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Theory of Prehomogeneous Vector Spaces, II, A Supplement

By

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This paper is a supplement of [G1]. In [G1, §§2 and 3], we have mainly studied D-modules Df^{α} generated by a complex power of a regular function, especially a relative invariant of a prehomogeneous vector space. Here we modify the argument so that we can include a more general D-modules such as $D(f^{\alpha}u)$, where u is a section of a regular holonomic D-module. The main results are (6.20)–(6.22). In (6.20), we determine the Fourier transform of $D(f^{\alpha}u)$, assuming that f is a relative invariant of a prehomogeneous vector space, and that Du is an integrable connection of rank one satisfying certain additional assumptions. As its corollary, we get (6.21) and (6.22). The latter will be used in a study of character sums associated to prehomogeneous vector spaces over a finite field.

Convention and Notation. We denote by $\mathbf{Z}$ the rational integer ring, and by $\mathbf{C}$ the complex number field. As for $\mathscr{D}$-modules, we shall work in the algebraic category unless otherwise stated. We define the de Rham functor $\mathrm{DR}(-)$ so that $\mathrm{DR}(\mathcal{O}_X)=\mathbf{C}_X$, where $\mathcal{O}_X$ is the structure sheaf. For a morphism $F: X\to Y$ between varieties, and for an $\mathcal{O}_Y$-module $\mathscr{M}$, F^* denotes the usual $\mathcal{O}$-module pull-back; $F^*\mathscr{M}=\mathcal{O}_X\otimes_{F^{-1}\mathcal{O}_Y} F^{-1}\mathscr{M}$. We shall refer to [G1, (a,b,c)] etc. simply as (a,b,c) etc.

§5. *D*-Modules

The content of this section is a supplement of §2.

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5.1. Notation. Let X be a non-singular irreducible algebraic variety over the complex number field $\mathbf{C}$, $\mathcal{O}=\mathcal{O}_X$ the sheaf of regular functions, and $\mathscr{D}=\mathscr{D}_X$ the sheaf of algebraic differential operators. If X is an affine variety, we put $\mathbf{C}[X]:=\Gamma(X,\mathcal{O}_X)$ and $D=D_X:=\Gamma(X,\mathscr{D}_X)$. More generally, for a $\mathbf{C}[X]$-module we denote the corresponding quasi-coherent sheaf on X by the corresponding script letter, and vice versa. For any $\mathbf{C}$-algebra A, we put $\mathscr{D}_A=\mathscr{D}_{X,A}:=\mathscr{D}_X\otimes_{\mathbf{C}}A$, and $D_A=D_{X,A}:=D_X\otimes_{\mathbf{C}}A$. In particular, when A is the polynomial ring $\mathbf{C}[s]$, we often write $\mathscr{D}[s]=\mathscr{D}_X[s]$ and $D[s]=D_X[s]$ for $\mathscr{D}_{\mathbf{C}[s]}$ and $D_{\mathbf{C}[s]}$, respectively. We need the $\mathbf{C}$-algebra $\mathbf{C}[s,t]$ given in (2.3.5), namely, the $\mathbf{C}$-algebra defined by the relation $ts=(s+1)t$. Put $\mathscr{D}[s,t]=\mathscr{D}_X[s,t]:=\mathscr{D}_X\otimes_{\mathbf{C}}\mathbf{C}[s,t]$ and $D[s,t]=D_X[s,t]:=D_X\otimes_{\mathbf{C}}\mathbf{C}[s,t]$.

5.2. $\mathscr{D}$-Modules $\mathscr{D}_X[s](f^s\underline{u}\mid V)$ and $\mathscr{D}_X(f^\alpha\underline{u}\mid V)$. We fix $0\neq f\in\Gamma(X,\mathcal{O}_X)$. Let $X_0:=X\backslash f^{-1}(0)$, V be a Zariski open subset of X_0, $\mathscr{M}$ a coherent $\mathscr{D}_V$-module, and $\underline{u}=(u_1,\cdots,u_p)$ a p-tuple of elements of $\Gamma(V,\mathscr{M})$. Consider the left $\mathscr{D}_X[s]$-submodule $\mathscr{I}$ of $\mathscr{D}_X[s]^p$ consisting of $(P_1(s),\cdots,P_p(s))\in\mathscr{D}_X[s]^p$ such that $\Sigma_{i=1}^p(f^{m-s}P_i(s)f^s)u_i=0$ holds in $\mathbf{C}[s]\otimes_{\mathbf{C}}\mathscr{M}$ whenever $m\in\mathbf{Z}$ is sufficiently large. Put $\mathscr{N}:=\mathscr{D}_X[s]^p/\mathscr{I}$. Denote by $(f^s\underline{u})_i\mid V$ the element $((0,\cdots0,1,0,\cdots,0)$ mod $\mathscr{I})$, where 1 appears at the i-th place. Then $\mathscr{N}=\Sigma_{i=1}^p\mathscr{D}_X[s]((f^s\underline{u})_i\mid V)$. Put $f^s\underline{u}\mid V:=((f^s\underline{u})_1\mid V,\cdots,(f^s\underline{u})_p\mid V)$. We write $\mathscr{N}=\mathscr{D}_X[s](f^s\underline{u}\mid V)$. For a complex number α, put $\mathscr{N}(\alpha):=\mathscr{N}/(s-\alpha)\mathscr{N}$, and $f^\alpha\underline{u}\mid V=((f^\alpha\underline{u})_1\mid V,\cdots,(f^\alpha\underline{u})_p\mid V)$ $:=(f^s\underline{u}\mid V \bmod (s-\alpha)\mathscr{N})$. Then $\mathscr{N}(\alpha)=\mathscr{D}_X(f^\alpha\underline{u}\mid V)=\Sigma_{i=1}^p\mathscr{D}_X((f^\alpha\underline{u})_i\mid V)$. If X is an affine variety, we define $D_X[s](f^s\underline{u}\mid V)$ and $D_X(f^\alpha\underline{u}\mid V)$ in the same way. If $V=X_0$, we sometimes write $f^s\underline{u}$ and $f^\alpha\underline{u}$ for $f^s\underline{u}\mid X_0$ and $f^\alpha\underline{u}\mid X_0$. It is easy to see that

