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On Algebraic #-Cones in Topological Tensor Algebras II. Closed Hulls and Extremal Rays

By

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Abstract

The investigations on the structure of alg-# cones $\{F, \#\}$ in topological tensor algebras are continued, and they are aimed at the closed hulls and the extremal rays of such cones. Among others, it is proven that the elements of the closed hulls of a large class of alg-# cones with respect to some intermediate l.c. topologies are explicitely given by (infinite) sums of elements of $\{F, \#\}$. Furthermore, a Krein-Milman like theorem is shown for some alg# cones, i.e., it is shown that there are enough extremal rays in $\{\tilde{F}, \#\}$ so that every element of $\widetilde{\{F, \#\}}$ is an (infinite) sum of extremals of $\{\tilde{F}, \#\}$.

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§1. Introduction

The present paper is aimed at a further investigation of algebraic #-cones (alg-# cones) in tensor algebras $E_{\otimes}$ such as their closed hulls and their extremal rays. It is the second paper of a series of papers on the structure of alg-# cones. In the following the first paper of this series is referred to as [I], and theorems and formulae taken from [I] are denoted, e.g., by [I: Theorem 3.3] and [I: (4)], respectively. In a subsequent paper the structure of linear functionals that are positive on some given alg-# cone will be considered.

Let us recall that the concept of alg-# cones was introduced as a generalization of the cones such as, e.g., the well-known cone of positivity, the cone of reflexion positivity ([28], [33]) and certain cones of α-positivity ([21], [3], [19], [20]), all of which are of some special interest within the algebraic approach to general (axiomatic) quantum field theory (QFT). Let us further mention that within the present investigations the basic space $E[t]$ of the tensor algebra $E_{\otimes}$ is chosen as general as possible so that applications to general QFT such as QFT on $\mathscr{S}_{\alpha}^{\beta}$-spaces ([27], [34]) and QFT on curved space-time ([37], [30], [10]) are also covered.

One motivation for considering the closed hull of alg-# cones comes from general QFT given in terms of Wightman functionals $\mathscr{W}$. In order to get continuity of $\mathscr{W}$ it was first observed by Wyss ([39]) and further discussed by Yngvason ([40]) that it is sufficient to demand positivity of $\mathscr{W}$ on the closed hull $\overline{\mathscr{S}_{\otimes}^{+}}$ of the cone of positivity of the completed tensor algebra $\widetilde{\mathscr{S}}_{\otimes}$ over the basic space $\mathscr{S}(\mathbb{R}^4)$ (Schwartz-space of basic (rapidly dimishing) functions). Afterwards some papers on the closed hull of the cone of positive elements $E_{\otimes}^{+}$ in a tensor algebra $E_{\otimes}$ were published ([5], [12], [13], [32], [1]).

The present paper extends those results in the following ways: i) The cone of positive elements $E_{\otimes}^{+}$ is replaced by the more general concept of alg-# cones $\{F, \#\}$ in $E_{\otimes}$, ii) the closed hulls of finitely many alg-# cones are considered, iii) the results of the present paper apply to more classes of topological tensor algebras.

Let us mention that i), ii) are motivated by respectively, the Osterwalder-Schrader and Hegerfeldt axioms (cf [28], [11], Example 3.11, below). With respect to iii), let us stress that all the investigations in the present paper consider as well graded as non-graded topologies (see [I, §1]). Concerning graded topologies, it seems to be of interest also to consider l.c. topologies $\tau(\Gamma, ., .,)$ with $\Gamma \neq \mathbf{R}^{N^*}$ since : 1) the interesting considerations on non-commutative moment problems, which are due to Dubois-Violette, Alcantara and Yngvason ([9, 2]), are based on l.c. topologies τ such that the τ-completed hull $\widetilde{E_\otimes^+}$ satisfies

$$\widetilde{E_\otimes^+} \neq \overline{E_\otimes^+},$$

where the closed hull $\overline{E_\otimes^+}$ is considered in $E_\otimes$, 2) the investigations of Dubin and Hennings ([8]) are especially concerned with such topologies.

For studying the closed hulls of alg-# cones, the key is to distinguish locally convex (l.c.) topologies τ (see Definition 3.1) such that

$$\overline{\{F, \#\} \cap E^n}^{\tau} = \overline{\{F, \#\}}^{\tau} \cap E^n \tag{1}$$

n = 0, 1, 2,..., where E^n denotes the n-th truncated tensor algebra (see Chapter 2.1). Using (1), assertions on the sequence closure of $\{F, \#\}$ and on the equality of the closed hulls with respect to several l.c. topologies are implied (see Corollary 3.2, Proposition 3.3). Using dual spaces, (1) is characterized by the property that the set of all canonical projections of positive and τ-continuous linear functionals from $(E_\otimes)'$ in $(E^{2n})'$ is weakly dense in the set of all the $(\{F, \#\} \cap E^{2n})$-positive and τ-continuous linear functionals on E^{2n} (cf Lemma 3.6, below). As a consequence, a generalizations of Yngvason's density lemma is obtained (cf [41, p. 17], Cor. 3.7, below).

Further, if the basic space $E[t]$ is a nuclear (LF)-space and if the defining mapping # of $\{F, \#\}$ is t-continuous on the basic $E[t]$ and satisfies a further algebraic condition (see (5) below), then the closed hull of $\{F, \#\}$ is explicitely given for a large family of l.c. topologies by

$$\overline{\{F, \#\}}^{\varepsilon_\infty} = \overline{\{F, \#\}}^{\varepsilon_\otimes} = \overline{\{F, \#\}}^{\tilde{\varepsilon}} = \{\sum_{t=1}^{\infty} f^{(t)\#} f^{(t)} ; f^{(t)} \in F\},$$

where $\varepsilon_\otimes$, ε_∞ and $\tilde{\varepsilon}$ are given below in §§ 2.2 and 3.2, espectively.

Furthermore, questions related to the Theorem of Krein-Milman and the representation of convex cones as the closed convex hull of their extremal rays are considered. Because alg-# cones do not contain (topological) interior points with respect to all the intermediate topologies τ, $\varepsilon_P \prec \tau \prec \iota_\otimes$ (see [I, Theorem 2.3e)]), the well-known theory of Krein-Milman and Klee ([23, §25]) does not apply. The special structure of alg-# cones however allows to prove similar results. Along these lines it will be shown that under rather mild topological assumption (the truncated tensor algebras $\tilde{E}^n[\varepsilon^n]$, $n=1, 2, 3, \ldots$, have to be nuclear semi-Montel spaces) for a large class of alg-# cones $\{F, \#\}$ there are enough extremal rays in $\{\tilde{F}, \#\}$ so that each $0 \neq k \in \overline{\{F, \#\}}^{\varepsilon_\otimes}$ can be given as a (possibly infinite) sum of extremal elements of $\{\tilde{F}, \#\}$.

Let us mention that the last result is a generalization of a theorem due to Alcantara ([1, Theorem 2]) since our assumptions are weaker and we prove that the extremal decomposition under consideration is given by a sum, and not only by a weak integral as in [1], [35].

The pattern of the present note is as follows. For the reader's convenience, the prerequisites needed for the following investigations are briefly recalled in §2. While the definition of alg-# cones and some of their properties frequently used in the following are given in §2.1, l.c. topologies are introduced on $E_\otimes$ in §2.2. §3 is devoted to the closed hull of alg-# cones. In §3.1 the basic definition of the closure condition with respect to some alg-# cone is given, and then the theorem on the closed hull of alg-# cones in truncated tensor algebras (Theorem 3.1) is stated and shown. Some immediate consequences of Theorem 3.1 and a discussion of its assumptions are also contained in §3.1. An explicite description of the elements of the closed hulls for a large class of a alg-# cones is given in §3.2 (Theorems 3.10, 3.12). The closed hull of finitely many alg-# cones is investigated in §3.3, and it is shown that under rather mild assumptions the closed hull of the convex hull of finitely many alg-# cones coincides with the convex hull of the closed hulls of the alg-# cones considered (Theorem 3.13). In Example 3.14 the closed hull of the convex hull of the cone of positivity $\widehat{\mathcal{S}}^+_\otimes$ and the cone of reflexion positivity are considered in $\widetilde{\mathcal{S}}_\otimes$, and it is shown that Theorem 3.13 applies to this example being of some interest within the axiomatic approach to Euclidean QFT ([11], [28]). §4 is devoted to an investigation of the extremal rays of involutive cones. In the case of homogeneous elements of $E_\otimes$ a characterization of the extremal rays is given in Lemma 4.2. Under assumptions listed above the main theorem of that § (Theorem 4.4) states that there are enough extremal rays in some given alg-# cones $\{F, \#\}$ so that the $\varepsilon_\otimes$-completed hull of $\{F, \#\}$ coincides with a (possibly infinite) sum of extremal elements of $\{\tilde{F}, \#\}$. Furthermore, those decompositions are uniquely defined, if and only if dim $(F)=1$. These results are finally applied to the cone of positivity $\mathbb{C}^+_\otimes$ of the tensor algebra $\mathbb{C}_\otimes$ (algebra of polynomials) in §4.2.

§2. Preliminaries

Let us briefly recall some of the prerequisites needed in the following. For details the reader is refered to [I].

§2.1. Some Facts on Alg-# Cones

For the following let us be given a vector space E over the field of complex numbers $\mathbb{C}$, and let

$$E_n = E \otimes E \otimes \ldots \otimes E \text{ (n copies)}$$

stand for the n-fold algebraic tensor product of E by itself, n $\in \mathbb{N}$. The tensor algebra $E_\otimes$ over the basic space E is then defined by

$$E_\otimes = \mathbb{C} \oplus E_1 \oplus E_2 \oplus \ldots \text{(direct sum)},$$

i.e., the elements $f \in E_\otimes$ are terminating sequences

$$f = (0,\ldots,0, f_M, f_{M+1},\ldots,f_N, 0, 0,\ldots), \tag{2}$$

where $f_n \in E_n$, n$=0,1,2,\ldots$, ($E_0, = \mathbb{C}$, $E_1 = E$), and f_n is called the *n-th homogeneous component* of f. If $f_M \neq 0$, $f_N \neq 0$ in (2), then

$$\text{grad}(f) = M \text{ and } \text{Grad}(f) = N$$

are called the *lower grade* and the *upper grade* of f, respectively. Recall also that $E_\otimes$ becomes on (associative) *-algebra with unity $\mathbb{1} = (1,0,0,0,\ldots)$, where the algebraic operations are defined as usual (see, e.g., [I]).

For the following let $Q_n : E_\otimes \to \oplus_{i=0}^n E_i \ (\equiv E^n)$ denote the canonical projections, where the *truncated tensor algebras* E^n are considered as subspaces of $E_\otimes$, i.e,

$$Q_n(f) = (f_0,\ldots,f_n, 0, 0, \ldots) \in E^n \subset E_\otimes,$$

and f is taken from (2). For each $f_n \in E_n$ let

$$\check{f}_n = (0,\ldots,f_n, 0, 0,\ldots) \in E_\otimes.$$

Let us recall the definition of the class of alg-# cones which fits the algebraic structure of $E_\otimes$ very well. Let us now be given a subspace

$$F=\oplus_{i=0}^{\infty} F_i$$

of $E_\otimes$, where $F_i \subset E_i$. Further let us consider an antilinear mapping $\# : E_\otimes \to E_\otimes$ which satisfies

$$f^{\#\#}=f, \tag{3}$$
$$(Q_n f)^{\#}=Q_n(f^{\#}) \tag{4}$$

for all $f \in E_\otimes$, $n=0, 1, 2, \ldots$. Let us put

$$\{F, \#\}=\left\{\sum_{i=1}^{M} f^{(i)\#} f^{(i)};\ f^{(i)} \in F,\ M \in \mathbb{N}\right\}.$$

Such cones $\{F, \#\}$ were called *alg-# cones* in [I]. Let us distinguish an interesting class of alg-# cones, where the above mapping # also satisfies

$$(x^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(n)})^{\#}=x^{(\pi(1))\#} \otimes x^{(\pi(2))\#} \otimes \ldots \otimes x^{(\pi(n))\#} \tag{5}$$

for all $x^{(1)},\ldots,x^{(n)} \in E$, and $\pi(.)$ is a fixed permutation of $\{1,2,\ldots,n\}$, $n=2,3,\ldots$

If # additionally satisfies

$$(fg)^{\#}=g^{\#} f^{\#} \tag{6}$$

for all $f, g \in E_\otimes$, then $\{F, \#\}$ was denoted as *involutive cone*. For important and interesting examples of alg-# cones and some of their fundamental properties the reader is referred to [I].

For proving deeper results on the structure of alg-# cones, estimates between the homogeneous components of their elements were established in [I]. Along these lines it is important to distinguish the class of (in general nonlinear) functionals $\pounds : E_\otimes \to \mathbb{C}$ satisfying *property* (A) with respect to some given alg-# cone $\{F, \#\}$ and some sequence of reals $(\omega_j)_{j=0}^{\infty}$, $\omega_j>0$ (see [I : Definition 3.1]). Remember also that there are interesting examples of functionals $\pounds$ satisfying property (A) such as $\{F, \#\}$-positive linear functionals on $E_\otimes$, and ε_∞ — (resp. σ_∞) — continuous semi norms on $E_\otimes$ (see [I : Examples 3.4]). For each $\sum_{i=1}^{M} f^{(i)\#} f^{(i)} \in \{F, \#\}$, $f^{(i)} \in F(i=1, 2, \ldots M)$, the matrix $\mathbb{A}=(a_{rs})_{r,s=0}^{\infty}$ was considered, where

$$a_{rs}=\sum_{i=1}^{M} f_r^{(i)\#}\otimes f_s^{(i)}.$$

Then, the diagonal elements

$$L_n=\sqrt{\pounds(\check{a}_{nn})}$$

are of some special interest because there are i) estimates from below and ii) estimates from above.

ad i). Recall the definitions of the universal estimate-sequences

$$(\beta_m^n(c,(\omega_i)))_{m=n}^{\infty},\ (\beta_m(c,(\omega_i)))_{m=1}^{\infty}$$

introduced in [I : (18)...(20)]. Along these lines remember also that if $c\leq 1$, $\omega_i\geq 1$ $(i=0, 1, 2,...)$, then

$$\beta_m\geq\frac{1}{2}(m^{m^m})^{-1}$$

$m=1,2,...,$ (cf [4, eq. (1.3)]).

Then there is the following theorem which is the key for the following considerations on the closed hulls of alg-# cones.

Theorem 2.1. *Let us be given an alg-# cone* $\{F, \#\}$ *and a functional* $\pounds$ *satisfying* (A) *with respect to* $\{F, \#\}$ *and a sequence* $(\omega_i)_{i=0}^{\infty}$, $\omega_i>0$, *and an element* $\sum_{i=1}^{M} f^{(i)\#}f^{(i)}\in\{F, \#\}$, $f^{(i)}\in F$ $(i=1,2,...,M)$ *such that* $L_n\leq 1$ *for all* $n=0, 1, 2, ...$ *If there are an even index* $n_0=2s_0$ *and two constants* $c>0$, $1>\delta>0$ *such that*

$$c=(L_{s_0})^2\geq\delta^{-1}|\pounds(\sum_{r+s=2s_0}\check{a}_{rs})|,$$

then there is an even index $2m>n_0$ *with*

$$2|\pounds(\sum_{r+s=2m}\check{a})_{rs})|>\beta_{2m}^{(n_0)}((1-\delta)c,(w_i))\geq\beta_{2m}((1-\delta)c,(w_i)).$$

Proof. See [I : Theorem 3.3c)].

ad ii). Let us be given two sequences $(\omega_n)_{n=0}^{\infty}$, $(d_n)_{n=0}^{\infty}$ of reals ω_n, $d_n>0$, and consider the diagonalized matrix $\mathbb{D}=\operatorname{diag}[d_0, d_1, d_2,...]$. Then, let $\mathscr{A}_{(w,d)}$ denote the set of all the sequences $(\alpha_n)_{n=0}^{\infty}$ with $\alpha_n\geq 0$, $\alpha_{2s+1}=0$ $(s=0, 1, 2, ...)$ such that the matrix inequality $\mathbb{G}\geq\mathbb{D}$ holds, where $\mathbb{G}=(g_{ij})_{i,j=0}^{\infty}$ and

$$g_{ij}=\begin{cases} \alpha_{2i} : i=j \\ -\omega_i\omega_j\alpha_{i+j} : i\neq j. \end{cases}$$

For every sequence $(\alpha_n)\in\mathscr{A}_{(\omega,d)}$ let us introduce the semi norm

$$f\to\|f\|_{(\pounds,(\alpha_n))}=\sum_{n=0}^{\infty}\alpha_{2n}|\pounds(\check{f}_{2n})|,$$

$f\in E_{\otimes}$. In [I : 3.6] there it is shown that $\mathscr{A}_{(\omega,d)}\neq\emptyset$.

Theorem 2.2. *Let us be given an alg-# cone* $\{F, \#\}$, *sequences* (ω_n), (d_n) *as above and a functional* $\pounds$ *satisfying (A) with respect to* $\{F, \#\}$ *and* (ω_n). *If* $(\alpha_n)_{n=0}^{\infty}\in\mathscr{A}_{(\omega,d)}$, *then*

$$\sum_{n=0}^{\infty} d_n(L_n)^2\leq\|\sum_{i=1}^{M} f^{(i)\#}f^{(i)}\|_{(\pounds,(\alpha_n))}$$

for $f^{(i)}\in F$, $i=1, 2,...,M$ $(M\in\mathbb{N})$.

Proof. See [I : Theorem 3.5a)].

§2.2. Locally Convex Topologies on Tensor Algebras

Assume now that $E[t]$ is an l.c. vector space. Then, there is the following concept to introduce l.c. topologies on $E_{\otimes}$ in a canonical way, see [I , §1], [18]. First, on E_n let us consider the class of *compatible* l.c. topologies t_n, i.e.,

$$\varepsilon_n \prec t_n \prec \iota_n,$$

where ε_n and ι_n denote the *injective* and *inductive* topologies on the tensor products E_n (n=2, 3, ...), respectively (cf [23]). (For any two l.c. topologies τ, τ', let $\tau \prec \tau'$ denote that τ' is finer (not necessarily strictly finer) than τ.) Secondly, let $E_\otimes$ be furnished with an *intermediate* l.c. topology τ, i.e.,

$$\tau_{|E_m} = t_m \, (m=0, 1, 2, \ldots),$$

where $\tau_{|E_n}$ denotes the topology inherited by τ on the subspace $\check{E}_m \subset E_\otimes$, $t_1 = t$ and t_0 denotes the Euclidean topology on $E_0 = \mathbb{C}$. Further, let us also put

$$\tau_{|E^m} = t^m \, (m=0, 1, 2, \ldots).$$

Let us now be given $E_n[t_n]$, $n=0, 1, 2, \ldots$. In order to define intermediate topologies τ on $E_\otimes$, the algebraic structure of $E_\otimes$ defines the weakest intermediate and the finest intermediate l.c. topology on $E_\otimes$ denoted by $\tau_{P,(t_n)}$ and $\tau_{\otimes,(t_n)}$, respectively. Recall that $\tau_{\otimes,(t_n)}$ is the topology of the direct sum $\bigoplus_{n=0}^\infty E_n[t_n]$, and $\tau_{P,(t_n)}$ is the topology which is induced by the topology of the direct product $\prod_{n=0}^\infty E_n[t_n]$ on its subspace $E_\otimes$. Hence, an l.c. topology τ on $E_\otimes$ is an intermediate one, if and only if

$$\tau_{P,(t_n)} \prec \tau \prec \tau_{\otimes,(t_n)}.$$

If $t_m = \varepsilon_m$ $(m=2, 3, 4, \ldots)$, then let us write ε_P and $\varepsilon_\otimes$ instead of $\tau_{P,(t_n)}$ and $\tau_{\otimes,(t_n)}$, respectively.

