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Necessary Conditions for Fredholmness of Partial Differential Operators of Irregular Singular Type

By

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§1. Introduction

The convergence of formal power series solutions of ordinary differential equations are extensively studied by many authors in connection with irregularities. (cf. [6] and [10].) In case of partial differential equations of regular singular type, several sufficient conditions are known. (cf. [4] and [1].) In the preceeding paper [9], we gave sufficient conditions for the Fredholmness of partial differential operators of irregular singular type of two independent variables in analytic and Gevrey spaces. Then we deduced the convergence of formal power series solutions from the Fredholmness of the operators. These conditions are expressed in terms of Toeplitz symbols, and they are equivalent to a Riemann-Hilbert factorization condition. (cf. [3] and (2.5), (2.6), (2.7) which follow.) In this paper, we shall show the necessity of these sufficient conditions. More precisely, we will prove that the Riemann-Hilbert factorization conditions (2.6) and (2.7) are necessary and sufficient for the partial differential operators of irregular singular type to be Fredholm operators on certain analytic and Gevrey spaces.

The proof of our theorem is based on the analysis of the main (principal) part of irregular singular type operators via Toeplitz operators on the torus $\mathbf{T}^2 := \mathbf{R}^2/2\pi\mathbf{Z}^2$. In fact, we will show that the essential parts of these operators in studying Fredholm properties are precisely Toeplitz operators. This

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enables us to give necessary and sufficient conditions for Fredholmness in terms of Toeplitz symbols.

§2. Statement of the Results

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{C}$ be the set of complex numbers. Let $\mathbb{C}[[x]]$ be the set of all formal power series

$$\mathbb{C}\,[[x]] := \Big\{u(x)\,;\,u(x) = \sum_{\eta\in\mathbb{N}^2} u_\eta \frac{x^\eta}{\eta!},\quad u_\eta \in \mathbb{C}\Big\}.$$

Let $w_j > 0\ (j = 1, 2)$ and $s > 0$, and let us set $w = (w_1, w_2)$. We denote by $\mathcal{O}(\{|x_1| < w_1\} \times \{|x_2| < w_2\})$ the set of holomorphic functions in a domain $\{|x_1| < w_1\} \times \{|x_2| < w_2\} \subset \mathbb{C}^2$. We define the space $G^s_w(\mu)$ $(\mu \in \mathbb{R})$ by

$$(2.1)\qquad G^s_w(\mu) := \Big\{u = \sum_\eta u_\eta \frac{x^\eta}{\eta\,!}\,;\,\|u\| := \Big(\sum_\eta\Big(|u_\eta|\frac{w^\eta}{(|\eta|-\mu/s)\,!^s}\Big)^2\Big)^{1/2} < \infty\Big\},$$

where the factorial is understood as the gamma function, $r! := \Gamma(r+1)$ for $r \geq 0$ and where we set $(|\eta|-\mu/s)\,! = 1$ if $|\eta|-\mu/s \leq 0$. The space $G^s_w(\mu)$ is a Hilbert space with the norm $\|\cdot\|$. We note that if $s = 1$, the space $G^s_w(\mu)$ is a Hardy space.

We denote by ∂_1^{-1} the integration with respect to x_1, $\partial_1^{-1}u(x) = \int_0^{x_1} u(y_1, x_2)\,dy_1$. The operator ∂_2^{-1} is defined similarly. For $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$, we set $\partial^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}$, where if $\beta_j < 0$ we understand that $\partial_j^{\beta_j} = (\partial_j^{-1})^{-\beta_j}$.

Let $P \equiv P(x, \partial)$ be an integro-differential operator of finite order with holomorphic coefficients in a neighborhood of the origin of $\mathbb{C}^2$,

$$(2.2)\qquad P(x, \partial) = \sum_{\beta\in\mathbb{Z}^2} \partial^\beta a_\beta(x),$$

where $a_\beta(x)$'s are analytic functions of x in some neighborhood of the origin and the summation with respect to β is a finite sum.

By substituting the Taylor expansion of $a_\beta(x)$, $a_\beta(x) = \sum_\gamma a_{\gamma\beta}\,x^\gamma$ in (2.2), we have

$$(2.3)\qquad P(x, \partial) = \sum_{\gamma\in\mathbb{N}^2,\beta\in\mathbb{Z}^2} a_{\gamma\beta}\partial^\beta x^\gamma.$$

For $\partial^\beta x^\gamma\cdot$ we define the s-Gevrey order $\mathrm{ord}_s\, \partial^\beta x^\gamma$ of $\partial^\beta x^\gamma\cdot$ by

$$\mathrm{ord}_s\, \partial^\beta x^\gamma \cdot := |\beta| + (1-s)(|\gamma| - |\beta|). \tag{2.4}$$

Then the s-Gevrey order of P in (2.3) is defined by

$$\mathrm{ord}_s\, P := \sup_{\gamma,\beta} \{|\beta| + (1-s)(|\gamma| - |\beta|)\ ;\ a_{\gamma\beta} \neq 0\}.$$

Here and in what follows we always assume

(A.1) the s - Gevrey order of $P(x, \partial)$ is finite.

This implies that P is of polynomial coefficients in case $s < 1$.

We easily see that

$$P(x, \partial) : G^s_w(\mu + m) \rightarrow G^s_w(\mu), \qquad m = \mathrm{ord}_s\, P$$

is a bounded mapping for every $s > 0$ and every $\mu \geq 0$ if w is sufficiently small because $P(x, \partial)$ is an analytic partial differential operator. (cf. Lemma 3.2 which follows). We assume

(A.2) If $s = 1$ $a_\beta(x)$ is a polynomial in x for every β in (2.2) such that $|\beta| = \mathrm{ord}_1\, P$.

We define the torus $\mathbf{T}^2$ by $\mathbf{T}^2 = \{(z_1, z_2)\ ;\ z_j = e^{i\theta_j}, 0 \leq \theta_j \leq 2\pi, j = 1,2\}$. Then we define the Toeplitz symbol associated with $P(x, \partial)$ by

$$L(z\,; \xi) := \sum_{|\beta| + (1-s)(|\alpha| - |\beta|) = \mathrm{ord}_s\, P} a_{\alpha\beta} z^{\alpha-\beta} w^{\alpha-\beta} \xi^\alpha, \quad \xi \in \mathbf{R}^2, z \in \mathbf{T}^2. \tag{2.5}$$

Theorem 2.1. *Assume that* (A.1) *and* (A.2) *are satisfied. Let* $s > 0$ *and* $\mu \geq 0$ *and* w *be sufficiently small as above, and let* m *be the s-Gevery order of* P. *Then* $P(x, \partial)$ *is a Fredholm operator on* $G^s_w(\mu + m)$ *into* $G^s_w(\mu)$ *if and only if the following conditions are satisfied.*

$$L(z, \xi) \neq 0\ \forall (z_1, z_2) \in \mathbf{T}^2, \quad \forall \xi \in \mathbf{R}^2, \quad |\xi| = 1,\ \xi \geq 0. \tag{2.6}$$

$$\mathrm{ind}_1\, L = \mathrm{ind}_2\, L = 0. \tag{2.7}$$

Here $\mathrm{ind}_1\, L$ (*resp.* $\mathrm{ind}_2\, L$) *is defined by*

$$\text{ind}_1 L = \frac{1}{2\pi i} \oint_{|\zeta_1|=1} d \log L(\zeta_1, z_2, \xi). \tag{2.8}$$

Remarks. (a) We note that the right-hand side of (2.8) is an integer-valued continuous function of z_2 and ξ. Because the sets $\{z_2 \in \mathbb{C}; |z_2|=1\}$ and $\{\xi \in \mathbb{R}^2; |\xi|=1\}$ are connected, the integral (2.8) is constant. Hence the right-hand side is independent of z_2 and ξ. We write this quantity by $\text{ind}_1 L$. We similarly define $\text{ind}_2 L$. We note that the conditions (2.6) and (2.7) are independent of $\mu \geq 0$.

(b) In [9] it was proved that if the conditions (2.6) and (2.7) are satisfied then the operator $P(x, \partial) : G^s_w(\mu+m) \to G^s_w(\mu)$ is a Fredholm operator of index zero for every $s>0$ and $\mu \geq 0$. (cf. Theorem 4.3 of [9].) Hence, the novelity of the above theorem lies in the necessity of (2.6) and (2.7).

§3. Preliminary Lemmas

Let X_j $(j = 1,2)$ be a positive number and set $X = (X_1, X_2)$. We denote by $\mathcal{O}(|x| \leq X)$ the set of holomorphic functions on $\{x \in \mathbb{C}^2 ; |x_j| < X_j, j = 1,2\}$ and continuous on its closure. For $a(x) \in \mathcal{O}(|x| \leq X)$, we put $\|a\|_X := \max_{|x_j| \leq X_j} |a(x)|$. Then we have

Lemma 3.1. *Let $s \geq 1$. Assume that $a(x) \in \mathcal{O}(|x| \leq \rho w)$ $(\rho > 1)$. Then for any $U(x) \in G^s_w(\mu)$, we have $a(x)U(x) \in G^s_w(\mu)$ and there exists a constant C depending only on μ such that*

$$\|aU\| \leq C\left(\frac{\rho}{\rho-1}\right)^2 \|a\|_{\rho w} \|U\|. \tag{3.1}$$

Proof. We put $a(x) = \sum A_\gamma x^\gamma / \gamma! \in \mathcal{O}(|x| \leq \rho w)$. Then by Cauchy's integral formula, we have $|A_\gamma| \leq \|a\|_{w\rho}\, \gamma! / (\rho w)^\gamma$ $(\gamma \in \mathbb{N}^2)$. We put $a(x)U(x) = \sum V_\beta x^\beta / \beta!$. Then we have

$$V_\beta = \sum_{0 \leq \gamma \leq \beta} A_\gamma U_{\beta-\gamma} \frac{\beta!}{(\beta-\gamma)!\,\gamma!}.$$

