}\}$ be dual base of $V_{\tau*}$ and $V_{\eta*}$ respectively. Then

$$\phi_{\pi,\tau,i}(g) = \sum_{m,n} \langle \pi(g) i(v_n^\tau), j^*(w_m^{\eta*}) \rangle v_m^\eta \otimes w_m^{\tau*}.$$

Definition 4.1. *We call* $\phi_{\pi,\tau,i}$ *an* (η, τ)-*spherical function of* H_π. *If multiplicities of* η, τ *in* $H_\pi|_K$ *are one, the dimension of* (η, τ)-*spherical functions of* H_π *is one.*

The function $\phi_{\pi,\tau,i}$ is determined by its restriction to A by the Cartan decomposition. We denote its restriction on A by the same symbol $\phi_{\pi,\tau,i}$. Since $A \simeq \mathfrak{a} = RH_1 \oplus RH_2 \simeq \mathbf{R}^2$, we consider $\phi_{\pi,\tau,i}$ as a function on $\mathbf{R}^2$ by setting $\phi_{\pi,\tau,i}(x_1, x_2) = \phi_{\pi,\tau,i}(\exp(x_1 H_1 + x_2 H_2))$.

For a differential operator D on G, we define the differential operator $R(D)$ on A by $(D\phi)|_A = R(D)(\phi|_A)$ for any $\phi \in C_{\eta,\tau}^\infty(K\backslash G/K)$. We call $R(D)$ the radial part of D.

Hereafter we consider the radial parts of differential operators which act on $C_{\eta,\tau}^\infty(K\backslash G/K)$ as scalars in cases of $\pi = \pi_0$ or π_1.

4.2 The Symmetry Condition of Spherical Functions

We have some symmetry conditions of spherical functions from the action of the Weyl group.

Lemma 4.2 (1-dimensional case). *We assume* $\eta = (k,k)$, $\tau = (l,l) \in \hat{K}$ (*This means* $\dim V_\eta = \dim V_\tau = 1$ *and* $\tau^* = (-l, -l)$). *For* $\phi \in C_{\eta,\tau}^\infty(K\backslash G/K)$, *we have the following.*

  (i) *If* $k - l \equiv 1 \bmod 2$, *then* $\phi(x_1, x_2) = 0$.
  (ii) $\phi(x_2, x_1) = \phi(x_1, x_2)$.
  (iii) $\phi(x_1, -x_2) = \begin{cases} \phi(x_1, x_2) & k \equiv l \bmod 4, \\ -\phi(x_1, x_2) & otherwise. \end{cases}$

Proof. For $d_2 \in M$, $\mathrm{Ad}(d_2) \mathrm{diag}(x_1, x_2, -x_1, -x_2) = \mathrm{diag}(x_1, x_2, -x_1, -x_2)$ holds, and $d_2 = \exp(\pi \sqrt{-1} H_2')$ and $\tau_{(j,j)}(H_2') = j$. So we have

$$\phi(x_1, x_2) = \phi(\mathrm{diag}(x_1, x_2, -x_1, -x_2))$$

$$\begin{aligned}
&= \phi(\mathrm{Ad}(d_2)\mathrm{diag}(x_1, x_2, -x_1, -x_2)) \\
&= \eta(d_2) \otimes \tau^*(d_2) \phi(\mathrm{diag}(x_1, x_2, -x_1, -x_2)) \\
&= e^{\pi\sqrt{-1}(k-l)} \phi(x_1, x_2),
\end{aligned}$$

This proves (i).

The generators of the Weyl group are

$$w_L = \exp\left(\frac{\pi}{2}\sqrt{-1}H_2'\right) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right),$$

$$w_S = \exp\left(\frac{\pi}{2}(X - \bar{X})\right) = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & 1 \\ & & -1 & 0 \end{array}\right),$$

whose actions on $\mathfrak{a}$ are $\mathrm{Ad}(w_L)\,\mathrm{diag}(x_1, x_2, -x_1, -x_2) = \mathrm{diag}(x_1, -x_2, -x_1, x_2)$, $\mathrm{Ad}(w_S)\mathrm{diag}(x_1, x_2, -x_1, -x_2) = \mathrm{diag}(x_2, x_1, -x_2, -x_1)$. Therefore we have

$$\begin{aligned}
\phi(x_1, -x_2) &= \phi(\mathrm{Ad}(w_L)\mathrm{diag}(x_1, x_2, -x_1, -x_2)) \\
&= (\eta(w_L) \otimes \tau^*(w_L))\phi(x_1, x_2) \\
&= e^{\frac{\pi}{2}\sqrt{-1}(k-l)}\phi(x_1, x_2), \\
\phi(x_2, x_1) &= \phi(\mathrm{Ad}(w_S)\mathrm{diag}(x_1, x_2, -x_1, -x_2)) \\
&= (\eta(w_S) \otimes \tau^*(w_S))\phi(x_1, x_2) \\
&= \phi(x_1, x_2),
\end{aligned}$$

because $\tau_{(j,j)}(H'_2) = j$, $\tau_{(j,j)}(X) = \tau_{(j,j)}(\bar{X}) = 0$.

Thus (ii) and (iii) are proved. $\qquad\square$

Lemma 4.3 (2-dimensional case). *We assume* $\eta = (k, k-1)$, $\tau = (l, l-1)$ $\in \hat{K}$ *(This means* $\dim V_\eta = \dim V_\tau = 2$ *and* $\tau^* = (-l+1, -l)$*). For* $\phi = \phi_{00}v_0^\eta \otimes v_0^{\tau*} + \phi_{01}v_0^\eta \otimes v_1^{\tau*} + \phi_{10}v_1^\eta \otimes v_0^{\tau*} + \phi_{11}v_1^\eta \otimes v_1^{\tau*} \in C_{\eta,\tau}^\infty(K\backslash G/K)$, *we have the following.*

   (i) *If* $k-l \equiv 0 \bmod 2$, *then* $\phi_{00}(x_1, x_2) = \phi_{11}(x_1, x_2) = 0$.
        If $k-l \equiv 1 \bmod 2$, *then* $\phi_{01}(x_1, x_2) = \phi_{10}(x_1, x_2) = 0$.

  (ii) $\phi_{11}(x_2, x_1) = \phi_{00}(x_1, x_2)$.
       $\phi_{10}(x_2, x_1) = -\phi_{01}(x_1, x_2)$.

(iii) *If* $k-l \equiv 0 \bmod 4$, *then* $\begin{cases} \phi_{01}(x_1, -x_2) = \phi_{01}(x_1, x_2), \\ \phi_{10}(x_1, -x_2) = \phi_{10}(x_1, x_2). \end{cases}$

         If $k-l \equiv 2 \bmod 4$, *then* $\begin{cases} \phi_{01}(x_1, -x_2) = -\phi_{01}(x_1, x_2), \\ \phi_{10}(x_1, -x_2) = -\phi_{10}(x_1, x_2). \end{cases}$

         If $k-l \equiv 1 \bmod 4$, *then* $\begin{cases} \phi_{00}(x_1, -x_2) = -\phi_{00}(x_1, x_2), \\ \phi_{11}(x_1, -x_2) = \phi_{11}(x_1, x_2). \end{cases}$

If $k-l \equiv 3 \bmod 4$, then $\begin{cases} \phi_{00}(x_1, -x_2) = \phi_{00}(x_1, x_2), \\ \phi_{11}(x_1, -x_2) = -\phi_{11}(x_1, x_2). \end{cases}$

Proof. This can be proved in the same way as Lemma 4.2.   □

§5. The Casimir Operator of $Sp(2, \mathbf{R})$

The Casimir operator $L \in U(\mathfrak{g})$ is given by $L = H_1^2 + H_2^2 - 4H_1 - 2H_2$ $+ 2E_{e_1-e_2}E_{-e_1+e_2} + 4E_{2e_1}E_{-2e_1} + 2E_{e_1+e_2}E_{-e_1-e_2} + 4E_{2e_2}E_{-2e_2}$. Here we define $H_{2e_i} = H_i$ $(i=1, 2)$, $H_{e_1 \pm e_2} = H_1 \pm H_2$. Then, $[E_\alpha, E_{-\alpha}] = H_\alpha$ $(^\forall \alpha \in \Delta^+)$.
For $X, Y \in U(\mathfrak{k})$, $H \in U(\mathfrak{a})$, $a \in A$ and $\phi \in C_{\eta,\tau}^\infty(K \backslash G / K)$,

$$(\mathrm{Ad}(a^{-1})X \cdot H \cdot Y\phi)(a)$$

$$= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \phi(a \cdot \exp s \, \mathrm{Ad}(a^{-1})X \exp tH \exp uY)$$

$$= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \phi(\exp sX \cdot a \cdot \exp tH \exp uY)$$

$$= \frac{\partial^3}{\partial s \partial t \partial u}\bigg|_{s=t=u=0} \eta(\exp sX) \otimes \tau^*(\exp uY)^{-1}\phi(a \cdot \exp tH)$$

$$= \frac{d}{dt}\bigg|_{t=0} \eta(X) \otimes (-\tau^*(Y))\phi(a \cdot \exp tH).$$

Further, for $H_i \in \mathfrak{a}$ $(i=1, 2)$ and $a = \exp(x_1H_1 + x_2H_2) \in A$, $(H_i\phi)(x_1, x_2) = \frac{d}{dt}\bigg|_{t=0} \phi(\exp(x_1H_1 + x_2H_2 + tH_i)) = \frac{\partial}{\partial x_i}\phi(x_1, x_2)$. Therefore, to get the radial part of L, it is necessary to write L in the form $\sum_{X, Y \in U(\mathfrak{k}), H \in U(\mathfrak{a})}$ $\mathrm{Ad}(a^{-1})X \cdot H \cdot Y$.

For $a \in A$ and $\alpha \in \Delta$, we put $a^\alpha = \exp \alpha(\log a)$ and define

$$A' = \{a \in A \mid a^{2\alpha} \neq 1 \text{ for all } \alpha \in \Delta\}.$$

Lemma 5.1. *For any* $a \in A'$ *and any* $\alpha \in \Delta^+$,

$$E_\alpha E_{-\alpha} = \frac{1}{(a^\alpha - a^{-\alpha})^2}(\mathrm{Ad}(a^{-1})X_\alpha)^2 - \frac{a^\alpha + a^{-\alpha}}{(a^\alpha - a^{-\alpha})^2}(\mathrm{Ad}(a^{-1})X_\alpha)X_\alpha$$

$$+ \frac{a^\alpha}{a^\alpha - a^{-\alpha}}H_\alpha + \frac{1}{(a^\alpha - a^{-\alpha})^2}X_\alpha^2.$$

Here, $X_\alpha = E_\alpha - E_{-\alpha} \in \mathfrak{k}$.