Let us recall the definition of the important topologies of general character ε_∞, π_∞ introduced by G. Lassner in the special case of $\mathscr{S}_\otimes$ ([24]). Let $\mathscr{P}(t) = \{p_\alpha ; \alpha \in \mathrm{A}\}$, A is a directed set of indices, by a system of semi norms defining t on E. Recall that

$$\mathscr{P}(\varepsilon_n) = \{p_\alpha^{(n)} \equiv \underbrace{p_\alpha \otimes_\varepsilon \cdots \otimes_\varepsilon p_\alpha}_{n \text{ copies}} \; ; \alpha \in \mathrm{A}\}$$

defines ε_n on E_n. Consider now the semi norms

$$p_{(\gamma_n),\alpha}(f) = \sum_{n=0}^\infty \gamma_n p_\alpha^{(n)}(f_n),$$

$$p_{(\gamma_n),(\alpha_n)}(f) = \sum_{n=0}^\infty \gamma_n p_{\alpha_n}^{(n)}(f_n),$$

$$q_{n,\alpha}(f) = p_\alpha^{(n)}(f_n),$$

where $f = (f_0, f_1, \ldots, f_N, 0, 0, \ldots) \in E_\otimes$, $p_\alpha^{(0)}(f_0) = |f_0|$, $(\gamma_n) \in \mathbf{R}_+^{\mathrm{N}^*}$ (set of all the sequences of positive reals). The topologies ε_P, ε_∞, $\varepsilon_\otimes$ are then explicitely given by the following systems of semi norms

$$\mathscr{P}(\varepsilon_P) = \{q_{n,\alpha}\,;\, \alpha \in \mathrm{A},\ n \in \mathbb{N}^*\}$$
$$\mathscr{P}(\varepsilon_\infty) = \{p_{(\gamma_n),\alpha}\,;\, (\gamma_n) \in \mathbb{R}_+^{\mathbb{N}^*},\ \alpha \in \mathrm{A}\},$$
$$\mathscr{P}(\varepsilon_\otimes) = \{p_{(\gamma_n),(\alpha_n)}\,;\, (\gamma_n) \in \mathbb{R}_+^{\mathbb{N}^*},\ \alpha \in \mathrm{A}^{\mathbb{N}^*}\},$$

where $\mathrm{A}^{\mathbb{N}^*}$ denotes the set of all the sequence $(\alpha_n)_{n=0}^\infty$, $\alpha_n \in \mathrm{A}$. Obviously,

$$\varepsilon_P \prec \varepsilon_\infty \prec \varepsilon_\otimes.$$

Considering the *projective* topologies π_n on E_n, $n=2, 3,\ldots$, the definitions of π_P, π_∞, $\pi_\otimes$ are analogously. Along these lines recall that nuclearity of the basic space $E[t]$ implies $\varepsilon_n = \pi_n (n=2, 3, \ldots)$, $\varepsilon_P = \pi_P$, $\varepsilon_\infty = \pi_\infty$, $\varepsilon_\otimes = \pi_\otimes$.

Let us recall that the topology ε_∞ is of some special interest since it is well adopted as well i) to the algebraic structure of $E_\otimes$ (see Lemma 2.3) as ii) to the semi-ordering induced by an alg-# cone $\{F, \#\}$ (e.g., see [I, Example 4.6, Remark a)]). The interplay between i) and ii) is reflected by Lemma 2.4 used in the sequal. Let us mention that Lemma 2.3 a) was first shown for $E_\otimes = \mathscr{S}_\otimes$ in [24], and Lemma 2.4 under the additional assumption of the nuclearity of $E[t]$ in [1, Lemma 1 (iv)].

Lemma 2.3. a) *The multiplication*

$$m : E_\otimes[\varepsilon_\infty] \times E_\otimes[\varepsilon_\infty] \to E_\otimes[\varepsilon_\infty],$$

$m(f, g) = fg$, $f, g \in E_\otimes$, is jointly continuous.
b) *The linear mappings*

$$M : E_\otimes[\pi_\infty] \otimes E_\otimes[\pi_\infty] \to E_\otimes[\pi_\infty \otimes_\pi \pi_\infty],$$
$$M_n : E^n[\pi^n] \otimes E^n[\pi^n] \to E^{2n}[\pi^n \otimes_\pi \pi^n]$$

are continuous, where $\pi^n = \pi_{\otimes|E^n} = \pi_{P|E^n}$, $M_n = M_{|E^n}$, $M(f \otimes g) = fg$.

Proof. a) : Noticing that for each sequence $(\gamma_n)_{n=0}^\infty \in \mathbb{R}_+^{\mathbb{N}^*}$ there is some $(\delta_n)_{n=0}^\infty \in \mathbb{R}_+^{\mathbb{N}^*}$ satisfying

$$\gamma_n \leq \min_{k+l=n} \{\delta_k \delta_l\},$$

$n=0, 1, 2,\ldots$, the assertion under consideration follows from

$$p_{(\gamma n),\alpha}(fg)=\sum_{n=0}^{\infty}\gamma_n p_n^{(\alpha)}\Big(\sum_{k+l=n} f_k\otimes g_l\Big)\le\sum_{n=0}^{\infty}\gamma_n\sum_{k+l=n}p_k^{(\alpha)}(f_k)\,p_l^{(\alpha)}(g_l)$$
$$\le\sum_{n=0}^{\infty}\sum_{k+l=n}(\delta_k p_k^{(\alpha)}(f_k))\,(\delta_l p_l^{(\alpha)}(g_l))=p_{(\delta n),\alpha}(f)\,p_{(\delta n),\alpha}(g),$$

$f, g\in E_{\otimes}$.

b): Let us be given $\mathscr{P}(t)=\{p_\alpha\,;\,\alpha\in \mathrm{A}\}$, A is a directed set of indices, and

$$\mathscr{P}(\pi_n)=\{p_\alpha^{(n)}\equiv\underbrace{p_\alpha\otimes_\pi\ldots\otimes_\pi p_\alpha}_{n\ copies}\,;\,\alpha\in\mathrm{A}\}$$

defining t on E and π_n on E_n, respectively. For each

$$p_{(\gamma n),\alpha}(f)=\sum_{n=0}^{\infty}\gamma_n p_\alpha^{(n)}(f_n),$$

$f\in E_{\otimes}$, $(\gamma_n)\in\mathbf{R}_+^{\mathbf{N}^*}$, $\alpha\in\mathrm{A}$, choose $(\gamma_n)\in\mathbf{R}_+^{\mathbf{N}^*}$ as in a). Considering $\zeta=\sum_{i<\infty} f^{(i)\#}\otimes g^{(i)}\in E_{\otimes}\otimes E_{\otimes}$, it follows

$$p_{(\gamma n),\alpha}(M(\zeta))=\sum_{r=0}^{\infty}\gamma_r p_\alpha^{(r)}\Big(\sum_{i<\infty}\sum_{s+t=r} f_s^{(i)}\otimes g_t^{(i)}\Big)$$
$$\le\sum_{r=0}^{\infty}\sum_{i<\infty}\sum_{s+t=r}(\delta_s p_\alpha^{(s)}(f_s^{(i)}))\,(\delta_t p_\alpha^{(t)}(g_t^{(i)}))$$
$$=\sum_{i<\infty}p_{(\delta n),\alpha}(f^{(i)})\,p_{(\delta n),\alpha}(g^{(i)}),$$

and consequently the first statement under consideration is implied since

$$p_{(\gamma n),\alpha}(M(\zeta))\le\inf\Big\{\sum_{i<\infty}p_{(\delta n),\alpha}(x^{(i)})\,p_{(\delta n),\alpha}(y^{(i)});\ \zeta=\sum_{i<\infty}f^{(i)\#}\otimes g^{(i)}\in E_{\otimes}\otimes E_{\otimes}\Big\}$$
$$=(p_{(\delta n),\alpha}\otimes_\pi p_{(\delta n),\alpha})(\zeta).$$

The second statement under consideration now follows from the first one and the fact that the topologies π^n are equivalent to the topologies of the direct products $\bigoplus_{j=0}^{n} E_j[\pi_j]$.

Lemma 2.4. *Let us be given an alg-# cone $\{F, \#\}$ satisfying* (5) *in $E_\otimes$. Further let the mapping $\# : E[t] \to E[t]$ induced by the involution # on $E[t]$ be continuous. Then, there is a system of semi norms $\mathcal{P}(\varepsilon_\infty)$ defining ε_∞ such that for each $p \in \mathcal{P}(\varepsilon_\infty)$, the following hold:*

i) $p(g^\#) = p(g)$ *for all* $g \in E_\otimes$,
ii) $p(k+k') \geq p(k)$ *holds for all* $k, k' \in \overline{\{F, \#\}}^{\varepsilon_\infty}$,
iii) *there are two semi norms* $p', p'' \in \mathcal{P}(\varepsilon_\infty)$ *such that*

$$p(f) \leq \sqrt{p'(f^\# f)} \leq p''(f)$$

for all $f \in F$.

Proof. i): Using the continuity of $\# : E[t] \to E[t]$, there is a system of semi norms $\mathcal{P}(t)$ defining t and satisfying

$$p(f) = p(f^\#) \tag{7}$$

for all $f \in E$, $p \in \mathcal{P}(t)$. Considering $g_n = \sum_{i=1}^M f^{(i,1)} \otimes \ldots \otimes f^{(i,n)} \in E_n$, $M \in \mathbb{N}$, $p \in \mathcal{P}(t)$, it follows

$$\begin{aligned} p\otimes_\varepsilon \ldots \otimes_\varepsilon p(g_n^\#) &\overset{(+)}{=} \sup\{|\sum_{i=1}^M \prod_{j=1}^n T^{(j)}(f^{(i,j)\#})| \,;\, T^{(j)} \in U_p^0\} \\ &= \sup\{|\sum_{j=1}^M \prod_{j=1}^n \overline{T^{(j)\#}(f^{(i,j)})}| \,;\, T^{(j)} \in U_p^0\} \overset{(*)}{=} p\otimes_\varepsilon \ldots \otimes_\varepsilon p(g_n), \end{aligned} \tag{8}$$

where

$$T^\#(f) = \overline{T(f^\#)},$$

$T \in E'$, $f \in E$. ((+) follows from (5) and the definition of ε-topologies. (*): Due to (7) it is $T \in U_p^0$ if and only if $T^\# \in U_p^0$. (*) is now evident.) (8) implies the continuity of $\# : E_\otimes[\varepsilon_\infty] \to E_\otimes[\varepsilon_\infty]$, and i) is now evident. ii): The continuity of # also yields that the projections

$$P_j : E_\otimes[\varepsilon_\infty] \to h(E_\otimes, \#)[\varepsilon_\infty],$$

$j=1, 2$, are continuous, where the topology induced by ε_∞ on the Hermitian part $h(E_\otimes, \#) = \{g \in E_\otimes \,;\, g = g^\#\}$ is also denoted by ε_∞, and $P_1(f) = \frac{1}{2}(f + f^\#)$, $P_2(f)$

$=\frac{i}{2}(f^{\#}-f)$, $f\in E_{\otimes}$. Using the closedness of $h(E_{\otimes},\#)$ in $E_{\otimes}[\varepsilon_{\infty}]$, the topological isomorphism

$$E_{\otimes}[\varepsilon_{\infty}]\cong h(E_{\otimes},\#)[\varepsilon_{\infty}]\oplus ih(E_{\otimes},\#)[\varepsilon_{\infty}] \tag{9}$$

follows. Noticing that the Remark to [I: Example 4.6] and [31, V.3.1] imply the ε_{∞}-normality of $\overline{\{F,\#\}}^{\varepsilon_{\infty}}$ and that furthermore the closedness of $h(E_{\otimes},\#)$ in $E_{\otimes}[\varepsilon_{\infty}]$ yields $\overline{\{F,\#\}}^{\varepsilon_{\infty}}\subset h(E_{\otimes},\#)$, it is implied that there is a system of semi norms $\mathscr{P}'$ defining ε_{∞} on the real vector space $h(E_{\otimes},\#)$ and satisfying $p(k+k')\geq p(k)$ for all $p\in\mathscr{P}'$, $k, k'\in\overline{\{F,\#\}}^{\varepsilon_{\infty}}$. Because of (9) it follows that

$$\mathscr{P}(\varepsilon_{\infty})=\{\tilde{p},\tilde{\tilde{p}};\, p\in\mathscr{P}'\}$$

defines ε_{∞} on $E_{\otimes}$, where $\tilde{p}(f)=p(f+f^{\#})$, $\tilde{\tilde{p}}(f)=p(i(f^{\#}-f))$, $f\in E_{\otimes}$. It is immediate that $\mathscr{P}(\varepsilon_{\infty})$ satisfies i), ii).

iii): Let $\mathscr{P}(t)=\{q_1^{(\beta)};\,\beta\in\mathrm{A}\}$, A is a directed set of indices, be the system of semi norms given above and defining t on E such that (7) applies. Then,

$$\widetilde{\mathscr{P}}(\varepsilon_{\infty})=\{g\rightarrow Q_{(\gamma_n),\beta}(g)\ ;\ (\gamma_n)\in\mathbf{R}_+^{\mathbf{N}^*},\,\beta\in\mathrm{A}\},$$

$Q_{(\gamma_n),\beta}(g)=\sum_{n=0}^{\infty}\gamma_n q_n^{(\beta)}(g_n)$, $q_m^{(\beta)}=q_1^{(\beta)}\otimes_{\varepsilon}\ldots\otimes_{\varepsilon}q_1^{(\beta)}$ (m copies), $m=1, 2, \ldots$, $q_0^{(\beta)}=|g_0|$, defines ε_{∞} on $E_{\otimes}$. Note that there is a cofinal subsystem $\widetilde{\widetilde{\mathscr{P}}}(\varepsilon_{\infty})$ of $\widetilde{\mathscr{P}}(\varepsilon_{\infty})$ such that each $Q\in\widetilde{\widetilde{\mathscr{P}}}$ satisfies condition (A). Let us now be given some semi norm $p\in\mathscr{P}(\varepsilon_{\infty})$. Then there are a constant $c>0$ and a semi norm $Q_{(\gamma_n),\beta}\in\widetilde{\widetilde{\mathscr{P}}}(\varepsilon_{\infty})$ such that

$$p(g)\leq cQ_{(\gamma_n),\beta}(g)$$

for all $g\in E_{\otimes}$. For $g\in E_{\otimes}$, $f\in F$, consider

$$L_n=\sqrt{q_{2n}^{(\beta)}(f_n^{\#}\otimes f_n)},\ n=0, 1, 2,\ldots,$$

$$\pounds(g)=\sum_{n=0}^{\infty}q_n^{(\beta)}(g_n).$$

Applying Theorem 2.2, there is a sequence $(\alpha_n)_{n=0}^{\infty}\in\mathbf{R}_+^{\mathbf{N}^*}$ such that

$$(Q_{(\gamma_n),\beta}(f))^2\overset{(*)}{\leq}\sum_{n=0}^{\infty}c_n(\gamma_n q_n^{(\beta)}(f_n))^2=\sum_{n=0}^{\infty}c_n(\gamma_n)^2q_{2n}^{(\beta)}(f_n^{\#}\otimes f_n)$$

$$\leq \|f^{\#} f\|_{(\pounds,(\alpha_n))} = \sum_{n=0}^{\infty} \alpha_{2n} q_{2n}^{(\beta)}((f^{\#} f)_{2n}) \leq Q_{(\alpha_n),\beta}(f^{\#} f),$$

where $c_n = 1 + \frac{1}{6}(\pi^2 + n(n+1)(2n+1))$. Hence, $p' = Q_{(\alpha_n),\beta}$ satisfies the first inequality of iii). The second inequality of iii) is a consequence of Lemma 2.3a) and i). ((*) holds because for each sequence $(x_n)_{n=0}^{\infty} \in d$ (vector space of terminating sequences) the inequality

$$\left(\sum_{n=0}^{\infty} x_n\right)^2 \leq \sum_{n=0}^{\infty} (1 + A_{n,2}) x_n^2$$

is satisfied, where $A_{n,2} = \sum_{r=1}^{\infty} r^{-2} + \sum_{r=1}^{n} r^2 = \frac{\pi^2}{6} + \frac{1}{6}(n(n+1)(2n+1))$.

In the sequal, Lemma 2.4 will be applied together with the following (cf [26]).

Lemma 2.5. *Let us be given a nuclear vector space $X[\tau]$ and some τ-normal cone K. If $k = \sum_{i=1}^{\infty} k^{(i)} \in E$, $k^{(i)} \in K$, $\sum \cdot$ is convergent with respect to τ, then for every τ-continuous semi norm p there is a τ-continuous semi norm q such that*

$$\sum_{i=1}^{\infty} p(k^{(i)}) \leq q(k).$$

Proof. Recalling that the nuclearity of $X[\tau]$ implies that for p there are a τ-continuous semi norm r and sequences $(c_n)_{n=1}^{\infty} \in l_1$, $c_n \geq 0$, $(T_n)_{n=1}^{\infty}$, $T_n \in U_r^0$, such that

$$p(f) \leq \sum_{n=1}^{\infty} c_n |T_n(f)|,$$

$f \in X$ (see [29, Satz 4.1.4]), and that the τ-normality of K yields that for r there is a τ-continuous semi norm s such that for each $T_n \in U_r^0$ there are positive linear functionals $S_n^{(m)} \in U_s^0$ $(m=1,\ldots,4)$, satisfying

$$T_n = S_n^{(1)} - S_n^{(2)} + i(S_n^{(3)} - S_n^{(4)})$$

(see [31, V.3.3, Cor. 1]), it follows

$$\sum_{i=1}^{\infty} p(k^{(i)}) \leq \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} c_n |T_n(k^{(i)})| = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} c_n |(S_n^{(1)} - S_n^{(2)} + i(S_n^{(3)} - S_n^{(4)}))(k^{(i)})|$$
$$\leq \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} c_n (\sum_{m=1}^{4} S_n^{(m)}(k^{(i)})) = \sum_{n=1}^{\infty} c_n (\sum_{m=1}^{4} S_n^{(m)}(k))$$
$$\leq 4cr(k),$$

where $c = \sum_{n=1}^{\infty} c_n < \infty$. Setting $q = 4cr$, the proof is completed.