Hence we have, for $C_1 > 0$

$$\sum_\beta\Big(|V_\beta|\frac{w^\beta}{(|\beta|-\mu/s)!^s}\Big)^2\le\|a\|^2_{\rho w}\sum_\beta\Big(\sum_{0\le\gamma\le\beta}|U_{\beta-\gamma}|\frac{1}{(\rho w)^\gamma}\frac{\beta!}{(\beta-\gamma)!}\frac{w^\beta}{(|\beta|-\mu/s)!^s}\Big)^2$$
$$\le C_1\|a\|^2_{\rho w}\sum_\beta\Big(\sum_{0\le\gamma\le\beta}|U_{\beta-\gamma}|\frac{1}{\rho^{|\gamma|}}\frac{w^{\beta-\gamma}}{(|\beta|-\mu/s-|\gamma|)!^s}\Big)^2$$

$$\le C_1\|a\|^2_{\rho w}\sum_\beta\Big(\sum_\gamma\frac{1}{\rho^{|\gamma|}}\Big)\Big(\sum_\gamma\frac{1}{\rho^{|\gamma|}}\Big(|U_{\beta-\gamma}|\frac{w^{\beta-\gamma}}{(|\beta|-\mu/s-|\gamma|)!^s}\Big)^2\Big)$$
$$\le C_1\Big(\frac{\rho}{\rho-1}\Big)^2\|a\|^2_{\rho w}\sum_\gamma\frac{1}{\rho^{|\gamma|}}\sum_{\beta\ge\gamma}\Big(|U_{\beta-\gamma}|\frac{w^{\beta-\gamma}}{(|\beta-\gamma|-\mu/s)!^s}\Big)^2$$
$$\le C_1\Big(\frac{\rho}{\rho-1}\Big)^4\|a\|^2_{\rho w}\|U\|^2.\ \blacksquare$$

Lemma 3.2. *Let $\mu=|\beta|+(1-s)(|\alpha|-|\beta|)$ be the s-Gevrey order of $\partial^\beta x^\alpha$. Then the map $\partial^\beta x^\alpha : G^s_w(\mu)\to G^s_w(0)$ is continuous. Moreover, for every $\varepsilon>0$ the map $\partial^\beta x^\alpha : G^s_w(\mu+\varepsilon)\to G^s_w(0)$ is a compact operator.*

Proof. For the sake of simplicity we omit the suffices of $G^s_w(\mu)$ and write it by $G(\mu)$. We first show that for every $\kappa<\mu$ the injection $\iota: G(\mu)\to G(\kappa)$ is compact. Let $B\subset G(\mu)$ be a bounded set in $G(\mu)$. If we write $u=\sum_\eta u_\eta x^\eta/\eta!\in B$, then for each fixed η the set $\{u_\eta\ ;\ u\in B\}$ is bounded. Hence, by the diagonal argument, we can choose a sequence $\{u^{(k)}\}\subset B$, $u^{(k)}(x)=\sum_\eta u^{(k)}_\eta x^\eta/\eta!$ such that for each η, $u^{(k)}_\eta\to u_\eta$ when $k\to\infty$.
Moreover we have that

$$\sum_{|\eta|\ge N}\Big(|u^{(k)}_\eta|\frac{w^\eta}{(|\eta|-\kappa/s)!^s)}\Big)^2\le\max_{|\eta|\ge N}\frac{(|\eta|-\mu/s)!^{2s}}{(|\eta|-\kappa/s)!^{2s}}\sum_{|\eta|\ge N}\Big(|u^{(k)}_\eta|\frac{w^\eta}{(|\eta|-\mu/s)!^s}\Big)^2$$
$$\le K\max_{|\eta|\ge N}\frac{(|\eta|-\mu/s)!^{2s}}{(|\eta|-\kappa/s)!^{2s}}\to0\ (N\to\infty),$$

where $K>0$ depends only on B. This proves that the sequence $\{u^{(k)}\}$ converges in $G(\kappa)$.

In order to complete the proof it is sufficient to show that the map $x^\alpha\partial^\beta : G(\mu)\to G(0)$ is continuous because the commutator $[x^\alpha,\partial^\beta]:=x^\alpha\partial^\beta-\partial^\beta x^\alpha : G(\mu)\to G(0)$ is compact. Indeed, we have $\mathrm{ord}_s[x^\alpha,\partial^\beta]<\mu.$. By simple calculations we have

$$x^\alpha\partial^\beta\sum_\eta u_\eta\frac{x^\eta}{\eta!}=\sum u_\eta\frac{x^{\eta+\alpha-\beta}}{(\eta-\beta)!}=\sum u_{\eta+\beta-\alpha}\frac{x^\eta}{(\eta-\alpha)!}. \tag{3.2}$$

Hence we have

(3.3)
$$\sum_\eta \Big(|u_{\eta+\beta-\alpha}| \frac{w^\eta}{(|\eta|)!^s} \frac{\eta!}{(\eta-\alpha)!} \Big)^2 = \sum_\eta \Big(|u_\eta| w^{\eta-\beta+\alpha} \frac{1}{(|\eta|-|\beta|+|\alpha|)!^s} \frac{(\eta-\beta+\alpha)!}{(\eta-\beta)!} \Big)^2.$$

If η is sufficiently large the term $(\eta-\beta+\alpha)!/(\eta-\beta)!$ can be estimated from the above by constant times $|\eta|^{|\alpha|}$. Therefore we have

(3.4)
$$\frac{(|\eta|-\mu/s)!^s}{(|\eta|-|\beta|+|\alpha|)!^s} \frac{(\eta-\beta+\alpha)!}{(\eta-\beta)!} \leq C|\eta|^{s(|\beta|-|\alpha|)-\mu}|\eta|^{|\alpha|} = C|\eta|^{s(|\beta|-|\alpha|)+|\alpha|-\mu}$$

for some constant C independent of η. Because $s(|\beta|-|\alpha|)+|\alpha|-\mu=0$ by assumption the right-hand side of (3.4) is bounded when $|\eta|$ tends to infinity. By (3.2), (3.3) and (3.4) we see that the map $x^\alpha\partial^\beta : G(\mu) \to G(0)$ is continuous. ∎

Let $p(\eta)$ be a function on $\mathbb{N}^2$ such that

$$|p(\eta)| \leq C|\eta|^m, \ \forall \eta \in \mathbb{N}^2 \tag{3.5}$$

for some $C>0$ and $m \geq 0$ independent of η. Then we define the Euler type pseudodifferential operator $p(\delta)$ on $G^s_w(\mu)$ by

$$p(\delta)u := \sum_\eta u_\eta \, p(\eta) \, x^\eta/\eta!, \quad u = \sum_\eta u_\eta \, x^\eta/\eta! \in G^s_w(\mu), \tag{3.6}$$

where we set $\delta=(\delta_1, \delta_2)$, $\delta_j = x_j(\partial/\partial x_j)$, $j=1, 2$. We note that if $p(\eta)=\eta_1+\eta_2$, the operator $p(\delta)=\delta_1+\delta_2$ is a so-called Euler type differential operator. Then we have

Lemma 3.3. *Let $p(\eta)$ be a function on $\mathbb{N}^2$ such that $\sup_{|\eta|\geq N}|p(\eta)| \to 0$ when $N \to \infty$. Then the map $p(\delta) : G^s_w(\mu) \to G^s_w(\mu)$ is a compact operator for every $\mu \geq 0$.*

The proof of this lemma follows exactly the same arguments of the former half of the proof of Lemma 3.2. Therefore we omit the proof.

In the following we give basic properties of Fredholm operators on a Hilbert space H with norm $\|\cdot\|$. We denote by $\mathscr{L}(H)$ the space of linear continuous operators on H. An operator $L \in \mathscr{L}(H)$ is said to be a Fredholm operator if the range LH of L is closed in H, and the kernel and cokernel of L are of finite dimension, i.e., dim $Ker L < \infty$ and dim $Coker L < \infty$, where $Coker\ L = H/LH$. We denote the set of Fredholm operators by $\Psi(H)$. For $L \in \Psi(H)$ an index of L is defined by

$$\text{ind } L := \dim Ker\ L - \dim Coker\ L.$$

Let $C_\infty(H)$ be the space of compact operators on H, and let I denote the identity operator on H. Then the following two lemmas are well known. (cf. [3]).

Lemma 3.4. *An operator $L \in \mathscr{L}(H)$ is a Fredholm operator if and only if there exist linear continuous operators $R_1 \in \mathscr{L}(H)$, $R_2 \in \mathscr{L}(H)$ and compact operators $K_1 \in C_\infty(H)$, $K_2 \in C_\infty(H)$ such that*

$$R_1 L = I + K_1,\ LR_2 = I + K_2.$$

Here the operators R_1 and R_2 are called left and right regularizers, respectively.

Lemma 3.5. *The set $\Psi(H)$ is an open subset of $\mathscr{L}(H)$ and the index is constant on each connected component of $\Psi(H)$. If $L \in \Psi(H)$ and $K \in C_\infty(H)$ the operator $L+K$ is in $\Psi(H)$ and ind $(L+K)$ = ind L.*

Lemma 3.6. *Let $L \in \mathscr{L}(H)$ and let L^* denote the adjoint operator of L. Then $L \in \Psi(H)$ if and only if there exist a positive constant A, a positive integer n and K_j, $\tilde{K}_j \in C_\infty(H)$, $(j = 1,2,\dots,n)$ such that the following inequalities hold:*

$$\|Lf\| + \sum_{j=1}^{n} \|K_j f\| \geq A\,\|f\|, \qquad \forall f \in H, \tag{3.7}$$

$$\|L^* f\| + \sum_{j=1}^{n} \|\tilde{K}_j f\| \geq A\,\|f\|, \qquad \forall f \in H. \tag{3.8}$$

Moreover, if $L \in \Psi(H)$ we can take $n = 1$ in (3.7) *and* (3.8).

Proof. Suppose that L is a Fredholm operator. Let $H = H_1 \oplus H_2$ and $H = H_1' \oplus H_2'$ be direct decompositions of H, where $H_1 = Ker\ L$, $H_1' = Coker\ L$. We denote by K the projection onto H_1. Clearly $K \in C_\infty(H)$ and $L: H_2 \rightarrow H_2'$ is continuous and bijective. Hence the closed graph theorem implies that L is a homeomorphism. Therefore, if we write $f = f_1 + f_2 \in H_1 \oplus H_2$ we have

$$(3.9) \qquad \|Lf\| = \|Lf_2\| \geq A\|f_2\| = A\|f\| - A\|f_1\| \geq A\|f\| - \|Kf\|$$

for some $A>0$. This proves (3.7). The inequality (3.8) will be proved similarly because L^* is also a Fredholm operator.

Conversely, let us suppose that (3.7) and (3.8) are satisfied. By (3.7) we have that $\sum_{j=1}^{n}\|K_j f\| \geq A\|f\|$ if $f \in Ker\ L$. Suppose that $Ker\ L$ is of infinite dimension. Then we can find a sequence $f_k \in Ker\ L$ such that $\|f_k\|=1$ and f_k weakly converges to zero when $k \to \infty$. Because K_j's are compact, the sequence $K_j f_k$ strongly converges to zero when $k \to \infty$ for each j. This contradicts to the above inequality. Therefore it follows that $Ker\ L$ is of finite dimension. Similarly (3.8) implies that $Ker\ L^*$ is of finite dimension.