(5.2.1) *f is not a zero divisor of $\mathscr{D}_X[s](f^s\underline{u}\mid V)$*

and

(5.2.2) *$\mathscr{D}_X[s](f^s\underline{u}\mid V)$ is $\mathbf{C}[s]$-flat (i.e., $\mathbf{C}[s]$-torsion free).*

5.3. b-Function of $\mathscr{D}_X[s](f^s\underline{u}\mid V)$. Assume that $\mathscr{D}_V\underline{u}$ is holonomic and the inclusion mapping $j_V: V\to X_0$ is an affine morphism. Then there exists a non-zero polynomial $b(s)\in\mathbf{C}[s]$ such that

$$b(s)\mathscr{D}_X[s](f^s\underline{u}\mid V)\subset\mathscr{D}_X[s](f^{s+1}\underline{u}\mid V). \tag{5.3.1}$$

(*Proof.* Since $\Gamma(X_0,(j_V)_*\mathscr{M})=\Gamma(V,\mathscr{M})\ni u_i$ and since $(j_V)_*\mathscr{M}$ is a holonomic

$\mathscr{D}_{X_0}$-module, we may assume $X_0=V$ from the beginning. Then the proof goes in the same way as [Ka2, Theorem 2.7].) Let $b(s,\mathscr{N})$ be the monic polynomial of minimal degree satisfying (5.3.1). Put

$$A_+(\mathscr{N}):=\{\alpha\in\mathbf{C}\mid b(\alpha+j,\mathscr{N})\neq 0 \quad \text{for } j=0,1,2,\cdots\} \text{ and}$$
$$A_-(\mathscr{N}):=\{\alpha\in\mathbf{C}\mid b(\alpha-j,\mathscr{N})\neq 0 \quad \text{for } j=1,2,\cdots\}.$$

The content of (5.4)–(5.6) seems to be standard (cf. [Gi]) and the proof is omitted. (The detail will be included in the proceeding of the conference on prehomogeneous vector spaces held in Kyoto in April, 1996, which will appear in RIMS Kokyuroku.)

Lemma 5.4. *Let V be a Zariski open subset of $X_0=X\setminus f^{-1}(0)$ such that inclusion mapping $j_V\colon V\to X_0$ is an affine morphism, let $j\colon X_0\to X$ denote the inclusion mapping, and assume that $\mathscr{D}_V\underline{u}=\Sigma_{i=1}^p\mathscr{D}_V u_i$ is a regular holonomic $\mathscr{D}_V$-module. Then*

(1) *$\mathscr{D}_X(f^\alpha\underline{u}\mid V)$ is a regular holonomic $\mathscr{D}_X$-module,*
(2) *$\mathrm{DR}(\mathscr{D}_X(f^\alpha\underline{u}\mid V))=Rj_*\mathrm{DR}(\mathscr{D}_{X_0}(f^\alpha\underline{u}\mid V))$ if $\alpha\in A_-(\mathscr{D}_X[s](f^s\underline{u}\mid V))$, and*
(3) *$\mathrm{DR}(\mathscr{D}_X(f^\alpha\underline{u}\mid V))=j_!\mathrm{DR}(\mathscr{D}_{X_0}(f^\alpha\underline{u}\mid V))$ if $\alpha\in A_+(\mathscr{D}_X[s](f^s\underline{u}\mid V))$.*

Remark 5.4.1. In the above theorem, the regularity assumption for $\mathscr{D}_V u$ can not be removed even for (2) or (3).

5.5. $\mathscr{D}_X$-Modules $(f^\alpha,\mathscr{M})_*$ and $(f^\alpha,\mathscr{M})_!$. Let $\mathscr{M}$ be a regular holonomic $\mathscr{D}_{X_0}$-module. If $\mathscr{M}$ is generated by global sections $\underline{u}=(u_1,\cdots,u_p)$ $(u_i\in\Gamma(X_0,\mathscr{M}))$, then we can define $\mathscr{D}_X(f^\alpha\underline{u})$ as in (5.2). Let $\underline{v}=(v_1,\cdots,v_q)$ be another global generator system of the $\mathscr{D}_{X_0}$-module $\mathscr{M}$. Then for $m\in\mathbf{Z}$,

$$\mathrm{DR}_X(\mathscr{D}_X(f^{\alpha+m}\underline{u}))=\mathrm{DR}_X(\mathscr{D}_X(f^{\alpha+m}\underline{v}))=\begin{cases}Rj_*(\mathbf{C}f^{-\alpha}\otimes\mathrm{DR}_{X_0}(\mathscr{M})) & \text{if } m\ll 0\\ j_!(\mathbf{C}f^{-\alpha}\otimes\mathrm{DR}_{X_0}(\mathscr{M})) & \text{if } m\gg 0\end{cases}$$

by (5.4). Hence the natural isomorphism $\mathscr{D}_{X_0}(f^{\alpha+m}\underline{u})\simeq\mathscr{D}_{X_0}(f^{\alpha+m}\underline{v})$ $(\simeq\mathscr{D}_{X_0}f^\alpha\otimes_{\mathscr{O}_{X_0}}\mathscr{M})$ uniquely extends to $\mathscr{D}_X(f^{\alpha+m}\underline{u})\simeq\mathscr{D}_X(f^{\alpha+m}\underline{v})$ if $m\gg 0$ or $m\ll 0$. By the same reason, $\mathscr{D}_X(f^{\alpha+m}\underline{u})$ is independent of a special choice of $m\in\mathbf{Z}$ as far as $m\gg 0$ or $m\ll 0$.