Corollary 2.6. *Let us be given an alg-# cone $\{F, \#\}$ in a tensor algebra $E_\otimes$ with nuclear basic space $E[t]$, and $k = \sum_{i=1}^{\infty} f^{(i)\#} f^{(i)} \in \widetilde{E_\otimes}[\varepsilon_\infty]$, $f^{(i)} \in \tilde{F}$. Then for every ε_∞-continuous semi norm p there is some ε_∞-continuous semi norm q such that*

$$\sum_{i=1}^{\infty} (p(f^{(i)}))^2 \leq q(k).$$

Proof. Recalling that $E_\otimes[\varepsilon_\infty]$ is nuclear, and every alg-# cone is ε_∞-normal ([I, Ex.4.6, Remark a)]), Lemmata 2.4, 2.5 readily yield the Corollary under consideration.

For further investigations of the topological structure of tensor algebras, the reader is refered to [16], [18].

§3. On the Closed Hulls of Alg-# Cones

§3.1. On the Closed Hulls of Alg-# Cones in Truncated Tensor Algebras

Definition 3.1. Let us be given a tensor algebra $E_\otimes$ endowed with an l.c. topology τ, $\varepsilon_P \prec \tau \prec \iota_\otimes$, and an alg-# cone $\{F, \#\}$. Then it is said that τ satisfies the *closure condition with respect to* $\{F, \#\}$, if there is a system of semi norms $\mathscr{P}(\tau)$ defining τ on $E_\otimes$ and satisfying the following properties:

i) For each semi norm $p \in \mathscr{P}(\tau)$ there is a sequence $(\omega_n)_{n=0}^{\infty}$, $\omega_n > 0$, such that $f \to p(f)$, $f \in E_\otimes$, satisfies property (A) with respect to $\{F, \#\}$ and $(\omega)_{n=0}^{\infty}$.

ii) For each $p \in \mathscr{P}(\tau)$, $p(f) \geq p(Q_{2n}(f))$ holds for all $f \in E_\otimes$ and $n = 0, 1, 2, \ldots$

iii) For each $p \in \mathscr{P}(\tau)$ and every constant $c > 0$ there are a τ-continuous semi norm p' and a sequence $(\alpha_n)_{n=0}^{\infty} \in \mathscr{A}_{(\omega,d)}$, where $(\omega_n)_{n=0}^{\infty}$ is taken from i) and $d_n=1$ (n=0, 1, 2, ...), such that

$$p(f) \leq p'(f) \tag{10}$$
$$\|f\|_{(p,(\alpha_n))} \leq p'(f) \tag{11}$$
$$p'(\check{f}_{2n}) \leq p'(f) \tag{12}$$
$$p(\check{f}_{2n}) \leq \min\{2c, \beta_{2n}(c, (\omega_i))\}\, p'(\check{f}_{2n}) \tag{13}$$

are satisfied for all $f \in E_{\otimes}$ and n=0, 1, 2, ...

Remarks. a) If p and p' are graded semi norms, then ii) and (12) follow immediately.

b) Assume that there is a system of semi norms $\mathscr{P}(\tau)$ defining τ on $E_{\otimes}$ such that each $p \in \mathscr{P}(\tau)$ is graded and satisfies property (A) with respect to $\{F, \#\}$ and a sequence $(\omega_n)_{n=0}^{\infty}$, $\omega_n > 0$, depending on p. If for every constant $c>0$ and each $p \in \mathscr{P}(\tau)$ there is a sequence $(\alpha_n)_{n=0}^{\infty} \in \mathscr{A}_{(\omega,d)}$ with $d_n=1$ (n=0, 1, 2, ...) such that the semi norm

$$f \to p'(f) = \sum_{n=0}^{\infty} \gamma_n p_n(f_n)$$

is also τ-continuous, where $p_n(f_n) = p(\check{f}_n)$, $\gamma_{2n} = \max\{(\beta_{2n}(c, (\omega_i)))^{-1}, \alpha_{2n}\}$, $\gamma_{2n+1} = 1$ ($n=0, 1, 2, \ldots$), then the closure condition is satisfied for τ with respect to $\{F, \#\}$.

c) An immediate consequence of b) is that if the assumptions of [I: Example 3.4b), (resp. Example 3.4c))] are fulfilled, then ε_∞ (resp. σ_∞) satisfies the closure condition for every alg-# cone $\{F, \#\}$.

The definition given above is motivated by the following theorem.

Theorem 3.1. *Let us be given an alg-# cone $\{F, \#\}$ in some tensor algebra $E_{\otimes}$ and an l.c. topology τ which satisfies the closure condition with respect to $\{F, \#\}$. Then,*

$$\text{a) } \overline{\{F, \#\}}^{\tau} \cap E^{2n} = \overline{\{F, \#\} \cap E^{2n}}^{\tau},$$

b) *if additionally $\widetilde{E_{\otimes}}[\tau] \subset \prod_{n=0}^{\infty} \widetilde{E_n}[t_n]$, $t_n = \tau_{|E_n}$, is satisfied, then*

$$\overline{\{\widetilde{F, \#}\}}^{\tau} \cap \overline{\widetilde{E^{2n}}}^{t_{2n}} = (\{F, \#\} \widetilde{\cap} E^{2n})^{t_{2n}},$$

for $n=0, 1, 2, \ldots$, *where* $\tilde{}$ *denotes the completed hull.*

Proof. a) : Obviously, $\overline{\{F, \#\}}^{\tau} \cap E^{2n} \supset \overline{\{F, \#\} \cap E^{2n}}^{\tau}$, $n=0, 1, 2, \ldots$ Assuming that there is an index $n_0 \in \mathbf{N}^*$ ($\equiv \{0, 1, 2, \ldots\}$) such that the assertion of the theorem under consideration does not hold, there is a.

$$g=(g_0, g_1, \ldots, g_{2n_0}, 0, 0, \ldots) \in (\overline{\{F, \#\}}^{\tau} \cap E^{2n_0}) \setminus \overline{\{F, \#\} \cap E^{2n_0}}^{\tau}. \qquad (14)$$

Further, let $\mathscr{P}(\tau)$ be a system of semi norms defining τ on $E_{\otimes}$ and satisfying Definition 3.1. Due to (14) there is a semi norm $p \in \mathscr{P}(\tau)$ and some constant κ with

$$0<\kappa<\frac{1}{2} \qquad (15)$$

such that

$$p(g-\sum_{i=1}^{M} f^{(i)\#} f^{(i)})>\kappa \qquad (16)$$

for all $f^{(i)} \in (F \cap E^{2n_0})$, $i=1, 2, \ldots, M$ ($M \in \mathbf{N}$). Choose now a sequence $(\omega_n)_{n=0}^{\infty}$ such that i) of Definition 3.1 is satisfied for p. Set

$$\tilde{\omega}=\max\{\tilde{\omega}_n\,;\ n=n_0+1, n_0+2, \ldots, 2n_0\},$$
$$c=(\kappa/2\sqrt{2}\,(n_0)^2\omega)^2,$$

where $\tilde{\omega}_n=\max\{\omega_i\omega_j\,;\ i+j=n,\ i\neq j\}$. Choose then a sequence $(\alpha_n)_{n=0}^{\infty}$ and a τ-continuous semi norm p' such that iii) of Definition 3.1 is fulfilled. Without loss of generality assume that

$$\|g\|_{(p,(\alpha_n))}<\frac{1}{2} \qquad (17)$$

(otherwise, if (17) does not apply, then consider $g'=\lambda g$ and $\kappa'=\lambda\kappa$ with $\lambda=(2\|g\|)^{-1}$ instead of g and κ in (15) and (16)). Since $g \in \overline{\{F, \#\}}^{\tau} \cap E^{2n_0}$ there is an

$$h=\sum_{i=1}^{M'} b^{(i)\#} b^{(i)} \in \{F, \#\},$$

$b^{(i)}=(b_0^{(i)},\dots,b_{N(i)}^{(i)},0,0,\dots)\in F$, $M'\in\mathbb{N}$, such that $p'(g-h)\leq\kappa/2$. Using (10),

$$p(g-h)\leq p'(g-h)\leq\kappa/2 \tag{18}$$

follow. Setting $(L_n)^2=p(\sum_{i=1}^{M'}\check{b}_n^{(i)\#}\check{b}_n^{(i)})$, $n=0,1,2,\dots$,

$$\sum_{n=0}^{\infty}(L_n)^2\overset{(*)}{\leq}\|h\|_{(p,(\alpha n))}\leq\|g\|_{(p,(\alpha n))}+\|g-h\|_{(p,(\alpha n))}\overset{(**)}{\leq}\frac{1}{2}+p'(g-h)\leq\frac{1}{2}+\frac{\kappa}{2}\leq 1$$

imply that

$$L_n\leq 1 \tag{19}$$

for n=0, 1, 2, ... ((*) is a consequence of Theorem 2.2. (**) is satisfied because of (17) and (11).) Further, there are an index r_0, $n_0<r_0\leq 2n_0$, and an L_{μ_0}, $n_0<\mu_0\leq 2n_0$, such that the following estimates hold

$$\begin{aligned}
\frac{\kappa}{2}&\leq\kappa-p(g-h)\overset{(+)}{\leq}p(g-\sum_{i=1}^{M'}(Q_{n_0}(b^{(i)}))^{\#}(Q_{n_0}(b^{(i)})))-p(g-Q_{2n_0}(h))\\
&\leq p([\sum_{i=1}^{M'}(Q_{n_0}(b^{(i)}))^{\#}Q_{n_0}(b^{(i)})]-Q_{2n_0}(h))=p(\sum_{r=n_0+1}^{2n_0}\sum_{\substack{\mu+\nu=r\\ \mu\geq n_0+1}}\sum_{i=1}^{M'}(b_\mu^{(i)\#}b_\nu^{(i)}+b_\nu^{(i)\#}b_\mu^{(i)}))\\
&\leq n_0 p(\sum_{\substack{\mu+\nu=r_0\\ \mu\geq n_0+1}}\sum_{i=1}^{M'}(b_\mu^{(i)\#}b_\nu^{(i)}+b_\nu^{[i]\#}b_\mu^{(i)}))\leq n_0\sum_{\substack{\mu+\nu=r_0\\ \mu\geq n_0+1}}\omega_\mu\omega_\nu L_\mu L_\nu\\
&\leq n_0(r_0-n_0)\tilde{\omega}L_{\mu_0}\leq(n_0)^2\tilde{\omega}L_{\mu_0}
\end{aligned}$$

((+) holds because of (17) and $p(g-Q_{2n_0}(h))=p(Q_{2n_0}(g-h))\leq p(g-h)$ due to Definition 3.1 ii).) Hence,

$$L_{\mu_0}\geq\frac{\kappa}{2(n_0)^2\tilde{\omega}}=\sqrt{2c}. \tag{20}$$

Using (19) and ii) of Definition 3.1, Theorem 2.1 applies to $\pounds=p$. Thus, either

$$p(\check{h}_{2\mu_0})\geq c \tag{21}$$

or there is an index $m>\mu_0$ with

$$p(\check{h}_{2m}) > \frac{1}{2}\beta_{2m}^{(\mu_0)}(c,(\omega_\iota)) \geq \frac{1}{2}\beta_{2m}(c,(\omega_\iota)). \tag{22}$$

Using (21) (resp. (22)), it follows

$$p'(g-h) \geq p'(((g-h)_{2\mu_0})^\vee) = p'(\check{h}_{2\mu_0}) \geq \frac{p(\check{h}_{2\mu_0})}{\min\{2c, \beta_{2\mu_0}(c,(\omega_i))\}} \geq \frac{c}{2c} = \frac{1}{2}$$

(resp. $p'(g-h) \geq p'(\check{h}_{2m}) \geq \frac{p(\check{h}_{2m})}{\beta_{2m}(c,(\omega_\iota))} \geq \frac{1}{2}$). This is a contradiction to (15) and (18). The proof of a) is completed. b) follows analogously.

For every set $M \subset E_\otimes$ and every l.c. topology τ let us distinguish between the τ-closed hull $\overline{M}^\tau$ and the sequence-closed hull $\overline{M}^{f,\tau}$ which is obtained by adding all the limit points of τ-convergent sequences to M. Obviously,

$$\overline{M}^{f,\tau} \subset \overline{M}^\tau. \tag{23}$$

Corollary 3.2. *Assuming that the assumptions of Theorem* 3.1 *are satisfied, the following hold.* a) *If τ' denotes a further l.c. topology on $E_\otimes$ such that $\tau \prec \tau'$ and $\tau'_{|E^n} = \tau_{|E^n} = t^n$ $(n=0, 1, 2, \ldots)$, then*

$$\overline{\{F,\#\}}^\tau = \overline{\{F,\#\}}^{\tau'}.$$

b) *If the truncated tensor algebras $E^n[t^n]$, $(n=0, 1, 2, \ldots)$ are metrizable l.c. vector spaces, then*

$$\overline{\{F,\#\}}^\tau = \overline{\{F,\#\}}^{f,\tau}.$$

Proof. a) Using Theorem 3.1, the assertion under consideration follows from

$$\overline{\{F,\#\}}^{\tau'} \subset \overline{\{F,\#\}}^\tau = \bigcup_{n=0}^{\infty} (\overline{\{F,\#\}}^\tau \cap E^{2n})$$

$$= \bigcup_{n=0}^{\infty} \overline{\{F,\#\} \cap E^{2n}}^\tau = \bigcup_{n=0}^{\infty} \overline{\{F,\#\} \cap E^{2n}}^{\tau'} \subset \overline{\{F,\#\}}^{\tau'},$$

b) The assertion to be shown is a consequence of (23) and

$$\overline{\{F,\#\}}^\tau = \bigcup_{n=0}^{\infty} (\overline{\{F,\#\}}^\tau \cap E^{2n}) = \bigcup_{n=0}^{\infty} \overline{\{F,\#\} \cap E^{2n}}^{t^{2n}} \overset{(*)}{=}$$

$$= \bigcup_{n=0}^{\infty} \overline{\{F,\#\} \cap E^{2n}}^{f,t^{2n}} \subset \overline{\{F,\#\}}^{f,\tau},$$

where (*) is a consequence of the metrizability of $E^{2n}[t^{2n}]$.

Proposition 3.3. *Let us be given a tensor algebra $E_{\otimes}$ and an alg-# cone $\{F, \#\}$ satisfying* (5). *Let further the basic space $E[t]$ be a metrizable l.c. vector space, and let # induce a continuous mapping on $E[t]$. Then,*

$$\overline{\{F, \#\}}^{f,\varepsilon_{\otimes}} = \overline{\{F, \#\}}^{\varepsilon_{\infty}}$$

follows.

Proof. Recalling that ε_{∞} satisfies the closure-condition with respect to $\{F, \#\}$ (see Remark c) on Definition 3.1), the assertion to be shown follows from

$$\overline{\{F, \#\}}^{f,\varepsilon_{\otimes}} \overset{(\dagger)}{\subset} \overline{\{F, \#\}}^{\varepsilon_{\infty}} \overset{(\ddagger)}{=} \bigcup_{n=0}^{\infty} \overline{\{F, \#\} \cap E^{2n}}^{f,\varepsilon^{2n}} \overset{(\dagger\ddagger)}{\subset} \overline{\{F, \#\}}^{f,\varepsilon_{\otimes}}.$$

(($\dagger$) is a consequence of (23) and $\varepsilon_{\infty} \prec \varepsilon_{\otimes}$. ($\ddagger$): Noticing that $E^{n}[\varepsilon^{n}] = \oplus_{m=0}^{n} E_m[\varepsilon_m]$ ($n=0, 1, 2,\ldots$) are metrizable l.c. vector spaces, ($\ddagger$) follows from Theorem 3.1. ($\dagger\ddagger$) holds because of $\varepsilon_{\infty|E^{2n}} = \varepsilon_{\otimes|E^{2n}} = \varepsilon^{2n}$.)

Remark. If Theorem 3.1b) applies, then similar statements as those of Corollary 3.2 and Proposition 3.3 follow for the completed hulls.

The following two lemmas are aimed at a discussion of the assumptions of Theorem 3.1 and Proposition 3.3. In Lemma 3.4 it will be shown that the assertions of Theorem 3.1 and Proposition 3.3 are a consequence of the special algebraic structure of alg-# cones because there are sets $\mathcal{M}$ not being alg-# cones and not satisfying the above assertions. The aim of Lemma 3.5 and its remark is to indicate that the assumption that τ has to satisfy the closure condition is not redundant in Theorem 3.1. It will be shown on one hand that ε_p does not satisfy the closure condition and on the other hand that there are involutive cones not satisfying the assertions of Theorem 3.1.

Let us now be given a metrizable l.c. vector space $E[t]$ such that there is a continuous norm p on E. Setting

$$p_n = p \otimes_{\varepsilon} \ldots \otimes_{\varepsilon} p \ (n \text{ copies})$$

let us consider the set

$$\mathcal{M}=\{\check{f}_1+\check{f}_n \in E_\otimes \,;\, n(p(f_1)+p_n(f_n))>1\} \subset E_\otimes.$$

Lemma 3.4. *It holds* $\overline{\mathcal{M}}^{f,\varepsilon\otimes}=\overline{\mathcal{M}}^{f,\varepsilon_\infty}\underset{\neq}{\subset}\overline{\mathcal{M}}^{\varepsilon\otimes}$.

Proof. The proof uses the following assertion (*) which is a well-known fact of the theories of l.c. direct sums and topological tensor algebras [17].

(*) *Let us be given a sequence* $(f^{(i)})_{i=1}^\infty$, $f^{(i)}\in E_\otimes$, *and a* $g\in E_\otimes$ *such that* $f^{(i)}\longrightarrow g$ *as* $i\longrightarrow\infty$ *with respect to* ε_∞. *Then there is an index* $M\in\mathbf{N}$ *such that* Grad $(f^{(i)})\leq M$ *for all* $i\in\mathbf{N}$.