Proof. See [W Proposition 9.1.2.11].   □

From this lemma and the above consideration, the radial part of $E_\alpha E_{-\alpha}$ is

given by

$$R(E_\alpha E_{-\alpha}) = \frac{a^\alpha}{a^\alpha - a^{-\alpha}} \frac{\partial}{\partial \alpha} + \frac{1}{(a^\alpha - a^{-\alpha})^2} \eta(X_\alpha)^2$$

$$+ \frac{a^\alpha + a^{-\alpha}}{(a^\alpha - a^{-\alpha})^2} \eta(X_\alpha) \otimes \tau^*(X_\alpha) + \frac{1}{(a^\alpha - a^{-\alpha})^2} \tau^*(X_\alpha)^2.$$

For $H_\alpha = \epsilon_1 H_1 \pm \epsilon_2 H_2$, we set $\dfrac{\partial}{\partial \alpha} = \epsilon_1 \dfrac{\partial}{\partial x_1} \pm \epsilon_2 \dfrac{\partial}{\partial x_2}$.

Using the definition of η, $\tau \in \hat{K}$ as in section 2.1, we can easily have the radial part of L in 1-dimensional case and 2-dimensional case.

Proposition 5.2 (1-dimensional case). *For* $\eta = (k, k)$, $\tau = (l,l) \in \hat{K}$ *with* $k \equiv l \bmod 2$ *and* $\phi \in C^\infty_{\eta,\tau}(K \backslash G / K)$, *the radial part* $R(L)$ *of* L *is given by*

$$R(L)\phi(x_1, x_2) = \{L_0 - (k^2 + l^2)(\mathrm{sh}^{-2} 2x_1 + \mathrm{sh}^{-2} 2x_2)$$
$$+ 2kl(\mathrm{ch}\, 2x_1 \cdot \mathrm{sh}^{-2} 2x_1 + \mathrm{ch}\, 2x_2 \cdot \mathrm{sh}^{-2} 2x_2)\} \phi(x_1, x_2),$$

where

$$L_0 = \partial^2_{x_1} + \partial^2_{x_2} + \{2 \coth 2x_1 + \coth(x_1 + x_2) + \coth(x_1 - x_2)\} \partial_{x_1}$$
$$+ \{2 \coth 2x_2 + \coth(x_1 + x_2) - \coth(x_1 - x_2)\} \partial_{x_2}$$

and $\partial_{x_i} = \dfrac{\partial}{\partial x_i}$ $(i = 1, 2)$.

Proposition 5.3 (2-dimensional case). *For* $\eta = (k, k-1)$, $\tau = (l, l-1) \in \hat{K}$ *with* $k \equiv l \bmod 2$ *and* $\phi = \phi_{01} v^\eta_0 \otimes v^{\tau*}_1 + \phi_{10} v^\eta_1 \otimes v^{\tau*}_0 \in C^\infty_{\eta,\tau}(K \backslash G / K)$, *the radial part* $R(L)$ *of* L *is given by*

$$R(L)\phi(x_1, x_2)$$
$$= [\{L_0 - \mathrm{sh}^{-2}(x_1 + x_2) - \mathrm{sh}^{-2}(x_1 - x_2) - ((k-1)^2 + (l-1)^2)\mathrm{sh}^{-2} 2x_1$$
$$- (k^2 + l^2)\mathrm{sh}^{-2} 2x_2 + 2(k-1)(l-1)\mathrm{ch}\, 2x_1 \cdot \mathrm{sh}^{-2} 2x_1$$
$$+ 2kl\, \mathrm{ch}\, 2x_2 \cdot \mathrm{sh}^{-2} 2x_2\} \phi_{01}(x_1, x_2)$$
$$- \{\mathrm{ch}(x_1 + x_2) \cdot \mathrm{sh}^{-2}(x_1 + x_2)$$
$$+ \mathrm{ch}(x_1 - x_2) \cdot \mathrm{sh}^{-2}(x_1 - x_2)\} \phi_{10}(x_1, x_2)] v^\eta_0 \otimes v^{\tau*}_1$$

$$+ [\{L_0 - \mathrm{sh}^{-2}(x_1 + x_2) - \mathrm{sh}^{-2}(x_1 - x_2) - (k^2 + l^2)\mathrm{sh}^{-2} 2x_1$$
$$- ((k-1)^2 + (l-1)^2)\mathrm{sh}^{-2} 2x_2 + 2kl\, \mathrm{ch}\, 2x_1 \cdot \mathrm{sh}^{-2} 2x_1$$
$$+ 2(k-1)(l-1)\mathrm{ch}\, 2x_2 \cdot \mathrm{sh}^{-2} 2x_2\} \phi_{10}(x_1, x_2)$$
$$- \{\mathrm{ch}(x_1 + x_2) \cdot \mathrm{sh}^{-2}(x_1 + x_2)$$
$$+ \mathrm{ch}(x_1 - x_2) \cdot \mathrm{sh}^{-2}(x_1 - x_2)\} \phi_{01}(x_1, x_2)] v^\eta_1 \otimes v^{\tau*}_0,$$

where L_0 *is the same as in* Lemma 4.5.

§6. Shift Operators

6.1 Schmid Operators

First, we define the function space $C_\tau^\infty(G/K)$ on which Schmid operators act.

$$C_\tau^\infty(G/K) = \{ f : G \to V_\tau | f \text{ is a } C^\infty\text{-function},$$
$$f(gk) = \tau^*(k^{-1})f(g), \ {}^\forall k \in K, \ {}^\forall g \in G \},$$

Definition 6.1 (the Schmid operator). *Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form. Then the Schmid operator ∇_τ is defined by*

$$\nabla_\tau : C_\tau^\infty(G/K) \ni \phi \to \sum_i R_{X_i}\phi \otimes X_i \in C_{\tau \otimes \mathrm{Ad}_\mathfrak{p}}^\infty(G/K).$$

Here, $\phi \in C_\tau^\infty(G/K)$, $\mathfrak{g}_C^\beta = \mathbb{C}X_\beta$, $R_X\phi(x) = \dfrac{d}{dt}\Big|_{t=0} f(x \exp tX)$, ∇_τ is independent of the choice of an orthonormal basis of $\mathfrak{p}$. This kind of operator is originally defined in [S].

For $X_\beta(\beta \in \Sigma(\mathfrak{g}, \mathfrak{h}))$ defined in Section 1,

$$\left\{ C|\beta|(X_\beta + X_{-\beta}), \frac{C|\beta|}{\sqrt{-1}}(X_\beta - X_{-\beta}) | \beta \in \Sigma_n^+ \right\}$$

is an orthonormal basis of $\mathfrak{p}$ for some $C > 0$. Using this basis, we have a decomposition $\nabla_\tau = 8C(\nabla_\tau^+ + \nabla_\tau^-)$ with

$$\nabla_\tau^\pm : C_\tau^\infty(G/K) \ni \phi \to \frac{1}{4}\sum_{\beta \in \Sigma_n^+} |\beta|^2 R_{X_{\pm\beta}}\phi \otimes X_{\mp\beta} \in C_{\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}^\infty(G/K).$$

6.2 The Shift Operators

We will construct shift operators from Schmid operators after [MO1] in this subsection.

Definition 6.2 (Shift operators for 1-dimensional case). *Set $\tau = (l, l)$, then $\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm} \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}$ has $\tau_\pm = (l \pm 2, l \pm 2)$ as an irreducible component with multiplicity 1 from Lemma 2.5 respectively. We define* $\mathrm{pr}_1^\pm$ *by*

$$\mathrm{pr}_1^\pm : C_{\tau \otimes \mathrm{Ad}_{\mathfrak{p}_\pm} \otimes \mathrm{Ad}_{\mathfrak{p}_\pm}}^\infty(G/K) \to C_{\tau_\pm}^\infty(G/K) : projection,$$

and we call the following differential operators of order 2 shift operators.

$$\begin{cases} D^+ = \mathrm{pr}_1^+ \circ \nabla_{\tau \otimes \mathrm{Ad}_{\mathfrak{p}^+}}^+ \circ \nabla_\tau^+ \ : \ C_{(l,l)}^\infty(G/K) \to C_{(l+2,\,l+2)}^\infty(G/K) \\ D^- = \mathrm{pr}_1^- \circ \nabla_{\tau \otimes \mathrm{Ad}_{\mathfrak{p}^-}}^- \circ \nabla_\tau^- \ : \ C_{(l,l)}^\infty(G/K) \to C_{(l-2,\,l-2)}^\infty(G/K) \end{cases}$$

By Lemma 2.6 and Lemma 2.8, we have $D^\pm \in U(\mathfrak{g})$ and they are of the form

$$(6.1) \qquad\qquad D^+ = X_{2\beta_1} X_{2\beta_2} + X_{2\beta_2} X_{2\beta_1} - \frac{1}{2} X_{\beta_1+\beta_2}^2,$$

$$(6.2) \qquad\qquad D^- = X_{-2\beta_1} X_{-2\beta_2} + X_{-2\beta_2} X_{-2\beta_1} - \frac{1}{2} X_{-\beta_1-\beta_2}^2,$$

We denote $D_l^\pm$ for $D^\pm|_{C_{(l,l)}^\infty(G/K)}$.

Remark 6.3. $D_{l-2}^+ \circ D_l^-$ is a map from $C_{(l,l)}^\infty(G/K)$ to $C_{(l,l)}^\infty(G/K)$. Especially for any $\eta \in \widehat{K}$ and $\tau = (l, l)$, this is a map from $C_{\eta,\tau}^\infty(K \backslash G/K)$ to $C_{\eta,\tau}^\infty(K \backslash G/K)$.

Definition 6.4 (Shift operators for 2-dimensional case). Set $\tau = (l, l-1)$, then $\tau \otimes \mathrm{Ad}_{\mathfrak{p}^\pm}$ has $\tau_+ = (l+1, l)$, $\tau_- = (l-1, l-2)$ as an irreducible component with multiplicity 1 from Lemma 2.5 respectively. We define $\mathrm{pr}_2^\pm$ by

$$\mathrm{pr}_2^\pm : \ C_{\tau \otimes \mathrm{Ad}_{\mathfrak{p}^\pm}}^\infty(G/K) \to C_{\tau_\pm}^\infty(G/K) : \ projection.$$

and we call the following matrices whose entries are differential operators of order 1 shift operators.

$$\begin{cases} E^+ = \mathrm{pr}_2^+ \circ \nabla_\tau^+ \ : \ C_{(l,l-1)}^\infty(G/K) \to C_{(l+1,\,l)}^\infty(G/K) \\ E^- = \mathrm{pr}_2^- \circ \nabla_\tau^- \ : \ C_{(l,l-1)}^\infty(G/K) \to C_{(l-1,\,l-2)}^\infty(G/K) \end{cases}$$

By Lemma 2.7, we see that $E^\pm$ are of the form

$$(6.3) \qquad \begin{pmatrix} \phi_0^+ \\ \phi_1^+ \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} X_{\beta_1+\beta_2} & -X_{2\beta_1} \\[2mm] X_{2\beta_2} & \dfrac{1}{2} X_{\beta_1+\beta_2} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix},$$

$$(6.4) \qquad \begin{pmatrix} \phi_0^- \\ \phi_1^- \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} X_{-\beta_1-\beta_2} & -X_{-2\beta_2} \\[2mm] X_{-2\beta_1} & -\dfrac{1}{2} X_{-\beta_1-\beta_2} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

for $\phi_0 v_0^{\tau*} + \phi_1 v_1^{\tau*} \in C_\tau^\infty(G/K)$, $\phi_0^\pm v_0^{\tau_\pm^*} + \phi_1^\pm v_1^{\tau_\pm^*} \in C_{\tau_\pm}^\infty(G/K)$.
We denote $E_l^\pm$ for $E^\pm|_{C_{(l,l-1)}^\infty(G/K)}$.