Generally, let $X=\bigcup_i U_i$ be a finite open covering, $\underline{u}^{(i)}\in\Gamma(U_i,\mathscr{M})^{p_i}$ $(p_i\in\mathbf{Z}_{\geq 0})$ a finite generator system of $\mathscr{M}\mid U_i$, and consider $\mathscr{D}_{U_i}(f^{\alpha+m}\underline{u}^{(i)})$ $(m\gg 0$ or $m\ll 0)$. By what we have seen above, these $\mathscr{D}_{U_i}$-modules patch together. In other words, there uniquely exist regular holonomic $\mathscr{D}_X$-modules $(f^\alpha,\mathscr{M})_* =(f^\alpha,\mathscr{M})_{*,X}$ and $(f^\alpha,\mathscr{M})_! =(f^\alpha,\mathscr{M})_{!,X}$ such that

$$\mathscr{D}_{U_i}(f^{\alpha+m}\underline{u}^{(i)})=\begin{cases}(f^\alpha,\mathscr{M})_*\mid U_i & \text{if } m\ll 0\\ (f^\alpha,\mathscr{M})_!\mid U_i & \text{if } m\gg 0.\end{cases} \tag{5.5.1}$$

Then

$$\mathrm{DR}_X((f^\alpha,\mathscr{M})_*)=Rj_*(\mathbf{C}f^{-\alpha}\otimes\mathrm{DR}_{X_0}(\mathscr{M})),\ \text{and} \tag{5.5.2}$$

$$\mathrm{DR}_X((f^\alpha,\mathscr{M})_!)=j_!(\mathbf{C}f^{-\alpha}\otimes\mathrm{DR}_{X_0}(\mathscr{M})). \tag{5.5.3}$$

The functorial properties of $(f^\alpha,\mathscr{M})_*$ and $(f^\alpha,\mathscr{M})_!$ follow from (5.5.2) and (5.5.3).

Lemma 5.6. (1) *If $\mathscr{M}$ is a regular holonomic $\mathscr{D}_{X_0}$-module, then $\underline{\mathrm{ch}}(f^\alpha,\mathscr{M})_*=\underline{\mathrm{ch}}(f^\alpha,\mathscr{M})_!$ and it is independent of $\alpha\in\mathbf{C}$.* (2) *If further $\mathscr{M}$ is locally $\mathscr{O}_{X_0}$-free of rank r, then*

$$\underline{\mathrm{ch}}(f^\alpha,\mathscr{M})_*=\underline{\mathrm{ch}}(f^\alpha,\mathscr{M})_!=r\cdot\underline{\mathrm{ch}}(\mathscr{D}_Xf^\alpha).$$

(Here $\underline{\mathrm{ch}}$ denotes characteristic cycle.) (3) *If further $\mathscr{M}=\mathscr{D}_{X_0}\underline{u}$, then*

$$\underline{\mathrm{ch}}(\mathscr{D}_X[s](f^s\underline{u}))=r\cdot\underline{\mathrm{ch}}(\mathscr{D}_X[s]f^s).$$

Cf. (2.4.6) *for the right hand side.*

5.7. A trick to study $\mathscr{D}[s]$-modules. In order to study $\mathscr{D}[s]$-modules, the following trick is useful. Let $\mathbf{K}$ be an algebraic closure of $\mathbf{C}(s)$, where s is an indeterminate over $\mathbf{C}$.

5.7.1. *For any subfield k of $\mathbf{C}$ whose cardinality is countable, there is an isomorphism $\mathbf{K}\to\mathbf{C}$ which preserves every element of k invariant.*

This simple remark enables us to apply the results obtained so far to $\mathscr{D}_{\mathbf{K}}$-modules. (Here and below, we put $\mathscr{D}_{\mathbf{K}}:=\mathscr{D}\otimes_{\mathbf{C}}\mathbf{K}$ and $\mathscr{D}_{\mathbf{K}}:=D\otimes_{\mathbf{C}}\mathbf{K}$. More generally, we indicate $\otimes_{\mathbf{C}}\mathbf{K}$ by the suffix $\mathbf{K}$.)

Lemma 5.8. *Let $\mathscr{M}$ be a quasi-coherent $\mathscr{D}_{X_0}$-module, $u_i\in\Gamma(X_0,\mathscr{M})$, $\underline{u}:=(u_1,\cdots,u_p)$, σ an indeterminate over $\mathbf{K}$, and $\mathscr{N}:=\mathscr{D}_{X,\mathbf{K}}[\sigma](f^\sigma\underline{u})$. Then we can naturally identify $\mathscr{N}/(\sigma-s)\mathscr{N}=\mathscr{D}_X[s](f^s\underline{u})\otimes_{\mathbf{C}[s]}\mathbf{K}$.*

Proof. For the sake of simplicity, we assume that $p=1$ ($\underline{u}=:u$), and X

is an affine variety. Then we have a natural surjection

$$\varphi:\mathcal{N}=(\mathbf{K}\otimes_{\mathbf{C}}\mathscr{D})[\sigma](f^{\sigma}u)\ni\xi:=\sum_j a_j(s)\otimes P_j(\sigma)(f^{\sigma}u)$$

$$\to\sum_j a_j(s)P_j(s)(f^s u)\in\mathscr{D}_X[s](f^s u)\otimes_{\mathbf{C}[s]}\mathbf{K}.$$

It suffices to show that $\ker\varphi\subset(\sigma-s)\mathcal{N}$. If $\varphi(\xi)=0$, then the second term of

$$\xi=\sum_j(a_j(s)\otimes P_j(\sigma)-1\otimes a_j(\sigma)P_j(\sigma))(f^{\sigma}u)+\sum_j 1\otimes a_j(\sigma)P_j(\sigma)(f^{\sigma}u)$$

vanishes, and hence $\xi\in(\sigma-s)\mathcal{N}$. □

This lemma enables us to apply the results concerning $\mathscr{D}_X(f^{\alpha}\underline{u})$ ($\alpha\in\mathbf{C}$) to $\mathscr{D}_{X,\mathbf{K}}(f^s\underline{u}):=\mathscr{D}_{X,\mathbf{K}}[\sigma](f^{\sigma}\underline{u})/(\sigma-s)\mathscr{D}_{X,\mathbf{K}}[\sigma](f^{\sigma}\underline{u})=\mathscr{D}_X[s](f^s\underline{u})\otimes_{\mathbf{C}[s]}\mathbf{K}$. For example, we get the following lemma from (5.4).