Using (*) and $\varepsilon_{\otimes|E^{2n}}=\varepsilon_{\infty|E^{2n}}=\varepsilon^{2n}$, $n\in\mathbf{N}$, $\overline{\mathcal{M}}^{f,\varepsilon\otimes}=\overline{\mathcal{M}}^{f,\varepsilon_\infty}$ follows. The second part of the lemma under consideration follows from

i) $0\notin\overline{\mathcal{M}}^{f,\varepsilon_\infty}$,
ii) $0\in\overline{\mathcal{M}}^{\varepsilon\otimes}$

Proof of i). Assuming that i) is not satisfied, there is a sequence $(f^{(m)})_{m=1}^\infty$, $f^{(m)}\in\mathcal{M}$, such that $f^{(m)}\longrightarrow 0$ as $m\longrightarrow\infty$ with respect to ε_∞. Hence,

$$\sum_{n=0}^{\infty}\gamma_n p_n(f_n^{(m)})\longrightarrow 0 \tag{24}$$

as $m\longrightarrow\infty$ for every sequence $(\gamma_n)_{n=0}^\infty$, $\gamma_n>0$, which is a contradiction to (*).

Proof of ii). Take some system of semi norms $\mathcal{P}(t)=\{p^{(n)}\,;\,n=1,2,3,\ldots\}$ which defines the l.c. topology t on E. Consider the semi norms

$$p_m^{(n)}=p^{(n)}\otimes_\varepsilon\ldots\otimes_\varepsilon p^{(n)}\ (m\ copies)$$

on E_m, $m=1,2,3,\ldots$. Notice that $\varepsilon_\otimes$ is defined by the system of semi norms

$$\mathcal{P}(\varepsilon_\otimes)=\{p_{(\gamma_n),(k_n)}\,;\,(\gamma_n)\in\mathbf{R}_+^{\mathbf{N}^*},\ \gamma_n>0,\ (k_n)\in\mathbf{N}^{\mathbf{N}^*}\},$$

where $p_{(\gamma_n),(k_n)}(f)=\sum_{n=0}^\infty\gamma_n p_n^{(k_n)}(f_n)$, $f=(f_0,f_1,\ldots)\in E_\otimes$. Let us now be given $\delta>0$ and some semi norm $p_{(\gamma_n)(k_n)}\in\mathcal{P}(\varepsilon_\otimes)$. Choose an $f_1\in E$ such that

$$0<p_1^{(k_1)}(f_1)<\frac{\delta}{2\gamma_1},$$

and then an $n' \in \mathbb{N}$ with $n' > 1/p_1(f_1)$. Choose further an $f_{n'} \in E_{n'}$ such that

$$0 < p_{n'}^{(kn')}(f_{n'}) < \frac{\delta}{2\gamma_{n'}}$$

Setting $f = \check{f}_1 + \check{f}_{n'} \in E_{\otimes}$, it follows that $f \in \mathcal{M}$ since

$$p_{(\gamma n)(kn)}(f) = \gamma_1 p_1^{(k_1)}(f_1) + \gamma_{n'} p_{n'}^{(kn')}(f_{n'}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence, every 0-neighborhood of $\varepsilon_{\otimes}$ contains an $f \in \mathcal{M}$ proving ii). The proof of Lemma 3.4 is completed.

Lemma 3.5. *Let us be given a tensor algebra* $E_{\otimes}$ *and an involutive cone* $\{E_{\otimes}, \#\}$. *Let* ε_P *be metrizable.* a) *Then,*

$$\overline{\{E_{\otimes},\ \#\}}^{\varepsilon_P} \supset \{g = (g_0, \ldots, g_N, 0, 0, \ldots) \in E_{\otimes}\ ;\ g_0 \geq 0,\ g_n \in h(E_n, \#),\ n = 1,2,3,\ldots\},$$

where $h(E_n, \#) = \{f_n \in E_n\ ;\ f_n^{\#} = f_n\}$.

b) *If* $\#$ *is continuous on the basic space* $E[t]$, *then the assertion of Theorem* 3.1 *does not apply to* $\{E_{\otimes}, \#\}$ *with respect to* ε_P.

Proof. a) Let us be given some $f_n \in h(E_n, \#)$, $n \in \mathbb{N}$. For each $m \in \mathbb{N}$ let us recursively define a sequence $(\zeta_r^{(m)})_{r=0}^{\infty}$ of reals by

$$\zeta_0^{(m)} = \frac{1}{m},\ \zeta_1^{(m)} = m,\ 2\zeta_0^{(m)}\zeta_r^{(m)} + \sum_{\substack{i+j=r \\ i,j\geq 1}} \zeta_i^{(m)}\zeta_j^{(m)} = 0,$$

$r = 2, 3, 4, \ldots$. Setting $f_{ns} = f_n \otimes \ldots \otimes f_n$, s copies, consider

$$f^{(m)} = \zeta_0^{(m)} \mathbb{1} + \zeta_1^{(m)} \check{f}_n + \zeta_2^{(m)} \check{f}_{2n} + \ldots + \zeta_m^{(m)} \check{f}_{mn} \in E_{\otimes}.$$

Noting that (6) implies $\mathbb{1}^{\#} = \mathbb{1}$,

$$f^{(m)\#} f^{(m)} = \frac{1}{m^2}\mathbb{1} + 2\check{f}_n + \sum_{\mu=1}^{m} \Big(\sum_{\substack{r+s=m+\mu \\ r,s\geq\mu}} \zeta_r^{(m)}\zeta_s^{(m)} \Big) \check{f}_{(m+\mu)n}$$

follow. Hence, $f^{(m)\#}f^{(m)}\to 2\check{f}_n$ as $m\to\infty$ with respect to ε_P. Thus, $\check{f}_n \in \overline{\{E_\otimes, \#\}}^{\varepsilon_P}$. Noticing also that obviously $g_0 \in \mathbb{C}$, $g_0 \geq 0$, implies $\check{g}_0 \in \overline{\{E_\otimes, \#\}}^{\varepsilon_P}$, the proof of a) is completed.

b) Using a), there are $n_0 \in \mathbb{N}$ and

$$0 \neq \check{f}_{2n_0-1} \in \overline{\{E_\otimes, \#\}}^{\varepsilon_P} \cap \check{E}_{2n_0-1} \subset \overline{\{E_\otimes, \#\}}^{\varepsilon_P} \cap E^{2n_0}.$$

Assume now that

$$\check{f}_{2n_0-1} \in \overline{\{E_\otimes, \#\} \cap E^{2n_0}}^{\varepsilon_P}. \tag{25}$$

Using the metrizability of ε_P, there is a sequence $(g^{(m)})_{m=1}^{\infty}$ with

$$g^{(m)} \in \{E_\otimes, \#\} \cap E^{2n_0}, \tag{26}$$

$m=1, 2, 3,\ldots$, such that

$$g^{(m)} \to \check{f}_{2n_0-1} \tag{27}$$

as $m\to\infty$ with respect to ε_P. Because of (26) it follows

$$g^{(m)} = \sum_{i=1}^{M_m} a^{(i,m)\#} a^{(i,m)},$$

where $a^{(i,m)} \in E_\otimes$ and Grad $(a^{(i,m)}) \leq n_0$, $i = 1,2,\ldots,M_m$, $(M_m \in \mathbb{N})$, $m = 1,2,3,\ldots$. As $m\to\infty$, (27) implies that

$$\begin{aligned} g^{(m)}_{2n_0} &\to 0 \text{ with respect to } \varepsilon_{2n_0}, \\ g^{(m)}_{2n_0-1} &\to f_{2n_0-1} \text{ with respect to } \varepsilon_{2n_0-1}, \\ g^{(m)}_{2n_0-2} &\to 0 \text{ with respect to } \varepsilon_{2n_0}. \end{aligned} \tag{28}$$

Because of the continuity of # on $E[t]$ there is a system of semi norms $\mathcal{P}(t)$ defining t on E such that $p(x)=p(x^\#)$ for each $p \in \mathcal{P}(t)$ and all $x \in E$. Setting

$$p_n = p \otimes_\varepsilon \ldots \otimes_\varepsilon p \ (n \text{ copies}),$$

the systems of semi norms $\mathcal{P}(\varepsilon_n) = \{p_n \,;\, p \in \mathcal{P}(t)\}$ define the injective topologies ε_n on E_n, $n=1, 2, 3,\ldots$. Recall that

$$f \to P(f) = \sum_{n=0}^{\infty} p_n(f_n),$$

$f \in E_\otimes$, $p_0(f_0) = |f_0|$, satisfies property (A) with $(\omega_i)_{i=0}^{\infty}$, $\omega_i = 1$, $i = 0, 1, 2, \ldots$ (see [I : Example 3.4c)]). Setting

$$L_{n_0-1}^{(m)} = \sqrt{p_{2n_0-2}\left(\sum_{i=1}^{M_m} a_{n_0-1}^{(i,m)\#} \otimes a_{n_0-1}^{(i,m)}\right)}$$

and using [I : Theorem 3.5] and (27), the existence of an index $m_0 \in \mathbb{N}$ follows such that $L_{n_0-1}^{(m)} < 1$ for all $m > m_0$. Now,

$$p_{2n_0-1}(g_{2n_0-1}^{(m)}) = p_{2n_0-1}\left(\sum_{i=1}^{M_m} (a_{n_0}^{(i,m)\#} \otimes a_{n_0-1}^{(i,m)} + a_{n_0-1}^{(i,m)\#} \otimes a_{n_0}^{(i,m)})\right)$$
$$\leq 2L_{n_0-1}^{(m)} p_{2n_0}(g_{2n_0}^{(m)}) \to 0$$

as $m \to \infty$ for all $p \in \mathcal{P}(t)$ yield a contradiction to (28) since ε_{2n_0-1} is a Hausdorff topology.

Remark. Noting that ε_P does not satisfy property (A_{iii}) with respect to $\{E_\otimes, \#\}$, ε_P does not fulfil the closure condition. Hence, Lemma 3.5b) shows that the assumption on the closure condition is not redundant in Theorem 3.1.

The following is aimed at a characterization of (1) using dual spaces. For each $n \in \mathbb{N}$ and every l.c. topology τ on $E_\otimes$, let us consider the following subsets of $(E_\otimes[\tau])'$:

$$\mathcal{X}_\tau^{2n} = \{Q'_{2n}S;\ S \in (E_\otimes[\tau])',\ S(\{F, \#\}) \geq 0\},$$
$$\mathcal{Y}_\tau^{2n} = \{T \in (E_\otimes[\tau])';\ T = Q'_{2n}T;\ T(\{F, \#\} \cap E^{2n}) \geq 0\},$$

where $Q'_{2n}T(f) = T(Q_{2n}f)$, $f \in E_\otimes$. Obviously, $\mathcal{X}_\tau^{2n} \subset \mathcal{Y}_\tau^{2n}$. Setting $E^{(2n)} = \oplus_{i=0}^{2n} E_i$ and using the canoical isomorphism

$$j : E^{(2n)} \to E^{2n},$$

let $\tau^{(2n)}$ denote the l.c. topology defined by j^{-1} and $\tau_{|E^{2n}}$ (l.c. topology inherited by τ on the subspace E^{2n}) on $E^{(2n)}$. Put further $\mathcal{X}_\tau^{(2n)} = j'\mathcal{X}_\tau^{2n}$, $\mathcal{Y}_\tau^{(2n)} = j'\mathcal{Y}_\tau^{2n}$, where $j'T(x) = T(jx)$, $x \in E^{(2n)}$. Let us also consider the duality $(E^{(2n)}, (E^{(2n)}[\tau^{(2n)}])')$ and the corresponding weak topology $\sigma = \sigma(E^{(2n)}, (E^{(2n)}[\tau^{(2n)}])')$.

A Hahn-Banach like extension theorem for positive functionals (cf [38, Theorem 2.15]) then yields the following.

Lemma 3.6. *For each $n \in \mathbf{N}$ and every l.c. topology τ on $E_\otimes$ the following are equivalent:*

(i) $$\overline{\{F, \#\}}^{\tau} \cap E^{2n} = \overline{\{F, \#\} \cap E^{2n}}^{\tau}$$

(ii) *$\mathcal{X}_\tau^{(2n)}$ is σ-dense in $\mathcal{Y}_\tau^{(2n)}$.*

Corollary 3.2 a) and Lemma 3.6 yield the following.

Corollary 3.7. *If τ satisfies the closure condition and $\tau_{|E^n} = t^n$, $n \in \mathbf{N}$, then $\mathcal{X}_\tau^{2n}$ is σ-dense in $\mathcal{Y}_{\tau_\otimes, t}^{2n}$.*

Remark. Noticing that $\tau \prec \tau'$ implies $\mathcal{X}_\tau^{(2n)} \subset \mathcal{X}_{\tau'}^{(2n)}$ and that there are graded topologies $\tau(\Gamma_{,..})$ satisfying the assumptions of Corollary 3.7 and $\Gamma \neq \mathbf{R}^{\mathbf{N}^*}$, Corollary 3.7 thus yields a generalization of Yngvason's density-lemma ([41, p. 17]).

§3.2. On the Closed Hulls of Alg-# Cones

The following lemma relates the closed hull $\overline{\{F, \#\}}$ (resp. completed hull $\widetilde{\{F, \#\}}$) of some given alg-# cone to its basic space F.

Lemma 3.8. *Let us be given an alg-# cone $\{F, \#\}$ in some tensor algebra $E_\otimes[\tau]$ such that the multiplication $m: F^\# \times F \to E_\otimes$ is jointly τ-continuous, and # is τ-continuous. Then, $\widetilde{\{F, \#\}} = \{\tilde{F}, \#\}$ and $\overline{\{F, \#\}} = \overline{\{\bar{F}, \#\}}$, where the completed hulls $\tilde{\ }$ are concerning τ and the closure is in $E_\otimes[\tau]$.*

Proof. The assertions on # and m imply that there is a system of seminorms $\mathcal{P}(\tau)$ defining τ such that

$$p(f^\#) = p(f),$$

$p \in \mathcal{P}(\tau)$, and for each $p \in \mathcal{P}(\tau)$ there is some $q \in \mathcal{P}(\tau)$ such that

$$p(f^\# g) \leq q(f)\, q(g) \tag{29}$$

for all $f, g \in F$. Let us now be given a net $\{f^{(\alpha)}\}_{\alpha \in A}$, $f^{(\alpha)} \in F$, A is a directed set of indices, such that

$$f^{(\alpha)} \to f$$

with respect to τ. Further, let us be given $p \in \mathscr{P}(\tau)$ and $\delta > 0$. Choose then $q \in \mathscr{P}(\tau)$ satisfying (29), and an $\alpha_0 \in$ A such that

$$q(f - f^{(\alpha)}) < (\sqrt{q(f)^2 + \delta^2} - q(f))/2 =: \rho$$

for all $\alpha > \alpha_0$. The lemma under consideration then follows from

$$\begin{aligned} p(f^{\#} f - f^{(\alpha)\#} f^{(\alpha)}) &\leq q(f^{(\alpha)\#} - f^{\#})\, q(f^{(\alpha)}) + q(f^{\#})\, q(f^{(\alpha)} - f) \\ &\leq \rho(q(f) + \rho) + q(f)\rho) \leq 2\rho(q(f) + \rho) \\ &= \delta^2/2. \end{aligned}$$

The following theorem on the representation of the elements of respectively, the closed and completed hulls of alg-# cones by infinite sums of "squares" was first shown for the cone of positivity of the completed tensor algebra $\widetilde{\mathscr{S}_{\otimes}}\, [\varepsilon_{\otimes}]$ used in the Wightman-axioms of general QFT (see [12], [13], [5]). Later, in [32], [1] the proofs were generalized for the cones of positivity in completed tensor algebras $\widetilde{E_{\otimes}}\, [\varepsilon_{\otimes}]$, where the basic spaces $E[t]$ are nuclear (F)- and nuclear (LF)-spaces, respectively. The following key-lemma is a minor generalization of a lemma due to Schmüdgen ([32, Lemma 2]).

Lemma 3.9. *Let us be given a truncated tensor algebra $E^N[\varepsilon^N]$ endowed with a continuous, antilinear and involutive bijection #, and $F^N = \bigoplus_{n=0}^N F_n \subset E_n$, such that $F^N[t]$, $t = \varepsilon^N_{\restriction F^N}$, is nuclear and metrizable. Considering $K = \{\sum_{i=1}^M f^{(i)\#} \otimes f^{(i)};\ f^{(i)} \in F,\ M \in \mathbb{N}\} \subset E^N \otimes E^N$ and the injective topology $\varepsilon = t \otimes_\varepsilon t$ on $E^N \otimes E^N$, it then follows that*

$$\tilde{K}^{\varepsilon} = \{\sum_{i=1}^{\infty} f^{(i)\#} \otimes f^{(i)};\ f^{(i)} \in \tilde{F},\ \sum \text{ is convergent in } (E^N \otimes E^N)^{\tilde{}}[\varepsilon]\},$$

Proof. For simplicity let us write F and H instead of F^N and E^N, respectively. Using that $\tilde{F}[t]$, $\widetilde{F^{\#}}[t]$ are nuclear Frechet-spaces, it follows $t \otimes_\varepsilon t = t \otimes_\pi t$ on $F^{\#} \otimes F$ ([29]). Furthermore, for each

$$\xi \in \widetilde{F^{\#}}\ \tilde{\otimes}_\pi \tilde{F} = \widetilde{F^{\#}}\ \tilde{\otimes}_\varepsilon \tilde{F},$$

there are sequences $\{x_n^\#\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, x_n, $y_n \in \tilde{F}$, $\{\lambda_n\}_{n=1}^{\infty} \in l_1$ satisfying

$$t-\lim_{n\to\infty} x_n^\# = 0,\ t-\lim_{n\to\infty} y_n = 0,$$

$$\xi = \sum_{n=1}^{\infty} \lambda_n\, x_n^\# \otimes y_n,$$

where $\sum\ldots$ is convergent with respect to $t \otimes_\pi t$ (cf. [31, Theorem III.6.4]). Assume now that there are $\xi \in \widetilde{F^\#}\,\tilde{\otimes}_\varepsilon \tilde{F}$ and

$$\xi_r = \sum_{i=1}^{Mr} x^{(r,i)\#} \otimes x^{(r,i)} \in \widetilde{F^\#} \otimes \tilde{F}$$

such that

$$\xi_r \to \xi \tag{30}$$

as $r\to\infty$ with respect to ε, where $x^{(r,i)} \in \tilde{F}$, $i=1, 2,\ldots, M_r (M_r \in \mathbb{N})$. For $S \in \tilde{F}$ let us define $S^\# \in (\tilde{F}^\#)'$ by $S^\#(f^\#) = \overline{S(f)}$, $f \in \tilde{F}$. Recall that $p \in \mathscr{P}(t)$ implies that $S \in U_p^0$ if and only if $S^\# \in U_p^0$, where the polar sets are considered in the dualities $(\tilde{F}, \tilde{F}')$ and $(\tilde{F}^\#, (\tilde{F}^\#)')$, respectively. Hence, (30) implies

$$\sup\{|S^\# \otimes T(\xi_r - \xi)|\,;\, S,\, T \in U_p^0\} \to 0 \tag{31}$$

as $r\to\infty$ for each $p \in \mathscr{P}(t)$. On $\tilde{F}'$ let us now define a semi-scalar product by

$$\langle S, T\rangle = \xi(S^\#, T) = \lim_{r\to\infty}((S^\# \otimes T)(\xi_r)) = \lim_{r\to\infty}(\sum_{i=1}^{Mr} \overline{S(x^{(r,i)})}\, T(x^{(r,i)})),$$

S, $T \in \tilde{F}'$. Consider then the pre-Hilbert space

$$\widehat{\tilde{F}'} = \tilde{F}'/(\ker(\|.\|)),$$

$\ker(\|.\|) = \{T \in \tilde{F}'\,;\, \|T\| = \sqrt{\langle T, T\rangle} = 0\}$, and the canonical mapping $\mu : \tilde{F}' \to \widehat{\tilde{F}'}$.

