

On Tuples of Commuting Compact Operators

By

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Abstract

The joint spectrum of commuting operators, as introduced by Taylor, has been shown by Cho and Takaguchi to be the set of joint eigenvalues in the case of matrices. These joint eigenvalues can be read off from their simultaneous uppertriangularization. We prove here a similar result for compact operators on Banach spaces.

§ 1. Introduction

Let $\sigma_T(\mathbf{A})$ and $\sigma_{pt}(\mathbf{A})$ denote respectively the Taylor joint spectrum and the joint point spectrum of a commuting tuple $\mathbf{A} = (A_1, \dots, A_n)$ of bounded linear operators on a Banach space X . We shall briefly recapitulate their definitions and basic properties in Section 3, but for details see [3] and [4]. If the Banach space X is finite-dimensional with $\dim(X) = N$, say, then there exist subspaces L_0, L_1, \dots, L_N of X such that

- (i) $\{0\} = L_0 \subseteq L_1 \subseteq \dots \subseteq L_N = X$,
- (ii) L_k is k -dimensional ($k=1, \dots, N$),
- (iii) each L_k is simultaneously invariant under A_1, \dots, A_n .

A family of subspaces $\{L_1, \dots, L_N\}$, which has the properties (i), (ii) and (iii) above, determines an upper-triangular representation of A_1, \dots, A_n . One can choose a basis $\mathcal{E} = \{x_1, \dots, x_N\}$ of X which has the properties:

- (i) each x_j lies in L_j but not in L_{j-1} ,
- (ii) the matrix of the operators A_1, \dots, A_n with respect to the basis \mathcal{E} are upper-triangular.

With respect to the basis \mathcal{E} , the matrices for A_1, \dots, A_n are of the form

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$$\left(\begin{matrix} \lambda_1^{(1)} & a_{12}^{(1)} & \cdots & a_{1N}^{(1)} \\ 0 & \lambda_2^{(1)} & \cdots & a_{2N}^{(1)} \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_N^{(1)} \end{matrix} \right), \dots, \left(\begin{matrix} \lambda_1^{(n)} & a_{12}^{(n)} & \cdots & a_{1N}^{(n)} \\ 0 & \lambda_2^{(n)} & \cdots & a_{2N}^{(n)} \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_N^{(n)} \end{matrix} \right)$$

The *joint diagonal coefficients* are then defined to be the n -tuples $\lambda_i, i = 1, \dots, N$, consisting of the i th. diagonal entries in the matrices of A_1, \dots, A_n i.e. $\lambda_i = (\lambda_i^{(1)}, \dots, \lambda_i^{(n)})$. In this case, it is well-known that $\sigma_T(\mathbb{A}) = \sigma_{pr}(\mathbb{A}) =$ the set of all joint diagonal coefficients of \mathbb{A} . See e.g., [2], [3, page 401] and [6, Prop. 7]. Here we will obtain an extension of this to commuting compact linear operators acting on infinite-dimensional Banach spaces. For the reader's convenience, the known results for a single operator are summarised in Section 2. This will also serve to establish notations used later. In Section 3 we obtain some spectral properties of commuting compact tuples of Banach space operators. In Section 4 we show that such tuples can be simultaneously reduced to a triangular form, and that their joint spectrum can be read off from that form.

§2. Preliminaries

Throughout the paper, X stands for a complex infinite-dimensional Banach space. The set \mathcal{L} of all closed subspaces of X is a partially ordered set under inclusion. A totally ordered subset of this set is called a *chain*. The class \mathcal{C} of all chains is again a partially ordered set by the inclusion relation on the subsets of \mathcal{L} . Let \mathcal{C}_0 be a completely ordered subset of \mathcal{C} . If we define $\mathcal{F}_0 = \cup \{ \mathcal{F} : \mathcal{F} \in \mathcal{C}_0 \}$, then it follows easily that \mathcal{F}_0 is a chain. \mathcal{F}_0 is obviously an upper bound for the class \mathcal{C}_0 . Moreover if \mathcal{G} is any other upper bound, then $\mathcal{F}_0 \subseteq \mathcal{G}$. So \mathcal{F}_0 is the least upper bound of the class \mathcal{C}_0 . So each totally ordered subset of \mathcal{C} has a least upper bound. It follows from Zorn's lemma that \mathcal{C} contains maximal elements, which we call *maximal chains*. Every chain is contained in at least one maximal chain.