Remark 6.5. $E_{l-1}^+ \circ E_l^-$ is a map from $C_{(l,l-1)}^\infty(G/K)$ to $C_{(l,l-1)}^\infty(G/K)$. Especially for any $\eta \in \widehat{K}$ and $\tau = (l, l-1)$, this is a map from $C_{\eta,\tau}^\infty(K \backslash G/K)$ to $C_{\eta,\tau}^\infty(K \backslash G/K)$.

Proposition 6.6 (1-dimensional case). *For $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$ with $\eta = (k,$
$k)$, $\tau = (l, l)$, $k \equiv l \bmod 2$, the radial parts of $D_l^\pm$ are given as follows.*

(i) $R(D_l^+)\phi(x_1, x_2) = [2\partial_{x_1}\partial_{x_2}$
$\qquad + \{-2l \coth 2x_2 + 2k \, \mathrm{sh}^{-1}2x_2 + \coth(x_1+x_2) - \coth(x_1-x_2)\}\partial_{x_1}$
$\qquad + \{-2l \coth 2x_1 + 2k \, \mathrm{sh}^{-1}2x_1 + \coth(x_1+x_2) + \coth(x_1-x_2)\}\partial_{x_2}$
$\qquad + 2(l \coth 2x_1 - k \, \mathrm{sh}^{-1}2x_1)(l \coth 2x_2 - k \, \mathrm{sh}^{-1}2x_2)$
$\qquad - (l \coth 2x_2 - k \, \mathrm{sh}^{-1}2x_2)(\coth(x_1+x_2) + \coth(x_1-x_2))$
$\qquad - (l \coth 2x_1 - k \, \mathrm{sh}^{-1}2x_1)(\coth(x_1+x_2) - \coth(x_1-x_2))]\phi(x_1, x_2).$

(ii) $R(D_l^-)\phi(x_1, x_2) = [2\partial_{x_1}\partial_{x_2}$
$\qquad + \{2l \coth 2x_2 - 2k \, \mathrm{sh}^{-1}2x_2 + \coth(x_1+x_2) - \coth(x_1-x_2)\}\partial_{x_1}$
$\qquad + \{2l \coth 2x_1 - 2k \, \mathrm{sh}^{-1}2x_1 + \coth(x_1+x_2) + \coth(x_1-x_2)\}\partial_{x_2}$
$\qquad + 2(l \coth 2x_1 - k \, \mathrm{sh}^{-1}2x_1)(l \coth 2x_2 - k \, \mathrm{sh}^{-1}2x_2)$
$\qquad + (l \coth 2x_2 - k \, \mathrm{sh}^{-1}2x_2)(\coth(x_1+x_2) + \coth(x_1-x_2))$
$\qquad + (l \coth 2x_1 - k \, \mathrm{sh}^{-1}2x_1)(\coth(x_1+x_2) - \coth(x_1-x_2))]\phi(x_1, x_2).$

Proof. By Lemma 1.1 and the proof of Lemma 5.1, we can rewrite X_β with
$\mathfrak{a}_C$, $\mathfrak{k}_C$ and $\mathrm{Ad}(a^{-1})\mathfrak{k}_C$ for $a \in A'$.

$$X_{\pm 2\beta_1} = \mp\sqrt{-1}\mathrm{sh}^{-1}2x_1 \, \mathrm{Ad}(a^{-1})X_{2e_1}$$
$$\qquad \mp \sqrt{-1}T_1 \pm \sqrt{-1}(\coth 2x_1 + 1)X_{2e_1} + H_1$$
$$\qquad = \pm \mathrm{sh}^{-1}2x_1 \, \mathrm{Ad}(a^{-1})H_1' + H_1 \mp \coth 2x_1 H_1'$$
$$X_{\pm 2\beta_2} = \mp\sqrt{-1}\mathrm{sh}^{-1}2x_2 \, \mathrm{Ad}(a^{-1})X_{2e_2}$$
$$\qquad \mp \sqrt{-1}T_2 \pm \sqrt{-1}(\coth 2x_2 + 1)X_{2e_2} + H_2$$
$$\qquad = \pm \mathrm{sh}^{-1}2x_2 \, \mathrm{Ad}(a^{-1})H_2' + H_2 \mp \coth 2x_2 H_2'$$
$$X_{\pm(\beta_1+\beta_2)} = -\mathrm{sh}^{-1}(x_1-x_2)\mathrm{Ad}(a^{-1})X_{e_1-e_2}$$
$$\qquad \mp \sqrt{-1}\mathrm{sh}^{-1}(x_1+x_2)\mathrm{Ad}(a^{-1})X_{e_1+e_2}$$
$$\qquad + \coth(x_1-x_2)X_{e_1-e_2} \pm \sqrt{-1}\coth(x_1+x_2)X_{e_1+e_2}$$

Here we use $X_{2e_i} = T_i = \sqrt{-1}H_i'$ and $X_{e_1\pm e_2} = \mp X - \bar{X}$.
   Hence we have

$$X_{\pm 2\beta_1}X_{\pm 2\beta_2} = \mathrm{sh}^{-1}2x_1 \, \mathrm{sh}^{-1}2x_2 \, \mathrm{Ad}(a^{-1})H_1' \cdot \mathrm{Ad}(a^{-1})H_2'$$
$$\qquad \pm \mathrm{sh}^{-1}2x_1 \, \mathrm{Ad}(a^{-1})H_1' \cdot H_2 - \mathrm{sh}^{-1}2x_1 \coth 2x_2 \, \mathrm{Ad}(a^{-1})H_1' \cdot H_2'$$
$$\qquad \pm \mathrm{sh}^{-1}2x_2 H_1 \cdot \mathrm{Ad}(a^{-1})H_2' + H_1 \cdot H_2 \mp \coth 2x_2 H_1 \cdot H_2'$$
$$\qquad - \coth 2x_1 \, \mathrm{sh}^{-1}2x_2 H_1' \cdot \mathrm{Ad}(a^{-1})H_2' \mp \coth 2x_1 H_1' \cdot H_2$$
$$\qquad + \coth 2x_1 \coth 2x_2 H_1' \cdot H_2'.$$

Since $[H_1, \mathrm{Ad}(a^{-1})H_2'] = [H_1', \mathrm{Ad}(a^{-1})H_2'] = [H_1', H_2] = 0$ and $\eta(H_i') = k$, $\tau^*(H_i')$
$= -l$, we get the following.

$$R(X_{\pm 2\beta_1}X_{\pm 2\beta_2}) = \partial_{x_1}\partial_{x_2} \pm (-l \coth 2x_2 + k \, \mathrm{sh}^{-1}2x_2)\partial_{x_1}$$
$$\qquad \pm (-l \coth 2x_1 + k \, \mathrm{sh}^{-1}2x_1)\partial_{x_2}$$
$$\qquad + (-l\coth 2x_1 + k \, \mathrm{sh}^{-1}2x_1)(-l\coth 2x_2 + k \, \mathrm{sh}^{-1}2x_2).$$

Similarly, we obtain

$$R(X_{\pm 2\beta_2}X_{\pm 2\beta_1})=R(X_{\pm 2\beta_1}X_{\pm 2\beta_2}).$$

On the other hand, since $\eta(X_{e_1\pm e_2})=\eta(\mp X-\bar{X})=0$ and $\tau(X_{e_1\pm e_2})=\tau(\mp X-\bar{X})=0$,

$$\begin{aligned}
R(X^2_{\pm(\beta_1+\beta_2)})=R(&-\mathrm{sh}^{-1}(x_1-x_2)\coth(x_1-x_2)X_{e_1-e_2}\,\mathrm{Ad}(a^{-1})X_{e_1-e_2}\\
&+\mathrm{sh}^{-1}(x_1+x_2)\coth(x_1+x_2)X_{e_1+e_2}\mathrm{Ad}(a^{-1})X_{e_1+e_2}\\
&\mp\sqrt{-1}\,\mathrm{sh}^{-1}(x_1+x_2)\coth(x_1-x_2)X_{e_1-e_2}\mathrm{Ad}(a^{-1})X_{e_1+e_2}\\
&\mp\sqrt{-1}\,\mathrm{sh}^{-1}(x_1-x_2)\coth(x_1+x_2)X_{e_1+e_2}\mathrm{Ad}(a^{-1})X_{e_1-e_2})
\end{aligned}$$

holds. From the commutation relations

$$\begin{aligned}
&[X_{e_1-e_2},\,\mathrm{Ad}(a^{-1})X_{e_1-e_2}]=-2\,\mathrm{sh}(x_1-x_2)(H_1-H_2),\\
&[X_{e_1+e_2},\,\mathrm{Ad}(a^{-1})X_{e_1+e_2}]=-2\,\mathrm{sh}(x_1+x_2)(H_1+H_2),\\
&[X_{e_1-e_2},\,\mathrm{Ad}(a^{-1})X_{e_1+e_2}]=2\,\mathrm{sh}^{-1}2x_1(\mathrm{sh}(x_1-x_2)X_{2e_1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\mathrm{sh}(x_1+x_2)\mathrm{Ad}(a^{-1})X_{2e_1})\\
&\qquad\qquad\qquad\qquad+2\,\mathrm{sh}^{-1}2x_2(\mathrm{sh}(x_1-x_2)X_{2e_2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\mathrm{sh}(x_1+x_2)\mathrm{Ad}(a^{-1})X_{2e_2}),\\
&[X_{e_1+e_2},\,\mathrm{Ad}(a^{-1})X_{e_1-e_2}]=2\,\mathrm{sh}^{-1}2x_1(-\mathrm{sh}(x_1+x_2)X_{2e_1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\mathrm{sh}(x_1-x_2)\mathrm{Ad}(a^{-1})X_{2e_1})\\
&\qquad\qquad\qquad\qquad+2\,\mathrm{sh}^{-1}2x_2(\mathrm{sh}(x_1+x_2)X_{2e_2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\mathrm{sh}(x_1-x_2)\mathrm{Ad}(a^{-1})X_{2e_2}),
\end{aligned}$$

we have

$$\begin{aligned}
R(X^2_{\pm(\beta_1+\beta_2)})=&2\{\coth(x_1-x_2)-\coth(x_1+x_2)\}\partial_{x_1}\\
&-2\{\coth(x_1-x_2)+\coth(x_1+x_2)\}\partial_{x_2}\\
&\pm 2(l\coth 2x_2-k\,\mathrm{sh}^{-1}2x_2)(\coth(x_1+x_2)+\coth(x_1-x_2))\\
&\pm 2(l\coth 2x_1-k\,\mathrm{sh}^{-1}2x_1)(\coth(x_1+x_2)-\coth(x_1-x_2)).
\end{aligned}$$