In order to show that the range of L is closed, we first show the following

(*) Every bounded sequence $f_k \in H$ such that Lf_k is strongly convergent has a strongly convergent subsequence.

Indeed, f_k contains a weakly convergent subsequence because f_k is bounded. Without loss of generality, we may assume that f_k weakly converges to some $f \in H$. By the continuity of L it follows that Lf_k weakly converges to Lf. Because Lf_k strongly converges by assumption, it follows that Lf_k strongly converges to Lf. On the other hand, by the compactness of K_j in (3.7) the sequence $K_j f_k$ strongly converges to $K_j f$ when $k \to \infty$. By (3.7) with f replaced by $f_k - f$ we see that the sequence $\{f_k\}$ strongly converges to f. Hence we have the assertion.

Let H_2 be an orthogonal complement of $Ker\ L =: H_1$. It then follows that

$$(3.10) \qquad \|f\| \leq C\|Lf\|, \quad f \in H_2,$$

for some $C>0$. In fact, otherwise there is a sequence $f_k \in H_2$ with $\|f_k\| = 1$ and $\|Lf_k\| \leq 1/k$. By the property (*), f_k contains a subsequence strongly convergent to f such that $f \in H_2$ with $\|f\| = 1$, $Lf = 0$, which is a contradiction proving the assertion. The closedness of the range of L is clear from (3.10). Since $LH = \overline{LH} = Ker\ L^{*\perp}$, it follows that $Coker\ L$ is of finite dimension. ∎

§4. Reduction to Toeplitz Operators

We set $\langle\eta\rangle := (1+|\eta|^2)^{1/2}$ and we denote by $\langle\delta\rangle$ the Euler type pseudo-differential operator with symbol $\langle\eta\rangle$. Let P be given by (2.3) and let m be an s-Gevrey order of P. Then we have

Proposition 4.1. *Let the operators P_0 and Q_0 be defined by*

$$Q_0 := P_0\langle\delta\rangle^{-m}, \quad P_0(x, \partial) := \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}\partial^\beta x^\alpha. \tag{4.1}$$

Then Q_0 maps $G^s_w(0)$ into itself. Moreover Q_0 is a Fredholm operator if and only if $P: G^s_w(m) \to G^s_w(0)$ is a Fredholm operator.

Proof. In the following, we omit the suffices of $G^s_w(\mu)$ and write it by $G(\mu)$. We write the operator P in the following form

$$\begin{aligned} P(x, \partial) &= \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}\partial^\beta x^\alpha + \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)<m} a_{\alpha\beta}\partial^\beta x^\alpha \\ &=: P_0(x, \partial) + P_1(x, \partial). \end{aligned} \tag{4.2}$$

Because the s-Gevrey order of each term $\partial^\beta x^\alpha$ in P_1 is smaller than m, it follows from Lemmas 3.1 and 3.2 that the map $P_1 : G(m) \to G(0)$ is compact. Therefore by Lemma 3.5 it follows that $P_0 : G(m) \to G(0)$ is a Fredholm operator if and only if $P: G(m) \to G(0)$ is a Fredholm operator.

Since we can easily see that $\langle\delta\rangle^{-m} : G(0) \to G(m)$ is an isomorphism, we conclude that $Q_0 : G(0) \to G(0)$ is a Fredholm operator if and only if $P_0 : G(m) \to G(0)$ is a Fredholm operator. ∎

Next we shall show that the Fredholmness of the operator $Q_0 : G^s_w(0) \to G^s_w(0)$ is equivalent to that of a certain Toeplitz operator on the two dimensional torus $\mathbf{T}^2$. Let us take the coordinate $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbf{T}^2$. Let $u = \sum u_\eta x^\eta / \eta! \in G^s_w(0)$. We set $v_\eta := u_\eta\, w^\eta / |\eta|!^s$. Then $u \in G^s_w(0)$ if and only if the sequence $\{v_\eta\}$ is in $\ell_2^+ := \ell_2(\mathbf{Z}^2_+)$, the set of square summable sequences on $\mathbf{Z}^2_+$, where $\mathbf{Z}_+$ is the set of nonnegative integers. Because the space ℓ_2^+ and the Hardy space $H^2(\mathbf{T}^2)$ are isomorphic, it follows that $u \in G^s_w(0)$ if and only if $\sum_\eta v_\eta e^{i\theta\eta}$ is in $H^2(\mathbf{T}^2)$. Because $H^2(\mathbf{T}^2)$ is a closed subspace of $L^2(\mathbf{T}^2)$, the space of square integrable functions, there is a projection π from $L^2(\mathbf{T}^2)$ onto $H^2(\mathbf{T}^2)$. By the correspondence between the spaces $G^s_w(0)$ and $H^2(\mathbf{T}^2)$ the Euler type operator $p(\delta)$ in (3.6) on $G^s_w(0)$ also defines a pseudodifferential operator $p(D_\theta)$ ($D_\theta = i^{-1}\partial/\partial\theta$) on $H^2(\mathbf{T}^2)$. We denote by $\lambda_\alpha(D_\theta)$ the pseudodifferential operator with symbol $\lambda_\alpha(\eta) := \eta^\alpha|\eta|^{-|\alpha|}$ ($\eta \neq 0$) and $\lambda_\alpha(0) = 0$. We define a Toeplitz operator on $H^2(\mathbf{T}^2)$ by

$$(4.3) \qquad T = \pi \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta}\, w^{\alpha-\beta} e^{i(\alpha-\beta)\theta} \lambda_\alpha(D_\theta) : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2).$$

We note that, by (2.5), $T = \pi L(e^{i\theta}, D_\theta/|D_\theta|)$.

Proposition 4.2. *The operator $Q_0 : G^s_w(0) \to G^s_w(0)$ is a Fredholm operator if and only if the Toeplitz operator $T : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)$ is a Fredholm operator.*

Proof. By the isomorphism between $G^s_w(0)$ and $H^2(\mathbb{T}^2)$ the projection π : $L^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)$ naturally induces a projection on the formal Laurent series

$$(4.4) \qquad \pi\, u := \sum_{\eta\in\mathbb{Z}^2_+} u_\eta x^\eta / \eta! \quad \text{for} \quad u = \sum_{\eta\in\mathbb{Z}^2} u_\eta x^\eta / \eta!.$$

Here we use the same notation π for the sake of simplicity. We note that, for $k \geq 0$, $n \geq 0$ and $m \geq 0$

$$(4.5) \qquad \left(\frac{\partial}{\partial t}\right)^k t^m t^n = (n+m)(n+m-1)\cdots(n+m-k+1)\, t^{n+m-k}$$

$$= \begin{cases} t^{n+m-k} \dfrac{\Gamma(n+m+1)}{\Gamma(n+m-k+1)}, & \text{if} \quad n+m-k \geq 0 \\ 0 & \text{if} \quad n+m-k < 0 \end{cases}$$

where Γ denotes the gamma function. Similarly, if $k < 0$ we have

$$\left(\frac{\partial}{\partial t}\right)^k t^m t^n = \frac{t^{n+m-k}}{(n+m+1)\cdots(n+m-k)} = t^{n+m-k} \frac{\Gamma(n+m+1)}{\Gamma(n+m-k+1)}.$$

If we define Euler type operator $p_{\alpha\beta}(\delta)$ on $G(\mu)$ $(\mu \geq 0)$ by

$$(4.6) \qquad p_{\alpha\beta}(\eta) = \begin{cases} \displaystyle\prod_{j=1}^{2} \frac{\Gamma(\eta_j + \alpha_j + 1)}{\Gamma(\eta_j + \alpha_j - \beta_j + 1)}, & \text{if} \quad \eta + \alpha - \beta \in \mathbb{Z}^2_+ \\ 0 & \text{if} \quad \eta + \alpha - \beta \notin \mathbb{Z}^2_+ \end{cases}$$

we have, for $\alpha \in \mathbb{Z}^2_+$ and $\beta \in \mathbb{Z}^2$

$$\partial^\beta x^\alpha u = x^{\alpha-\beta} p_{\alpha\beta}(\delta) u, \quad \text{for} \quad u \in G^s_w(\mu) \ (\mu \geq 0).$$

Therefore we have that

$$(4.7) \qquad P_0(x, \partial) = \sum_{|\beta|+(1-s)(|\alpha|-|\beta|)=m} a_{\alpha\beta} x^{\alpha-\beta} p_{\alpha\beta}(\delta).$$

By the definition of $p_{\alpha\beta}(\delta)$ in (4.6) we see that in the expression of $x^{\alpha-\beta}p_{\alpha\beta}(\delta)\langle\delta\rangle^{-m}u(u\in G(0))$ there appear no negative powers. Hence we have

$$(4.8) \qquad \begin{aligned} Q_0 &= \pi\sum a_{\alpha\beta}x^{\alpha-\beta}p_{\alpha\beta}(\delta)\langle\delta\rangle^{-m} \\ &= \pi\sum a_{\alpha\beta}x^{\alpha-\beta}\langle\delta\rangle^{(s-1)(|\alpha|-|\beta|)}\langle\delta\rangle^{-|\beta|}p_{\alpha\beta}(\delta), \end{aligned}$$

on $G(0)$.