Given a subfamily \mathcal{F}_0 of a chain \mathcal{F} the set $\cap \{ L : L \in \mathcal{F}_0 \}$ is a closed subspace of X , the same is true for $\overline{\cup \{ L : L \in \mathcal{F}_0 \}}$, where \overline{M} denotes the closure of M in norm. Given $M \in \mathcal{F}$, the immediate predecessor of M is defined to be the subspace

$$M_- = \overline{\cup \{ L \in \mathcal{F} : L \subsetneq M \}},$$

interpreting the right hand side as $\{0\}$ when there is no proper subspace of M in \mathcal{F} . The subspace M_- is not necessarily in \mathcal{F} .

A chain \mathcal{F} is called a *simple chain* if it satisfies the following conditions :

- (i) $\{0\}, X \in \mathcal{F}$,
- (ii) if \mathcal{F}_0 is a subfamily of \mathcal{F} then $\cap \{L: L \in \mathcal{F}_0\}$ and $\overline{\cup \{L: L \in \mathcal{F}_0\}}$ are in \mathcal{F} .
- (iii) for each $M \in \mathcal{F}$, $\dim M/M_-$ is at most one.

Condition (ii) implies that $M_- \in \mathcal{F}$ for each $M \in \mathcal{F}$. A *continuous chain* is a simple chain such that $M = M_-$ for each $M \in \mathcal{F}$. It is known that a chain is maximal if and only if it is simple. For a detailed proof of this, see [8].

A chain \mathcal{F} is called *invariant* under a compact operator T if each $M \in \mathcal{F}$ is an invariant subspace of T . It is a classical fact (see [1]) that any compact operator T has a non-trivial closed invariant subspace i.e., there exists a closed subspace which is neither $\{0\}$ nor X and which is left invariant by T . This shows the existence of non-trivial invariant chains. Let \mathcal{C}_i denote the class of all invariant chains of T . The usual Zorn's lemma argument applied to \mathcal{C}_i shows the existence of maximal elements of \mathcal{C}_i which we call *maximal invariant chains*. It is not apparent that maximal invariant chains are also maximal chains i.e., elements which are maximal in \mathcal{C}_i are maximal in \mathcal{C} . The following theorem (see [8], page 169) shows that this is indeed the case.

Theorem A. *For an invariant chain \mathcal{F} the following conditions are equivalent:*

- (i) \mathcal{F} is a maximal chain,
- (ii) \mathcal{F} is a maximal invariant chain,
- (iii) \mathcal{F} is simple.

This theorem implies that there is a simple chain \mathcal{F} of closed subspaces of X such that each L in \mathcal{F} is invariant under T . If $M \in \mathcal{F}$ then either $M = M_-$ or $M \not\subseteq M_-$ has dimension one. In the later case, suppose $z_M \in M \setminus M_-$ so that M is the linear span of $\{z_M\} \cup M_-$. Since M is invariant under T , $Tz_M \in M$ so that there exists a scalar α^M and a vector $y_M \in M_-$ such that

$$Tz_M = \alpha^M z_M + y_M.$$

The scalar α^M does not depend on the choice of z_M in $M \setminus M_-$. Since M_- is invariant under $T - \alpha^M$ and $(T - \alpha^M)z_M \in M_-$ it follows that

$$(T - \alpha^M)M \subseteq M_-.$$

α^M is defined to be 0 when $M = M_-$. In this way is associated with each M in \mathcal{F} a scalar α^M which is called the *diagonal coefficient* of T at M . Ringrose then proves the following theorem which gives a one-one correspondence between the eigenvalues of T and the diagonal coefficients of T . (Theorem 4.3.10 in [8].)

Theorem B. *Let T, X and \mathcal{F} be as above. Then*

- (i) *a non-zero scalar λ is an eigenvalue of T if and only if it is the diagonal*

coefficient of T at M for some $M \in \mathcal{F}$.