Thus we have the proposition. $\qquad\square$

Proposition 6.7 (2-dimensional case). *For* $\phi=\phi_{01}v_0^\eta\otimes v_1^{\tau^*}+\phi_{10}v_1^\eta\otimes v_0^{\tau^*}\in C^\infty_{\eta,\tau}(K\backslash G/K)$, $\phi^-=\phi_{00}^-v_0^\eta\otimes v_0^{\tau_-^*}+\phi_{11}^-v_1^\eta\otimes v_1^{\tau_-^*}\in C^\infty_{\eta,\tau_-}(K\backslash G/K)$ *with* $\eta=(k,\ k-1)$, $\tau=(l,l-1)$, $\tau_-=(l-1,\ l-2)$ *and* $k\equiv l \bmod 2$, *the radial parts of* E^+_{l-1}, E^-_l *are given as follows.*

(i) $\quad R(E^+_{l-1})\phi^-(x_1,\ x_2)=[\{\partial_{x_2}-(l-2)\coth 2x_2+k\,\mathrm{sh}^{-1}2x_2$

$$+\frac{1}{2}(\coth(x_1+x_2)-\coth(x_1-x_2))\}\phi_{00}^-(x_1,\ x_2)$$

$$+\frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2)+\mathrm{sh}^{-1}(x_1-x_2))\phi_{11}^-(x_1,\ x_2)]v_0^\eta\otimes v_1^{\tau^*}$$

$$+[-\{\partial_{x_1}-(l-2)\coth 2x_1+k\,\mathrm{sh}^{-1}2x_1$$

$$+\frac{1}{2}(\coth(x_1+x_2)+\coth(x_1-x_2)))\}\phi_{\bar{1}\bar{1}}(x_1,\ x_2)$$

$$-\frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2)-\mathrm{sh}^{-1}(x_1-x_2))\phi_{\bar{0}\bar{0}}(x_1,\ x_2)]v_1^{\eta}\otimes v_0^{\tau^*},$$

(ii) $R(E_l^-)\phi(x_1,\ x_2)=[-\{\partial_{x_2}+l\coth 2x_2-k\,\mathrm{sh}^{-1}2x_2$

$$+\frac{1}{2}(\coth(x_1+x_2)-\coth(x_1-x_2)))\}\phi_{01}(x_1,\ x_2)$$

$$-\frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2)-\mathrm{sh}^{-1}(x_1-x_2))\phi_{10}(x_1,\ x_2)]v_0^{\eta}\otimes v_0^{\tau^*}$$

$$+[\{\partial_{x_1}+l\coth 2x_1-k\,\mathrm{sh}^{-1}2x_1$$

$$+\frac{1}{2}(\coth(x_1+x_2)+\coth(x_1-x_2)))\}\phi_{10}(x_1,\ x_2)$$

$$+\frac{1}{2}(\mathrm{sh}^{-1}(x_1+x_2)+\mathrm{sh}^{-1}(x_1-x_2))\phi_{01}(x_1,\ x_2)]v_1^{\eta}\otimes v_1^{\tau^*},$$

Proof.   These are computed in the same way as Proposition 6.6. $\square$

§7.   Differential Equations Satisfied by Spherical Functions

7.1   Spherical Functions of H_{π_0}

In this section, we will find the system of differential equations satisfied by spherical functions of H_{π_0} and H_{π_1}.

On the principal series representation $H_{\pi_0}=C^{\infty}\text{-Ind}_P^G(\sigma\otimes a^{\mu+\rho}\otimes 1_N)$, $(\mu=(\mu_1,\ \mu_2))$, the Casimir operator L acts as a scalar $\mu_1^2+\mu_2^2-5$, which is the infinitesimal character of H_{π_0}. Therefore L acts on $\phi_{\pi_0,\tau,i}\in C_{\eta,\tau}^{\infty}(K\backslash G/K)$ as a scalar $\mu_1^2+\mu_2^2-5$.

It can be shown that $D^{\pm}\phi_{\pi_0,\tau,i}$ and $E^{\pm}\phi_{\pi_0,\tau,i}$ are also spherical functions attached to H_{π_0}. So are $D_{l-2}^+\circ D_l^-\phi_{\pi_0,\tau,i}$ and $E_{l-1}^+\circ E_l^-\phi_{\pi_0,\tau,i}$. As for D^+, using following two K-maps

$$p:\ V_{\tau^*}\otimes\mathrm{Ad}_{\mathfrak{p}-}\otimes\mathrm{Ad}_{\mathfrak{p}-}\longrightarrow Vv_{\tau^*}:\ \text{projection},$$

$$m:\ V_{\tau}\otimes\mathrm{Ad}_{\mathfrak{p}+}\otimes\mathrm{Ad}_{\mathfrak{p}+}\ni v\otimes X\otimes Y\to\pi(YX)i(v)\in H_{\pi_0},$$

we have $m\circ{}^t p\in\mathrm{Hom}_K(V_{\tau+},\ H_{\pi_0})$ and

$$D^+\phi_{\pi_0,\tau,i}(g)=\sum_{n,m}\langle\pi_0(g)m\circ{}^t p(v_n^{\tau}),\ j^*(w_m^{\eta*})\rangle v_m^{\eta}\otimes w_n^{\tau^*}$$

$$=8C\phi_{\pi_0,\tau,m\circ{}^t p}(g)$$

for $g\in G$ with the constant C appeared in Definition 6.1. From Proposition 3.2, the multiplicity of τ in $H_{\pi_0}|_K$ is one in both 1-dimensional and 2-dimensional cases, the dimension of spherical functions attached to H_{π_0} in $C_{\eta,\tau}^{\infty}(K\backslash G/K)$ is one. Therefore $D_{l-2}^+\circ D_l^-$ and $E_{l-1}^+\circ E_l^-$ act on $\phi_{\pi_0,\tau,i}$ as scalars. To get these

scalars, we use the same method as in [MO1].

Lemma 7.1 (1-dimensional case). *For $\eta=(k, k)$, $\tau=(l, l)$ with $k\equiv l$ mod 2, $D_{l-2}^+\circ D_{\bar{l}}^-$ acts on $C_{\eta,\tau}^\infty(K\backslash G/K)$ as a scalar $4\{\mu_1^2-(l-1)^2\}\{\mu_2^2-(l-1)^2\}$.*

Proof. Let f_l be the realization of $(l, l)\in\hat{K}$ defined in Lemma 2.2 and i be the injective K-map from $V_\tau=\boldsymbol{C}f_l$ to H_π. Then $i(f_l)(nak)=a^{\mu+\rho}f_l(k)$ and we denote $i(f_l)$ by the same f_l in short.

From Lemma 2.2, $T_jf_l=\sqrt{-1}H_j'f_l=\sqrt{-1}lf_l$ for $j=1, 2$ and $Xf_l=\bar{X}f_l=0$. And by the definition, for any $a\in A$,

$$(H_jf_l)(a)=\frac{d}{dt}\Big|_{t=0}f_l(a\exp tH_j)=\frac{d}{dt}\Big|_{t=0}\exp t(\mu+\rho)(H_j)f_l(a)$$
$$=(\mu+\rho)(H_j)f_l(a)\text{ for }j=1, 2,$$
$$(E_\alpha f_l)(a)=\frac{d}{dt}\Big|_{t=0}f_l(a\exp tE_\alpha)=\frac{d}{dt}\Big|_{t=0}f_l(a)=0\text{ for }{}^\forall\alpha\in\varDelta^+$$

hold. Thus, we have $(\mathfrak{gn}f_l)(a)=(\mathfrak{g}Xf_l)(a)=(\mathfrak{g}\bar{X}f_l)(a)=0$. Since N is normalized by A, $(\mathfrak{na}f_l)(a)=0$. If $i\neq j$, then $[T_i, E_{2e_j}]=[T_j, H_j]=0$. Hence,

$$(X_{\pm2\beta_1}X_{\pm2\beta_2}f_l)(a)$$
$$=\{(\mp\sqrt{-1}T_1+H_1\pm2\sqrt{-1}E_{2e_1})(\mp\sqrt{-1}T_2+H_2\pm2\sqrt{-1}E_{2e_2})f_l\}(a)$$
$$=\{(\mp\sqrt{-1}T_1+H_1)(\mp\sqrt{-1}T_2+H_2)f_l\}(a)$$
$$=(\pm l+\mu_1+2)(\pm l+\mu_2+1)f_l(a)$$

holds. Similarly, we have

$$(X_{\pm2\beta_2}X_{\pm2\beta_1}f_l)(a)$$
$$=(\pm l+\mu_2+1)(\pm l+\mu_1+2)f_l(a).$$

On the other hand, since $[\bar{X}, E_{e_1-e_2}+\sqrt{-1}E_{e_1+e_2}]=H_2-\sqrt{-1}T_2+2\sqrt{-1}E_{2e_2}$,

$$(X_{\beta_1+\beta_2}^2f_l)(a)$$
$$=4\{(\bar{X}+E_{e_1-e_2}+\sqrt{-1}E_{e_1+e_2})^2f_l\}(a)$$
$$=4\{\bar{X}(E_{e_1-e_2}+\sqrt{-1}E_{e_1+e_2})f_l\}(a)$$
$$=4([\bar{X}, E_{e_1-e_2}+\sqrt{-1}E_{e_1+e_2}]f_l)(a)$$
$$=4\{(H_2-\sqrt{-1}T_2+2\sqrt{-1}E_{2e_2})f_l\}(a)$$
$$=4(\mu_2+1+l)f_l(a)$$

holds. Similarly, we have

$$(X_{-(\beta_1+\beta_2)}^2f_l)(a)=4(\mu_2+1-l)f_l(a).$$

Since $D^\pm=X_{\pm2\beta_1}X_{\pm2\beta_2}+X_{\pm2\beta_2}X_{\pm2\beta_1}-\frac{1}{2}X_{\pm(\beta_1+\beta_2)}^2$, $D_l^\pm f_l(a)=2(\mu_1\pm l+1)(\mu_2\pm l+1)f_l(a)$ holds.

From the definition,

$$D^\pm : C^\infty_{\sigma_l}(M\backslash K) \longrightarrow C^\infty_{\sigma_l}(M\backslash K)$$

and $CD^\pm f_l \simeq V_{(l\pm2, l\pm2)}$. Since dim $\mathrm{Hom}_K(V_{(l\pm2, l\pm2)}, C^\infty_{\sigma_l}(K))=1$, $D^\pm f_l = C_\pm f_{l\pm2}$ for some $C_\pm \in C$. From the fact $f_l|_A = f_{l\pm2}|_A$ and the concrete description of the action of $D^\pm$ on f_l we have seen above, we conclude $D^\pm f_l = 2(\mu_1 \pm l + 1)(\mu_2 \pm l + 1)f_{l\pm2}$. Hence $D^- \circ D^+ f_l = 4\{\mu_1^2 - (l+1)^2\}\{\mu_2^2 - (l+1)^2\}f_l$.