We shall study the operators $\pi x^{\gamma}\langle\delta\rangle^{(s-1)|\gamma|}(\gamma=\alpha-\beta)$ and $\langle\delta\rangle^{-|\beta|}p_{\alpha\beta}(\delta)$. Let $u=\sum u_\eta x^\eta/\eta!\in G(0)$. We set $v_\eta := u_\eta w^\eta/|\eta|!^s$. Then we have

$$\begin{aligned} \pi x^{\gamma}\langle\delta\rangle^{(s-1)|\gamma|}\sum u_\eta\frac{x^\eta}{\eta!} &= \pi x^{\gamma}\langle\delta\rangle^{(s-1)|\gamma|}\sum v_\eta\, w^{-\eta}|\eta|!^s\frac{x^\eta}{\eta!} \\ &= \sum_{\eta+\gamma\ge 0,\eta\ge 0} v_\eta\, w^{-\eta}|\eta|!^s\,\langle\eta\rangle^{(s-1)|\gamma|}\frac{x^{\eta+\gamma}}{\eta!} \\ &= \sum v_{\eta-\gamma}\, w^{\gamma-\eta}\,(|\eta|-|\gamma|)!^s\,\langle\eta-\gamma\rangle^{(s-1)|\gamma|}\frac{\eta!}{(\eta-\gamma)!}\frac{x^\eta}{\eta!}. \end{aligned}$$

Therefore $\pi x^{\gamma}\langle\delta\rangle^{(s-1)|\gamma|}$ is given by

$$(4.9) \qquad \pi x^{\gamma}\langle\delta\rangle^{(s-1)|\gamma|} : \{v_\eta\}\in\ell_2^+ \rightarrow \left\{v_{\eta-\gamma}\, w^{\gamma}\frac{(|\eta|-|\gamma|)!^s}{|\eta|!^s}\langle\eta-\gamma\rangle^{(s-1)|\gamma|}\frac{\eta!}{(\eta-\gamma)!}\right\}\in\ell_2^+.$$

We define the pseudodifferential operator $A_\gamma(D_\theta)$ with symbol $A_\gamma(\eta)$ by

$$(4.10) \qquad A_\gamma(\eta) := \frac{|\eta|!^s\langle\eta\rangle^{(s-1)|\gamma|}}{(|\eta|+|\gamma|)!^s}\frac{(\eta+\gamma)!}{\eta!}.$$

Let S_γ be a multiplication operator by a function $e^{i\gamma\theta}$, $S_\gamma\sum u_\eta e^{i\eta\theta}=\sum u_\eta e^{i(\eta+\gamma)\theta}$. Then it follows from (4.9) that the operator $\pi x^{\gamma}\langle\delta\rangle^{(s-1)|\gamma|}$ corresponds to

$$(4.11) \qquad \pi S_\gamma A_\gamma(D_\theta)\, w^{\gamma}.$$

We have, assuming $\eta+\gamma\geq 0$

(4.12)
$$\frac{|\eta|!^s\langle\eta\rangle^{(s-1)|\gamma|}}{(|\eta|+|\gamma|)!^s}\frac{(\eta+\gamma)!}{\eta!}=\tilde{\lambda}_\gamma(\eta)+r_\gamma(\eta),\ \tilde{\lambda}_\gamma(\eta)=\frac{(\eta_1+1)^{\gamma_1}(\eta_2+1)^{\gamma_2}}{\langle\eta\rangle^{|\gamma|}},$$

where $r_\gamma(\eta)$ consists of terms such that $r_\gamma(\eta)\to 0$ when $|\eta|\to\infty$.

Indeed the quantity $|\eta|!^s\langle\eta\rangle^{s|\gamma|}(|\eta|+|\gamma|)!^{-s}$ tends to 1 when $|\eta|\to\infty$ and γ fixed. On the other hand, we get, assuming $\eta_j+\gamma_j\geq 0$,

$$\frac{(\eta_j+\gamma_j)!}{\eta_j!}\langle\eta\rangle^{-\gamma_j}=\left(\frac{\eta_j+1}{\langle\eta\rangle}\right)^{\gamma_j}+\Psi_j(\eta), \tag{4.13}$$

with $\Psi_j(\eta)$ satisfying $\Psi_j(\eta)\to 0$ when $|\eta|\to\infty$. By these estimates we have (4.12).

It follows from (4.11), (4.12) and the definition of π that

$$\pi S_\gamma A_\gamma(D_\theta)w^\gamma=\pi S_\gamma\tilde{\lambda}_\gamma(D_\theta)w^\gamma+\pi R_\gamma(D_\theta) \tag{4.14}$$

where $R_\gamma(D_\theta):=S_\gamma r_\gamma(D_\theta)w^\gamma$, with $r_\gamma(D_\theta)$ being the psedodifferential operator with symbol $r_\gamma(\eta)$. We note that by Lemma 3.3 the operator $\pi R_\gamma(D_\theta)$ is a compact operator.

Next we consider the operator $\langle\delta\rangle^{-|\beta|}p_{\alpha\beta}(\delta)$. In view of the relation

$$\langle\delta\rangle^{-|\beta|}p_{\alpha\beta}(\delta)\sum_{\eta\geq 0}u_\eta\frac{x^\eta}{\eta!}=\sum_{\eta\geq\beta-\alpha}v_\eta w^{-\eta}|\eta|!^s\langle\eta\rangle^{-|\beta|}p_{\alpha\beta}(\eta)\frac{x^\eta}{\eta!},$$

the operator $\langle\delta\rangle^{-|\beta|}p_{\alpha\beta}(\delta)$ corresponds to the pseudodifferential operator $\langle D_\theta\rangle^{-|\beta|}p_{\alpha\beta}(D_\theta)$ on H^2. If $\eta+\alpha-\beta\geq 0$ we have

$$\langle\eta\rangle^{-|\beta|}p_{\alpha\beta}(\eta)=\tilde{\lambda}_\beta(\eta)+\tilde{r}_{\alpha\beta}(\eta),$$

where $\tilde{r}_{\alpha\beta}(\eta)$ satisfies that $\sup_{|\eta|\geq n}|\tilde{r}_{\alpha\beta}(\eta)|\to 0$ when n tends to infinity. Therefore the operator $\langle\delta\rangle^{-|\beta|}p_{\alpha\beta}(\delta)$ in (4.8) corresponds to $\tilde{\lambda}_\beta(D_\theta)+\tilde{r}_{\alpha\beta}(D_\theta)$.

By (4.14) with $\gamma=\alpha-\beta$ we get from (4.8) that Q_0 corresponds to the operator

$$\pi\sum a_{\alpha\beta}\,(S_{\alpha-\beta}\,\tilde{\lambda}_{\alpha-\beta}\,w^{\alpha-\beta}+R_{\alpha-\beta})\,(\tilde{\lambda}_\beta+\tilde{r}_{\alpha\beta}) \tag{4.15}$$

$$= \pi \sum a_{\alpha\beta}\, S_{\alpha-\beta}\, w^{\alpha-\beta}\, \tilde{\lambda}_{\alpha-\beta}\, \tilde{\lambda}_{\beta} + \pi \sum a_{\alpha\beta}\, K_{\alpha\beta}$$

$$= \pi \sum a_{\alpha\beta}\, S_{\alpha-\beta}\, \tilde{\lambda}_{\alpha}\, w^{\alpha-\beta} + \pi \sum a_{\alpha\beta}\, K_{\alpha\beta},$$

where

$$K_{\alpha\beta} = S_{\alpha-\beta}\, \tilde{\lambda}_{\alpha-\beta}\, w^{\alpha-\beta}\, \tilde{r}_{\alpha\beta} + R_{\alpha-\beta}\, \tilde{\lambda}_{\beta} + R_{\alpha-\beta}\, \tilde{r}_{\alpha\beta}. \tag{4.16}$$

For each α and β, $K_{\alpha\beta}$ is a compact operator by the definition of symbols $R_{\alpha-\beta}$ and $\tilde{r}_{\alpha\beta}$ and Lemma 3.3. Since the sum $\sum a_{\alpha\beta} K_{\alpha\beta}$ is a finite sum, the second term in the right-hand side of (4.15) is a compact operator. Because $\tilde{\lambda}_{\alpha} - \eta^{\alpha}/|\eta|^{|\alpha|}$ defines a compact pseudodifferential operator on H^2 the right-hand side of (4.15) is equal to T modulo compact operators. ■

In view of Propositions 4.1 and 4.2 it is sufficient to show the following theorem.

Theorem 4.3. *Suppose that the Toeplitz operator* $T: H^2(\mathbf{T}^2) \to H^2(\mathbf{T}^2)$ *in* (4.3) *is a Fredholm operator. Then the conditions* (2.6) *and* (2.7) *are necessary.*

We shall prove this theorem in Sections 5 and 6.

§5. Proof of Theorem 4.3. — the Necessity of (2.6)

In this section we shall prove the necessity of (2.6) in Theorem 4.3. We use the same notations as in the previous sections unless otherwise is stated. We define the projections π_1 and π_2 by

$$\pi_j\, u = \sum_{\eta_j \geq 0} u_\eta\, e^{i\eta\theta}, \quad \text{for} \quad u = \sum_{\eta} u_\eta\, e^{i\eta\theta} \in L^2(\mathbf{T}^2), \quad (j = 1,2). \tag{5.1}$$

Then the spaces H_j^2,

$$H_j^2 \equiv H_j^2(\mathbf{T}^2) := \pi_j\, L^2(\mathbf{T}^2), \quad (j = 1,2),$$

are closed subspaces of $L^2(\mathbf{T}^2)$. It is obvious that $\pi = \pi_1 \pi_2$. Recall that $L(z, \xi)$ denotes the Toeplitz symbol defined in (2.5). Let $\mathscr{L}(\theta, D_\theta)$ be the pseudodifferential operator of order zero on $\mathbb{T}^2$ with symbol $L(z, \xi/|\xi|)$ where $z = (e^{i\theta_1}, e^{i\theta_2})$. Then it follows from (4.3) that

$$T = \pi\mathscr{L}(\theta, D_\theta)\ ;\ H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2). \tag{5.2}$$

We define Toeplitz operators $T_j (j = 1,2)$ by

$$T_j := \pi_j\mathscr{L}(\theta, D_\theta)\ ;\ H_j^2(\mathbb{T}^2) \to H_j^2(\mathbb{T}^2), \tag{5.3}$$

and the multiplication operators U_j $(j = 1,2)$ by

$$U_j u := \sum_\eta u_\eta e^{i\eta\theta + i\theta_1 + i\theta_2 - i\theta_j}, \quad \text{for} \quad u = \sum_\eta u_\eta\, e^{i\eta\theta} \in L^2(\mathbb{T}^2). \tag{5.4}$$

Clearly, U_j are unitary operators on L^2 and $[U_1, U_2] = 0$, where $[A, B] := AB - BA$. Moreover we have that $[U_j, \pi_j] = 0$, which is crucial in the following arguments.