(ii) the diagonal multiplicity of λ is equal to its algebraic multiplicity as an eigenvalue of T .

(iii) If $\sigma(T)$ denotes the spectrum of T , then $\sigma(T) = \{0\}$ if and only if $T(M) \subseteq M_-$ for all $M \in \mathcal{F}$.

§3. Spectral Properties

In this section we shall see that the spectral properties of a compact n -tuple resemble those of an n -tuple of matrices to a large extent. First we recapitulate very briefly the definition of the Taylor joint spectrum. Let Λ_n be the exterior algebra on n generators with identity $e_0 = 1$. This is the algebra of forms in e_1, \dots, e_n with complex coefficients, subject to the collapsing property $e_i e_j + e_j e_i = 0$ ($1 \leq i, j \leq n$). The algebra Λ_n is graded: $\Lambda_n = \bigoplus_{k=1}^n \Lambda_n^k$ with $\{e_{i_1} \cdots e_{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$ as the basis for Λ_n^k . Let $E_i : \Lambda_n \rightarrow \Lambda_n$ be given by

$$E_i \xi = e_i \xi, \quad i = 1, \dots, n, \quad \xi \in \Lambda_n. \tag{3.1}$$

Clearly $E_i E_j + E_j E_i = 0$ ($1 \leq i, j \leq n$). If X is a vector space, we define $\Lambda_n(X) \stackrel{\text{def}}{=} X \otimes \Lambda_n$. Then for $\mathbb{A} = (A_1, \dots, A_n)$ the operator $D_{\mathbb{A}} : \Lambda_n(X) \rightarrow \Lambda_n(X)$ is defined by

$$D_{\mathbb{A}} \stackrel{\text{def}}{=} \sum_{i=1}^n A_i \otimes E_i. \tag{3.2}$$

If \mathbb{A} is a commuting tuple, then

$$D_{\mathbb{A}}^2 = \sum_{i,j=1}^n A_j A_i \otimes E_j E_i = \sum_{i < j} A_i A_j \otimes (E_i E_j + E_j E_i) = 0, \tag{3.3}$$

If $\mathcal{R}(D_{\mathbb{A}})$ and $\mathcal{N}(D_{\mathbb{A}})$ denote the range and kernel respectively of the operator $D_{\mathbb{A}}$, then the equation (3.3) says that $\mathcal{R}(D_{\mathbb{A}}) \subseteq \mathcal{N}(D_{\mathbb{A}})$. Consider a chain complex $K(\mathbb{A}, X)$, called the *Koszul complex* as follows:

$$K(\mathbb{A}, X) : 0 \rightarrow \Lambda_n^0(X) \xrightarrow{D_{\mathbb{A}}^0} \Lambda_n^1(X) \xrightarrow{D_{\mathbb{A}}^1} \cdots \xrightarrow{D_{\mathbb{A}}^{k-1}} \Lambda_n^k(X) \xrightarrow{D_{\mathbb{A}}^k} \cdots \xrightarrow{D_{\mathbb{A}}^{n-1}} \Lambda_n^n(X) \rightarrow 0. \tag{3.4}$$

(Here $\Lambda_n^k(X)$ is $X \otimes \Lambda_n^k$ and $D_{\mathbb{A}}^k \stackrel{\text{def}}{=} D_{\mathbb{A}}|_{\Lambda_n^k(X)}$.) Then the *Taylor joint spectrum* of \mathbb{A} on X is defined as

$$\sigma_T(\mathbf{A}) = \{\lambda \in \mathbb{C}^n : K(\mathbf{A} - \lambda, X) \text{ is not exact}\}. \tag{3.5}$$

We say that \mathbf{A} is *nonsingular* on X if $0 \notin \sigma_T(\mathbf{A})$. An important subset of the Taylor joint spectrum is the joint point spectrum $\sigma_{pt}(\mathbf{A})$ which is the set of all joint eigenvalues. A joint eigenvalue of \mathbf{A} is an n -tuple of scalars $\lambda = (\lambda_1, \dots, \lambda_n)$ for which there exists a non-zero vector $x \in X$ satisfying $A_1x = \lambda_1x, \dots, A_nx = \lambda_nx$.