For $Z_1, Z_2 \in \mathfrak{g}$ and $\phi(g)v^\eta \otimes v^{\tau^*} \in C^\infty_{\eta,\tau}(K\backslash G/K)$,

$$\begin{aligned}
&Z_1 Z_2(\phi(g)v^\eta \otimes v^{\tau^*})\\
&= \frac{\partial}{\partial s \partial t}\Big|_{s=t=0} \phi(g \exp sZ_1 \exp tZ_2)v^\eta \otimes v^{\tau^*}\\
&= \frac{\partial}{\partial s \partial t}\Big|_{s=t=0} \eta(e) \otimes \tau^*(\exp(sZ_1)\exp(tZ_2))^{-1}(\phi(g)v^\eta \otimes v^{\tau^*})\\
&= \phi(g)v^\eta \otimes \tau^*(Z_2 Z_1)v^{\tau^*}.
\end{aligned}$$

We define an involution ι on $\mathfrak{g} \otimes \mathfrak{g}$ by $\iota(Z_1 \otimes Z_2) = Z_2 \otimes Z_1$ for $Z_1 \otimes Z_2 \in \mathfrak{g} \otimes \mathfrak{g}$. Since $D^\pm$ are in $\mathfrak{g} \otimes \mathfrak{g}$ and ι-invariant, taking f_{-l} as v^{τ^*}, we have

$$\begin{aligned}
&D^+_{l-2} \circ D^-_l(\phi(g)v^\eta \otimes v^{\tau^*})\\
&= \phi(g)v^\eta \otimes \tau^*(D^- \circ D^+)v^{\tau^*}\\
&= 4\{\mu_1^2 - (l-1)^2\}\{\mu_2^2 - (l-1)^2\}(\phi(g)v^\eta \otimes v^{\tau^*}).
\end{aligned}$$

Thus we have proved the lemma.   □

Lemma 7.2 (2-dimensional case). *For* $\eta=(k, k-1)$, $\tau=(l, l-1)$ *with* $k \equiv l \bmod 2$, $E^+_{l-1} \circ E^-_l$ *acts on* $C^\infty_{\eta,\tau}(K\backslash G/K)$ *as a scalar*

$$\begin{cases} -\{\mu_1^2 - (l-1)^2\} & \text{if } l : odd,\\ -\{\mu_2^2 - (l-1)^2\} & \text{if } l : even. \end{cases}$$

Proof. This can be shown in the same way as Lemma 7.1 using the realization of $(l, l-1)$ (Lemma 2.3) and Lemma 2.4. See [MO1] for details.

□

From the above lemmas, we have the system of differential equations satisfied by spherical functions of H_{π_0}.

Theorem 7.3 (1-dimensional case). *For* $\eta=(k,k)$, $\tau=(l,l)$ *with* $k \equiv l \bmod 2$, *the system of differential equations satisfied by spherical functions* $\phi \in C^\infty_{\eta,\tau}(K\backslash G/K)$ *of* H_{π_0} *is the following.*

(7.1) $R(L)\phi(x_1, x_2) = (\mu_1^2 + \mu_2^2 - 5)\phi(x_1, x_2),$

(7.2) $R(D^+_{l-2}) \circ R(D^-_l)\phi(x_1, x_2) = 4\{\mu_1^2 - (l-1)^2\}\{\mu_2^2 - (l-1)^2\}\phi(x_1, x_2).$

Theorem 7.4 (2-dimensional case). *For $\eta=(k, k-1)$, $\tau=(l, l-1)$ with $k \equiv l \bmod 2$, the system of differential equations satisfied by spherical functions $\phi \in C_{\eta,\tau}^{\infty}(K \backslash G/K)$ of H_{π_0} is the following.*

$$(7.3) \quad R(L)\phi(x_1, x_2)=(\mu_1^2+\mu_2^2-5)\phi(x_1, x_2),$$

$$(7.4) \quad R(E_{l-1}^+)\circ R(E_l^-)\phi(x_1, x_2)=\begin{cases} -\{\mu_1^2-(l-1)^2\}\phi(x_1, x_2) & \text{if } l : odd \\ -\{\mu_2^2-(l-1)^2\}\phi(x_1, x_2) & \text{if } l : even \end{cases}$$

7.2 Spherical Functions of H_{π_1}

For the case of a generalized principal series representation, the Casimir operator L acts on $H_{\pi_1}=C^{\infty}\text{-Ind}_{P_J}^G(\sigma_J\otimes a_J^{\nu_1+\rho_J}\otimes 1_{N_J})$ as the infinitesimal character $\nu_1^2+(l-1)^2-5$ and so does on $\phi_{\pi_1,\tau,i}\in C_{\eta,\tau}^{\infty}(K \backslash G/K)$.

On the other hand, for $\tau=(l,l)$, $D_l^-\phi_{\pi_1,\tau,i}$ is the spherical function attached to in H_{π_1} included in $C_{\eta,(l-2,l-2)}^{\infty}(K \backslash G/K)$, which must be 0 from Proposition 3.4. Therefore $D_l^-\phi_{\pi_1,\tau,i}=0$.

Similarly, for $\tau=(l,l-1)$, $E_l^-\phi_{H_{\pi_1}}=0$.

Thus we have the following theorems.

Theorem 7.5 (1-dimensional case). *For $\eta=(k,k)$, $\tau=(l,l)$ with $l\geq 2$ and $k \equiv l \bmod 2$, a spherical function $\phi\in C_{\eta,\tau}^{\infty}(K \backslash G/K)$ of H_{π_1} satisfies the following system of differential equations.*

$$(7.5) \qquad R(L)\phi(x_1,x_2)=\{\nu_1^2+(l-1)^2-5\}\phi(x_1, x_2),$$

$$(7.6) \qquad R(D_l^-)\phi(x_1, x_2)=0$$

Theorem 7.6 (2-dimensional case). *For $\eta=(k,k-1)$, $\tau=(l,l-1)$ with $l\geq 2$ and $k \equiv l \bmod 2$, a spherical function $\phi\in C_{\eta,\tau}^{\infty}(K \backslash G/K)$ of H_{π_1} satisfies the following system of differential equations.*

$$(7.7) \qquad R(L)\phi(x_1,x_2)=\{\nu_1^2+(l-1)^2-5\}\phi(x_1,x_2),$$

$$(7.8) \qquad R(E_l^-\phi(x_1,x_2)=0$$

Remark 7.7. (i) In the case $k=l=0$ and more than or equal two variables, the system of differential equations in Theorem 7.3 and Theorem 7.5 are defined in [DG1], [DG2] with more general parameters, which are generalizations of root multiplicities without using the geometry of G/K.

The polynomial solutions of the system (7.1), (7.2) with $k=l=0$ were given in [DG2] and the general solution of the system of (7.5) and (7.6) with $k=l=0$ are obtained in [DG1].

(ii) In the case $k=l=0$, the system of (7.1) and (7.2) are defined as a family of commuting differential operators invariant under the action of B_2-type Weyl

group in [OO] and the system of (7.5) and (7.6) are defined as the reducible system of that. Those systems have more parameters than the systems defined in [DG1], [DG2].

§8. Spherical Functions of H_{π_1}

8.1 Reduction to the Case $k = l = 0$

In this section, we will find the solutions of the system of differential equations in Theorem 7.5 and Theorem 7.6.

To do this, we use the reduction to the case $k = l = 0$. The reduction of the Casimir operator in 1-dimensional case was given in [H2], [Sh].

We set

$$\delta(x_1, x_2; k, l) = (\mathrm{ch}\, x_1 \cdot \mathrm{ch}\, x_2)^{\frac{k+l}{2}} (\mathrm{sh}\, x_1 \cdot \mathrm{sh}\, x_2)^{-\frac{k-l}{2}}.$$

Then we have the following propositions.

Proposition 8.1 (1-dimensional case). *If $\phi \in C^{\infty}_{\eta, \tau}(K \backslash G / K)$ is a solution of (7.5) and (7.6), then after the change of variables ; $y_i = -\mathrm{sh}^2 x_i$ $(i=1,2)$, $\psi(x_1, x_2) = \delta(x_1, x_2; k, l) \phi(x_1, x_2)$ satisfies the following system of differential equations equivalent to (7.5), (7.6) in Theorem 7.5.*

$$(8.1) \quad \left[\sum_{i=1}^{2} y_i(y_i - 1) \partial^2_{y_i} + \left\{ (2-l)y_1 - 1 - \frac{k-l}{2} + \frac{y_1(y_1 - 1)}{y_1 - y_2} \right\} \partial_{y_1} \right.$$

$$\left. + \left\{ (2-l)y_2 - 1 - \frac{k-l}{2} - \frac{y_2(y_2 - 1)}{y_1 - y_2} \right\} \partial_{y_2} - \frac{1}{4} \{ \nu_1^2 - (l-2)^2 \} \right] \psi = 0,$$

$$(8.2) \quad \left[\partial_{y_1} \partial_{y_2} - \frac{1}{2} \frac{1}{y_1 - y_2} \partial_{y_1} + \frac{1}{2} \frac{1}{y_1 - y_2} \partial_{y_2} \right] \psi = 0.$$

Here we set $\partial_{y_i} = \dfrac{\partial}{\partial_{y_i}}$.

Proof. By Proposition 5.2 and Proposition 6.6 (ii) and the following formulas, we have this proposition easily.

$$\begin{cases} \delta(x_1, x_2) \circ \partial_{x_i} \circ \delta(x_1, x_2)^{-1} = \partial_{x_i} + k\, \mathrm{sh}^{-1} 2x_i - l \coth 2x_i \\ \delta(x_1, x_2) \circ \partial^2_{x_i} \circ \delta(x_1, x_2)^{-1} = \partial^2_{x_i} + 2(k\, \mathrm{sh}^{-1} 2x_i - l \coth 2x_i)\partial_{x_i} \\ \qquad\qquad\qquad + (k^2 + (l+1)^2 - 1)\mathrm{sh}^{-2} 2x_i \\ \qquad\qquad\qquad - 2k(l+1)\mathrm{sh}^{-2} 2x_i\, \mathrm{ch}\, 2x_i + l^2 \\ \partial_{x_i} = -\mathrm{sh}\, 2x_i \partial_{y_i} \\ \partial^2_{x_i} = 4y_i(y_i - 1)\partial^2_{y_i} + 2(2y_i - 1)\partial_{y_i} \end{cases} \qquad\qquad \Box$$

Lemma 8.2. *If $\phi = \phi_{01} v_0^{\eta} \otimes v_1^{\tau^*} + \phi_{10} v_1^{\eta} \otimes v_0^{\tau^*} \in C^{\infty}_{\eta, \tau}(K \backslash G / K)$ is a solution of the system of differential equations (7.7) and (7.8), then*

$$\begin{cases} \phi_{01}(x_1,x_2) = \delta(x_1,x_2\,;\,k,l)\,\mathrm{ch}^{-1}\,x_1\phi_{01}(x_1,x_2) \\ \phi_{10}(x_1,x_2) = \delta(x_1,x_2\,;\,k,l)\,\mathrm{ch}^{-1}x_2\phi_{10}(x_1,x_2) \end{cases}$$

satisfy the following two differential equations.