For the later use we introduce the function class $\mathscr{A}_{\xi,\hat{\theta}}$ in the following way. Let $\xi \in \mathbb{R}_+^2$, $\xi \neq 0$, $\kappa > 0$ and $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2)$, $0 \leq \widehat{\theta}_j \leq 2\pi$ $(j = 1,2)$ are given. For $\varphi \in C^\infty(\mathbb{T}^2)$ with support contained in a small neighborhood of the origin we define

$$\mathscr{A}_{\xi,\hat{\theta}} := \left\{\varphi\left(\frac{\theta - \widehat{\theta}}{\kappa}\right) e^{i\theta k\xi}\ ;\ \kappa \in \mathbb{N}\right\}. \tag{5.5}$$

Then we have the following

Lemma 5.1. *Let ξ be given by* (5.5). *Then under the same assumptions as in Theorem* 4.3 *there exist a constant $C > 0$ and a pseudodifferential operator $\mathscr{K}$ of order -1 on $\mathbb{T}^2$ such that*

$$\|\mathscr{L}g\| + \|\mathscr{K}g\| \geq C\|g\| \quad \textit{for any} \quad \forall g \in \mathscr{A}_{\xi,\hat{\theta}}. \tag{5.6}$$

Proof. We shall divide the proof into 6 steps.

Step 1. By assumption there exist a compact operator K on H^2 and $A > 0$ such that

$$\|Tu\| + \|Ku\| \geq A\|u\|, \quad \forall u \in H^2(\mathbb{T}^2). \tag{5.7}$$

We put $u = \pi g = \pi_1\pi_2 g$, $g \in H_1^2$ and $Q_j = I - \pi_j (j = 1,2)$ in (5.7). Because $\pi_1 g = g$ for $g \in H_1^2$, we have,

$$\|\pi\mathscr{L}(\theta, D_\theta)\ \pi g\| + \|\pi K \pi g\| + A\|Q_2 g\| \geq A\|g\|, \quad \forall g \in H_1^2. \tag{5.8}$$

For a positive integer n we get, from (5.8) that

$$\|\pi\mathscr{L}(\theta, D_\theta)\ \pi U_1^n g\| + \|\pi K \pi U_1^n g\| + A\|Q_2 U_1^n g\| \geq A\|U_1^n g\|, \quad \forall g \in H_1^2. \tag{5.9}$$

Because U_1 is an isometry on H_1^2 we have

(5.10)
$$\|U_1^{-n}\ \pi\mathscr{L}(\theta, D_\theta)\ \pi U_1^n g\| + \|\pi K \pi U_1^n g\| + A\|U_1^{-n}\ Q_2 U_1^n g\| \geq A\|g\|, \ \forall g \in H_1^2.$$

Similarly we replace g in (5.10) with $\pi_1 U_2^n\ g\ (g \in L^2)$. Then, by noting that $Q_1 = I - \pi_1$ we have

(5.11)
$$\begin{aligned}\|U_1^{-n}\ \pi_1\pi_2\mathscr{L}\pi_1\pi_2 U_1^n\ \pi_1 U_2^n g\| + \|\pi K \pi U_1^n \pi_1 U_2^n g\| + A\|U_1^{-n} Q_2 U_1^n \pi_1 U_2^n g\| \\ + A\|Q_1 U_2^n g\| \geq A\|U_2^n g\|, \ \forall g \in L^2.\end{aligned}$$

Step 2. We shall rewrite (5.11). We define the operators $V_{j,n}$ $(j = 1,2)$ by

$$V_{1,n} := U_1^{-n}\pi_2 U_1^n, \quad V_{2,n} := U_2^{-n}\pi_1 U_2^n. \tag{5.12}$$

Because $[U_j, \pi_j] = 0$, $[U_1, U_2] = 0$ on L^2, we have $[V_{2,n}, U_1^{\pm n}] = 0$ and $[V_{1,n}, U_2^{\pm n}] = 0$ for every integer n. In addition it is easy to see that $[V_{1,n}, V_{2,n}] = 0$ and $(V_{j,n})^2 = V_{j,n}$ for $j = 1,2$. It follows that

(5.13)
$$\begin{aligned}\|U_1^{-n}\pi_1\pi_2\ \mathscr{L}\pi_1\pi_2 U_1^n \pi_1 U_2^n g\| &= \|U_2^{-n} U_1^{-n}\pi_1\pi_2\ \mathscr{L}\pi_1\pi_2\ U_1^n\pi_1 U_2^n g\| \\ &= \|V_{1,n} U_1^{-n} V_{2,n} U_2^{-n}\ \mathscr{L} U_2^n V_{2,n} U_2^{-n}\ U_1^n V_{1,n} U_2^n V_{2,n}\ g\| \\ &= \|V_{1,n} V_{2,n} U_1^{-n} U_2^{-n}\ \mathscr{L} U_2^n U_1^n V_{1,n} V_{2,n} g\|.\end{aligned}$$

In the same way we have

$$\|\pi K \pi U_1^n \pi_1 U_2^n g\| = \|\pi_1\pi_2 K\ U_1^n U_2^n V_{1,n} V_{2,n} g\|, \tag{5.14}$$

$$\|U_1^{-n} Q_2 U_1^n \pi_1 U_2^n g\| = \|U_2^{-n} U_1^{-n} Q_2 U_1^n \pi_1 U_2^n g\| = \|(I - V_{1,n})\ V_{2,n} g\|, \tag{5.15}$$

$$\|Q_1 U_2^n g\| = \|U_2^{-n} Q_1 U_2^n g\| = \|(I - V_{2,n})\ g\|. \tag{5.16}$$

By inserting (5.13) - (5.16) to (5.11) we obtain the following

(5.17)
$$\|V_{1,n}V_{2,n}U_1^{-n}U_2^{-n}\mathcal{L}U_2^nU_1^nV_{1,n}V_{2,n}g\| + \|\pi_1\pi_2\ KU_1^nU_2^nV_{1,n}V_{2,n}g\| \\ + A\|(I - V_{1,n})V_{2,n}g\| + A\|(I - V_{2,n})g\| \geq A\|g\|,\ \forall g \in L^2.$$

Step 3. We shall estimate the limit when $n \to \infty$ in (5.17) for $g \in \mathscr{A}_{\xi,\hat{\theta}}$. To this end, we first calculate the Fourier coefficients of $g(\theta) := \varphi((\theta - \widehat{\theta})/\kappa)e^{i\theta k\xi}$. Because the support of φ is contained in some neighborhood of the origin, the support of the function $\varphi((\theta - \widehat{\theta})/\kappa)$ is contained in a neighborhood of $\widehat{\theta}$ if we take $\kappa > 0$ sufficiently small. Therefore, if κ is sufficiently small the Fourier coefficient $\hat{g}(\eta)$ of $g(\theta)$ is given by

$$\int_{\mathbb{T}^2} e^{-i\theta\eta}\varphi\left(\frac{\theta - \widehat{\theta}}{\kappa}\right)e^{i\theta k\xi}d\theta = \int_{\mathbb{R}^2} e^{-i(y\kappa+\widehat{\theta})(\eta - k\xi)}\kappa^2\varphi(y)\,dy = e^{-i\widehat{\theta}(\eta-k\xi)}\kappa^2\widehat{\varphi}((\eta - k\xi)\kappa)$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ. It follows that the right-hand side is rapidly decreasing in η when $\eta \to \infty$ in a sufficiently small conical neighborhood of ξ_0 such that $\xi_0 \neq \xi$ uniformly in k.

By the definition of $V_{2,n}$ in (5.12) we have

(5.18)
$$V_{2,n}g = \sum_{\eta_1 \geq -n} e^{i\eta\theta - i(\eta-k\xi)\widehat{\theta}}\kappa^2\widehat{\varphi}((\eta - k\xi)\kappa),\ g = \varphi((\theta - \widehat{\theta})/\kappa)e^{i\theta k\xi} \in \mathscr{A}_{\xi,\hat{\theta}}.$$

Therefore we have

(5.19)
$$\|(I - V_{2,n})g\| \to 0\ (n \to \infty),$$

where the limit is uniform with respect to $k, (k = 1,2,\ldots)$.

Similarly, by simple computations we have

(5.20)
$$(I - V_{1,n})V_{2,n}g = \sum_{\eta_1 \geq -n, \eta_2 < -n} e^{i\eta\theta - i\hat{\theta}(\eta-k\xi)}\kappa^2\widehat{\varphi}((\eta - k\xi)\kappa),\ g = \varphi\left(\frac{\theta - \widehat{\theta}}{\kappa}\right)e^{i\theta k\xi} \in \mathscr{A}_{\xi,\hat{\theta}}.$$

It follows that

(5.21)
$$\|(I - V_{1,n})V_{2,n}'g\| \to 0\ (n \to \infty),$$

uniformly with respect to k.

Step 4. We shall estimate the term $\|\pi_1\pi_2 KU_1^n U_2^n V_{1,n} V_{2,n}g\|$ in (5.17) when $n \to \infty$. We note

$$V_{1,n}V_{2,n}g = \sum_{\eta_1 \geq -n, \eta_2 \geq -n} e^{i\eta\theta - i(\eta - k\xi)\hat{\theta}} \kappa^2 \hat{\varphi}((\eta - k\xi)\kappa),$$

for $g = \varphi((\theta - \hat{\theta})/\kappa) e^{i\theta k\xi} \in \mathscr{A}_{\xi,\hat{\theta}}$. It follows that

$$\lim_{n \to \infty} \| V_{1,n} V_{2,n} g - g \| = 0 \quad \text{uniformly in} \quad g \in \mathscr{A}_{\xi,\hat{\theta}}. \tag{5.22}$$

By (5.22) and the definitions of $\mathscr{A}_{\xi,\hat{\theta}}$, U_1 and U_2 we easily see that $U_1^n U_2^n V_{1,n} V_{2,n}g$ weakly converges to zero uniformly in $g \in \mathscr{A}_{\xi,\hat{\theta}}$ when $n \to \infty$. In order to see this it is sufficient to show that $U_1^n U_2^n g$ weakly converges to zero uniformly in $g \in \mathscr{A}_{\xi,\hat{\theta}}$ when $n \to \infty$. This follows from the definition of U_1 and U_2 in (5.4).