Lemma 5.9. *If $\mathscr{D}_{X_0}\underline{u}$ is a regular holonomic $\mathscr{D}_{X_0}$-module, then*

$$\mathrm{DR}_{X_{\mathbf{K}}}(\mathscr{D}_{X,\mathbf{K}}(f^s\underline{u}))=R(j_{\mathbf{K}})_*(\mathbf{K}f^s\otimes\mathrm{DR}_{X_{0,\mathbf{K}}}(\mathscr{D}_{X_0,\mathbf{K}}\underline{u})).$$

Here we may replace $(j_{\mathbf{K}})_*$ with either of $(j_{\mathbf{K}})_!$ or $(j_{\mathbf{K}})_{!*}$. In particular, if $\mathscr{D}_{X_0}\underline{u}$ is a simple $\mathscr{D}_{X_0}$-module, then $\mathscr{D}_{X,\mathbf{K}}(f^s\underline{u})$ is a simple $\mathscr{D}_{X,\mathbf{K}}$-module.

5.10. Various b-functions. Let X be a connected non-singular variety over $\mathbf{C}$, $0\neq f\in\Gamma(X,\mathcal{O}_X)$, and $X_0:=X\backslash f^{-1}(0)$. For $x\in X$, put $A_x:=\mathcal{O}_{X,x}$, $\tilde{A}_x:=\mathcal{O}^{\mathrm{an}}_{X,x}$, and let $\hat{A}_x$ be the completion of A_x by the maximal ideal. Let $\mathcal{M}$ be a holonomic $\mathscr{D}_{X_0}$-module, $u_i\in\Gamma(X_0,\mathcal{M})$ $(1\leq i\leq p)$, and $\underline{u}=(u_1,\cdots,u_p)$. Let R be one of the rings A_x, $\tilde{A}_x$ or $\hat{A}_x$. Let $B_x(s,\underline{u})$, $\tilde{B}_x(s,\underline{u})$ or $\hat{B}_x(s,\underline{u})$ be the (monic) minimal polynomial of

$$s\in\mathrm{End}\left(\frac{R\otimes_{A_x}\mathscr{D}_{X,x}[s](f^s\underline{u})}{R\otimes_{A_x}\mathscr{D}_{X,x}[s](f^{s+1}\underline{u})}\right)$$

for the respective R, which we shall call the *b-function.* (If we admit the b-function to be zero, we do not need to assume the holonomicity of $\mathcal{M}$.) Let $B(s,\underline{u})$ be the minimal polynomial of

$$s\in\mathrm{End}\left(\frac{\mathscr{D}_X[s](f^s\underline{u})}{\mathscr{D}_X[s](f^{s+1}\underline{u})}\right),$$

which is also called the *b-function*. In the remainder of this section, we study a relation among these b-functions as a preliminary for (6.17). It is easy to see that

(5.10.1) *$B(s,\underline{u})$ is the least common multiple of* $\{B_x(s,\underline{u}) \mid x \in X\}$. (Cf. (2.5.2).)

5.11. b-Functions and group actions. Let γ be an automorphism of X such that

(5.11.1) $f(\gamma x) = \lambda f(x)$ *with some* $\lambda \in \mathbf{C}^\times$.

Then γ induces an automorphism of X_0. Let $\mathscr{M} = \mathscr{D}_{X_0}\underline{u}$ $(\underline{u} = (u_1, \cdots, u_p))$ be a $\mathscr{D}_{X_0}$-module such that

(5.11.2) *there exists a* $\mathscr{D}_{X_0}$*-isomorphism* $\varphi = \varphi_\gamma : \gamma^*\mathscr{M} \to \mathscr{M}$ *such that* $\Sigma_i \Gamma(X, \mathscr{O}_X) u_i = \Sigma_i \Gamma(X, \mathscr{O}_X) \varphi(\gamma^* u_i)$. *(Here* $\gamma^* u_i = 1 \otimes u_i$.*)*

Put $v_i := \varphi(\gamma^* u_i)$ and $\underline{v} = (v_i)_i$. Then it is easy to see that

(5.11.3)
$$\mathscr{D}_X[s](f^s\underline{u}) = \mathscr{D}_X[s](f^s\underline{v}) \underset{\varphi}{\overset{\approx}{\leftarrow}} \mathscr{D}_X[s](f^s \cdot \gamma^*\underline{u}),$$

as $\mathscr{D}_X[s,t]$-modules, where t is the operator $s \mapsto s+1$ (cf. (5.1)), and hence

(5.11.4)
$$B_x(s,\underline{u}) = B_x(s, \gamma^*\underline{u}) = B_{\gamma x}(s,\underline{u}) \qquad (x \in X).$$

Now assume that a group Γ acts on X. By (5.11.4), we get the following two assertions.

5.11.5. *If every element* $\gamma \in \Gamma$ *induces an automorphism of* X *satisfying* (5.11.1) *and* (5.11.2), *and if* $x_0 \in X$ *is contained in the Zariski closure of a* Γ*-orbit* Γx_1, *then* $B_{x_1}(s,\underline{u})$ *divides* $B_{x_0}(s,\underline{u})$. (Cf. (2.5.3).)

5.11.6. *Keep the assumption of* (5.11.5), *and further assume that* $x_0 \in X$ *is contained in the Zariski closure of every* Γ*-orbit, then* $B(s,\underline{u}) = B_{x_0}(s,\underline{u})$.

Lemma 5.12. *For any* $x \in X$,

$$B_x(s,\underline{u}) = \tilde{B}_x(s,\underline{u}) = \hat{B}_x(s,\underline{u}).$$