(8.3) $\quad [\sum_{i=1}^{2}\partial_{x_i}^2 + \{-2l\coth 2x_1 + 2(k+1)\mathrm{sh}^{-1}2x_1 + 2\,\mathrm{th}\,x_1$

$\qquad\qquad + \coth(x_1+x_2) + \coth(x_1-x_2)\}\partial_{x_1}$

$\qquad\quad + \{-2l\coth 2x_1 + 2(k+1)\mathrm{sh}^{-1}2x_2$

$\qquad\qquad\quad + \coth(x_1+x_2) - \coth(x_1-x_2)\}\partial_{x_2}$

$\qquad + \mathrm{th}\,x_1(\coth(x_1+x_2) + \coth(x_1-x_2))$

$\qquad\qquad - \mathrm{sh}^{-2}(x_1+x_2) - \mathrm{sh}^{-2}(x_1-x_2) + 2l^2 - 4l - 3]\phi_{01}$

$\qquad + \{\mathrm{ch}(x_1+x_2)\mathrm{sh}^{-2}(x_1+x_2)$

$\qquad\qquad + \mathrm{ch}(x_1-x_2)\mathrm{sh}^{-2}(x_1-x_2)\}\mathrm{ch}^{-1}x_1\,\mathrm{ch}\,x_2\phi_{10}$

$\qquad = \{\nu_1^2 + (l-1)^2 - 5\}\phi_{01}$

(8.4) $\quad \{\partial_{x_2} + \dfrac{1}{2}(\coth(x_1+x_2) - \coth(x_1-x_2))\}\phi_{01}$

$\qquad\quad - \dfrac{1}{2}\mathrm{ch}^{-1}x_1\,\mathrm{ch}\,x_2\{\mathrm{sh}^{-1}(x_1+x_2) - \mathrm{sh}^{-1}(x_1-x_2)\}\phi_{10} = 0$

Proof. These are computed in the same way as Proposition 8.1. $\qquad\square$

Proposition 8.3 (2-dimensional case). *For $i=1$, 2, under the change of variables; $y_i = -\mathrm{sh}^2 x_i$, ϕ_{01} defined in Lemma 8.2 satisfies the following system of differential equations.*

(8.5) $\quad \Big[\sum_{i=1}^{2}y_i(y_i-1)\partial_{y_i}^2 + \Big\{(2-l)y_1 - 1 - \dfrac{k-l}{2} + \dfrac{y_1(y_1-1)}{y_1-y_2}\Big\}\partial_{y_1}$

$\qquad + \Big\{-ly_2 - \dfrac{k-l}{2} - 3\dfrac{y_2(y_2-1)}{y_1-y_2}\Big\}\partial_{y_2} - \dfrac{1}{4}\{\nu_1^2 - (l-1)^2 - 2\}\Big]\phi_{01} = 0,$

(8.6) $\quad \Big[\partial_{y_1}\partial_{y_2} - \dfrac{1}{2}\dfrac{1}{y_1-y_2}\partial_{y_1} + \dfrac{3}{2}\dfrac{1}{y_1-y_2}\partial_{y_2}\Big]\phi_{01} = 0.$

Proof. From equations (8.3) and (8.4), ϕ_{10} can be eliminated and we have (8.5).

Exchanging x_1 and x_2 and using Lemma 4.3 (ii), we have

$$\Big\{\partial_{x_1} + \dfrac{1}{2}(\coth(x_1+x_2) + \coth(x_1-x_2))\Big\}\phi_{10}$$

$$- \dfrac{1}{2}\,\mathrm{ch}\,x_1\,\mathrm{ch}^{-1}x_2\{\mathrm{sh}^{-1}(x_1+x_2) + \mathrm{sh}^{-1}(x_1-x_2)\}\phi_{01} = 0$$

Eliminating ϕ_{10} from (8.4) and the above equation, we have a differential equation of order 2, which is reduced to (8.6) by changing variables. $\qquad\square$

8.2   Series Expansions and Integral Formulas

From the definition, in the case $k < l$, ψ and ψ_{01} are analytic and have zeros at the origin, and in the case $k \geq l$, they are analytic or have poles at $y_1 = y_2 = 0$ since ϕ and ϕ_{01} are analytic. Nevertheless, we will see that ψ and ψ_{01} can not have zeros when $k < l$ and they can not have poles when $k \leq l$ in Remark 8.5 below. Therefore, there exist no spherical functions attached to H_{π_1} in the case $k < l$ and $l \geq 2$, and we have only to find anlytic solutions of (8.1), (8.2) and (8.5), (8.6) in the case $k \geq l$.

The analytic solution of (8.1) and (8.2) (see Theorem 8.7) is found in [DG1] in a more general case. We can find the analytic solution of (8.5) and (8.6) in the same way as [DG1 Théorème 4, Théorème 7]. The key lemmas are the following Lemma 8.4 and Lemma 8.6. In the case $B_1 = B_2$, they are shown in [DG1 Théorème 3] and in [DG2 Lemma 2.6] respectively.

Lemma 8.4.   *When* Re B_1, Re $B_2 > 0$, *the function* $f(y_1, y_2)$ *which is analytic around the origin and satisfies*

$$\left[\partial_{y_1}\partial_{y_2} - B_2 \frac{1}{y_1 - y_2}\partial_{y_1} + B_1 \frac{1}{y_1 - y_2}\partial_{y_2} \right] f = 0$$

has the following series expansion and integral representation.

(i) $f(y_1, y_2) = \sum_{m_i \geq 0} \dfrac{(B_1)_{m_1}(B_2)_{m_2}\xi(m_1 + m_2)}{m_1! \, m_2!} y_1^{m_1} y_2^{m_2}$

Here we set

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)}$$

and ξ *is some function on* N.

(ii) $f(y_1, y_2) = \displaystyle\int_0^1 F(ty_1 + (1-t)y_2) t^{B_1 - 1}(1-t)^{B_2 - 1} dt$

Here F *is some analytic function around the origin.*

(iii)   *If functions in* (i) *and* (ii) *coincide with each other, then*

$$F(s) = \frac{\Gamma(B_1 + B_2)}{\Gamma(B_1)\Gamma(B_2)} \sum_{k \geq 0} \frac{(B_1 + B_2)_k \xi(k)}{k!} s^k$$

holds.

Proof.   (i)   We denote the differential operator $\partial_{y_1}\partial_{y_2} - B_2 \dfrac{1}{y_1 - y_2}\partial_{y_1}$ $+ B_1 \dfrac{1}{y_1 - y_2}\partial_{y_2}$ by Q_{B_1, B_2}. We put $f(y_1, y_2) = \sum_{m_i \geq 0} a_{m_1, m_2} y_1^{m_1} y_2^{m_2} \in \mathrm{Ker}\ Q_{B_1, B_2}$,

then we have

$$\sum_{m, \geq 0} \{m_1 m_2 a_{m_1, m_2}(y_1^{m_1} y_2^{m_2-1} - y_1^{m_1-1} y_2^{m_2})$$
$$- B_2 m_1 a_{m_1, m_2} y_1^{m_1-1} y_2^{m_2} + B_1 m_2 a_{m_1, m_2} y_1^{m_1} y_2^{m_2-1}\} = 0$$
$$\Leftrightarrow (m_1+1)(m_2+B_2) a_{m_1+1, m_2} = (m_1+B_1)(m_2+1) a_{m_1, m_2+1}$$
$$\text{for } {}^\forall m_1, \ m_2 \geq 0.$$

If we put

$$a_{m_1, m_2} = \frac{(B_1)_{m_1}(B_2)_{m_2} \xi(m_1, m_2)}{m_1! \, m_2!},$$

then this recurrence relation is reduced to $\xi(m_1+1, m_2) = \xi(m_1, m_2+1)$. This means ξ is a function of $m_1 + m_2$.

(ii)   If we set $F(z) = \sum_{k=0}^\infty \frac{\zeta(k)}{k!} z^k$, then

$$\int_0^1 F(ty_1 + (1-t)y_2) t^{B_1-1}(1-t)^{B_2-1} dt$$
$$= \sum_{k=0}^\infty \frac{\zeta(k)}{k!} \int_0^1 (ty_1 + (1-t)y_2)^k t^{B_1-1}(1-t)^{B_2-1} dt$$
$$= \sum_{k=0}^\infty \frac{\zeta(k)}{k!} \int_0^1 \sum_{l=0}^k \binom{k}{l} y_1^l y_2^{k-l} t^{l+B_1-1}(1-t)^{k-l+B_2-1} dt$$
$$= \sum_{k=0}^\infty \zeta(k) \sum_{l=0}^k \frac{y_1^l y_2^{k-l}}{(k-l)! \, l!} \frac{\Gamma(l+B_1)\Gamma(k-l+B_2)}{\Gamma(k+B_1+B_2)}$$
$$= \sum_{k=0}^\infty \zeta(k) \sum_{m_1+m_2=k, \, m, \geq 0} \frac{\Gamma(B_1)\Gamma(B_2)(B_1)_{m_1}(B_2)_{m_2}}{\Gamma(B_1+B_2)(B_1+B_2)_{m_1+m_2} m_1! \, m_2!} y_1^{m_1} y_2^{m_2}$$
$$= \sum_{m, \geq 0} \frac{\Gamma(B_1)\Gamma(B_2)\zeta(m_1+m_2)(B_1)_{m_1}(B_2)_{m_2}}{\Gamma(B_1+B_2)(B_1+B_2)_{m_1+m_2} m_1! \, m_2!} y_1^{m_1} y_2^{m_2}.$$

If we put

$$\frac{\Gamma(B_1)\Gamma(B_2)\zeta(m_1+m_2)}{\Gamma(B_1+B_2)(B_1+B_2)_{m_1+m_2}} = \xi(m_1+m_2),$$

then

$$f(y_1, y_2) = \int_0^1 F(ty_1 + (1-t)y_2) t^{B_1-1}(1-t)^{B_2-1} dt \in \mathrm{Ker} \, Q_{B_1, B_2}$$

holds from (i).

(iii)   It has already been shown in the proof of (ii).   □

Remark 8.5.   If $B_1, B_2 \notin \boldsymbol{Z}$ and $f(y_1, y_2) = \sum_{m, \geq N,} a_{m_1, m_2} y_1^{m_1} y_2^{m_2}$ is contained in Ker Q_{B_1, B_2} for non-negative integers N_1, N_2, then we can prove that $N_1 = N_2 = 0$ in the same way as the proof of Lemma 8.4 (i).