Theorem 3.1. *Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n -tuple of compact operators on a complex Banach space X and $\lambda = (\lambda_1, \dots, \lambda_n)$ be a non-zero n -tuple of scalars. If $\lambda \in \sigma_T(\mathbf{A})$ then λ is a joint eigenvalue of \mathbf{A} .*

Proof. We start by noting that the non-zero scalar tuple λ can not be a limit point of $\sigma_T(\mathbf{A})$. In fact the only possible limit point is $(0, 0, \dots, 0)$. Indeed if μ is a limit point of $\sigma_T(\mathbf{A})$, then we take a sequence $\mu_n = (\mu_n^{(1)}, \dots, \mu_n^{(m)})$ from $\sigma_T(\mathbf{A})$ such that $\|\mu_n - \mu\| \rightarrow 0$. So $|\mu_n^{(j)} - \mu^{(j)}| \rightarrow 0$ for each j . By the projection property of Taylor joint spectrum (see [4, Theorem 4.9]), $\mu_n^{(j)} \in \sigma(A_j)$. So $\mu^{(j)}$ is a limit point of $\sigma(A_j)$. Since each A_j is compact, it follows that $\mu_j = 0$ for each j .

Since λ is not a limit point of $\sigma_T(\mathbf{A})$, we can find a neighbourhood N of λ containing no other point of $\sigma_T(\mathbf{A})$. Let $f: \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic function, defined on some neighbourhood of $\sigma_T(\mathbf{A})$, which is 1 on N and 0 on $\sigma_T(\mathbf{A}) \setminus \{\lambda\}$. Then $f(\mathbf{A})$ is a projection which commutes with each A_j . So $X_\lambda \stackrel{\text{def}}{=} \mathcal{R}(f(\mathbf{A}))$ is a closed invariant subspace for each A_j . And the Taylor spectrum of the restriction of \mathbf{A} to this subspace consists only of λ . Again by the projection property, it follows that $\sigma(A_j|_{X_\lambda}) = \{\lambda_j\}$. Let λ_{j_0} be a non-zero component of λ . Then $A_{j_0}|_{X_\lambda}$ is an invertible compact operator. Consequently, X_λ is finite-dimensional. Since on a finite-dimensional Banach space points of the joint spectrum are joint eigenvalues, we conclude that λ is joint eigenvalue of $\mathbf{A}|_{X_\lambda}$, hence of \mathbf{A} . ■

We define the null space and the range space of a tuple as follows :

$$\mathcal{N}(\mathbf{A}) = \{x \in X : A_1x = \dots = A_nx = 0\} = \bigcap_{j=1}^n \mathcal{N}(A_j).$$

$$\begin{aligned} \mathcal{R}(\mathbf{A}) &= \{y \in X : \text{There exist } x_1, \dots, x_n \in X \\ &\quad \text{such that } y = A_1x_1 + \dots + A_nx_n\} \\ &= \mathcal{R}(A_1) + \dots + \mathcal{R}(A_n). \end{aligned}$$

Obviously $\mathcal{N}(\mathbf{A})$ is a closed subspace of X . Moreover, for any $0 \neq \lambda = (\lambda_1, \dots, \lambda_n)$, we have $\mathcal{N}(\mathbf{A} - \lambda)$ to be finite-dimensional.

The next concept useful in investigating the spectral properties of \mathbb{A} is the product of tuples. If \mathbb{B} is another n -tuple of commuting bounded operators on X , then define $\mathbb{A}\mathbb{B}$ to be the n^2 -tuple whose entries are $A_i B_j$, $1 \leq i, j \leq n$, arranged in lexicographic order. This product was introduced in [7]. Using this multiplication rule, one can successively define the powers $\mathbb{A}^2, \mathbb{A}^3, \dots$. Then \mathbb{A}^k is a tuple with n^k entries; these are the products $A_{i_1} \cdots A_{i_k}$ where the indices are chosen from $\{1, 2, \dots, n\}$ with repetitions allowed, and are then arranged lexicographically.