Proof. It suffices to prove that $B := B_x$ divides $\hat{B} := \hat{B}_x$. (Cf. (2.5.4).) Put $A := \mathbf{C}[X]$ and $A_0 := \mathbf{C}[X_0]$. For the sake of simplicity, we assume that $\underline{u}$

consists of only one section $u' \in \Gamma(X_0, \mathscr{M})$. Since the problem is local, we may assume X to be affine Put $M := D_{X_0}u'$, and let u be the image of u' in $A_x \otimes_A M$. Let $(x_1, \cdots, x_n)$ be a local coordinate system at x, $\partial_i := \partial/\partial x_i$, $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| = \Sigma_i \alpha_i$ for $\alpha = (\alpha_i, \cdots, \alpha_n) \in (\mathbf{Z}_{\geq 0})^n$. Let

$$P(f^{s+1}u) = \hat{B} \cdot f^s u \tag{5.12.1}$$

with $P \in \hat{A}_x[s] \otimes_A D_X$. Take $Q_1, \cdots, Q_N \in A_x[s] \otimes_A D_X$ so that $\Sigma_i (A_x[s] \otimes_A D_{X_0}) Q_i$ is equal to the annihilator $\operatorname{ann}(u;\, A_x[s] \otimes_A D_{X_0})$ of u in $A_x[s] \otimes_A D_{X_0}$. Since $\hat{A}_x[s]$ is faithfully $A_x[s]$-flat, we may regard u as an element of $\hat{A}_x[s] \otimes_{A_x[s]} (A_x[s] \otimes_A M) = \hat{A}_x[s] \otimes_A M$ (cf. [B, Chapter 1, §3, Proposition 9]), and then we have

$$\sum_i (\hat{A}_x[s] \otimes_A D_{X_0}) Q_i = \operatorname{ann}(u;\, \hat{A}_x[s] \otimes_A D_{X_0}).$$

Let

$$f^{-s} \cdot (P \cdot f - \hat{B}) \cdot f^s = f^{-l} \sum_i R_i Q_i \tag{5.12.2}$$

with $l \in \mathbf{Z}_{\geq 0}$ and $R_i \in \hat{A}_x[s] \otimes_A D_X$. (The left hand side is a product of operators.) Let

$$k := \max\{\operatorname{ord} P, \operatorname{ord} R_i, \operatorname{ord} Q_i \mid 1 \leq i \leq N\},$$

$$P = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha,$$

$$f^{-s} \cdot \partial^\alpha \cdot f^{s+1} = f^{-k+1} \sum_{|\beta| \leq k} c_{\alpha\beta} \partial^\beta \quad (|\alpha| \leq k),$$

$$R_i = \sum_{|\alpha| \leq k} d_{i\alpha} \partial^\alpha, \quad \text{and}$$

$$\partial^\alpha Q_i = \sum_{|\beta| \leq 2k} e_{\alpha i \beta} \partial^\beta \quad (|\alpha| \leq k),$$

where $a_\alpha \in \hat{A}_x[s]$, $c_{\alpha\beta} \in A_x[s]$, $d_{i\alpha} \in \hat{A}_x[s]$ and $e_{\alpha i \beta} \in A_x[s]$. Then (5.12.2) can be expressed as

$$f^{-k+1} \sum_{|\alpha|, |\beta| \leq k} X_\alpha c_{\alpha\beta} \partial^\beta - \hat{B} = f^{-l} \sum_{\substack{1 \leq i \leq N \\ |\alpha| \leq k \\ |\beta| \leq 2k}} Y_{i\alpha} e_{\alpha i \beta} \partial^\beta \tag{5.12.3}$$

with $X_\alpha = a_\alpha$ and $Y_{i\alpha} = d_{i\alpha}$. Thus $f^{k+l} \times$ (5.12.3) gives a system of linear equations for unknown variables X_α and $Y_{i\alpha}$ with $A_x[s]$-coefficients, and has a solution in $\hat{A}_x[s]$. Since $\hat{A}_x[s]$ is faithfully $A_x[s]$-flat, (5.12.3) has a solution in $A_x[s]$. (Cf. [B, Chapter 1, §3, Proposition 13].) Hence we can find $P \in A_x[s] \otimes_A D_X$ satisfying (5.12.1). Hence B divides $\hat{B}$. □

The following two lemmas are not used in the present note, but will become necessary in a forthcoming paper.

Lemma 5.13. *Let*

$$0 \to \mathscr{D}_{X_0}\underline{u}' \to \mathscr{D}_{X_0}\underline{u} \to \mathscr{D}_{X_0}\underline{u}'' \to 0$$

be an exact sequence of coherent $\mathscr{D}_{X_0}$-modules. Take $x \in X$. Put $b(s) := B_x(s, \underline{u})$, $b'(s) := B_x(s, \underline{u}')$ and $b''(s) := B_x(s, \underline{u}'')$. Then $b(s)$ divides $\Pi_{i=-m}^{m} b'(s+i) b''(s+i)$ if $m \gg 0$.

Proof. Consider $D_{X,x}[s,t]$-modules $N := \mathscr{D}_{X,x}[s](f^s \underline{u})$, $N' := \mathscr{D}_{X,x}[s](f^s \underline{u}')$, and $N'' := D_{X,x}[s]\ (f^s \underline{u}'')$. First let us consider the case where $\mathscr{D}_{X_0}\underline{u}'' = 0$. Since $N[t^{-1}] = N'[t^{-1}]$, and since N and N' are finitely generated $\mathscr{D}_{X,x}[s]$-modules, $t^k N' \subset N \subset t^{-l} N'$ for $k, l \gg 0$. Then

$$\begin{aligned} & b'(s+k) b'(s+k-1) \cdots b'(s-l) N \\ & \subset b'(s+k) b'(s+k-1) \cdots b'(s-l) t^{-l} N' \\ & = t^{-l} b'(s+k+l) \cdots b'(s) N' \subset t^{k+1} N' \subset tN. \end{aligned}$$

Hence $b(s)$ divides $b'(s+k) b'(s+k-1) \cdots b'(s-l)$.

Next consider the case where $\underline{u}''$ is the image of $\underline{u}$, and $N' \subset N$. Then we get morphisms $0 \to N' \xrightarrow{B} N \xrightarrow{C} N'' \to 0$, where B is the inclusion mapping, $CB = 0$ and C is surjective. Moreover, this sequence becomes exact after the localization by t^{-1}. Hence $b''(s) N \subset tN + (N'[t^{-1}] \cap N)$, and consequently $b''(s) N \subset tN + t^{-k} N'$ for $k \gg 0$. Then

$$\begin{aligned} & b'(s) \cdots b'(s-k+1) b'(s-k) b''(s) N \\ & \subset tN + t^{-k} b'(s+k) \cdots b'(s+1) b'(s) N' \\ & \subset tN + tN' = tN. \end{aligned}$$

Hence $b(s)$ divides $b'(s)\cdots b'(s-k)b''(s)$.