We want to show that $KU_1^n U_2^n V_{1,n} V_{2,n}g$ converges to zero strongly and uniformly on $\mathscr{A}_{\xi,\hat{\theta}}$ when $n \to \infty$. Let $\varepsilon > 0$ be arbitrarily given. Because the set $B := \{U_1^n U_2^n V_{1,n} V_{2,n}g \,;\, n = 1,2,\dots, g \in \mathscr{A}_{\xi,\hat{\theta}}\}$ is bounded in L^2, the set KB is precompact. Thus we can find a finite $g_j \in \mathscr{A}_{\xi,\hat{\theta}}$ and integers n_j $(j = 1,\dots,\ell)$ such that the set KB is covered by ε-neighborhoods of $h_j := KU_1^n U_2^n V_{1,n} V_{2,n}g$ $(n = n_j,\ j = 1,\dots,\ell)$. Let us write $h_j = \sum_\eta h_j(\eta) e^{i\eta\theta}$ and let $N \gg 1$. For every $h \in KB$, $h = \sum_\eta h(\eta) e^{i\eta\theta}$ we can find j such that $\|h - h_j\| < \varepsilon$. It follows that

$$\begin{aligned}\Big(\sum_{|\eta| \geq N} |h(\eta)|^2\Big)^{1/2} &\leq \Big(\sum_{|\eta| \geq N} |h(\eta) - h_j(\eta)|^2\Big)^{1/2} + \Big(\sum_{|\eta| \geq N} |h_j(\eta)|^2\Big)^{1/2} \\ &\leq \varepsilon + \Big(\sum_{|\eta| \geq N} |h_j(\eta)|^2\Big)^{1/2} \to \varepsilon \ (N \to \infty).\end{aligned}$$

Thus there exists $N \geq 1$ such that

$$\Big(\sum_{|\eta| \geq N} |h(\eta)|^2\Big)^{1/2} \leq 2\varepsilon \quad \text{for all} \quad \forall\, h \in KB. \tag{5.23}$$

For such an integer N we take a cutoff function $\phi(\eta)$ supported on $|\eta| < N$ such that $\phi \equiv 1$ on $|\eta| < N$, $\eta \in \mathbf{Z}^2$. Then for the pseudodifferential operator $\phi \equiv \phi(D_\theta)$ we have

$$KU_1^n U_2^n V_{1,n} V_{2,n}g = \phi KU_1^n U_2^n V_{1,n} V_{2,n}g + (1 - \phi) KU_1^n U_2^n V_{1,n} V_{2,n}g.$$

The second term in the right-hand side is smaller than 2ε. As to the first term we note that it strongly converges to zero uniformly with respect to $g \in \mathscr{A}_{\xi,\hat{\theta}}$ when $n \to \infty$ if and only if $KU_1^n U_2^n V_{1,n} V_{2,n} g$ weakly converges to zero uniformly with respect to $g \in \mathscr{A}_{\xi,\hat{\theta}}$. In view of the arguments in the above, we have the latter assertion.

Step 5. We shall estimate $\| V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} \mathscr{L} U_1^n U_2^n V_{1,n} V_{2,n} g \|$ when $n \to \infty$ for $g \in \mathscr{A}_{\xi,\hat{\theta}}$. We have

$$
\begin{aligned}
(5.24) \qquad & V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} \mathscr{L} U_1^n U_2^n V_{1,n} V_{2,n} g = \\
& V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} \mathscr{L} U_1^n U_2^n (V_{1,n} V_{2,n} - I) g + V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} \mathscr{L} U_1^n U_2^n g.
\end{aligned}
$$

By (5.22) and the boundedness of $V_{j,n}, U_j^{\pm n}$ and $\mathscr{L}$ we see that the first term in the right-hand side of (5.24) tends to zero strongly and uniformly for $g \in \mathscr{A}_{\xi,\hat{\theta}}$ when $n \to \infty$. The second term is equal to

$$
\begin{aligned}
(5.25) \qquad & V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} [\mathscr{L}, U_1^n U_2^n] g + V_{1,n} V_{2,n} \mathscr{L} g \\
& = V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} [\mathscr{L}, U_1^n U_2^n] g + (V_{1,n} V_{2,n} + I) \mathscr{L} g - \mathscr{L} g.
\end{aligned}
$$

Because $\mathscr{L}$ is a pseudodifferential operator of polynomial coefficients it follows from (5.22) that the second term of (5.25) tends to zero strongly and uniformly on $\mathscr{A}_{\xi,\hat{\theta}}$ when $n \to \infty$ for any $g \in \mathscr{A}_{\xi,\hat{\theta}}$, $(g \neq 0)$.

Step 6. Let $g \in \mathscr{A}_{\xi,\hat{\theta}}$, $(g \neq 0)$. We replace g in (5.17) with $g/\|g\|$. By (5.19), (5.21) and the arguments in the above we have, if n is sufficiently large

$$
\left\| \mathscr{L} \frac{g}{\|g\|} \right\| + \left\| V_{1,n} V_{2,n} U_1^{-n} U_2^{-n} [\mathscr{L}, U_1^n U_2^n] \frac{g}{\|g\|} \right\| \geq \frac{A}{2}.
$$

Because $\mathscr{K} := [\mathscr{L}, U_1^n U_2^n]$ is a pseudodifferential operator of order -1 we have (5.6). $\square$

Proof of the necessity of (2.6). Let $\xi \in \mathbb{Z}_+^2$ and $t \in \mathbb{N}$. We set

$$
(5.26) \qquad g_t(\theta) := \phi(\theta) e^{it\theta\xi}, \qquad \phi(\theta) = \varphi\left(\frac{\theta - \hat{\theta}}{\kappa}\right).
$$

Clearly, $g_t \in \mathscr{A}_{\xi,\hat{\theta}}$ and the Fourier coefficient $\hat{g}_t$ of g_t is given by $\hat{g}_t = \hat{\varphi}((\eta - t\xi)\kappa)\kappa^2 e^{-i\hat{\theta}(\eta - t\xi)}$.

By Lebesgue's dominated convergence theorem we have

$$
\lim_{t \to \infty} e^{-it\theta\xi} \mathscr{L} g_t = \lim_{t \to \infty} \sum_{\eta \in \mathbb{Z}^2} L\left(e^{i\theta}, \frac{\eta}{|\eta|}\right) \kappa^2 e^{-i\hat{\theta}(\eta - t\xi)} \hat{\varphi}((\eta - t\xi)\kappa) e^{i\theta(\eta - t\xi)}
$$

$$= \lim_{t\to\infty} \sum_{\zeta\in Z^2-t\xi=Z^2} L(e^{i\theta}, \frac{\zeta+t\xi}{|\zeta+t\xi|})\kappa^2 e^{-i\hat{\theta}\zeta}\widehat{\varphi}(\zeta\kappa)e^{i\theta\zeta}$$

$$= L(e^{i\theta}, \frac{\xi}{|\xi|})\sum_{\zeta\in Z^2}\kappa^2 e^{-i\hat{\theta}\zeta}\widehat{\varphi}(\zeta\kappa)e^{i\theta\zeta} = L(e^{i\theta}, \frac{\xi}{|\xi|})\phi(\theta).$$

Again, by Lebesgue's dominated convergence theorem we have

$$\lim_{t\to\infty}\|\mathscr{L}g_t\| = \|L(e^{i\cdot}, \frac{\xi}{|\xi|})\phi\|. \tag{5.27}$$

In order to estimate $\|\mathscr{K}g_t\|$ with $\mathscr{K}$ given in (5.6) we recall that $\mathscr{K}$ is a pseudodifferential operator of order -1. Therefore $\|Kg_t\|$ is bounded by

$$\begin{aligned}\sum_{\eta}|\widehat{\varphi}((\eta-t\xi)\kappa)|^2\langle\eta\rangle^{-2} &= \sum_{\zeta\in Z^2-t\xi}|\widehat{\varphi}(\zeta\kappa)|^2\langle\zeta+t\xi\rangle^{-2}\\ &= t^{-2}\sum|\widehat{\varphi}(\zeta\kappa)|^2\langle t^{-1}\zeta+\xi\rangle^{-2}.\end{aligned} \tag{5.28}$$

The last term tends to zero when $t\to\infty$. On the other hand, by definition we have $\|g_t\| = \|\phi\|$.

By substituting (5.27)-(5.28) into (5.6) we obtain

$$\|\phi\| \le C\|L(\cdot, \xi/|\xi|)\phi\|. \tag{5.29}$$

Here we note that the constant C in (5.29) is independent of ϕ. It follows that

$$\inf_{z\in\mathbf{T}^2}|L(z, \xi/|\xi|)| \ge C^{-1}. \tag{5.30}$$

Indeed, if otherwise there exist $z_\nu\in\mathbf{T}^2$ and $\xi_\nu\to\infty$ such that

$$|L(z_\nu, \xi_\nu/|\xi_\nu|)| \le C_0 < C^{-1}. \tag{5.31}$$

Passing to the subsequence, if necessary one may assume that $z_\nu\to z_0$ and $\xi_\nu/|\xi|\to\xi_0$ $(\nu\to\infty)$. Hence in a small neighborhood of $z_0\equiv e^{i\hat{\theta}}$ we have

$$|L(z, \xi_\nu/|\xi_\nu|) \le (C_0+C^{-1})/2. \tag{5.32}$$

If we take $\phi(\theta) := \varphi((\theta-\widehat{\theta})/\kappa)$ with support contained in a small neighborhood of $\widehat{\theta}$ by an appropriate choice of κ we have

$$(5.33)\quad \|L(\cdot,\ \xi_\nu/|\xi_\nu|)\phi\| = \Big(\int |L(e^{i\theta},\ \xi_\nu/|\xi_\nu|)\phi(\theta)|^2 d\theta\Big)^{1/2} \le \frac{1}{2}(C_0 + C^{-1})\|\phi\|.$$

This contradicts to (5.29). Hence we have proved (5.30). This proves (2.6). $\square$

§6. Proof of Theorem 4.3. — the Necessity of (2.7)

In this section we shall prove the necessity of (2.7) in Theorem 4.3. We use the same notations as in the previous sections unless otherwise is stated. We first prepare two lemmas. Let $\mathscr{L}(H_1^2)$ be the space of bounded linear operators on H_1^2 into H_1^2 and let $C_\infty \equiv C_\infty(H_1^2)$ be the space of linear compact operators on H_1^2. We denote the variable in $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ by $\theta = (\theta_1,\ \theta_2)$. For a continuous function $\phi(\theta_2)$ on $\mathbb{T}^2$ depending only on θ_2 we denote by $M(\phi)$ the multiplication operator by ϕ, $M(\phi)u := \phi u$ for $u \in L^2$. We denote by $I \otimes M(\phi)$ the tensor product of I and $M(\phi)$. For $0 \le \gamma \le 2\pi$ we define a pseudodifferential operator $\mathscr{L}^{2,\gamma}(\theta_1,\ D_\theta)$ by

$$(6.1)\qquad \mathscr{L}^{2,\gamma}(\theta_1,\ D_\theta) := \mathscr{L}(\theta_1,\ \gamma,\ D_\theta).$$

Let T_1 be defined by (5.3) with $j = 1$. Then we have

Lemma 6.1. *For any $\varepsilon > 0$ we can find a smooth function $\phi(\theta_2)$ satisfying $0 \le \phi \le 1$ and $\phi \equiv 1$ in some neighborhood of $\theta_2 = \gamma$ such that, for $\Psi = I \otimes M(\phi)$*

$$(6.2)\qquad \|(T_1 - \pi_1 \mathscr{L}^{2,\gamma}(\theta_1,\ D_\theta)\pi_1)\Psi\| < \varepsilon,$$

where the norm is understood in $L(H_1^2)/C_\infty(H_1^2)$.