For any $k \geq 1$, the n -tuple \mathbb{A} can also be regarded as an operator from X^k to X^{nk} , taking a vector $x = (x_1, \dots, x_k)$ to the nk -tuple $\mathbb{A}x = (A_1 x_1, \dots, A_n x_1, A_1 x_2, \dots, A_n x_k)$. Thought in this way there is a similarity with the single operator case. For any complex n -tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ and for any $k \geq 1$, we define the following subspaces :

$$\mathcal{N}_k(\mathbb{A} - \lambda) = \{x \in X : (\mathbb{A} - \lambda)^k x = 0\}.$$

The space $\mathcal{N}_1(\mathbb{A} - \lambda)$ is the same as the null space $\mathcal{N}(\mathbb{A} - \lambda)$. When there is no chance of confusion we shall simply write \mathcal{N}_k for $\mathcal{N}_k(\mathbb{A} - \lambda)$.

Lemma 3.2. $\mathcal{N}_i \subseteq \mathcal{N}_{i+1}$ for all $i = 1, 2, \dots$. If $\mathcal{N}_{i+1} = \mathcal{N}_i$ for some n , then $\mathcal{N}_{i+k} = \mathcal{N}_i$ for all $k = 1, 2, \dots$

Proof. The first statement is evident. Let $\mathcal{N}_{i+1} = \mathcal{N}_i$. Let $x \in \mathcal{N}_{i+2}$. Then $(\mathbb{A} - \lambda)^{i+2} x = 0$,
 or, $(\mathbb{A} - \lambda)^{i+1} (\mathbb{A} - \lambda) x = 0$,
 or, $(\mathbb{A} - \lambda_j)^{i+1} (A_j - \lambda_j) x = 0$ for all $j = 1, \dots, n$,
 or, $(A_j - \lambda_j) x \in \mathcal{N}_{i+1}$ for all $j = 1, \dots, n$.
 But $\mathcal{N}_{i+1} = \mathcal{N}_i$. So,
 $(A_j - \lambda_j) x \in \mathcal{N}_i$ for all $j = 1, \dots, n$,
 or, $(\mathbb{A} - \lambda)^i (A_j - \lambda_j) x = 0$ for all $j = 1, \dots, n$.
 or, $(\mathbb{A} - \lambda)^{i+1} x = 0$,
 So $x \in \mathcal{N}_{i+1}$. Now the rest follows by induction. □

Lemma 3.3. Let $\mathbb{A} = (A_1, \dots, A_n)$ be an n -tuple of commuting compact operators and λ a non-zero scalar n -tuple. Then $\mathcal{N}_k(\mathbb{A} - \lambda)$ is finite-dimensional for all $k \geq 1$. Moreover, there exists a positive integer ν such that

$$\{0\} = \mathcal{N}_0(\mathbb{A} - \lambda) \subseteq \mathcal{N}_1(\mathbb{A} - \lambda) \subseteq \dots \subseteq \mathcal{N}_\nu(\mathbb{A} - \lambda) = \mathcal{N}_{\nu+1}(\mathbb{A} - \lambda) = \dots$$

Proof. Since $\lambda \neq 0$, for at least one j , $\lambda_j \neq 0$. Assume without loss of generality that $\lambda_1 \neq 0$. By definition of the null space of a tuple,

$$\mathcal{N}_k(\mathbf{A}) \subseteq \mathcal{N}(A_1 - \lambda_1)^k.$$

Since A_1 is a compact operator and $\lambda_1 \neq 0$, it is known that $\dim \mathcal{N}(A_1 - \lambda_1)^k < \infty$ for all $k \geq 1$. Moreover, $\dim \mathcal{N}(A_1 - \lambda_1)^k$ is bounded as $k \rightarrow \infty$ (see [9], pages 278-280). That completes the proof in view of Lemma 3.2. ■

Thus the integer ν occurring in Lemma 3.3 is the smallest i such that $\mathcal{N}_i(\mathbf{A} - \lambda) = \mathcal{N}_{i+1}(\mathbf{A} - \lambda)$. We call the integer ν the *index* and $\dim(\mathcal{N}_\nu(\mathbf{A} - \lambda))$ the *algebraic multiplicity* of λ with respect to \mathbf{A} . Theorem 3.1 and Lemma 3.3 together show that any non-zero point in the Taylor spectrum of a commuting compact tuple is a joint eigenvalue with finite algebraic multiplicity.