In the general case, put $\tilde{N}':=t^mN'$ and $\tilde{b}'(s):=B_x(s,f^m\underline{u}')$. Then $\tilde{b}'(s)\tilde{N}' \subset t\tilde{N}'$, and $\tilde{N}' = t^mN' \subset N$. Hence $b'(s+m)\tilde{N}' = b'(s+m)t^mN' = t^mb'(s)N' \subset t^{m+1}N' = t\tilde{N}'$, i.e.,

(5.13.1) $$\tilde{b}'(s) \text{ divides } b'(s+m).$$

Let $\tilde{\underline{u}}''$ be the image of $\underline{u}$ in $\mathscr{D}_{X_0}\underline{u}''$, and put $\tilde{b}''(s)=B_x(s,\tilde{\underline{u}}'')$. By the first and the second steps, we can see that, if $k,l\gg 0$, then

(5.13.2) $$\tilde{b}''(s) \text{ divides } b''(s+k)\cdots b''(s-l), \text{ and}$$

(5.13.3) $$b(s) \text{ divides } \tilde{b}'(s)\cdots\tilde{b}'(s-k)\tilde{b}''(s).$$

By (5.13.1)–(5.13.3), we get the result. □

Lemma 5.14. *Let $\mathscr{D}_{X_0}\underline{u}$ be a regular holonomic $\mathscr{D}_{X_0}$-module such that $\mathrm{DR}_{X_0}(\mathscr{D}_{X_0}\underline{u})$ is locally constant and has a finite monodromy. Then for any $x\in X$, the zeros of $B_x(s,\underline{u})$ are rational numbers.*

Proof. As in [Ka1], we may assume $f^{-1}(0)$ to be normal crossing. By (5.10.1) and (5.12), we may assume that $X=\mathbf{C}^n$, $f(x)=x_1^{e_1}\cdots x_n^{e_n}$ $(e_i\in\mathbf{Z}_{\geq 0})$, $\mathrm{DR}(\mathscr{D}_{X_0}\underline{u})$ is locally constant and has a finite monodromy. By (5.13), we may assume that $\mathrm{DR}(\mathscr{D}_{X_0}\underline{u})$ is locally constant sheaf of rank one. By (5.13) again, we may assume that $\mathscr{D}_{X_0}\underline{u}=\mathscr{D}_{X_0}(x_1^{\alpha_1}\cdots x_n^{\alpha_n})$ $(\alpha_i\in\mathbf{C})$ and $\underline{u}=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, taking up possibly a different generator system. Because of the finiteness of the monodromy, $\alpha_i\in\mathbf{Q}$. Hence we get the result. □

5.15. Microlocal condition. Let $\pi: T^*X\to X$ be the projection, $\mathscr{M}$ a regular holonomic $\mathscr{D}_{X_0}$-module, $u\in\Gamma(X_0,\mathscr{M})$, and $p_0\in\mathrm{ch}(\mathscr{D}_X(f^\alpha u))$. Assume that $Q\in\mathscr{D}^{\mathrm{an}}_{X,\pi(p_0)}$ is invertible in $\mathscr{E}_{p_0}$ (= {microdifferential operators at p_0}), and that $Q(f^{s+1}u)=b_0(s)(f^su)$ with $b_0(s)\in\mathbf{C}[s]$. Then $b_0(s)$ is a non-zero constant multiple of $\tilde{B}_{x_0}(s,\underline{u})$. (Cf. (2.5.7).)

§6. *D*-Modules and Prehomogeneous Vector Spaces

The content of this section is a supplement of §1 and §3. The main results are (6.20)–(6.22). To fix notation, first we review the content of §1 and §3, with some supplement. Next we give an essential part of the proof of (6.20). Since the proof of (6.20) has a large overlap with that of (3.11), we have omitted the part which is essentially the same as (3.11).

6.1. Prehomogeneous vector space. Let G be a connected reductive group over $\mathbb{C}$, $\rho: G \to GL(V)$ a linear representation such that $O_0 = G \cdot v_0$ is open in V for some $v_0 \in V$. (The G-action is defined by $g \cdot v := \rho(g)v$ ($g \in G$, $v \in V$).) Such (G, ρ, V) is called a prehomogeneous vector space. We denote by the same letter ρ the Lie algebra homomorphism $\mathfrak{g} := \mathrm{Lie}(G) \to \mathfrak{gl}(V)$ induced by $\rho: G \to GL(V)$.

6.2. Relative invariant. Let $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ and $0 \neq f \in \mathbb{C}[V]$ such that $f(gv) = \phi(g) f(v)$ ($g \in G$, $v \in V$). Such f is called a relative invariant with the character ϕ.

6.3. Dual. Let $V^\vee$ be the dual space of V, and $\rho^\vee: G \to GL(V^\vee)$ be the contragradient representation. Then $(G, \rho^\vee, V^\vee)$ is also a prehomogeneous vector space. There exists a relative invariant $0 \neq f^\vee \in \mathbb{C}[V^\vee]$ whose character is ϕ^{-1}. Then $\dim V = \dim V^\vee =: n$ and $\deg f = \deg f^\vee =: d$. Let $\langle v, v^\vee \rangle = \langle v^\vee, v \rangle$ ($v \in V$, $v^\vee \in V^\vee$) be the natural pairing.

We fix f and $f^\vee$ as above in the remainder of this section.

6.4. Put $\Omega := V \backslash f^{-1}(0)$ and $\Omega^\vee := V^\vee \backslash f^{\vee -1}(0)$. There exists a unique G-orbit $O_1 = G \cdot v_1$ (resp. $O_1^\vee = G \cdot v_1^\vee$) which is closed in Ω (resp. $\Omega^\vee$). Define morphisms $F: \Omega \to V^\vee$ and $F^\vee: \Omega^\vee \to V$ by $F := \mathrm{grad} \log f$ and $F^\vee := \mathrm{grad} \log f^\vee$. Then F and $F^\vee$ are G-equivariant, $F(\Omega) = F(O_0) = F(O_1) = O_1^\vee$, and $F^\vee(\Omega^\vee) = F^\vee(O_0^\vee) = F^\vee(O_1^\vee) = O_1$. Moreover $F: O_1 \to O_1^\vee$ and $F^\vee: O_1^\vee \to O_1$ are isomorphisms which are the inverse of each other. In particular, $\dim O_1 = \dim O_1^\vee =: m$. Let $O_1 \xrightarrow{i} \Omega \xrightarrow{j} V$ and $O_1^\vee \xrightarrow{i^\vee} \Omega^\vee \xrightarrow{j^\vee} V^\vee$ be the inclusion mappings.