Proof. We note that pseudodifferential operators of negative order define compact operators on $L^2(\mathbb{T}^2)$. Because $\mathscr{L}$ is a pseudodifferential operator with trigonometric polynomial coefficients the operator $\mathscr{L}$ can be written in the following form modulo compact operators

$$(6.3)\qquad \mathscr{L}(\theta,\ D_\theta) = \sum_{k\in\mathbb{Z}}^{\text{finite}} a_k(\theta_2,\ D_\theta)\, e^{ik\theta_1}$$

for some pseudodifferential operators $a_k(\theta_2,\ D_2)$. In view of (6.1), a similar representation holds for $\mathscr{L}^{2,\gamma}(\theta_1,\ D_\theta)$ modulo compact operators. Therefore, by the definition of T_1 we have

(6.4)

$$(T_1 - \pi_1 \mathscr{L}^{2,\gamma}(\theta_1, D_\theta)\pi_1)\Psi = (\pi_1 \sum_k a_k(\theta_2, D_\theta) e^{ik\theta_1} \pi_1 - \pi_1 \sum_k a_k(\gamma, D_\theta) e^{ik\theta_1} \pi_1)\Psi$$
$$= \sum_k (a_k(\theta_2, D_\theta) - a_k(\gamma, D_\theta))\pi_1 e^{ik\theta_1} \pi_1 (I \otimes M(\phi)).$$

Because the summation in k in (6.4) is a finite sum it is sufficient to show that each term in (6.4) tends to zero if supp ϕ tends to a point $\theta_2 = \gamma$. Indeed, if we set $S_k = \pi_1 e^{ik\theta_1} \pi_1$ and write $a_k(\theta_2, D_\theta) = \sum_m a_{km}(D_\theta) e^{im\theta_2}$ we have, for $u \in L^2(\mathbf{T}^2)$

$$(6.5) \quad (a_k(\theta_2, D_\theta) - a_k(\gamma, D_\theta)) S_k (I \otimes M(\phi)) u = (a_k(\theta_2, D_\theta) - a_k(\gamma, D_\theta)) \phi(\theta_2) S_k u$$
$$= \sum_m a_{km}(D_\theta)(e^{im\theta_2} - e^{im\gamma})(\phi S_k u) = \sum_m a_{km}(D_\theta)((e^{im\theta_2} - e^{im\gamma})(\phi S_k u).$$

Because the summation with respect to m in (6.5) is a finite sum it is sufficient to estimate the term $a_{km}(D_\theta)((e^{im\theta_2} - e^{im\gamma})\phi S_k u)$. Because $a_{km}(D_\theta)$ is order zero and its L^2 norm is bounded, the L^2 norm of the term can be estimated by $A\|S_k u\|$, where $A \to 0$ if supp ϕ tends to a point $\theta_2 = \gamma$. Therefore it follows that, modulo compact operators,

(6.6)

$$\|a_{km}(D_\theta)((e^{im\theta_2} - e^{im\gamma})\phi S_k u)\| \le C\|(e^{im\theta_2} - e^{im\gamma})\phi S_k u\| \le CA\|S_k u\| \le CA\|u\|,$$

for some $C > 0$, where we have used

$$(6.7) \qquad \|S_k u\| = \|\pi_1 e^{ik\theta_1} \pi_1 u\| \le \|e^{ik\theta_1} \pi_1 u\| = \|\pi_1 u\| \le \|u\|.$$

This proves Lemma 6.1.

Let $\varphi(\theta_1) \in C^\infty(\mathbf{T}) \cap H^2(\mathbf{T})$ and $\psi(\theta_2) \in C^\infty(\mathbf{T})$. We define a function class $\mathscr{F}$ by

$$(6.8) \qquad \mathscr{F} := \{\varphi(\theta_1)\psi(\theta_2) e^{im\theta_2};\ m = 1,2,\dots\} \subset H_1^2.$$

Then we have

Lemma 6.2. *Under the same assumptions as in Theorem* 4.3 *there exist compact operators* K *and* $\mathscr{K}^*$ *on* H_1^2 *such that the following inequalities hold*

$$(6.9)\qquad ||\pi_1\mathscr{L}\pi_1 g|| + ||\mathscr{K}g|| \geq C||g||, \quad \forall g \in \mathscr{F},$$

$$(6.10)\qquad ||\pi_1\mathscr{L}^*\pi_1 g|| + ||\mathscr{K}^* g|| \geq C||g||, \quad \forall g \in \mathscr{F},$$

where $\mathscr{L}^*$ *denotes the adjoint operator on* L^2 *of* $\mathscr{L}$.

Proof. We shall prove (6.9). The inequality (6.10) will be proved similarly if we note Lemma 3.6. It follows from (5.10) that

(6.11)
$$||U_1^{-n}\pi_1\pi_2\mathscr{L}(\theta,D_\theta)\pi_1\pi_2 U_1^n g|| + ||\pi_1\pi_2 K\pi_1\pi_2 U_1^n g|| + A||U_1^{-n}Q_2 U_1^n g|| \geq A||g||, \ \forall g\in\mathscr{F}.$$

If $g \neq 0$ we set $h := g/||g||$ in (6.11). We obtain

(6.12)
$$||U_1^{-n}\pi_1\pi_2\mathscr{L}\pi_1\pi_2 U_1^n h|| + ||\pi_1\pi_2 K\pi_1\pi_2 U_1^n h|| + A||U_1^{-n}Q_2 U_1^n h|| \geq A \ \ \forall h \in \mathscr{F}, ||h|| = 1.$$

The third term in the left-hand side of (6.12) can be estimated by similar calculations as in (5.19). In fact, we have that, for any $\varepsilon > 0$ there exists n such that, for all $h \in \mathscr{F}$

$$(6.13)\qquad ||U_1^{-n}Q_2 U_1^n h|| < \varepsilon.$$

In order to estimate the first term we recall that $[U_1, \pi_1] = 0$. Hence we have

(6.14)
$$\begin{aligned} U_1^{-n}\pi_1\pi_2\mathscr{L}\pi_1\pi_2 U_1^n h &= V_{1,n}U_1^{-n}\pi_1\mathscr{L}\pi_1 U_1^n V_{1,n}h = V_{1,n}\pi_1 U_1^{-n}\mathscr{L}U_1^n\pi_1 V_{1,n}h \\ &= V_{1,n}\pi_1\mathscr{L}\pi_1 V_1^n h + V_{1,n}\pi_1 U_1^{-n}[\mathscr{L}, U_1^n]\,\pi_1 V_{1,n}h. \end{aligned}$$

The second term is a compact operator. As to the first term we have

$$(6.15)\quad V_{1,n}\,\pi_1\mathscr{L}\pi_1\,V_{1,n}h = (V_{1,n}-I)\,\pi_1\mathscr{L}\pi_1 V_{1,n}h + \pi_1\mathscr{L}\pi_1(V_{1,n}-I)\,h + \pi_1\mathscr{L}\pi_1 h.$$

The first and the second term of the right-hand side of (6.15) tend to zero uniformly in $h \in \mathscr{F}$ by the definition of $V_{1,n}$ because $\mathscr{L}$ is of trigonometric polynomial coefficients. Therefore (6.15) implies that

$$V_{1,n}\,\pi_1\mathscr{L}\pi_1 V_{1,n}h \to \pi_1\mathscr{L}\pi_1 h \quad \text{uniformly for} \quad h \in \mathscr{F}.$$

Hence it follows from (6.13) that, for sufficiently large n

$$\|\pi_1 \mathscr{L} \pi_1 h\| + \|\mathscr{K} h\| \geq C, \quad \forall h \in \mathscr{F}, \|h\| = 1, \tag{6.16}$$

where $\mathscr{K} : H_1^2 \to H_1^2$ is a compact operator. This proves (6.9). ∎

Proof of the necessity of (2.7). Let $a_k(\theta_2, \eta)$ be the symbol of $a_k(\theta_2, D_\theta)$ in (6. 3). We define a function $A_\gamma(\theta_1)$ *by*

$$A_\gamma(\theta_1) = \sum_k a_k(\gamma, 0, 1) e^{ik\theta_1}. \tag{6.17}$$

Namely we set $\theta_2 = \gamma$ and $\eta = (0,1)$. We note that $A_\gamma(\theta_1)$ is a trigonometric polynomial of θ_1. We then define a Toeplitz operator $\mathscr{T}(A_\gamma)$ with symbol $A_\gamma \equiv A_\gamma(\theta_1)$ by

$$\mathscr{T}(A_\gamma) := \pi_1 A_\gamma ; H^2(\mathbf{T}) \to H^2(\mathbf{T}). \tag{6.18}$$

We want to show that $\mathscr{T}(A_\gamma)$ is invertible. By the condition (2.6) and the definition of $\mathscr{T}(A_\gamma)$ we know that $\mathscr{T}(A_\gamma)$ is a Fredholm operator. (cf. Theorem 2.42 of [3]). Hence, in order to prove (2.7) it is sufficient to show that

$$Ker\, \mathscr{T}(A_\gamma) = Ker\, \mathscr{T}(A_\gamma)^* = \{0\},$$

where $\mathscr{T}(A_\gamma)^*$ is the adjoint operator of $\mathscr{T}(A_\gamma)$.

Let $\varphi \equiv \varphi(\theta_1) \in H^2(\mathbf{T})$ satisfy that $\mathscr{T}(A_\gamma)\varphi = 0$, that is $\pi_1 A_\gamma \varphi = 0$. We want to show that $\varphi(\theta_1)$ is smooth. Because A_γ is a trigonometric polynomial and $\varphi(\theta_1) \in H^2(\mathbf{T})$ the relation $\pi_1 A_\gamma \varphi = 0$ implies that $A_\gamma \varphi$ is a trigonometric polynomial with only negative Fourier coefficients. It follows that $g := A_\gamma \varphi \in C^\infty(\mathbf{T})$. Because $A_\gamma \neq 0$ on $\mathbf{T}$ by (2.6) and the definition of A_γ we see that $\varphi = g / A_\gamma$ is smooth.