Let us denote the norm induced by $\|.\|$ on $\widehat{\tilde{F}'}$ by $\|.\|$ also.

Considering the l.c. topology of precompact convergence $\tau_c(\tilde{F})$ on the dual space $\tilde{F}'$ (cf [23, §21.6]), the continuity of the identity mapping $\iota : \tilde{F}'[\tau_c] \to \tilde{F}'[\|.\|]$ follows from

$$\|T\|^2 = \xi(T^\#, T) = |\sum_{n=1}^{\infty} \lambda_n \overline{T(x_n)}\, T(y_n)| \leq (\sum_{n=1}^{\infty} |\lambda_n|)(\sup\{|T(x)|\,;\, x \in U\})^2$$
$$= C p_U(T)^2, \tag{32}$$

where $C = \sum_{n=1}^{\infty} |\lambda_n| < \infty$ and $U \subset \tilde{F}$ is some absolutely convex and precompact set containing the 0-sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$. Hence, the canonical mapping

$$\mu : \tilde{F}'[\tau_c] \to \widehat{\tilde{F}'}[\|.\|]$$

is continuous, too. Since for every t-continuous seminorm p, the polar set U_p is τ_c-compact ([23, §21.6 (3)]), the continuity of ι and μ imply that U_p and $\mu(U_p)$ are compact sets in $\tilde{F}'[\|.\|]$ and $\widehat{\tilde{F}'}[\|.\|]$, respectively. Using now the metrizability of t, there is a system of seminorms $\mathcal{P}(t) = \{p_n\,;\, n=1, 2, \ldots\}$ defining t such that

$$\tilde{F}' = \bigcup_{n=1}^{\infty} U_{p_n},$$

and consequently, $\widehat{\tilde{F}'}[\|.\|]$ is a separable pre-Hilbert space due to the compactness of $\mu(U_p)$.

Choosing a countable and total system $(e^{(j)})_{j=1}^{\infty}$ of ortho-normal vectors $e^{(j)} \in \tilde{F}'$, let us consider linear functionals

$$z^{(j)} : S \to \langle e^{(j)}, \mu(S) \rangle$$

on F', $j=1, 2, 3, \ldots$. Because of (32) there is a τ_c-continuous semi norm q_W^0 on F' such that

$$|z^{(j)}(S)| \leq \|e^{(j)}\|\, \|\mu(S)\| \leq q_W^0(S)$$

for each $S \in F'$. Hence, $z^{(j)} \in (F'[\tau_c])' = F$, $j=1, 2, 3\ldots$ (cf [23, §21.6 (1)]). It follows now that

$$\xi = \sum_{j=1}^{\infty} z^{(j)\#} \otimes z^{(j)}$$

due to

$$(p \otimes_{\varepsilon} p)\,(\xi - \sum_{j=1}^{k} z^{(j)\#} \otimes z^{(j)}) =$$

$$= \sup\{|\,(T^{\#} \otimes S)\,(\xi) - \sum_{j=1}^{k} T^{\#}(z^{(j)\#})\,S(z^{(j)})\,|\,;\; T^{\#},\, S \in U_p^0\} =$$

$$= \sup\{|\,\langle T, S\rangle - \sum_{j=1}^{k} \overline{T(z^{(j)})}\,S(z^{(j)})\;;\; T, S \in U_p^0\} \overset{(*)}{=}$$

$$= \sup\{|\sum_{j=k+1}^{\infty} \overline{T(z^{(j)})}\,S(z^{(j)})\;|\,;\; T, S \in U_p^0\} \leq \sum_{j=k+1}^{\infty} p(z^{(j)})^2$$

$$\overset{(**)}{\longrightarrow} 0$$

as $k \to \infty$. ((*) is a consequence of

$$\langle T, S\rangle = \langle \mu(T), \mu(S)\rangle = \langle \sum_{i=1}^{\infty} z^{(i)}(T)\,e^{(i)}, \sum_{j=1}^{\infty} z^{(j)}(S)\,e^{(j)}\rangle =$$

$$= \sum_{i=1}^{\infty} \overline{z^{(i)}(T)}\,z^{(i)}(S) = \sum_{i=1}^{\infty} \overline{T(z^{(i)})}\,S(z^{(i)}).$$

(**) Since nuclearity yields $\varepsilon^N \otimes_{\varepsilon} \varepsilon^N = \pi^N \otimes_{\pi} \pi^N$, the continuity of M_N follows from Lemma 2.3b.), and consequently, $p(\sum_{j=1}^{\infty} z^{(j)\#} z^{(j)}) < \infty$. (**) is now a consequence of Corollary 2.6) The proof is completed.

In the following let us be given a topological tensor algebra $E_{\otimes}[\tilde{\varepsilon}]$ and an alg-# cone $\{F, \#\}$, where $F = F_0 \oplus F_1 \oplus F_2 \oplus \ldots$, $\mathbf{1} \in F$, and $\tilde{\varepsilon}_{|E_n} = \varepsilon_n$ ($n = 0, 1, 2, \ldots$), $\varepsilon_1 = t$, ε_0 denotes the Euclidean topology on $E_0 = \mathbf{C}$. Further, let us consider

$$E^m[\varepsilon^m] = \oplus_{i=0}^m E_i[\varepsilon_i]\ (\subset E_\otimes),$$
$$F^m = \oplus_{i=0}^m F_i\ (\subset E_\otimes),$$
$$Y^{(m)} = (F^m)^\# \otimes F^m,$$
$$(F^m)^\# = \{f^\#; f \in F^m\},$$

$m=0, 1, 2, \ldots.$

Theorem 3.10. *Let us be given* $E_\otimes[\tilde{\varepsilon}]$ *with an alg-# cone* $\{F, \#\}$ *such that*

a) $\tilde{\varepsilon}$ *satisfies the closure condition with respect to* $\{F, \#\}$,

b) $\#$ *is continuous on* $E_1[\varepsilon_1]$ *and satisfies* (5),

c) $F_n[\varepsilon_n]$ *are metrizable and nuclear,* $n = 1, 2, \ldots.$

Then,

$$\widetilde{\{F, \#\}}^{\tilde{\varepsilon}} = \widetilde{\{\tilde{F}, \#\}}^{\tilde{\varepsilon}} = \widetilde{\{F, \#\}}^{\varepsilon\infty} = \widetilde{\{F, \#\}}^{\varepsilon\otimes} =$$
$$= \{\sum_{i=1}^{\infty} f^{(i)\#} f^{(i)};\ f^{(i)} \in \widetilde{F^n},\ i=1, 2, \ldots, \sum \text{ is convergent in } \widetilde{E^{2n}}[\varepsilon^{2n}]\ (n=0, 1, 2, \ldots)\},$$

and

$$\overline{\{F, \#\}}^{\tilde{\varepsilon}} = \overline{\{\tilde{F}, \#\}}^{\tilde{\varepsilon}} = \overline{\{F, \#\}}^{\varepsilon\infty} = \overline{\{F, \#\}}^{\varepsilon\otimes} =$$
$$= \{\sum_{i=1}^{\infty} f^{(i)\#} f^{(i)};\ f^{(i)} \in \tilde{F}^n,\ i=1, 2, \ldots, \sum \text{ is convergent in } E^{2n}[\varepsilon^{2n}]\ (n=0, 1, 2, \ldots)\},$$

Proof. Let us now consider the linear mappings

$$M_n: Y^{(n)} \to \{F, \#\} \cap E^{2n},$$

where $M_n(f^\# \otimes g) = f^\# g$, $f, g \in F^{(n)}$, $n=1, 2, 3, \ldots.$ Noticing that the nuclearity of $F_n[\varepsilon_n]$ implies $\pi^n \otimes_\pi \pi^n = \varepsilon^n \otimes_\varepsilon \varepsilon^n$, Lemma 2.3b) yields the continuity of

$$M_n: Y^{(n)}[\varepsilon^n \otimes_\varepsilon \varepsilon^n] \to E^{2n}[\varepsilon^{2n}].$$

The continuous extension of M_n onto the completed hull $(Y^{(n)})^{\sim}[\varepsilon^n \otimes_\varepsilon \varepsilon^n]$ will also be denoted by M_n.

Let us be given $n \in \mathbb{N}$ and a sequence $(v^{(r)})_{r=1}^{\infty}$ such that

$$v^{(r)}=\sum_{j=1}^{Nr} x^{(r,j)\#}\, x^{(r,j)},$$

$x^{(r,j)} \in F^{(n)}$, $j=1, 2,\dots, N_r$ ($N_r \in \mathbf{N}$) and

$$v=\lim_{r\to\infty} v^{(r)} \tag{33}$$

in $E^{2n}[\varepsilon^{2n}]$. Hence, for every ε^{2n}-continuous semi norm q there is a constant $C_q>0$ such that

$$\sup\{q(v^{(r)});\ r=1, 2,\dots\}<C_q.$$

Consider further

$$\xi_r=\sum_{j=1}^{Nr} x^{(r,j)\#} \otimes x^{(r,j)} \in Y^{(n)}.$$

Because of b), c) there is a system of semi norms $\mathcal{P}(\varepsilon_\infty)$ such that Lemma 2.4 applies and $\varepsilon^n=\varepsilon_{\infty|E^n}$, $n=1, 2,\dots$. Noting then Corollary 2.6 implies that for each ε^n-continuous semi norm p there is some semi norm $q \in \mathcal{P}(\varepsilon_\infty)$ such that

$$(p \otimes_\varepsilon p)(\xi_r) \leq \sum_{r=1}^{N_r} p(x^{(r,i)\#})\,p(x^{(r,i)}) \leq \sum_{r=1}^{N_r} (p(x^{(r,i)}))^2 \leq q(v^{(r)}) < C_q,$$

the boundedness of $\{\xi_r\ ;\ r\in \mathbf{N}\} \subset Y^{(n)}$ follows. Since $(Y^{(n)})^{\sim}[\varepsilon^n \otimes_\varepsilon \varepsilon^n]$ are nuclear (F)-spaces and consequently (FM)-spaces ([36, p 520]) there are $\xi \in (Y^{(n)})^{\sim}$ and a subsequence $(\xi_{r'})_{r'=1}^{\infty}$ of (ξ_r) such that

$$\lim_{r'\to\infty} \xi_{r'}=\xi \tag{34}$$

with respect to $\varepsilon^n \otimes_\varepsilon \varepsilon^n$. Using Lemma 3.9 there are $f^{(j)} \in F^n$ such that

$$\xi=\sum_{j=1}^{\infty} f^{(j)\#} \otimes f^{(j)}, \tag{35}$$

where $\sum$ is convergent with respect to $\varepsilon^n \otimes_\varepsilon \varepsilon^n$. Applying now the continuity of M_n, (33), (34), and (35), it follows that

$$v=\lim_{r'\to\infty} v^{(r')}=\lim_{r'\to\infty} M_n(\xi_{r'})=M_n(\xi)=\sum_{j=1}^{\infty} f^{(j)\#}f^{(j)}. \tag{36}$$

Because of assumption a), Theorem 3.1 and (36), it is implied that

$$\overline{\{F,\ \#\}}^{\tilde{\varepsilon}}=\bigcup_{n=0}^{\infty} \overline{\{F,\#\}\cap E^{(2n)}}^{\varepsilon(2n)}=$$

$$=\bigcup_{n=0}^{\infty}\{\sum_{i=1}^{\infty} f^{(i)\#}f^{(i)}\,;f^{(i)}\in\widetilde{F^n},\ \sum \text{ is convergent in } E^{2n}[\varepsilon^{2n}]\}=$$

$$=\{\sum_{i=1}^{\infty} f^{(i)\#}f^{(i)}\,;f^{(i)}\in\tilde{F},\ \sum \text{ is convergent in } E_{\otimes}[\varepsilon_{\otimes}]\}.$$

The remaining to be shown concerning the closed hulls are now a consequence of Remark c) on Definition 3.1, Corollary 3.2a), Lemmata 2.3 and 3.8. The assertions on the completed hulls follow analogously.

The aim of the following is to weaken the assumptions of Lemma 3.9 and Theorem 3.10. Let us be given $H[t]$ as in Lemma 3.9 and a subspace $F\subset H$. Consider further a subspace $G\subset F$, and put

$$K=\{\sum_{i=1}^{N} f^{(i)\#}\otimes f^{(i)}\,;f^{(i)}\in F,\ N\in\mathbb{N}\}\subset F^{\#}\otimes F,$$

$$C=\{\sum_{i=1}^{N} g^{(i)\#}\otimes g^{(i)}\,;g^{(i)}\in G,\ N\in\mathbb{N}\}\subset G^{\#}\otimes G.$$

On respectively, $G^{\#}\otimes G$ and $F^{\#}\otimes F$ introduce l.c topologies

$$\varepsilon'=t_{|G^{\#}}\otimes_{\varepsilon} t_{|G}$$

and

$$\varepsilon=t_{|F^{\#}}\otimes_{\varepsilon} t_{|F}.$$

Note $\varepsilon_{|G^{\#}\otimes G}=\varepsilon'$ (cf [22, p 348]). Consider again the linear mapping

$$M: E_{\otimes} \otimes E_{\otimes} \to E_{\otimes}$$

defined in Lemma 2.3.

Lemma 3.11. a) *If $F[t]$ is nuclear, then $\overline{C}^{\varepsilon'} = \overline{K}^{\varepsilon} \cap \overline{(G^{\#} \otimes G)}^{\varepsilon'}$.*
b) *Let us be given an alg-# cone $\{F, \#\} \subset E_{\otimes}$ such that # is ε_{∞}-continuons, and $F[\varepsilon_{\infty}]$ is nuclear. Then for every subspace $G \subset F$, it follows*

$$\overline{\{G, \#\}}^{\varepsilon_\infty} = \overline{\{F, \#\}}^{\varepsilon_\infty} \cap \overline{M(G^{\#} \otimes G)}^{\varepsilon_\infty}.$$

Proof. a) Letting L stand for G and F, respectively, consider the (real) vector spaces

$$(L^{\#} \otimes L)_h = \{\xi \in L^{\#} \otimes L\, ;\, \xi^{\dagger} = \xi\},$$

where $(f^{\#} \otimes g)^{\dagger} = g^{\#} \otimes f$, $g, f \in L$. Let further $\mathcal{N}(L^{\#}, L)$ denote the set of all nuclear bilinear forms b on $L^{\#} \times L$, and

$$(\mathcal{N}(L_n^{\#}, L_n))_h = \{b \in \mathcal{N}(L_n^{\#}, L_n)\, ;\, b(x^{\#}, y) = \overline{b(y^{\#}, x)},\ x, y \in L\}.$$

Using the nuclearity of $G^{\#} \otimes G[\varepsilon']$ and $F^{\#} \otimes F[\varepsilon]$ (cf [29, Satz 5.1.1, 5.4.1]), the kernel theorem ([22, §21.3.5]) implies that

$$((L_n^{\#} \otimes L_n)_h,\ (\mathcal{N}(L_n^{\#}, L_n))_h)$$

are dual pairs, and consequently,

$$\overline{C}^{\varepsilon'} = C^{00}, \tag{37}$$
$$\overline{K}^{\varepsilon} = K^{00} \tag{38}$$

due to the bipolar theorem ([23, §§20.7 (6), 20.8 (5)]), (where the above polars are taken in the appropriate dualities). Consider the set of matrices

$$\mathscr{A}_+ = \{\mathbf{A} = (a_{ij})_{i,j=1}^{\infty}\, ;\, a_{ij} \in \mathbf{C},\ \sum_{i,j=1}^{\infty} |a_{ij}| < \infty,\ \mathbf{A} \geq 0\},$$

and let $\mathscr{F}_p$ (resp. $\mathscr{G}_{p'}$) denote the set of all sequences $(T^{(i)})_{i=1}^{\infty}$ of linearly independent $T^{(i)} \in F'$ (resp. $T^{(i)} \in G'$) satisfying $p^0(T^{(i)}) = 1$ (resp. $p'^0(T^{(i)}) = 1$), $i = 1, 2, \ldots$, where p^0 denotes the Minkowski-functional of

$$U_p^0=\{T\in G_n';\ |T(x)|\leq p(x)\ \text{for all}\ x\in G_n\}$$

(resp. p'^0 that of $U_{p'}^0$). It then follows

$$C^0=\{-\sum_{i,j=1}^{\infty} a_{ij}T^{(i)\#}\otimes T^{(j)};\ (a_{i,j})_{i,j=1}^{\infty}\in\mathscr{A}_+,\ (T^{(i)})_{i=1}^{\infty}\in\mathscr{G}_{p'},\ p'\in\mathscr{P}(t_{|G})\}\tag{39}$$

$$K^0=\{-\sum_{i,j=1}^{\infty} a_{ij}T^{(i)\#}\otimes T^{(j)};\ (a_{ij})_{i,j=1}^{\infty}\in\mathscr{A}_+,\ (T^{(i)})_{i=1}^{\infty}\in\mathscr{F}_p,\ p\in\mathscr{P}(t_{|F})\}.\tag{40}$$

(The proofs will be given at the end of the present proof.)