§4. Upper-triangularization

The starting point of this section is the classical result by Lomonosov (see [5]) that any compact operator T on X has a non-trivial closed hyperinvariant subspace. A subspace is called hyperinvariant for T if it is invariant under any bounded operator commuting with T . It follows from this theorem that a commuting family of compact operators has a common non-trivial closed invariant subspace. This implies the existence of non-trivial chains consisting of subspaces simultaneously invariant under A_1, \dots, A_n . Let $\mathcal{C}_\mathbf{A}$ denote the class of all chains which are simultaneously invariant under each A_j . A straightforward application of Zorn's lemma shows the existence of maximal elements of $\mathcal{C}_\mathbf{A}$. We skip the details of the argument because it is the same as the one used in Section 2 to prove the existence of maximal chains. We call these maximal elements of $\mathcal{C}_\mathbf{A}$ the *maximal simultaneously invariant chains*. Our next lemma shows that these are, in fact, simple chains.

Lemma 4.1. *Each maximal simultaneously invariant chain is simple.*

Proof. Suppose \mathcal{F} is a maximal simultaneously invariant chain. Then obviously $\{0\}, X \in \mathcal{F}$. For any subfamily \mathcal{F}_0 of \mathcal{F} , let $N = \bigcap \{L : L \in \mathcal{F}_0\}$. Then N is a closed subspace of X . Since each L is simultaneously invariant under each A_j , the same is true for N . Let $M \in \mathcal{F}$. Since \mathcal{F} is totally ordered either $M \subseteq L$ for each $L \in \mathcal{F}_0$, and hence $M \subseteq N$, or $L \subseteq M$ for at least one L in \mathcal{F}_0 , and hence $N \subseteq M$. It follows that $\mathcal{F} \cup \{N\}$ is totally ordered by inclusion and is therefore a simultaneously invariant chain. Since \mathcal{F} is maximal, $N \in \mathcal{F}$.

It remains to show that M/M_- has dimension at most one for each $M \in \mathcal{F}$. Suppose $\dim M/M_- > 1$ for some $M \in \mathcal{F}$. Consider the Banach space M/M_- and

the n -tuple \mathbb{A}_0 of bounded operators on it defined by $(\mathbb{A}_0)_j(x+M_-) = A_jx+M_-$. Then \mathbb{A}_0 is an n -tuple of commuting compact operators. So there is a closed subspace N_0 of M/M_- such that $\{0\} \neq N_0 \neq M/M_-$ and $(\mathbb{A}_0)_j(N_0) \subseteq N_0$ for all $j=1, \dots, n$. It follows that if $N = \{x \in M : x+M_- \in N_0\}$ then N is a closed subspace of X , simultaneously invariant under each A_j and $M_- \subsetneq N \subsetneq M$. Given any subspace $L \in \mathcal{F}$, either $M \subseteq L$ and so $N \subsetneq L$, or $L \subsetneq M$ and so $L \subseteq M_- \subsetneq M$. Hence $N \notin \mathcal{F}$ and $\mathcal{F} \cup \{N\}$ is a chain. This violates the maximality of \mathcal{F} as a simultaneously invariant chain. So for each $M \in \mathcal{F}$, $\dim M/M_-$ is at most one. \square

We can now define a joint diagonal coefficient for a commuting tuple of operators.

Definition 4.2. *Let \mathcal{F} be a maximal simultaneously invariant chain for \mathbb{A} and let $M \in \mathcal{F}$. The joint diagonal coefficient α^M of \mathbb{A} at M is the scalar n -tuple $(\alpha_1^M, \dots, \alpha_n^M)$ where α_j^M is the diagonal coefficient of A_j at M .*

For the rest of the paper, we choose and fix a maximal simultaneously invariant chain \mathcal{F} for \mathbb{A} .