Lemma 8.6. *Let* $P=\sum_{i=1}^{2} y_i(y_i-1)\partial_{y_i}^2+\Big\{(A+B_1-B_2+1)y_1+B_2-C+$

$2B_2\dfrac{y_1(y_1-1)}{y_1-y_2}\Big\}\partial_{y_1}+\Big\{(A-B_1+B_2+1)y_2+B_1-C-2B_1\dfrac{y_2(y_2-1)}{y_1-y_2}\Big\}\partial_{y_2}-\lambda,\ L=$

$z(z-1)\dfrac{d^2}{dz^2}-\{C-(A+B_1+B_2+1)z\}\dfrac{d}{dz}-\lambda$ *and let the linear operator* T_{B_1,B_2}

on the functions which are analytic around the origin be

$$(T_{B_1,B_2}f)(y_1,\ y_2)=\int_0^1 f(ty_1+(1-t)y_2)t^{B_1-1}(1-t)^{B_2-1}dt$$

for $A,\ C,\ \lambda\in C$ *and* $\mathrm{Re}\ B_1,\ \mathrm{Re}\ B_2>0.$ *Then we have*

$$P\circ T_{B_1,B_2}=T_{B_1,B_2}\circ L.$$

Proof. This can be shown by change of the order of integration and differentiation.   □

Using these lemmas, we can show the following two theorems.

Theorem 8.7 (1-dimensional case).
(i) *The analytic solution of* (8.1) *and* (8.2) *has the following series expansion up to scalar.*

$$(8.7)\qquad \phi(y_1,\ y_2)=\sum_{m_i\geq 0}\frac{\left(\dfrac{1}{2}\right)_{m_1}\left(\dfrac{1}{2}\right)_{m_2}(\mu_+)_{m_1+m_2}(\mu_-)_{m_1+m_2}}{m_1!\,m_2!\,(1)_{m_1+m_2}\left(\dfrac{3+k-l}{2}\right)_{m_1+m_2}}y_1^{m_1}y_2^{m_2}$$

Here we set $\mu_\pm=-(l-2\pm\nu_1)/2.$
(ii) *The analytic solution of* (8.1) *and* (8.2) *has the following integral representation up to scalar.*

$$(8.8)\qquad \phi(y_1,\ y_2)=\int_0^1 {}_2F_1\left(\mu_+,\ \mu_-;\ \frac{3+k-l}{2};\ ty_1+(1-t)y_2\right)t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt$$

Here ${}_2F_1$ *is the classical Gaussian hypergeometric function which is analytic around the origin.*

Proof. This theorem is a consequence of Lemma 8.4 and Lemma 8.6 for the case

$$A=-l+1,\ B_1=B_2=1/2,\ C=(k-l+3)/2,\ \lambda=(\nu_1^2-(l-2)^2)/4.$$

□

Remark 8.8. When $\nu_1=\pm l$ (this means $\mu_\mp=1$), the solution given above

becomes Appell's hypergeometric function

$$F_1(-l+1,\ 1/2,\ 1/2,\ (3+k-l)/2\ ;\ y_1,\ y_2).$$

Theorem 8.9 (2-dimensional case). (i) *The analytic solution of* (8.5) *and* (8.6) *has the following series expansion up to scalar.*

$$(8.9) \qquad \psi_{01}(y_1,\ y_2) = \sum_{m,\geq 0} \frac{\left(\frac{3}{2}\right)_{m_1}\left(\frac{1}{2}\right)_{m_2}(\mu_+)_{m_1+m_2}(\mu_-)_{m_1+m_2}}{m_1!\,m_2!\,(2)_{m_1+m_2}\left(\frac{3+k-l}{2}\right)_{m_1+m_2}} y_1^{m_1} y_2^{m_2}$$

Here we set $\mu_\pm = -(l-2\pm\sqrt{\nu_1^2-2l+1})/2$.

(ii) *The analytic solution of* (8.5) *and* (8.6) *has the following integral representation up to scalar.*

$$(8.10) \qquad \psi_{01}(y_1,\ y_2) = \int_0^1 {}_2F_1\left(\mu_+,\ \mu_-\ ;\ \frac{3+k-l}{2}\ ;\ ty_1+(1-t)y_2\right) t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt$$

Here ${}_2F_1$ *is the classical Gaussian hypergeometric function which is analytic around the origin.*

Proof. This theorem is a consequence of Lemma 8.4 and Lemma 8.6 for the case

$$A=-l,\ B_1=3/2,\ B_2=1/2,\ C=(k-l+3)/2,\ \lambda=\{\nu_1^2-(l-1)^2-2\}/4.$$

$\square$

Remark 8.10. When $\nu_1 = \pm\sqrt{l^2+6l+3}$ (this means $\mu_-=2$), the solution given above is Appell's hypergeometric function

$$F_1(-l,\ 3/2,\ 1/2,\ (3+k-l)/2\ ;\ y_1,\ y_2).$$

Remark 8.11. The hypergeometric series defind by (8.7) or (8.9) is denoted by

$$F_{10}\begin{pmatrix} a & b & c_1 & c_2 \\ d & e & & \end{pmatrix}\ ;\ y_1,\ y_2 \Bigg)$$

$$= \sum_{m,\geq 0} \frac{(a)_{m_1+m_2}(b)_{m_1+m_2}(c_1)_{m_1}(c_2)_{m_2}}{m_1!\,m_2!\,(d)_{m_1+m_2}(e)_{m_1+m_2}} y_1^{m_1} y_2^{m_2}$$

in [T].

Lemma 8.12.

(8.11)
$$F_{10}\left(\begin{matrix} a & b & c_1 & c_2 \\ c_1+c_2 & e \end{matrix}\ ;\ y_1,\ y_2\right)$$
$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{m!(e)_m}y_2^m {}_2F_1(-m,\ c_1\ ;\ c_1+c_2\ ;\ 1-y_1/y_2).$$

holds.

   Proof. F_{10} can be written as

$$F_{10}\left(\begin{matrix} a & b & c_1 & c_2 \\ d & e \end{matrix}\ ;\ y_1,\ y_2\right)$$
$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(d)_m(e)_m}y_2^m\sum_{n=0}^{m}\frac{(c_1)_n(c_2)_{m-n}}{(m-n)!\,n!}(y_1/y_2)^n.$$

Since

$$\frac{1}{(m-n)!}=\frac{(-1)^n(-m)_n}{m!}\ \text{for}\ n\leq m,$$
$$(c_2)_{m-n}=\frac{\Gamma(c_2+m-n)}{\Gamma(c_2)}=\frac{(-1)^{m+n}\pi}{\sin c_2\pi\Gamma(c_2)\Gamma(1-c_2-m+n)}$$

hold, we have

$$\sum_{n=0}^{m}\frac{(c_1)_n(c_2)_{m-n}}{(m-n)!\,n!}(y_1/y_2)^n$$
$$=\frac{\pi(-1)^m}{\sin c_2\pi\Gamma(c_2)m!}\sum_{n=0}^{m}\frac{(-m)_n(c_1)_n}{n!\,\Gamma(1-c_2-m+n)}(y_1/y_2)^n$$
$$=\frac{\pi(-1)^m}{\sin c_2\pi\Gamma(c_2)\Gamma(1-c_2-m)m!}{}_2F_1(-m,\ c_1\ ;\ 1-c_2-m\ ;\ y_1/y_2).$$

Using the relation

$${}_2F_1(-m,\ c_1\ ;\ 1-c_2-m\ ;\ y_1/y_2)$$
$$=\frac{\Gamma(1-c_1-c_2)\Gamma(1-c_2-m)}{\Gamma(1-c_2)\Gamma(1-c_1-c_2-m)}{}_2F_1(-m,\ c_1\ ;\ c_1+c_2\ ;\ 1-y_1/y_2)$$

for any non-negative integer m,

$$\sum_{n=0}^{m}\frac{(c_1)_n(c_2)_{m-n}}{(m-n)!\,n!}(y_1/y_2)^n$$
$$=\frac{(-1)^m\Gamma(1-c_1-c_2)}{m!\,\Gamma(1-c_1-c_2-m)}{}_2F_1(-m,\ c_1\ ;\ c_1+c_2\ ;\ 1-y_1/y_2)$$
$$=\frac{(c_1+c_2)_m}{m!}{}_2F_1(-m,\ c_1\ ;\ c_1+c_2\ ;\ 1-y_1/y_2)$$

holds. Thus we have

$$F_{10}\begin{pmatrix} a & b & c_1 & c_2 \\ d & e & \end{pmatrix} \; ; \; y_1, y_2 \end{pmatrix}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(d)_m (e)_m} y_2^m \frac{(c_1+c_2)_m}{m!} {}_2F_1(-m, c_1; c_1+c_2; 1-y_1/y_2).$$

Setting $d = c_1 + c_2$, we have (8.11). $\qquad\qquad\square$

§9. Appell's Hypergeometric Functions

As is seen in the previous section, the kernel ψ of

$$(9.1) \quad P = \sum_{i=1}^{2} y_i (y_i - 1) \partial_{y_i}^2$$

$$+ \left\{ (A+B_1-B_2+1)y_1 + B_2 - C + 2B_2 \frac{y_1(y_1-1)}{y_1-y_2} \right\} \partial_{y_1}$$

$$+ \left\{ (A-B_1+B_2+1)y_2 + B_1 - C - 2B_1 \frac{y_2(y_2-1)}{y_1-y_2} \right\} \partial_{y_2} - \lambda$$

and

$$(9.2) \qquad\qquad Q = \partial_{y_1} \partial_{y_2} - B_2 \frac{1}{y_1-y_2} \partial_{y_1} + B_1 \frac{1}{y_1-y_2} \partial_{y_2}$$

has the integral representation such as

$$(9.3) \qquad \psi(y_1, y_2) = \int_0^1 {}_2F_1(\mu_+, \mu_-; C; ty_1 + (1-t)y_2) t^{B_1-1}(1-t)^{B_2-1} dt,$$

where $\mu_\pm$ are roots of $x^2 - (A+B_1+B_2)x - \lambda = 0$.

Since the Gaussian hypergeometric function has the following integral representation for $c_0 = \Gamma(C)/\Gamma(C-\mu_+)\Gamma(\mu_+)$:

$$_2F_1(\mu_+, \mu_-; C; z) = c_0 \int_1^\infty s^{\mu_- - C}(1-s)^{C-\mu_++1}(s-z)^{-\mu_-} ds,$$

we have

$$\psi(y_1, y_2)$$
$$= c_0 \int_0^1 dt \int_1^\infty ds \cdot s^{\mu_- - C}(1-s)^{C-\mu_++1}(s-ty_1-(1-t)y_2)^{-\mu_-} t^{B_1-1}(1-t)^{B_2-1}.$$

Setting $w_1 = 1 - y_1/y_2$, $w_2 = 1/y_2$, we obtain

$$(9.4) \quad \psi(y_1, y_2) = c_0(-w_2)^{\mu_-} \int_0^1 dt \int_1^\infty ds \cdot s^{\mu_- - C}(1-s)^{C-\mu_++1}$$
$$\times t^{B_1-1}(1-t)^{B_2-1}(1-tw_1-sw_2)^{-\mu_-},$$

whose integral part satisfies Appell's hypergeometric differential equations $R_1 F = R_2 F = 0$.