Noticing that for each seminorm $p'\in\mathscr{P}(t_{|G})$ there is a seminorm $p\in\mathscr{P}(t_{|F})$ such that $p(g)=p'(g)$, $g\in\mathscr{G}$, the Hahn-Banach theorem implies that for each $T^{(i)}\in G'$ satisfying $p'^0(T^{(i)})=1$ there is a functional $S^{(i)}\in F'$ such that

$$S^{(i)}(g)=T^{(i)}(g),$$
$$p^0(S^{(i)})=1,$$

$i=1, 2,\ldots$ Hence for each $\mathscr{T}\in C^0$ there is an $\mathscr{S}\in K^0$ such that

$$\mathscr{S}_{|G^\#\otimes G}=\mathscr{T},$$

and consequently,

$$C^0\subset\{\mathscr{S}_{|G^\#\otimes G};\ \mathscr{S}\in K^0\}.$$

Then,

$$K^{00}\cap\overline{(G^\#\otimes G)_h}=\{\xi\in\overline{(G^\#\otimes G)_h};\ \mathscr{S}(\xi)\leq 0\ \text{for all}\ \mathscr{S}\in K^0\}\subset$$

$$\subset\{\xi\in(G^\#\otimes G)_h;\ \mathscr{T}(\xi)\leq 0\ \text{for all}\ \mathscr{T}\in C^0\}=C^{00},$$

and

$$\overline{K}\cap\overline{(G^\#\otimes G)_h}\subset\overline{C}\tag{41}$$

follow due to (37), (38). Noticing that the converse of (41) is obviously true, the assertion under consideration is implied.

b) Due to the nuclearity of $F[\varepsilon_\infty]$ and Lemma 2.3 b), the continuity of

$$M: F^\#[\varepsilon_\infty]\otimes_\varepsilon F[\varepsilon_\infty]\rightarrow\{F,\#\}[\varepsilon_\infty]$$

follows. Noticing $M(K)=\{F,\#\}$, $M(C)=\{G,\#\}$, it is implied

$$\overline{\{G,\#\}}^{\varepsilon_\infty}=\overline{M(C)}^{\varepsilon_\infty}=M(\overline{C}^{\varepsilon'})=M(\overline{K}^{\varepsilon}\cap\overline{G^\#\otimes G})=\overline{\{F,\#\}}^{\varepsilon_\infty}\cap\overline{M(G^\#\otimes G)}^{\varepsilon_\infty},$$

where $\varepsilon'=(\varepsilon_{\infty|G^\#})\otimes_\varepsilon(\varepsilon_{\infty|G})$, $\varepsilon=\varepsilon_\infty\otimes_\varepsilon\varepsilon_\infty$.

Proof of (40). Recall that

$$(\mathcal{N}(F^\#,F))_h=\{\mathcal{T}=-\sum_{i,j=1}^{\infty}a_{ij}T^{(i)\#}\otimes T^{(j)};(a_{ij})_{i,j=1}^{\infty}\in\mathcal{A}_h,(T^{(i)})_{i=1}^{\infty}\in\mathcal{F}_p,p\in\mathcal{P}(t_{|F})\},$$

where $\mathcal{A}_h$ denotes the set of all hermitean matrices satisfying $\sum_{i,j=1}^{\infty}|a_{ij}|<\infty$ ([22, §21. 3]). Assuming $(a_{ij})_{i,j=1}^{\infty}\geq 0$ and considering $z_{il}=T^{(i)}(x^{(l)})$, $\xi=\sum_{l=1}^{N}x^{(l)\#}\otimes x^{(l)}$, $x^{(l)}\in F$, it follows

$$\sum_{i,j=1}^{\infty}|a_{ij}\sum_{l=1}^{N}\bar{z}_{il}z_{jl}|\leq\sum_{i,j=1}^{\infty}(|a_{ij}|\sum_{l=1}^{N}|\bar{z}_{il}z_{jl}|\leq$$

$$\leq(\sum_{l=1}^{N}p(x^{(l)\#})p(x^{(l)}))\sum_{i,j=1}^{\infty}|a_{ij}|<\infty$$

and

$$\mathcal{T}(\xi)=-\sum_{i,j=1}^{\infty}(a_{ij}(\sum_{l=1}^{N}\bar{z}_{il}z_{jl}))=-\sum_{l=1}^{N}\sum_{i,j=1}^{\infty}a_{ij}\bar{z}_{il}z_{jl}\leq 0.$$

Hence, $\mathcal{T}\in K_n^0$. On the other hand, assume now that $(a_{ij})\in\mathcal{A}_h$ and $(a_{ij})\not\geq 0$. Then there is a vector $z=(z_1,z_2,\ldots)^T$, $z_i\in\mathbb{C}$, $\max\{|z_i|;i\in\mathbb{N}\}\leq 1$ and $z_i\neq 0$ only for finitely many $i\in\mathbb{N}$ such that

$$c:=z^*(a_{ij})z<0,$$

where $z^*=(\bar{z}_1,\bar{z}_2,\ldots)$. Define on the subspace $\mathcal{L}=\text{span}\{T^{(1)},T^{(2)},\ldots\}\subset F'$ a linear functional $\tilde{x}$ by

$$\tilde{x}(T^{(i)})=z_i,$$

$i=1,2,3,\ldots$ Then,

$$p^{00}(\tilde{x}) = \sup\{\frac{|z_i|}{p^0(T^{(i)})}\ ;\ i=1,2,3,\ldots\} = \\ = \sup\{|z_i|\ ;\ i=1,2,3,\ldots\} \leq 1.$$

Using the Hahn-Banach theorem there is an $x \in F'' = F$ such that $x_{|L} = \tilde{x}$ and $p(x) \leq 1$. Settig $y = \frac{2x}{\sqrt{-c}}$,

$$\mathcal{T}(y^{\#} \otimes y) = \frac{4}{c} \sum_{i,j=1}^{\infty} a_{ij}\ \overline{T^{(i)}(x)}\ T^{(j)}(x) = \\ = \frac{4}{c} \sum_{i,j=1}^{\infty} a_{ij}\ \bar{z}_i z_j = 4$$

implies $\mathcal{T} \notin K^0$. (39) follows analogously.

Lemma 3.9 and Lemma 3.11 b) now yield the following.

Theorem 3.12. *Assume that the assumptions a), b) of Theorem* 3.10 *are satisfied. Let us further assume c'): Let $F[t]$ be nuclear, and let there be a sequence $(G_n)_{n=1}^{\infty}$ of subspaces $G_n \subset F$ such that*

$$G_1 \subset G_2 \subset \ldots,\ \bigcup_{n=1}^{\infty} G_n = F,$$

and $t_n := t_{|G_n}$ are metrizable, $n=1, 2, \ldots$. Then,

$$\overline{\{F,\#\}}^{\tilde{\varepsilon}} = \overline{\{F,\#\}}^{\varepsilon_\infty} = \overline{\{F,\#\}}^{\varepsilon\otimes} = \\ = \{\sum_{i=1}^{\infty} f^{(i)\#} f^{(i)}\ ;\ f^{(i)} \in \widetilde{G_m} \cap \widetilde{E^n},\ i=1,2,3,\ldots, \\ \sum \textit{is convergent in } E^{2n}[\varepsilon^{2n}],\ n, m=1,2,3,\ldots\}.$$

§3.3. On the Closed Hull of the Convex Hull of Finitely Many Alg-# Cones

In the following the convex hull

$$\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\}, \tag{42}$$

$\overset{(s)}{F} \subset E_{\otimes}$, of finitely many alg-# cones $\{\overset{(s)}{F}, \#_s\}$, $s=1, 2, 3, \ldots, l$, is considered. For a given functional $\pounds : E_{\otimes} \to \mathbb{C}$ and $\sum_{i=1}^{M_s} a^{(i,s)\#_s} a^{(i,s)} \in \{\overset{(s)}{F}, \#_s\}$, $a^{(i,s)} \in F^{(s)}$, $i=1, 2, \ldots, M_s$ ($M_s \in \mathbb{N}$), set

$$L_n^{(s)} = \sqrt{\pounds_{2n}\Big(\sum_{i=1}^{Ms} a_n^{(i,s)\#_s} \otimes a_n^{(i,s)}\Big)}\,, \tag{43}$$

$$L_n^{(1,2,\cdots,l)} = \sqrt{\pounds_{2n}\Big(\sum_{s=1}^{l}\sum_{i=1}^{Ms} a_n^{(i,s)\#_s} \otimes a_n^{(i,s)}\Big)}\,, \tag{44}$$

Remember that condition (A) is modified for the convex hull (42) to (A') such that [I : (32)] holds. Definition 3.1 is now generalized to the convex hull (42). It is said that some l.c. topology τ satisfies the *closure condition with respect to the convex hull* (42), if ii), iii) of Definition 3.1 are satisfied and i) is replaced by i') : For each semi norm $p \in \mathcal{P}(\tau)$ there is a sequence $(\omega_n)_{n=0}^{\infty}$, $\omega_n > 0$, such that p satisfies (A') with respect to (42) and $(\omega_n)_{n=0}^{\infty}$:

$$p_n\Big(\sum_{\substack{r+k=n\\ r\neq k}} \sum_{s=1}^{l}\sum_{i=1}^{Ms} a_n^{(i,s)\#_s} \otimes a_n^{(i,s)}\Big) \leq \sum_{\substack{r+k=n\\ r\neq k}} \omega_r \omega_k L_r^{(1,\cdots,l)} L_s^{(1,\cdots,l)},$$

where $p_n(f_n) = p(\check{f}_n)$, see [I : (32)].

Theorem 3.13. *Let us be given a topological tensor algebra $E_{\otimes}[\tau]$ with l alg-# cones*

$$\{\overset{(s)}{F}, \#_s\},\ s=1, 2, 3, \ldots, l\,(l \in \mathbb{N}).$$

Assume further that τ satisfies the closure condition with respect to the convex hull $\sum_{s=1}^{l}\{\overset{(s)}{F}, \#_s\}$.

a) *Then,*

$$\overline{\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\}}^{\tau} \cap E^{2n} = \overline{\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\} \cap E^{2n}}^{\tau}$$

for $n = 0, 1, 2, \ldots$.

b) *Assume furthermore that $E^n[t^n]$, (resp. $\widetilde{E^n}[t^n]$), $n = 0, 1, 2, \ldots$, are semi-Montel spaces, and for every $p \in \mathcal{P}(\tau)$ there is a sequence $(d_n)_{n=0}^{\infty}$, $d_n > 0$, such* *that*

$$\sum_{s=1}^{l} L_n^{(s)} \leq d_n L_n^{(1,2,\dots,l)}, \tag{45}$$

$n=0, 1, 2,\dots,$ *where* $L_n^{(\cdot)}$ *are defined in* (43), (44) *replacing* $\mathcal{L}$ *by* $p \in \mathcal{P}(\tau)$. *Then,*

$$\overline{\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\}}^{\tau} = \sum_{s=1}^{l} \overline{\{\overset{(s)}{F}, \#_s\}}^{\tau},$$

where the closed hulls are considered in $E_{\otimes}[\tau]$ (*resp.* $\widetilde{E_{\otimes}}[\tau]$). *Further, for every*

$$v = k^{(1)} + \dots + k^{(l)} \in \overline{\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\}}^{\tau},$$

$k^{(s)} \in \overline{\{\overset{(s)}{F}, \#_s\}}^{\tau}$, *with* Grad $(v) \leq 2N$, *it then follows that*

$$\text{Grad}\,(k^{(s)}) \leq 2N,\ s=1,\dots, l.$$

Proof. a) The proof is analogous to that of Theorem 3.1. b) Obviously,

$$\overline{\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\}}^{\tau} \supset \sum_{s=1}^{l} \overline{\{\overset{(s)}{F}, \#_s\}}^{\tau}.$$

For proving the converse, consider some $v \in \overline{\sum_{s=1}^{l} \{\overset{(s)}{F}, \#_s\}}^{\tau}$. There is then a net $(k^{(1,\beta)} + \dots + k^{(l,\beta)})_{\beta \in \mathrm{B}}$, B is a directed set of indices, such that $k^{(s,\beta)} \in \{\overset{(s)}{F}, \#_s\}$ ($s = 1, 2,\dots,l$) and

$$k^{(1,\beta)} + \dots + k^{(l,\beta)} \longrightarrow v \tag{46}$$

with respect to τ. Using a) there are indices $N \in \mathbb{N}$ and $\beta_0 \in \mathrm{B}$ such that

$$\text{Grad}\,(k^{(1,\beta)} + \dots + k^{(l,\beta)}) \leq 2N \tag{47}$$

for $\beta \in \mathrm{B}$ with $\beta > \beta_0$. For $p \in \mathcal{P}(\tau)$ and

$$k^{(s,\beta)} = \sum_{i=1}^{Ms} a^{(s,\beta,i)\#_s}\ a^{(s,\beta,i)} \in \{\overset{(s)}{F}, \#_s\}$$

consider

$$L_n^{(s;p,\beta)} = \sqrt{p((\sum_{i=1}^{Ms} a_n^{(s,\beta,i)\#_s} \otimes a_n^{(s,\beta,i)})\check{\ })},$$

$$L_n^{(1,2,\cdots,l,p,\beta)} = \sqrt{p((\sum_{s=1}^{l}\sum_{i=1}^{Ms} a_n^{(s,\beta,i)\#_s} \otimes a_n^{(s,\beta,i)})\check{\ })}.$$

Using (47), [I: Lemma 3.2], (45), it follows

$$L_n^{(1,2,\cdots;p,\beta)} = 0, \tag{48}$$

$$L_n^{(1;p,\beta)} = 0 \tag{49}$$

for each $p \in \mathscr{P}(\tau)$, $\beta > \beta_0$, and $n > N$. Hence, there is a constant $D > 0$ depending on p such that

$$p(k^{(1,\beta)}) < D \tag{50}$$

for $\beta > \beta_0$, because otherwise, due to

$$p(k^{(1,\beta)}) \le \sum_{n=0}^{\infty} p(\check{k}_n^{(1,\beta)}) \overset{(*)}{\le} \sum_{n=0}^{2N} \Big((L_{n/2}^{(1;p,\beta)})^2 + \sum_{\substack{r+s=n \\ r \neq s}} \omega_r \omega_s\, L_r^{(1;p,\beta)}\, L_s^{(1,p,\beta)} \Big),$$

there would be an $n_0 \in \mathbf{N}$, $0 \le n_0 \le N$, such that $(L_{n_0}^{(1,p,\beta)})_{\beta \in \mathrm{B}}$ is not bounded yielding a contradiction to

$$L_{n_0}^{(1;p,\beta)} \le d_{n_0} L_{n_0}^{(1,\cdots l;p,\beta)} \overset{(+)}{\le}$$

$$\le d_{n_0} \Big\| \sum_{s=1}^{l} \sum_{i=1}^{Ms} a^{(s,\beta,i)\#_s}\ a^{(s,\beta,i)} \Big\|_{(p,(\alpha n))},$$

the τ-continuity of $\|.\|_{(p,(\alpha_n))}$ (see (11)), (46) and $v \in E_{\otimes}$. Further, (49) implies

$$p((\sum_{i=1}^{M_1} a_n^{(1,\beta,i)\#_1} \otimes a_n^{(1,\beta,i)})\check{\ }) = 0$$

for all $n>N$, $p\in\mathcal{P}(\tau)$ and $\beta>\beta_0$. Since τ is a Hausdorff topology,

$$\sum_{i=1}^{M_1} a_n^{(1,\beta,i)\#_1}\otimes a_n^{(1,\beta,i)}=0$$

follows. Using [I : Theorem 2.3b)], $a_n^{(1,\beta,i)}=0$ is implied for $i=1, 2,\ldots, M_1$, $n>N$, and $\beta>\beta_0$. [I : Lemma 2.2a)] implies now that

$$\mathrm{Grad}(k^{(1,\beta)})\leq 2N \tag{51}$$

for $\beta>\beta_0$. Since $E^{(2N)}[t^{(2N)}]$ is a semi-Montel space, (50) yields that the set $\{k^{(1,\beta)}\,;\,\beta\in \mathrm{A},\ \beta>\beta_0\}$ is relatively compact in $E^{(2N)}[t^{(2N)}]$. Hence, there are a cofinal subset $\mathrm{B}'\subset\mathrm{B}$ and a $k^{(1)}\in E^{(2N)}$ such that $k^{(1,\beta)}\to k^{(1)}$ for $\beta\in\mathrm{B}'$. (46) thus implies

$$v\in\overline{\{\overset{(1)}{F},\#_1\}}^{\tau}+\overline{\sum_{s=2}^{l}\{\overset{s}{F},\#_s\}}^{\tau}.$$

The assertions to be shown now follow by induction. ((*) is a consequence of [I : (21)] and (49). (+) follows from Theorem 2.2.)

In the following let us discuss an interesting example for the convex hull of two alg-# cones used by G.Hegerfeldt in the axiomatic approach to Euclidean QFT([11]). See also [I: Examples 2.4, 3.7].

Example 3.14. Setting $\mathcal{S}=\mathcal{S}(\mathbb{R}^d)$, $d\in\mathbb{N}$, $d\geq 2$ (Schwartz space of basic (rapidly dimishing) functons), let us consider the $\varepsilon_\otimes$-completed hull

$$\widetilde{\mathcal{S}_\otimes}=\mathbb{C}\oplus\mathcal{S}\oplus\widetilde{\mathcal{S}}_2\oplus\widetilde{\mathcal{S}}_3\oplus\ldots,$$

where $\widetilde{\mathcal{S}}_m=\mathcal{S}(\mathbb{R}^{dm})$, $m=2, 3,\ldots$ The cone of positivity is then given by

$$\widetilde{\mathcal{S}}_\otimes^+=\{\sum_{i=1}^{M} f^{(i)*}f^{(i)}\,;\,f^{(i)}\in\widetilde{\mathcal{S}_\otimes},\ M\in\mathbb{N}\},$$

where for $f=(f_0, f_1,\ldots, f_N, 0, 0,\ldots)\in\widetilde{\mathcal{S}_\otimes}$ the involution is given by $f^*=(\overline{f_0}, f_1^*,\ldots, f_N^*, 0, 0,\ldots)$ with

$$f_n^*(x_1,x_2,\ldots,x_n)=\overline{f_n(x_n,\ x_{n-1},\ldots,x_1)},$$

$n=1,\ 2,\ldots.$

As in [I: Example 2.4b)] let us also consider the cone of reflexion positivity $\{\mathscr{F},\#\}$, where $\mathscr{F}=\oplus_{m=0}^{\infty}\mathscr{F}_m$, $\mathscr{F}_0=\mathbf{C}$ and

$$\mathscr{F}_n=\{f_n\in\widetilde{\mathscr{S}}_n\ ;\ \mathrm{supp}(f_n)\subset\{x\in\mathbf{R}^{dn}\ ;\ 0<x_1^0<x_2^0<\ldots<x_n^0\}\},$$

$x_i=(x_i^0,\ x_i^1,\ldots,x_i^{d-1})\in\mathbf{R}^d$ $(i=1,\ 2,\ldots,n)$, $n\in\mathbf{N}$, and the antilinear bijection # is defined by setting $f^{\#}=(\overline{f_0},\ f_1^{\#},\ f_2^{\#},\ldots)$,

$$(f_n)^{\#}(x_1,\ldots,x_n)=\overline{f_n(\tilde{x}_n,\ldots,\tilde{x}_1)},$$

$\tilde{x}_i=(-x_i^0,\ x_i^1,\ldots,x_i^{d-1})\in\mathbf{R}^d$, $\bar{\cdot}$ denotes the complex value of $\cdot$. Recall that $\widetilde{\mathscr{S}}_{\otimes}^{+}$ and $\{\mathscr{F},\#\}$ are involutive cones.