Lemma 4.3. *If λ is a joint eigenvalue then it is a joint diagonal coefficient.*

Proof. There exists a non-zero vector x such that $A_jx = \lambda_jx$ for all $j=1, \dots, n$. We define $M = \cap \{L \in \mathcal{F} : x \in L\}$. Then it follows from Lemma 4.3.7 of [8] that λ_j is the diagonal coefficient of A_j at M . So λ is the joint diagonal coefficient of \mathbb{A} at M . \square

Lemma 4.4. *If $\alpha^M \neq 0$ is the joint diagonal coefficient of \mathbb{A} at M for some $M \in \mathcal{F}$, then α^M is a joint eigenvalue of \mathbb{A} .*

Proof. In the proof of this lemma we shall use the definition of the Taylor joint spectrum and Theorem 3.1. One of the necessary conditions for the Koszul complex $K(\mathbb{A}, X)$ to be exact is

$$\mathcal{R}(D_{\mathbb{A}}^{n-1}) = A_n^n(X). \tag{4.1}$$

(See (3.4).) $A_n^{n-1}(X)$ consists of vectors of the form $\sum_{i=1}^n x_i \otimes e_1 \dots \hat{e}_i \dots e_n$, where \hat{e}_i means that e_i is omitted. So $\mathcal{R}(D_{\mathbb{A}}^{n-1}) = \{(\sum_{i=1}^n A_i x_i) \otimes e_1 \dots e_n : x_1, \dots, x_n \in X\} = \mathcal{R}(\mathbb{A}) \otimes A_n$. Hence (4.1) is equivalent to saying that

$$\mathcal{R}(\mathbb{A}) = X. \tag{4.2}$$

Let \mathbb{A}_M be the restriction of the tuple \mathbb{A} to the subspace M . Then \mathbb{A}_M is a

Let \mathbf{A}_M be the restriction of the tuple \mathbf{A} to the subspace M . Then \mathbf{A}_M is a commuting tuple of compact operators on the space M . By definition of a joint diagonal coefficient, $\mathcal{R}(A_j - \alpha_j^M) \subseteq M_-$ for all $j=1, \dots, n$. So $\mathcal{R}(\mathbf{A}_M - \alpha^M) \subseteq M_-$. Since $\alpha^M \neq 0$ we have $M_- \neq M$. Now (4.2) implies that the Koszul complex $K(\mathbf{A}_M - \alpha^M, M)$ is not exact. Thus $\alpha^M \in \sigma_T(\mathbf{A}_M)$. By Theorem 3.1, α^M is a joint eigenvalue of \mathbf{A}_M , hence of \mathbf{A} . ■

Definition 4.5. Let α^M be the joint diagonal coefficient of \mathbf{A} at M . Consider the set $\{M \in \mathcal{F} : \alpha^M \text{ is the joint diagonal coefficient of } \mathbf{A} \text{ at } M\}$. The diagonal multiplicity of α^M is the cardinality of this set.

Now we are in a position to state and prove our final theorem :

Theorem 4.6. Suppose $\mathbf{A} = (A_1, \dots, A_n)$ is an n -tuple of commuting compact linear operators on a complex Banach space X , and \mathcal{F} is a maximal chain of closed subspaces of X , each of which is invariant under each $A_j, j= 1, \dots, n$. Then

- (i) a non-zero scalar n -tuple λ is a joint eigenvalue of \mathbf{A} if and only if there is an $M \in \mathcal{F}$ such that λ is the joint diagonal coefficient of \mathbf{A} at M ,
- (ii) the diagonal multiplicity of λ is equal to its algebraic multiplicity as a joint eigenvalue of \mathbf{A} .
- (iii) $\sigma_T(\mathbf{A}) = \{(0, \dots, 0)\}$ if and only if $A_j M \subseteq M_-$ for all $j=1, \dots, n$ and all $M \in \mathcal{F}$.

Proof. The above two lemmas prove the first part of the theorem.