6.5. (Cf. (1.18).) Let $(TO_1^\vee)^\perp$ be the conormal bundle of $O_1^\vee$ i.e.,

$$(TO_1^\vee)^\perp = \{(v, v^\vee) \in V \times V^\vee \mid v^\vee \in O_1^\vee,\ v \perp T_{v^\vee} O_1^\vee \ (\subset V^\vee)\}.$$

Then the following diagram is commutative.

$$\begin{array}{ccc} (TO_1^\vee)^\perp & \xrightarrow{\simeq \Phi} & \Omega \\ & \text{projection} \searrow \quad \swarrow F & \\ & O_1^\vee, & \end{array}$$

where $\Phi(v,v^\vee):=v+F^\vee(v^\vee)$. The inverse morphism of Φ is given by $\Psi(v):=(v-F^\vee F(v), F(v))$ $(v\in\Omega)$. Interchanging the roles of V and $V^\vee$, we get $\Phi^\vee$, $\Psi^\vee$, and a similar diagram.

6.6. For a local coordinate system $\{z_1,\cdots,z_m\}$ of O_1, put

$$\omega^2:=\det\left(\left\langle F_*\left(\frac{\partial}{\partial z_i}\right),\frac{\partial}{\partial z_j}\right\rangle\right)\cdot(dz_1\wedge\cdots\wedge dz_m)^{\otimes 2}. \tag{6.6.1}$$

(Here $\partial/\partial z_i$ denotes the vector field defined by z_i.) Then ω^2 is independent of the choice of the local coordinate, and gives rise to a global section of the line bundle $(\wedge^m T^*O_1)^{\otimes 2}$ which is everywhere non-vanishing. Let $\pi:\tilde{O}_1\to O_1$ be the two-fold covering of O_1 determined by $\omega:=\sqrt{\omega^2}$. The m-form ω on O_1 is defined only locally (with respect to the classical topology), but its pull-back $\pi^*\omega=:\tilde{\omega}$ is defined globally on $\tilde{O}_1$. Define $\omega^{\vee 2}$, $\omega^\vee$, $\tilde{\omega}^\vee$, and $\pi^\vee:\tilde{O}_1^\vee\to O_1^\vee$, replacing O_1 and f with $O_1^\vee$ and $f^\vee$. Consider the cartesian squares

$$\begin{array}{ccc} \tilde{\Omega}^\vee & \xrightarrow{\tilde{F}^\vee} & \tilde{O}_1 \\ {\scriptstyle\pi^\vee}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Omega^\vee & \xrightarrow{F^\vee} & O_1 \end{array} \quad\text{and}\quad \begin{array}{ccc} \tilde{\Omega} & \xrightarrow{\tilde{F}} & \tilde{O}_1^\vee \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^\vee} \\ \Omega & \xrightarrow{F} & O_1^\vee \end{array} \tag{6.6.2}$$

By (3.15), we get the cartesian squares

$$\begin{array}{ccc} \tilde{O}_1 & \xrightarrow{\tilde{i}} & \tilde{\Omega} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ O_1 & \xrightarrow{i} & \Omega \end{array} \quad\text{and}\quad \begin{array}{ccc} \tilde{O}_1^\vee & \xrightarrow{\tilde{i}^\vee} & \tilde{\Omega}^\vee \\ {\scriptstyle\pi^\vee}\downarrow & & {\scriptstyle\pi^\vee}\downarrow \\ O_1^\vee & \xrightarrow{i^\vee} & \Omega^\vee \end{array} \tag{6.6.3}$$

such that $\tilde{F}\circ\tilde{i}:\tilde{O}_1\to\tilde{O}_1^\vee$ is the isomorphism constructed in (3.15) (and similarly for $\tilde{F}^\vee\circ\tilde{i}^\vee$). We consider $\tilde{O}_1$ (resp. $\tilde{O}_1^\vee$) as a closed subvariety of $\tilde{\Omega}$ (resp. $\tilde{\Omega}^\vee$) by $\tilde{i}$ (resp. $\tilde{i}^\vee$), and we write $\tilde{F}$ (resp. $\tilde{F}^\vee$) for $\tilde{i}\circ\tilde{F}$ (resp. $\tilde{i}\circ\tilde{F}^\vee$) if there is no fear of confusion.

6.7. Let $\mathscr{D}_{\Omega}\underline{u}_0$ be a regular holonomic $\mathscr{D}_{\Omega}$-module such that $L:=\mathrm{DR}(\mathscr{D}_{\Omega}\underline{u}_0)$ is a locally constant sheaf of rank r. Let us apply the results of §5 to $\mathscr{D}_{\Omega}\underline{u}_0$. By (5.4),

(6.7.1) $$\mathrm{DR}(\mathscr{D}_V(f^{\alpha}\underline{u}_0))=\begin{cases}Rj_*(\mathbf{C}f^{-\alpha}\otimes L) & \text{if } \mathrm{Re}(\alpha)\ll 0,\\ j_!(\mathbf{C}f^{-\alpha}\otimes L) & \text{if } \mathrm{Re}(\alpha)\gg 0.\end{cases}$$

By (5.9),

(6.7.2) $$\mathrm{DR}(\mathscr{D}_{V,\mathbf{K}}(f^s\underline{u}_0))=(j_{\mathbf{K}})_{!*}(\mathbf{K}f^{-s}\otimes L_{\mathbf{K}}),$$

where $\mathscr{D}_{V,\mathbf{K}}(f^s\underline{u}_0)=\mathscr{D}_V[s](f^s\underline{u}_0)\otimes_{\mathbf{C}[s]}\mathbf{K}$, and hence