On $\widetilde{\mathscr{S}}_n$ the semi norms $f_n\to p_n^{(m)}(f_n)$, $f_n\in\widetilde{\mathscr{S}}_n$, are considered, where

$$p_n^{(m)}(f_n)=\sup_{x\in\mathbf{R}^{dn}}\max_{r_i^j\leq m}\left|\prod_{i=1}^{n}\prod_{j=0}^{d-1}(1+(x_i^j)^2)^m(\partial/\partial x_i^j)^{r_i^j}f_n(x_1,\ldots,x_n)\right|,$$

$m,\ n=0,\ 1,\ 2,\ldots,p_0(f_0)=|f_0|$. Setting

$$p_{(\gamma_n),m}(f)=\sum_{n=0}^{\infty}\gamma_n p_n^{(m)}(f_n),$$

$f\in\widetilde{\mathscr{S}}_{\otimes}$, the system of semi norms $\{p_{(\gamma_n),m}\ ;\ (\gamma_n)\in\mathbf{R}_+^{\mathbf{N}^*},\ m=0,\ 1,\ 2,\ldots\}$ then defines the topology ε_∞ on $\mathscr{S}_{\otimes}$. Because of Remark c) on Definition 3.1, it follows that ε_∞ satisfies the closure condition with respect to $\widetilde{\mathscr{S}}_{\otimes}^{+}$ and $\{\mathscr{F},\#\}$.

For every $p_{(\gamma_n),m}$ and each $\sum_{i=1}^{M}a^{(i)*}a^{(i)}\in\widetilde{\mathscr{S}}_{\otimes}^{+}$, $\sum_{j=1}^{M'}b^{(j)*}b^{(j)}\in\{\mathscr{F},\#\}$ $(M,\ M'\in\mathbf{N})$, let us put

$$L_n^{(*)}=\sqrt{p_{2n}^{(m)}\ \left(\sum_{i=1}^{M}a_n^{(i)*}\otimes a_n^{(i)}\right)},\tag{52}$$

$$L_n^{(\#)}=\sqrt{p_{2n}^{(m)}\ \left(\sum_{i=1}^{M'}b_n^{(i)*}\otimes b_n^{(i)}\right)},\tag{53}$$

$$L_n^{(*,\#)}=\sqrt{p_{2n}^{(m)}\left(\sum_{i=1}^{M} a_n^{(i)*}\otimes a_n^{(i)}+\sum_{i=1}^{M'} b_n^{(i)*}\otimes b_n^{(i)}\right)}. \tag{54}$$

Recall that

$$L_n^{(*)}+L_n^{(\#)}\leq(1+\sqrt{2})\,L_n^{(*,\#)}, \tag{55}$$

$n=0, 1, 2,\ldots$ (see [I: Example 3.7. Lemma]). [I: Lemma 3.6] implies now that ε_∞ satisfies the closure condition with respect to $\widetilde{\mathcal{S}}_\otimes^+ +\{\mathcal{F},\#\}$. Since $\widetilde{\mathcal{S}}^n[\varepsilon^n]$, $n=0, 1, 2,\ldots$, are Montel spaces, Theorem 3.13 and Corollary 3.2a) imply

$$\overline{\widetilde{\mathcal{S}}_\otimes^+ +\{\mathcal{F},\#\}}^{\varepsilon_\infty}=\overline{\widetilde{\mathcal{S}}_\otimes^+}^{\varepsilon_\otimes}+\overline{\{\mathcal{F},\#\}}^{\varepsilon_\otimes}.$$

Finally,

$$\overline{\widetilde{\mathcal{S}}_\otimes^+}^{\varepsilon_\otimes}=\{\sum_{i=1}^{\infty} f^{(i)*}f^{(i)}\,;\, f^{(i)}\in\widetilde{\mathcal{S}_\otimes},\ \textstyle\sum \text{ is convergent in } \widetilde{\mathcal{S}_\otimes}[\varepsilon_\otimes]\},$$

$$\overline{\{\mathcal{F},\#\}}^{\varepsilon_\otimes}=\{\sum_{i=1}^{\infty} h^{(i)\#}h^{(i)}\,;\, h^{(i)}\in\widetilde{\mathcal{F}},\ \textstyle\sum \text{ is convergent in } \widetilde{\mathcal{S}_\otimes}[\varepsilon_\otimes]\}$$

follow.

§4. On the Extremal Rays of Alg-# Cones in Topological Tensor Algebras

§4.1. A Theorem on the Representation of Alg-# Cones as the Sum of Their Extremal Rays

Let us start with some definitions and notions of the theory of extremal rays and convex sets (e.g., [23, §25]). Let X denote some vector space. For each $x\in X$, $x\neq 0$, let us consider the ray

$$\rho(x)=\{y\in X\,;\, y=\mu x,\ 0<\mu<\infty\}.$$

Let further$K$ be some convex cone with apex 0 in X. Then, $\rho(x)\subset K$ is called *extremal ray*, if every open intervall contained in K and intersecting $\rho(x)$ is completely contained in $\rho(x)$. The set of all the extremal rays of K is denoted by $\mathcal{ER}(K)$. In the following let

$$(a,b)=\{y\in X;\ y=\mu a+(1-\mu)\,b,\ 0<\mu<1\}$$

and

$$[a,\ b]=\{y\in X;\ \text{there are } k^{(i)}\in K\ (i=1,2)\ \text{with } y=a+k^{(1)}=b-k^{(2)}\}$$

denote the *open intervall* and *order intervall* between $a, b\in X$, respectively, There is the following characterization of extremal rays.

Lemma 4.1. *Let us be given a vector space X, a convex cone $K\subset X$ with apex 0, and some $0\neq x\in K$. The following are then equivalent:*

i) $\rho(x)\in\mathscr{ER}(K)$,
ii) *for each $k\in[0,x]$, $k\neq 0$, there exists $0<\mu\leq 1$ with $k=\mu x$,*
iii) *if $k^{(i)}\in K$ and $x=k^{(1)}+k^{(2)}$, then there are μ_i, $0<\mu_i\leq 1$, such that $k^{(i)}=\mu_i x$, $i=1,2$.*

The *proof* is straightforward.

The following two lemmas are the prerequisites for the proof of the main theorem of this section. There is the following characterization of extremal rays in the case of homogeneous elements $f\in F$ (Grad (f) = grad (f)).

Lemma 4.2. *Let $\{F,\#\}$ be an alg-# cone in some tensor algebra $E_\otimes$. Let further $E_\otimes[\varepsilon_\otimes]$ be nuclear and the involution $\#: E_\otimes[\varepsilon_\otimes]\to E_\otimes[\varepsilon_\otimes]$ be continuous. If furthermore*

$$\{\widetilde{F,\#}\}^{\varepsilon\otimes}\cap\widetilde{E^{2n}}=\{\sum_{i=1}^{\infty}g^{(i)\#}g^{(i)};\ g^{(i)}\in\tilde{F}\cap\widetilde{E^{n}},\ \textstyle\sum \textit{ is convergent in } \widetilde{E^{2n}}[\varepsilon^{2n}]\},$$

$n=0,1,2,\ldots$, then for each $f\in F$ with $\mathrm{Grad}(f)=\mathrm{grad}(f)$ *it follows that*

$$\rho(f^{\#}f)\in\mathscr{ER}(\{\widetilde{F,\#}\}^{\varepsilon\otimes}).$$

Proof. In the proof an idea of Brauer is used (see [7]). Assume that there are $k, k'\in\{\widetilde{F,\#}\}^{\varepsilon\otimes}$, $k\neq 0$, $k'\neq 0$, such that

$$f^{\#}f=k+k'. \tag{56}$$

Since there is an $M\in\mathbf{N}$ with $k, k'\in\widetilde{E^{2M}}$, the assumptions of the lemma under consideration imply the existence of $g^{(i)}, g'^{(i)} \in \tilde{F} \cap \widetilde{E^M}$ such that

$$k=\sum_{i=1}^{\infty} g^{(i)\#} g^{(i)},$$

$$k'=\sum_{i=1}^{\infty} g'^{(i)\#} g'^{(i)}, \tag{57}$$

and $\sum$ is convergent in $\widetilde{E^M}[\varepsilon^M]$. The assumption Grad $(f)=$ grad $(f)=$ N, $f \in F$, implies

$$\mathrm{Grad}\,(f^{\#}f)=\mathrm{grad}\,(f^{\#}f)=2N. \tag{58}$$

Then, $\max\{\mathrm{Grad}\,(k), \mathrm{Grad}\,(k')\}=2N$, because if

$$2N>\max\{\mathrm{Grad}\,(k), \mathrm{Grad}\,(k')\} \overset{[I\cdot(2)]}{\geq} \mathrm{Grad}\,(k+k')=2N$$

there would be a contradiction, and if $2M'=\max\{.\}>2N$ there would be some $i_0 \in \mathbb{N}$ such that $g_{M'}^{(i_0)} \neq 0$ or $g'^{(i_0)}_{M'} \neq 0$, and

$$\sum_{i=1}^{\infty} g_{M'}^{(i)\#} \otimes g_{M'}^{(i)}+\sum_{i=1}^{\infty} g'^{(i)\#}_{M'} \otimes g'^{(i)}_{M'}=0$$

yielding a contradiction to $(g_{M'}^{(i)\#} \otimes g_{M'}^{(i)})\check{}$, $(g'^{(i)\#}_{M'} \otimes g'^{(i)}_{M'}) \in \{F, \#\}$ $(i=1, 2, 3, \ldots)$ and the fact that $\widetilde{\{F, \#\}}^{\varepsilon_\infty}$ is a (proper) cone. By the same reasoning, min $\{\mathrm{grad}\,(k), \mathrm{grad}\,(k')\}=2N$ follows. Hence,

$$\mathrm{Grad}\,(k)=\mathrm{Grad}\,(k')=\mathrm{grad}\,(k)=\mathrm{grad}\,(k')=2N,$$

implying

$$\mathrm{Grad}\,(g^{(i)})=\mathrm{Grad}\,(g'^{(i)})=\mathrm{grad}\,(g^{(i)})=\mathrm{grad}\,(g'^{(i)})=N \tag{59}$$

by [I : Lemma 2.2a)], $i=1, 2, 3, \ldots$.

For every continuous functional T_N on $E_N[\varepsilon_N]$ with $T_N(f_N)=0$ and each $j \in \mathbb{N}$, it follows

$$|T_N(g_N^{(j)})|^2+|T_N(g'^{(j)}_N)|^2 \leq \sum_{i=1}^{\infty} |T_N(g_N^{(i)})|^2+\sum_{i=1}^{\infty} |T_N(g'^{(i)}_N)|^2$$
$$= T_N^{\#} \otimes T_N(\sum_{i=1}^{\infty} g_N^{(i)\#} \otimes g_N^{(i)}) + T_N^{\#} \otimes T_N(\sum_{i=1}^{\infty} g'^{(i)\#}_N \otimes g'^{(i)}_N)$$
$$= T_N^{\#} \otimes T_N(f_N^{\#} \otimes f_N) = |T_N(f_N)|^2=0,$$

and consequently, $g_N^{(j)}=\mu^{(j)} f_N$, $g'^{(j)}_N=\mu'^{(j)} f_N$, $\mu^{(j)}, \mu'^{(j)} \in \mathbf{C}$. (59) and (56) imply

$$\sum_{i=1}^{\infty} |\mu^{(i)}|^2+\sum_{i=1}^{\infty} |\mu'^{(i)}|^2=1,$$
$$k=(\sum_{i=1}^{\infty} |\mu^{(i)}|^2)f^{\#}f,$$
$$k'=(\sum_{i=1}^{\infty} |\mu'^{(i)}|^2)f^{\#}f.$$

The assertion to be shown is now a consequence of Lemma 4.1.

In the following let δ_{ij} denote *Kronecker's delta*.

Lemma 4.3. *Let us be given a tensor algebra* $E_{\otimes}$ *with nuclear basic space* $E[t]$, *continuous mapping* #, *and an alg-# cone* $\{F, \#\}$ *satisfying* (5). *If* $f^{\#} f= \sum_{i=1}^{\infty} g^{(i)\#} g^{(i)}$ *with* Grad$(f)=M$, $f, g^{(i)} \in \tilde{F}^{\varepsilon_\infty}$, *then there is a unitary matrix* $\mathbf{U}=(u_{ij})_{i,j=1}^{\infty}$, $u_{ij} \in \mathbf{C}$, *such that*

$$h^{(i)}=\sum_{j=1}^{\infty} u_{ij}g^{(j)} \in \tilde{F} \cap \widetilde{E^M},\ h_M^{(i)}=\delta_{1i}f_M\ (i=1, 2, 3, \ldots),$$
$$f^{\#}f=\sum_{i=1}^{\infty} h^{(i)\#} h^{(i)},$$

where all the sums are convergent in $\widetilde{E_{\otimes}}[\varepsilon_\infty]$.

Proof. In the proof there are used some ideas of H.J.Borchers (see [5, VI.1, VI.3]). The proof is subdivided into two steps i), ii).

i) Let $\mathbf{U}=(u_{ij})_{i,j=1}^{\infty}$ *be any unitary matrix and* $\sum_{i=1}^{\infty} g^{(i)\#} g^{(i)}$, $g^{(i)} \in \tilde{F}$, *be convergent in* $\widetilde{E}_{\otimes}[\varepsilon_\infty]$. *Then,* $\sum_{j=1}^{\infty} u_{ij}g^{(j)} = h^{(i)}$, $\sum_{i=1}^{\infty} h^{(i)\#} h^{(i)}$ *are summing in* $\widetilde{E_{\otimes}}[\varepsilon_\infty]$. *Furthermore,*

$$\sum_{i=1}^{\infty} g^{(i)\#} g^{(i)}=\sum_{i=1}^{\infty} h^{(i)\#} h^{(i)}.$$

Proof of i). As in the proof of Lemma 4.2, $f^{\#} f=\sum_{i=1}^{\infty} g^{(i)\#} g^{(i)}$ implies $g^{(i)} \in \tilde{F}^{\varepsilon}\otimes \cap \widetilde{E^{M}}$, $i=1, 2, 3,\ldots$. Corollary 2.6 implies that for every ε_{∞} semi norm p there is some ε_{∞}-continuous seminorm q such that

$$\sum_{j=1}^{\infty} p(u_{ij}\, g^{(j)}) \leq \sum_{j=1}^{\infty} |u_{ij}| p(g^{(j)}) \leq \sqrt{(\sum_{j=1}^{\infty} |u_{ij}|^2)(\sum_{j=1}^{\infty} p(g^{(j)})^2)} = \sqrt{\sum_{j=1}^{\infty} p(g^{(j)})^2}$$
$$\leq \sqrt{q(f^{\#} f)} < \infty,$$

and consequently,

$$h^{(i)}=\sum_{j=1}^{\infty} u_{ij}\, g^{(j)} \in \tilde{E}^{M}[\varepsilon^{M}]$$

and $\sum \cdot$ is absolutely summing. Further, using Lemma 2.4 and Corollary 2.6, there are $q', p' \in \mathscr{P}(\varepsilon_{\infty})$ such that

$$\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} p(\bar{u}_{ij}\, u_{il}\, g^{(j)\#} g^{(l)}) \leq \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} |u_{ij}\, u_{il}|\, p'(g^{(j)}) p'(g^{(l)})$$
$$\leq \sqrt{(\sum_{j=1}^{\infty} |u_{ij}|^2)(\sum_{j=1}^{\infty} p'(g^{(j)})^2)} \sqrt{(\sum_{l=1}^{\infty} |u_{il}|^2)(\sum_{l=1}^{\infty} p'(g^{(l)})^2)}$$
$$\leq q(f^{\#} f)$$

proving that for each $i \in \mathbb{N}$, $h^{(i)\#} h^{(i)}=\sum_{j=1}^{\infty} \sum_{j=1}^{\infty} \bar{u}_{ij} u_{il}\, g^{(j)\#} g^{(l)}$ is also absolutely summing.

Setting $h^{(i,n)}=\sum_{j=1}^{n} u_{ij}\, g^{(j)}$, notice that for n fixed,

$$\sum_{i=1}^{\infty} p(h^{(i,n)\#} h^{(i,n)})=\sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{k=1}^{n} p(\bar{u}_{ji}\, u_{ik} g^{(j)\#} g^{(k)}) \leq d\sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{k=1}^{n} |u_{ji}\, u_{ik}|$$
$$=d\sum_{i=1}^{\infty} (\sum_{j=1}^{n} |u_{ji}| \sum_{k=1}^{n} |u_{ik}|) \leq d \sqrt{\sum_{i=1}^{\infty} (\sum_{j=1}^{n} |u_{ji}|)^2 \sum_{l=1}^{\infty} (\sum_{k=1}^{n} |u_{lk}|)^2}$$
$$\leq 2^n d \sqrt{(\sum_{i=1}^{\infty} \sum_{j=1}^{n} |u_{ji}|^2)(\sum_{l=1}^{\infty} \sum_{k=1}^{n} |u_{lk}|^2)} = 2^n\, nd,$$

implies that $\sum_{i=1}^{\infty} h^{(i,n)\#} h^{(i,n)}$ is absolutely summing, where $p \in \mathscr{P}(\varepsilon_\infty)$,

$$d=\max\{p(g^{(j)\#} g^{(k)})\ ;\ j, k=1, 2,\dots, n\},$$

and $\sum_{i=1}^{\infty} \bar{u}_{ji}\, u_{ik}=\delta_{jk}$. Now,

$$p(\sum_{i=1}^{\infty} (h^{(i,n)\#} h^{(i,n)}) = p(\sum_{i=1}^{\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{u}_{ji}\, u_{ik}\, g^{(j)\#} g^{(k)} \leq p(\sum_{j=1}^{n} \sum_{k=1}^{n} (\sum_{i=1}^{\infty} (\bar{u}_{ji}\, u_{ik})\, g^{(j)\#} g^{(k)}$$
$$\leq \sum_{j=1}^{n} p'(g^{(j)})^2 \leq q'(f^{\#} f)$$

implies that $\sum h^{(i)\#} h^{(i)}$ is summing in $E^{2M}[\varepsilon^{2M}]$. Noting $h^{(i)} \in \tilde{F}$, Lemma 2.5 implies that $\sum h^{(i)\#} h^{(i)}$ is absolutely summing, too. $\sum_{i=1}^{\infty} g^{(i)\#} g^{(i)} = \sum_{i=1}^{\infty} h^{(i)\#}\, h^{(i)}$ is now obvious.

ii) For $\check{f}_M \in F$, Lemma 4.2 implies

$$g_M^{(i)} = \mu_i\, f_M, \tag{60}$$

$\mu_i \in \mathbf{C}$, $\sum_{i=1}^{\infty} |\mu_i|^2 = 1$. Now, choose a unitary matrix $\mathbf{V} = (v_{ij})_{i,j=1}^{\infty}$, $v_{ij} \in \mathbf{C}$, with $v_{1j} = \bar{\mu}_j$, $j=1, 2, 3,\dots$ Using i), it follows

$$f^{\#} f = \sum_{i=1}^{\infty} h^{(i)\#}\, h^{(i)}$$

for $h^{(i)} = \sum_{j=1}^{\infty} v_{ij}\, g^{(j)}$. (60), the definition of v_{1j} and the unitarity of $\mathbf{V}$ then imply

$$h_M^{(1)} = \sum_{j=1}^{\infty} v_{1j}\, g_M^{(j)} = \sum_{j=1}^{\infty} |\mu_j|^2\, f_M = f_M,$$
$$h_M^{(s)} = \sum_{j=1}^{\infty} v_{sj}\, g_M^{(j)} = (\sum_{j=1}^{\infty} v_{sj}\, \bar{v}_{1j})\, f_M = 0,$$

$s=2, 3, 4,\dots$ The proof of the lemma under consideration is comleted.