For a smooth function $\psi(\theta_2)$ on $\mathbf{T}$ which will be determined later we set $\Psi = I \otimes M(\psi)$. We consider the function class $\mathscr{F}$, (6.8) defined by these φ and ψ. Then we have

$$T_1 \Psi = \mathscr{T}_\gamma \Psi + (T_1 - \mathscr{T}_\gamma)\Psi,$$

where T_1 is given by (5.3) and we set $\mathscr{T}_\gamma = \pi_1 \mathscr{L}^{2,\gamma} \pi_1$ for the sake of simplicity. If we set $u = \phi(\theta_1) e^{im\theta_2}$ $(m = 1,2,\ldots)$, we see that

$$g := \Psi u = \phi \psi e^{im\theta_2} \in \mathscr{F}. \tag{6.19}$$

It follows from (6.9) that

$$\|\mathcal{T}_\gamma g\| + \|(T_1 - \mathcal{T}_\gamma) g\| + \|\mathcal{K} g\| \geq C\|g\|, \quad g = \Psi u = \phi\psi e^{im\theta_2}. \tag{6.20}$$

By Lemma 6.1 we can estimate the second term in the left-hand side of (6.20). In fact, for any $\varepsilon > 0$ there exists a smooth function $\psi \neq 0$ on $\mathbb{T}$ such that $\Psi := I \otimes M(\psi)$ satisfies, modulo compact operators

$$\|(T_1 - \mathcal{T}_\gamma)\ \Psi\| < \varepsilon. \tag{6.21}$$

We shall estimate the first term in (6.20). Let $\phi(\theta_1) = \sum_j \widehat{\phi}_j\, e^{ij\theta_1}$ and $\psi(\theta_2) = \sum_\ell \widehat{\psi}_\ell\, e^{i\ell\theta_2}$ be Fourier expansions of ϕ and ψ, respectively. For $A > 0$ chosen later at (6.27) and a positive integer m we set

$$\phi_m(\theta_1) = \sum_{0 \leq j \leq mA} \widehat{\phi}_j\, e^{ij\theta_1}, \quad \psi_m(\theta_2) = \sum_{|\ell| \leq m/2} \widehat{\psi}_\ell\, e^{i\ell\theta_2}. \tag{6.22}$$

Then we have

$$\begin{aligned} \mathcal{T}_\gamma g &= \mathcal{T}_\gamma \phi\psi e^{im\theta_2} = \mathcal{T}_\gamma(\phi_m + \phi - \phi_m)(\psi_m + \psi - \psi_m) e^{im\theta_2} \\ &= \mathcal{T}_\gamma(\phi_m\psi_m + R_m) e^{im\theta_2} = \mathcal{T}_\gamma \phi_m\psi_m\, e^{im\theta_2} + \mathcal{T}_\gamma\, R_m\, e^{im\theta_2}, \end{aligned} \tag{6.23}$$

where

$$R_m = (\phi - \phi_m)\ \psi_m + \phi_m(\psi - \psi_m) + (\phi - \phi_m)(\psi - \psi_m).$$

We shall estimate $\mathcal{T}_\gamma R_m\, e^{im\theta_2}$. We first note that $\|R_m\|$ is rapidly decreasing when $m \to \infty$, namely, for $k = 1,2,\dots$

$$\|R_m\|\, m^k \to 0 \qquad (m \to \infty). \tag{6.24}$$

Since $\mathcal{T}_\gamma$ is L^2 bounded it follows that

$$\|\mathcal{T}_\gamma\, R_m\, e^{im\theta_2}\| \leq C\|R_m\, e^{im\theta_2}\| = C\|R_m\| \tag{6.25}$$

for some $C > 0$.

We shall estimate $\mathcal{T}_\gamma \phi_m\psi_m\, e^{im\theta_2}$. In what follows we denote constants independent of m by C for the sake of simplicity. Because $m \geq 0$ and $\phi \in H^2(\mathbb{T})$ we have

(6.26)

$$\mathcal{T}_\gamma \phi_m(\theta_1)\, \psi_m(\theta_2)\, e^{im\theta_2} = \pi_1 \sum_{k;\text{finite}} a_k(\gamma, D_\theta)\, e^{ik\theta_1}\, \pi_1\phi_m(\theta_1)\, \psi_m(\theta_2)\, e^{im\theta_2}$$

$$= \pi_1 \sum_k a_k(\gamma, D_\theta) e^{ik\theta_1} \phi_m(\theta_1) \psi_m(\theta_2) e^{im\theta_2}$$

$$= \pi_1 \sum_{0 \le j \le mA, |\ell| \le m/2, k} \widehat{\phi}_j \widehat{\psi}_\ell a_k(\gamma, D_\theta) e^{i(k+j)\theta_1 + i(m+\ell)\theta_2}$$

$$= \pi_1 \sum a_k(\gamma, k+j, \ell+m) \widehat{\phi}_j \widehat{\psi}_\ell e^{i(k+j)\theta_1 + i(m+\ell)\theta_2}.$$

We note that $T(A_\gamma) \phi_m(\theta_1) \psi_m(\theta_2) e^{im\theta_2}$ has a similar expression as (6.26) where $a_k(\gamma, k+j, \ell+m)$ is replaced by $a_k(\gamma, 0, 1)$. By the homogenity of $a_k(\gamma, \eta)$ in η, for any $\varepsilon > 0$ there exist an $m_0 > 0$ and a $A > 0$ such that,

$$|a_k(\gamma, k+j, \ell+m) - a_k(\gamma, 0, 1)| < \varepsilon \quad 0 \le \forall j \le mA, \ \forall |\ell| \le m/2, \ \forall m \ge m_0. \tag{6.27}$$

It follows from (6.17), (6.18), (6.26) and (6.27) that, for $m \ge m_0$

(6.28)

$$\|(\mathscr{T}_\gamma - \mathscr{T}(A_\gamma)) \phi_m \psi_m e^{im\theta_2}\| = \|\pi_1 \sum_k (a_k(\gamma, D_\theta) - a_k(\gamma, 0, 1)) e^{ik\theta_1} \phi_m \psi_m e^{im\theta_2}\|$$

$$\le \sum_k \|(a_k(\gamma, D_\theta) - a_k(\gamma, 0, 1)) e^{ik\theta_1} \phi_m \psi_m e^{im\theta_2}\| \le C\varepsilon \|\phi\| \, \|\psi\|$$

for some $C > 0$. On the other hand, since $\mathscr{T}(A_\gamma)\phi = 0$ it follows that

$$\|\mathscr{T}(A_\gamma) \phi_m \psi_m e^{im\theta_2}\| = \|\psi_m e^{im\theta_2} \mathscr{T}(A_\gamma) \phi_m\| \tag{6.29}$$

$$= \|\psi_m e^{im\theta_2} \mathscr{T}(A_\gamma)(\phi_m - \phi)\| \le C\|\psi\| \|\phi_m - \phi\| \le C\varepsilon \|\psi\|,$$

for some $C > 0$ if m is sufficiently large. Therefore it follows from (6.28) and (6.29) that

(6.30)

$$\|\mathscr{T}_\gamma \phi_m \psi_m e^{im\theta_2}\| \le \|(\mathscr{T}_\gamma - \mathscr{T}(A_\gamma)) \phi_m \psi_m e^{im\theta_2}\| + \|\mathscr{T}(A_\gamma) \phi_m \psi_m e^{im\theta_2}\|$$

$$\le C\varepsilon \|\phi\| \, \|\psi\| + C\varepsilon \|\psi\|.$$

Hence it follows from (6.23), (6.25) and (6.30) that

$$\|\mathscr{T}_\gamma g\| \le \|\mathscr{T}_\gamma R_m e^{im\theta_2}\| + \|\mathscr{T}_\gamma \phi_m \psi_m e^{im\theta_2}\| \le Cm^{-1} + C\varepsilon \|\phi\| \, \|\psi\| + C\varepsilon \|\psi\|. \tag{6.31}$$

By (6.21) there exists a compact operator $\widetilde{\mathscr{K}}$ such that for $g = \Psi u = \phi\psi e^{im\theta_2}$, $u = \phi(\theta_1)e^{im\theta_2}$

$$\|(T_1 - \mathscr{T}_\gamma)g\| \leq \varepsilon\|u\| + \|\widetilde{\mathscr{K}}u\| = \varepsilon\|\phi\| + \|\widetilde{\mathscr{K}}\phi(\theta_1)e^{im\theta_2}\|.$$

Therefore, by (6.20), (6.31) and the relations $g = \Psi u = \phi\psi e^{im\theta_2}$, $u = \phi(\theta_1)e^{im\theta_2}$ we have

$$C\|g\| \leq \|\mathscr{K}g\| + \varepsilon\|\phi\| + \|\widetilde{\mathscr{K}}\phi(\theta_1)e^{im\theta_2}\| + Cm^{-1} + C\varepsilon\|\phi\|\,\|\psi\| + C\varepsilon\|\psi\|. \tag{6.32}$$

Since g and $\phi(\theta_1)e^{im\theta_2}$ weakly converge to zero when $m \to \infty$ and $\mathscr{K}$, $\widetilde{\mathscr{K}}$ are compact, it follows that $\mathscr{K}g$ and $\widetilde{\mathscr{K}}\phi(\theta_1)e^{im\theta_2}$ strongly converge to zero when $m \to \infty$. Noting that $\|g\| = \|\phi\|\,\|\psi\|$ and letting $m \to \infty$ in (6.32) we obtain

$$C\|\phi\|\,\|\psi\| \leq \varepsilon\|\phi\| + \varepsilon\|\phi\|\,\|\psi\| + C\varepsilon\|\psi\|.$$

Because $\varepsilon > 0$ is arbitrary we have $\|\phi\|\,\|\psi\| = 0$. Since $\|\psi\| \neq 0$ we have $\phi = 0$. This proves that the kernel of $\mathscr{T}(A_\gamma)$ is trivial. If we make the same argument to the adjoint operator $\mathscr{T}(A_\gamma)^*$ we can show that *Cokernel* $\mathscr{T}(A_\gamma) = \{0\}$. This proves that $\mathscr{T}(A_\gamma)$ is invertible. By the well-known result of Toeplitz operators (cf. Theorem 2.42 of [3]) we have $\mathrm{ind}_1\, A_\gamma = 0$. By the definition of A_γ this proves that $\mathrm{ind}_1\, L = 0$. By changing the parts of θ_1 and θ_2 we can prove that $\mathrm{ind}_2 L = 0$. Hence we have proved (2.7). The proof of Theorem 4.3 is complete. ⊠

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