To prove part (ii) let λ be a non-zero joint diagonal coefficient of \mathbf{A} . Let d denote the diagonal multiplicity, m the algebraic multiplicity and ν the index of λ relative to \mathbf{A} . Then \mathcal{N}_ν is m dimensional and

$$\{0\} = \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \dots \subseteq \mathcal{N}_\nu = \mathcal{N}_{\nu+1} = \dots.$$

We first reduce the problem to the case of $\nu=1$. For this purpose define the operator n^ν -tuple \mathbf{B} and the scalar n^ν -tuple μ by

$$\mathbf{B} - \mu = (\mathbf{A} - \lambda)^\nu.$$

Then $\mu = \lambda^\nu$ and \mathbf{B} , being a polynomial in the A_j 's, is a tuple of compact operators. For the same reason the invariant subspaces of \mathbf{A} in \mathcal{F} are also invariant under the tuple \mathbf{B} . Since $\mathbf{B} - \mu$ and $(\mathbf{B} - \mu)^2$ have the same null space $\mathcal{N}_\nu (= \mathcal{N}_{2\nu})$ of dimension m , it follows that μ is a joint eigenvalue of \mathbf{B} with index one and multiplicity m . So without loss of generality we can assume ν to be 1.

Suppose $d > m$. Then there exist subspaces $M(0), M(1), \dots, M(m)$ in \mathcal{F} satisfying $M(0) \subseteq M(1) \subseteq \dots \subseteq M(m)$ and λ is the joint diagonal coefficient

of \mathbb{A} at $M(k)$. Since $M(k-1) \subseteq M(k)$ it follows that $M(k-1) \subseteq M(k)_-$ for all $k=1, \dots, m$. There exist vectors x_0, x_1, \dots, x_m such that $A_j x_k = \lambda_j x_k$ and $x_k \in M(k) \setminus M(k)_-$; $k=1, \dots, m$. The vectors x_0, x_1, \dots, x_m lie in the m dimensional null space of $\mathbb{A} - \lambda$ and are, therefore, linearly dependent. Hence some x_k is a linear combination of x_0, x_1, \dots, x_{k-1} . But $x_0, x_1, \dots, x_{k-1} \in M(k-1) \subseteq M(k)_-$. So $x_k \in M(k)_-$. That is a contradiction. So $d \leq m$.

Suppose $m > d$. There are exactly d distinct subspaces $M(1), \dots, M(d)$, say, such that λ is the joint diagonal coefficient of \mathbb{A} at $M(k)$, $k=1, \dots, d$. Each $M(k)/M(k)_-$ has dimension 1 and therefore there is a continuous linear functional ϕ_k on $M(k)$ with kernel $\phi_k^{-1}(0)$ equal to $M(k)_-$. Extend ϕ_k to the whole of X . Call the extension φ_k . Then $M(k)_- = \{x \in M(k) : \varphi_k(x) = 0\}$. If $m > d$ there is a non-zero vector x_0 in the m dimensional space \mathcal{N}_1 satisfying the d linear conditions

$$\varphi_k(x) = 0, \quad k=1, \dots, d.$$

If $M = \cap \{L \in \mathcal{F} : x_0 \in L\}$ then $M \in \mathcal{F}$, $x_0 \in M \setminus M_-$ and the diagonal coefficient of \mathbb{A} at M is λ . So $M = M(k)$ for some k and hence $x_0 \in M(k) \setminus M(k)_-$ with $\varphi_k(x_0) = 0$. That is a contradiction. So $m \leq d$. Hence $m = d$.

The proof of part (iii) is straightforward. If $0 \neq \lambda \in \sigma_T(\mathbb{A})$ then λ is a joint eigenvalue and hence joint diagonal coefficient of \mathbb{A} at M for some $M \in \mathcal{F}$. Since $\lambda \neq 0$, it follows from the definition of joint diagonal coefficients that for at least one j , $A_j M \not\subseteq M_-$. Conversely, if $A_j M \not\subseteq M_-$ for some $M \in \mathcal{F}$, then α^M (the joint diagonal coefficient of \mathbb{A} at M) is non-zero. Any joint diagonal coefficient is a joint eigenvalue. Hence in this case $\sigma_T(\mathbb{A})$ contains at least one non-zero element viz. α^M . That completes the proof of the theorem. \square

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