We are now able to state and prove the main theorem of this section.

Theorem 4.4. *Let us be given an alg-# cone* $\{F, \#\}$ *satisfying* (5) *in some* *tensor algebra* $E_\otimes$. *Further let the following be satiesfied*:

a) *mapping* $\#$ *is* $\varepsilon_\otimes$*-continuous,*
b) $\widetilde{E^n}[\varepsilon^n]$, $n=1, 2, 3,\ldots (\varepsilon^1=t)$, *are nuclear semi-Montel spaces,*
c) *it holds*

$$\{\widetilde{F,\#}\}^{\varepsilon\otimes} \cap \widetilde{E^{2n}} = \{\sum_{i=1}^{\infty} g^{(i)\#} g^{(i)} ; g^{(i)} \in \tilde{F} \cap \widetilde{E^n}, \sum \text{ is summing in } \widetilde{E_\otimes}[\varepsilon_\otimes]\}$$

$n=0, 1, 2,\ldots.$

Then,
(I) *for each* $k \in \{\widetilde{F,\#}\}^{\varepsilon\otimes}$, *then there is a sequence* $(z^{(i)})_{i=1}^{\infty}$, $z^{(i)} \in \tilde{F}$, *such that*

$$\rho(z^{(i)\#} z^{(i)}) \in \mathscr{E}\mathfrak{R}(\{\widetilde{F,\#}\}^{\varepsilon\otimes}) \ (i=1, 2, 3,\ldots),$$

$$k=\sum_{i=1}^{\infty} z^{(i)\#} z^{(i)},$$

where $\sum \cdot$ *is summing in* $\widetilde{E_\otimes}[\varepsilon_\otimes]$,
(II) *the decomposition of* k *given in* (I) *is unique* (*up to rearrangements of summands*), *if and only if* dim $(F)=1$.

Proof. (I) : The proof is subdivided into four steps $i)$,..., $iv)$, and it is based on an application of Zorn's lemma. In the following let $\mathscr{E}\mathfrak{R}(\{\widetilde{F,\#}\}^{\varepsilon\otimes})$ be abbreviated by $\mathscr{E}\mathfrak{R}$.

i) *For* $0\neq f \in F$, $M=\mathrm{Grad}(f)$, *consider*

$\mathscr{X}_f=\{x \in F ; x_M=f_M$, *there is some* $k_x \in \{\widetilde{F,\#}\}^{\varepsilon\otimes}$ *with* $f^\# f=x^\# x+k_x\}$,
$\mathscr{Y}_f=\{k_x \in \{\widetilde{F,\#}\}^{\varepsilon\otimes}$; *there is some* $x \in \mathscr{X}_f$ *with* $f^\# f=x^\# x+k_x\}$.

Then there is some $z \in \mathscr{X}_f$ *with* $\rho(z^\# z) \in \mathscr{E}\mathfrak{R}$.

Proof of i). If $M=0$, then i) is satisfied since $f \in \mathscr{X}_f$ and $\rho(f^\# f) \in \mathscr{E}\mathfrak{R}$. Assume $M>1$ now. Since $f \in \mathscr{X}_f$, $\mathscr{X}_f \neq \emptyset$ follows. If $x \in \mathscr{X}_f$, then

$$\mathrm{Grad}(x)=M, \ \mathrm{Grad}(k_x) \leq 2M-2$$

follow since $\{\widetilde{F,\#}\}$ is a (proper) cone. Consider $\mathscr{P}(\varepsilon_\infty)$ introduced in Lemma 2.4. The boundedness of $\mathscr{X}_f \subset \tilde{E}^{2M}[\varepsilon^{2M}]$ and $\mathscr{Y}_f \subset \tilde{E}^{2M-2}[\varepsilon^{2M-2}]$ follow, since for each $p \in \mathscr{P}(\varepsilon_\infty)$ there is some $p' \in \mathscr{P}(\varepsilon_\infty)$ such that

$$p(x) \leq \sqrt{p'(x^\# x)} \leq \sqrt{p'(f^\# f)} =: c_p < \infty \tag{61}$$

$$p(k_x) \leq p(f^\# f) < \infty.$$

Let us now introduce a semi-ordering "$\prec$" in $\mathfrak{X}_f$ by setting $x \prec y$, x, $y \in \mathfrak{X}_f$, if there is a $k_{x,y} \in \overline{\{F, \#\}}^{\varepsilon \otimes}$ with

$$x^\# x = y^\# y + k_{x,y}. \tag{62}$$

$x \prec y$ then imply

$$\begin{aligned} f^\# f &= x^\# x + k_x = y^\# y + k_{x,y} + k_x = y^\# y + k_y \\ k_x &= k_{x,y} + k_x, \end{aligned}$$

and consequently

$$\begin{aligned} p(x^\# x) &\geq p(y^\# y), \\ p(k_y) &\geq p(k_x), \end{aligned} \tag{63}$$

for all $p \in \mathscr{P}(\varepsilon_\infty)$ (see Lemma 2.4ii)). Let $\{g^{(\alpha)}\}_{\alpha \in A}$ be a linearly ordered subset of $\mathfrak{X}_f$. Since $\mathfrak{X}_f$ and $\mathfrak{Y}_f$ are bounded and consequently relatively compact due to assumption c), there are cofinal subnets $\{g^{(\alpha')}\}_{\alpha' \in A'}, \{k^{(\alpha')}\}_{\alpha' \in A'}$, $A' \subset A$, $k^{(\alpha')} := k_{g^{(\alpha')}}$, such that

$$\begin{aligned} g^{(\alpha')} &\to g \in \widetilde{\mathfrak{X}}_f \cap \widetilde{E^{2M}}, \\ k^{(\alpha)} &\to k \in \widetilde{\{F, \#\}} \cap \widetilde{E^{2M-2}}, \end{aligned}$$

where $f^\# f = g^{(\alpha)\#} g^{(\alpha)} + k^{(\alpha)}$. Notice that $g_M^{(\alpha)} = f_M$ implies $g_M = f_M$. Further, $f^\# f = g^\# g + k$ holds because of

$$\begin{aligned} p(f^\# f - g^\# g - k) &\leq p(f^\# f - g^{(\alpha')\#} g^{(\alpha')} - k^{(\alpha')}) + p(k - k^{(\alpha')}) + p(g^\# g - g^{(\alpha')\#} g^{(\alpha')}) \\ &\leq p(k - k^{(\alpha')}) + p'(g^{(\alpha')\#} - g^\#) p'(g) + p'(g^{(\alpha')\#}) p'(g - g^{(\alpha')}) \\ &\leq p(k - k^{(\alpha')}) + 2c_p p'(g - g^{(\alpha')}) \to 0, \end{aligned}$$

where p, $p' \in \mathscr{P}(\varepsilon_\infty)$ are taken from Lemma 2.4, and c_p from (61). Hence, $g \in \mathfrak{X}_f$. Noticing that $g \succ g^{(\alpha)}$ for all $\alpha \in A$, Zorn's lemma applies. Then, there is some maximal element $z \in \mathfrak{X}_f$ that satisfies

$$\rho(z^\# z) \in \mathscr{E}\mathfrak{R}$$

because otherwise Lemma 4.1 would imply the existence of $a^{(i)} \in F \cap E^M$ and two indices $\nu, \mu \in \mathbb{N}$ such that $z^\# z = \sum_{i=1}^\infty a^{(i)\#} a^{(i)}$ and $\rho(a^{(\nu)\#} a^{(\nu)}) \neq \rho(a^{(\mu)\#} a^{(\mu)})$. Using Lemma 4.3, there would be $b \in F \cap E^M$ with $b_M = z_M$, $\rho(b^\# b) \neq \rho(z^\# z)$, and $z^\# z = b^\# b + k_b$, $0 \neq k_b \in \{\widetilde{F, \#}\}^{\varepsilon\otimes}$ implying $b \in \mathfrak{X}_f$ and $b \succ z$. This is a contradiction to the maximality of z.

ii) *Let* $f^\# f = \sum_{i=1}^\infty a^{(i)\#} a^{(i)}$, $a^{(i)\#} a^{(i)} = \sum_{j=1}^\infty b^{(i,j)\#} b^{(i,j)}$, $i = 1, 2, 3, \ldots$, *with* $a^{(i)}$, $b^{(i,j)} \in F$ *be satisfied. Using the bijective mapping* $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *given by*

$$s = \frac{i^2 - 3i + 2ij + j^2 - j + 2}{2},$$

set $g^{(s)} = b^{(i,j)}$. *Then,*

$$f^\# f = \sum_{i=1}^\infty \sum_{j=1}^\infty b^{(i,j)\#} b^{(i,j)} = \sum_{s=1}^\infty g^{(s)\#} g^{(s)}.$$

Proof of ii). Using Corollary 2.6, for each $p \in \mathcal{P}(\varepsilon_\infty)$ there are $p', q', \bar{p}, q'' \in \mathcal{P}(\varepsilon_\infty)$ such that

$$\sum_{j=1}^\infty p(b^{(i,j)\#} b^{(i,j)}) \leq \sum_{i=1}^\infty p'(b^{(i,j)})^2 \leq q'(a^{(i)\#} a^{(i)}),$$

$$\sum_{i=1}^\infty q'(a^{(i)\#} a^{(i)}) \leq \sum_{i=1}^\infty \bar{p}(a^{(i)})^2 \leq q''(f^\# f).$$

Consequently, $\sum_{i=1}^\infty \sum_{j=1}^\infty b^{(i,j)\#} b^{(i,j)}$ is absolutely summing, and thus, ii) follows.

iii) *For each* $0 \neq f \in \tilde{F}$ *there is a sequence* $(g^{(i)})_{i=1}^\infty$, $g^{(i)} \in \tilde{F}$, *such that* $\rho(g^{(i)\#} g^{(i)}) \in \mathcal{ER}$ *and*

$$f^\# f = \sum_{i=1}^\infty g^{(i)\#} g^{(i)}.$$

Proof of iii). The proof is inductively given with respect fo Grad(.). Let

$$n_0 = \min\{\mathrm{Grad}(v) \,;\, 0 \neq v \in F\}.$$

Noticing that $f \in F$ and Grad $(f) = n_0$ imply grad $(f) = n_0$, $f^\# f \in \mathcal{ER}$ follows from Lemma 4.2. Assume now that there is some $n \in \mathbb{N}$, $n \geq n_0$, such that for each $0 \neq f \in F$ with $\mathrm{Grad}(f) \leq n$, the assertion under consideration is satisfied. Setting

$$\nu_n = \min\{\mathrm{Grad}(v) \ ; \ v \in F, \ \mathrm{Grad}(v) > n\},$$

choose some $y \in F$ with $\mathrm{Grad}(y) = \nu_n$. Due to i) there are $z \in \mathfrak{X}_y$, $k_z \in \{\widetilde{F, \#}\}$ such that $\rho(z^\# z) \in \mathscr{E}\mathfrak{R}$ and

$$y^\# y = z^\# z + k_z.$$

It further follows $\mathrm{Grad}(k_z) < 2n$, and due to assumption d) there is a sequence $(a^{(i)})_{i=1}^{\infty}$, $a^{(i)} \in \tilde{F} \cap \widetilde{E^n}$, with

$$k_z = \sum_{i=1}^{\infty} a^{(i)\#} a^{(i)}.$$

Using the above assumption, there are sequences $(b^{(i,j)})_{j=1}^{\infty}$, $b^{(i,j)} \in \tilde{F} \cap \widetilde{E^n}$, with $\rho(b^{(i,j)\#} b^{(i,j)}) \in \mathscr{E}\mathfrak{R}$ and

$$a^{(i)\#} a^{(i)} = \sum_{j=1}^{\infty} b^{(i,j)\#} b^{(i,j)},$$

$i = 1, 2, 3, \ldots$ Now, ii) implies

$$f^\# f = z^\# z + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b^{(i,j)\#} b^{(i,j)} = z^\# z + \sum_{s=1}^{\infty} g^{(s)\#} g^{(s)}$$

completing the proof of iii).

iv) $0 \neq k \in \{\widetilde{F, \#}\}$ and assumption c) yield the existence of some $M \in \mathbf{N}$ and some sequence $(a^{(i)})_{i=1}^{\infty}$, $a^{(i)} \in \tilde{F} \cap \widetilde{E^M}$, such that $\mathrm{Grad}(k) = 2M$ and

$$k = \sum_{i=1}^{\infty} a^{(i)\#} a^{(i)}.$$

Applying iii), there are sequences $(b^{(i,j)})_{j=1}^{\infty}$, $b^{(i,j)} \in \tilde{F} \cap \widetilde{E^M}$ with $\rho(b^{(i,j)\#} b^{(i,j)}) \in \mathscr{E}\mathfrak{R}$ and

$$a^{(i)\#}a^{(i)}=\sum_{j=1}^{\infty}b^{(i,j)\#}b^{(i,j)},$$

$i=1, 2, 3,\ldots$. Setting $z^{(s)}=g^{(s)}=b^{(i,j)}$ as in ii), it follows $k=\sum_{s=1}^{\infty}z^{(s)\#}z^{(s)}$ completing the proof of Theorem 4.4. (I).

Proof of (II) : ($\Leftarrow$) : is obvious. ($\Rightarrow$) : Assuming that $\dim(F)\geq 2$ there are $x^{\#}x, y^{\#}y\in\mathscr{E}\mathfrak{R}$ and x, y are linearly independent. Consider

$$k=2x^{\#}x+2y^{\#}y=(x+y)^{\#}(x+y)+(x-y)^{\#}(x-y). \tag{64}$$

If $(x+y)^{\#}(x+y)$, $(x-y)^{\#}(x-y)\in\mathscr{E}\mathfrak{R}$, (64) yields the non-uniqueness of the decomposition of k. Otherwise, if $(x+y)^{\#}(x+y)\notin\mathscr{E}\mathfrak{R}$ (or $(x-y)^{\#}(x-y)\notin\mathscr{E}\mathfrak{R}$), consider the decomposition of the right hand side of (64) into extremal rays given in (I). Noticing that this decomposition contains at least three summands, the non-uniqueness of the decomposition of k into extremal rays follows also.

§4.2. An Example: The Extremal Rays of $\mathbb{C}_{\otimes}^{+}$

Let us shortly illustrate the preceeding for the simplest examples : the tensor algebra

$$\mathbb{C}_{\otimes}=\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\ldots,$$

and its cone of positivity

$$\mathbb{C}_{\otimes}^{+}=\{\sum_{i=1}^{M}c^{(i)*}c^{(i)}\ ;\ c^{(i)}\in\mathbb{C}_{\otimes},\ M\in\mathbb{N}\}.$$

Recall that there is a *-isomorphism π between $\mathbb{C}_{\otimes}$ and the algebra of polynomial $\mathbb{C}[t]$ with complex coefficients and one varile $t\in\mathbb{R}$ given by

$$\pi(c)=\sum_{n=0}^{\infty}c_n t^n,$$

$c=(c_0, c_1,\ldots, c_N, 0, 0,\ldots)\in\mathbb{C}_{\otimes}$.

Theorem 4.5. a) *It is* $p \in \mathbf{C}_{\otimes}^{+}$, *if and only if* $\pi(p)(t) \geq 0$ *for all* $t \in \mathbf{R}$.

b) *It is* $\mathbf{C}_{\otimes}^{+} = \overline{\mathbf{C}_{\otimes}^{+}}^{\varepsilon\otimes}$.

c) *It holds* $\rho(c^*c) \in \mathscr{E}\mathscr{R}(\mathbf{C}_{\otimes}^{+})$, *if and only if* $\pi(c)(t)$ *has real roots only.*

d) *Assuming that* $\pi(c)(t)$ *has* $r \neq 0$ *non-real roots* $a_j + ib_j$ $(0 \neq b_j \in \mathbf{R},\ a_j \in \mathbf{R})$, *and* $n-r$ *real roots* a_l $(j=1,\dots,r;\ l=r+1,\dots,n)$, *a decomposition of* c^*c *into extremal rays is given by*

$$\pi(c^*c) = \prod_{j=1}^{n} x_j + \sum_{s=1}^{r} b_s^2 \prod_{\substack{j=1 \\ j\neq s}}^{n} x_j + \sum_{\substack{s_1,s_2=1 \\ s_1<s_2}}^{r} b_{s_1}^2 b_{s_2}^2 \prod_{\substack{j=1 \\ j\neq s_1,s_2}}^{n} x_j + \dots + \sum_{\substack{s_1,\dots,s_{r-1} \\ s_1<\dots<s_{r-1}}}^{r} (b_{s_1}\dots b_{s_{r-1}})^2 \prod_{\substack{j=1 \\ j\neq s_1\dots s_{r-1}}}^{n} x_j + \prod_{j=1}^{r} b_j^2 \prod_{l=r+1}^{n} x_l, \tag{65}$$

where $x_\nu = (t-a_\nu)^2$, $\nu = 1, 2, \dots, n$.

Proof. a),b) are shown in [25]. *c)*: see [7, Satz 5.1], [14].
d) : Using

$$\pi(c)(t) = \prod_{j=1}^{r} (t-(a_j+ib_j)) \prod_{l=r+1}^{n} (t-a_l),$$

(65) is implied. Since each summand on the right-hand side of (65) is positive for all $t \in \mathbf{R}$ and is the square of polynomials with real roots only, a) and c) imply that the preimages of all of these summands are in $\mathscr{E}\mathscr{R}(\mathbf{C}_{\otimes}^{+})$.

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