This is pointed out by H. Ochiai.

Appell's hypergeometric differential operators mentioned above are defined by

$$R_1 = w_1(1-w_1)\partial_{w_1}^2 - w_1 w_2 \partial_{w_1}\partial_{w_2}$$
$$+ \{\gamma - (\alpha+\beta+1)w_1\}\partial_{w_1} - \beta w_2 \partial_{w_2} - \alpha\beta,$$
$$R_2 = w_2(1-w_2)\partial_{w_2}^2 - w_1 w_2 \partial_{w_1}\partial_{w_2}$$
$$+ \{\gamma' - (\alpha+\beta'+1)w_2\}\partial_{w_2} - \beta' w_1 \partial_{w_1} - \alpha\beta',$$

whose analytic kernel is usually written by $F_2(\alpha \ ; \ \beta, \ \beta' \ ; \ \gamma, \ \gamma' \ ; \ w_1, \ w_2)$.

Lemma 9.1. *Let P, Q, R_1 and R_2 be as above. If we set $w_1 = 1 - y_1/y_2$, $w_2 = 1/y_2$, then we have*

(9.5) $\quad (-w_2)^{-k} \circ P \circ (-w_2)^k = \dfrac{2 - 2w_1 - 2w_2 + w_1 w_2}{w_1} R_1 + w_2 R_2$

(9.6) $\quad (-w_2)^{-k} \circ Q \circ (-w_2)^k = -\dfrac{w_2^2}{w_1} R_1,$

with

$$\begin{cases} \alpha = k, \\ \beta = B_1, \ \beta' = k+1-C, \\ \gamma = B_1 + B_2, \ \gamma' = 2k+1-A-B_1-B_2, \\ k = \mu_+ \ \text{or} \ k = \mu_-. \end{cases}$$

Proof. An easy computation.   □

As is seen in Theorem 8.7 and Theorem 8.9, the function (9.3) is a hypergeometric function

$$F_{10}\begin{pmatrix} \mu_+ & \mu_- & B_1 & B_2 \\ B_1+B_2 & C & & \end{pmatrix} \ ; \ y_1, \ y_2 \end{pmatrix}.$$

On the other hand, the kernel of R_1 and R_2 for γ, $\gamma' \notin \mathbf{Z}$ is

$$CF_2(\alpha \ ; \ \beta, \ \beta' \ ; \ \gamma, \ \gamma' \ ; \ w_1, \ w_2)$$
$$\oplus C(-w_1)^{1-\gamma} F_2(\alpha+1-\gamma \ ; \ \beta+1-\gamma, \ \beta' \ ; \ 2-\gamma, \ \gamma' \ ; \ w_1, \ w_2)$$
$$\oplus C(-w_2)^{1-\gamma'} F_2(\alpha+1-\gamma' \ ; \ \beta, \ \beta'+1-\gamma' \ ; \ \gamma, \ 2-\gamma' \ ; \ w_1, \ w_2)$$
$$\oplus C(-w_1)^{1-\gamma}(-w_2)^{1-\gamma'}$$
$$\times F_2(\alpha+2-\gamma-\gamma' \ ; \ \beta+1-\gamma, \ \beta'+1-\gamma' \ ; \ 2-\gamma, \ 2-\gamma' \ ; \ w_1, \ w_2).$$

Then, from Lemma 9.1, there exist constants $p_1, p_2, p_3, p_4 \in C$ such that

$$(9.7) \quad (-y_2)^k F_{10}\begin{pmatrix} \mu_+ & \mu_- & B_1 & B_2 \\ B & C & & \end{pmatrix} ; \quad y_1, y_2 \end{pmatrix}$$

$$= p_1 F_2(k ; B_1, k-C+1 ; B, 2k-D+1 ; 1-y_1/y_2, 1/y_2)$$
$$+ p_2(y_1/y_2-1)^{1-B}$$
$$\times F_2(k-B+1 ; -B_2+1, k-C+1 ; 2-B, 2k-D+1 ; 1-y_1/y_2, 1/y_2)$$
$$+ p_3(-y_2)^{2k-D} F_2(D-k ; B_1, D-C-k+1 ; B, D-2k+1 ; 1-y_1/y_2, 1/y_2)$$
$$+ p_4(y_1/y_2-1)^{1-B}(-y_2)^{2k-D}$$
$$\times F_2(A-k+1 ; -B_2+1, D-C-k+1 ; 2-B, D-2k+1 ; 1-y_1/y_2, 1/y_2)$$

for $B = B_1 + B_2$ and $D = A + B$. Constants p_1, p_2, p_3 and p_4 are determined as follows.

Theorem 9.2. *Appell's hypergeometric function F_2 and hypergeometric function F_{10} have the following relation for $\mu_+ - \mu_-$, $B \bar{\in} \mathbf{Z}$ and $C \bar{\in} \{0, -1, -2, \cdots\}$*

$$(9.8) \quad F_{10}\begin{pmatrix} \mu_+ & \mu_- & B_1 & B_2 \\ B & C & & \end{pmatrix} ; \quad y_1, y_2 \end{pmatrix}$$

$$= \frac{\Gamma(\mu_- - \mu_+)\Gamma(C)}{\Gamma(\mu_-)\Gamma(C-\mu_+)}(-y_2)^{-\mu_+}$$
$$\times F_2(\mu_+ ; B_1, \mu_+ - C+1 ; B, \mu_+ - \mu_- + 1 ; 1-y_1/y_2, 1/y_2)$$
$$+ \frac{\Gamma(\mu_+ - \mu_-)\Gamma(C)}{\Gamma(\mu_+)\Gamma(C-\mu_-)}(-y_2)^{-\mu_-}$$
$$\times F_2(\mu_- ; B_1, \mu_- - C+1 ; B, \mu_- - \mu_+ + 1 ; 1-y_1/y_2, 1/y_2).$$

Here $(-y_2)^{-\mu_+}$ and $(-y_2)^{-\mu_-}$ are defined in $|\arg(-y_2)| < \pi$.

Proof. Setting $f(z_1, z_2) = \sum_{m=0}^{\infty} \frac{(\mu_+)_m(\mu_-)_m}{m!(C)_m} z_1^m {}_2F_1(-m, B_1 ; B ; z_2)$, we get

$$f(z_1, z_2) = \sum_{m \geq n \geq 0}^{\infty} \frac{(\mu_+)_m(\mu_-)_m(-m)_n(B_1)_n}{m!(C)_m n!(B)_n} z_1^m z_2^n$$

$$= \sum_{m \geq n \geq 0}^{\infty} \frac{(\mu_+)_m(\mu_-)_m(B_1)_n}{n!(m-n)!(B)_n(C)_n} z_1^m (-z_2)^n$$

Putting $m = n + k$, we have

$$\sum_{m \geq n \geq 0}^{\infty} \frac{(\mu_+)_m (\mu_-)_m (B_1)_n}{n!\,(m-n)!\,(B)_n (C)_n} z_1^m (-z_2)^n$$

$$= \sum_{k,n=0}^{\infty} \frac{(\mu_+)_{k+n} (\mu_-)_{k+n} (B_1)_n}{n!\,k!\,(B)_n (C)_{k+n}} z_1^{k+n} (-z_2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(\mu_+)_n (\mu_-)_n (B_1)_n}{n!\,(B)_n (C)_n} z_1^n (-z_2)^n \sum_{k=0}^{\infty} \frac{(\mu_+ + n)_k (\mu_- + n)_k}{k!\,(C+n)_k} z_1^k$$

$$= \sum_{n=0}^{\infty} \frac{(\mu_+)_n (\mu_-)_n (B_1)_n}{n!\,(B)_n (C)_n} z_1^n (-z_2)^n {}_2F_1(\mu_+ + n,\ \mu_- + n\ ;\ C+n\ ;\ z_1)$$

Using formulas

$$
{}_2F_1(\alpha,\beta\ ;\ \gamma\ ;\ z_1) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-z_1)^{-\alpha}{}_2F_1(\alpha,\alpha+1-\gamma\ ;\ \alpha+1-\beta\ ;\ 1/z_1)
$$

$$
+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z_1)^{-\beta}{}_2F_1(\beta,\beta+1-\gamma\ ;\ \beta+1-\alpha\ ;\ 1/z_1),
$$

$$
F_2(\alpha\ ;\ \beta,\ \beta'\ ;\ \gamma,\ \gamma'\ ;\ z_1,\ z_2) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m!\,(\gamma)_m} z_1^m {}_2F_1(\alpha+m,\ \beta'\ ;\ \gamma'\ ;\ z_2)
$$

and (8.11), we obtain (9.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 9.3. (i) The equation (9.7) is reduced to the relation between Appell's hypergeometric functions F_1 and F_2 :

$$F_1(A\ ;\ B_1,\ B_2\ ;\ C\ ;\ y_1,\ y_2)$$

$$= \frac{\Gamma(B-A)\Gamma(C)}{\Gamma(B)\Gamma(C-A)}$$

$$\times (-y_2)^{-A} F_2(A\ ;\ B_1,\ A-C+1\ ;\ B,\ A-B+1\ ;\ 1-y_1/y_2,\ 1/y_2)$$

$$+ \frac{\Gamma(A-B)\Gamma(C)}{\Gamma(A)\Gamma(C-B)}$$

$$\times (-y_2)^{-B} F_2(B\ ;\ B_1,\ B-C+1\ ;\ B\ ;\ -A+B+1\ ;\ 1-y_1/y_2,\ 1/y_2),$$

for $A-B \not\in \mathbf{Z}$, $C \not\in \{0,\ -1,\ -2,\ \cdots\}$ when $\lambda = -AB$.

Moreover, this relation is reduced to the famous relation of the Gaussian hypergeometric function:

$$
{}_2F_1(A,\ B\ ;\ C\ ;\ x)
$$

$$= \frac{\Gamma(B-A)\Gamma(C)}{\Gamma(B)\Gamma(C-A)}(-x)^{-A}{}_2F_1(A,\ A-C+1\ ;\ A-B+1\ ;\ 1/x)$$

$$+ \frac{\Gamma(A-B)\Gamma(C)}{\Gamma(A)\Gamma(C-B)}(-x)^{-B}{}_2F_1(B,\ B-C+1\ ;\ B-A+1\ ;\ 1/x)$$

by setting $y_1 = y_2 = x$,

(ii) The equation (9.8) can be also obtained by using connection formulas of F_2 given in [T Proposition 2.1 (5)], where the relation between F_1 and F_2 is not mentioned.

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