(6.7.3) *$\mathscr{D}_{V,\mathbf{K}}(f^s\underline{u}_0)$ is a simple $\mathscr{D}_{V,\mathbf{K}}$-module, if $\mathscr{D}_{\Omega}\underline{u}_0$ is a simple $\mathscr{D}_{\Omega}$-module.*

(See (5.7) for $\mathbf{K}$.) By (5.6),

(6.7.4) $$\underline{\mathrm{ch}}\mathscr{D}_V(f^{\alpha}\underline{u}_0)=r\cdot\underline{\mathrm{ch}}\mathscr{D}_V f^{\alpha}, \text{ and}$$

(6.7.5) $$\underline{\mathrm{ch}}\mathscr{D}_V[s](f^s\underline{u}_0)=r\cdot\underline{\mathrm{ch}}\mathscr{D}_V[s]f^s.$$

6.8. $\mathfrak{g}$-Action as differential operators. If a $\mathfrak{g}$-action is given on a non-singular variety over a field of characteristic zero, then each $A\in\mathfrak{g}$ gives a vector field on X, which we regard as a differential operator of first order. Hence, for (a local) section u of a $\mathscr{D}_X$-module and for $A\in\mathfrak{g}$, we can consider Au. Let ν be a Lie algebra character of $\mathfrak{g}$. Then the definition of the relative invariance of u_ν (=a symbol) with respect to $\mathfrak{g}$

(6.8.1) $$-Au_\nu=\nu(A)u_\nu \quad (A\in\mathfrak{g})$$

can be regarded as a system of linear differential equations for u_ν. Denote the corresponding $\mathscr{D}_X$-module by $\mathscr{D}_X u_\nu$. If X consists of a finite number of G-orbits, then $\mathscr{D}_X u_\nu$ becomes holonomic [Ka3, 5.1.12]. If X consists of a single G-orbit, then $\mathscr{D}_X u_\nu$ becomes a locally free $\mathscr{O}_X$-module of rank ≤ 1. (*Remark.* This rank can become zero. For example, consider the case where $\mathfrak{g}$ acts trivially on X and $\nu\neq 0$. For a less trivial example, see (6.11.10).) Assume that we are given $\mathfrak{g}$-actions on non-singular varieties X and Y, a $\mathfrak{g}$-equivariant morphism $\xi: X\to Y$, and a $\mathscr{D}_Y$-module $\mathscr{M}$. For a local section $u\in\mathscr{M}$, put $\xi^*u:=1\otimes u\in\mathscr{O}_X\otimes_{\xi^{-1}\mathscr{O}_Y}\xi^{-1}\mathscr{M}=:\xi^*\mathscr{M}$. Then we can see that

(6.8.2) $$A(\xi^*u)=\xi^*(Au) \quad (A\in\mathfrak{g},\ u\in\mathscr{M}).$$

(*Proof.* We may consider locally with respect to the classical topology. Let $X=\{x=(\cdots,x_i,\cdots)\}$, $Y=\{y=(\cdots,y_j,\cdots)\}$, $\xi(x)=(\cdots,\xi_j(x),\cdots)$, and $\Sigma a_i(x)\frac{\partial}{\partial x_i}$, (resp. $\Sigma_j b_j(y)\frac{\partial}{\partial y_j}$) be the vector field on X (resp. Y) defined by $A\in\mathfrak{g}$. For any C^∞-function ψ on Y, $A(\psi(\xi(x)))=\frac{d}{dt}\psi(\xi(e^{-tA}x))|_{t=0}=\frac{d}{dt}\psi(e^{-tA}\xi(x))|_{t=0}=(A\psi)(\xi(x))$. From this relation follows that $\Sigma_i a_i(x)\frac{\partial\xi_j}{\partial x_i}=b_j(\xi(x))$. Hence

$$\begin{aligned}A(\xi^*u)=A(1\otimes u)&=\sum_i a_i(x)\frac{\partial}{\partial x_i}(1\otimes u)\\&=\sum_{i,j}a_i(x)\frac{\partial\xi_j(x)}{\partial x_i}\otimes\frac{\partial}{\partial y_j}u\\&=\sum_j b_j(\xi(x))\otimes\frac{\partial}{\partial y_j}u=1\otimes Au=\xi^*(Au).\quad\Box)\end{aligned}$$

Remark 6.8.3. Let G be a simply connected semisimple group, Γ a finite subgroup, and $X:=G/\Gamma$. Since $\mathfrak{g}:=\mathrm{Lie}(G)$ has no non-trivial character, $\mathrm{DR}_X(\mathscr{D}_X u_\nu)\simeq\mathbf{C}_X$. On the other hand, the locally constant sheaves of rank one on X are in one-to-one correspondence with χ's in $\mathrm{Hom}(\Gamma,\mathbf{C}^\times)$. Hence the locally constant sheaf of rank one associated to $\chi\neq 1$ can not be obtained as $\mathrm{DR}_X(\mathscr{D}_X u_\nu)$. The author does not know whether all the locally constant sheaves of rank one on O_1 can be obtained as $\mathrm{DR}(\mathscr{D}u_\nu)$, or not.

6.9. Keep the notation and the assumption of (6.8). We further assume that there is an algebraic group action on X of a connected linear algebraic group G such that $\mathfrak{g}=\mathrm{Lie}(G)$ and that the G-action induces the $\mathfrak{g}$-action on X given in (6.8). For $\gamma\in G$, define an isomorphism

(6.9.1) $$\varphi=\varphi_\gamma:\gamma^*\mathscr{D}_X\to\mathscr{D}_X$$

so that $\varphi(P)h=\gamma^*P\gamma^{*-1}h$ for $h\in\mathcal{O}_{X,x}$ and for $P\in(\gamma^*\mathscr{D}_X)_x=\mathscr{D}_{X,\gamma x}$. (Here and below, we denote by γ the morphism $X\to X$, $x\mapsto\gamma x$.) Since $\gamma^*A\gamma^{*-1}=\gamma^{-1}A\gamma$ for $\gamma\in G$ and $A\in\mathfrak{g}$ (the right hand side is the adjoint action), and since $\nu(\gamma^{-1}A\gamma)=\nu(A)$, we have

$$\varphi(\gamma^*\sum_{A\in\mathfrak{g}}\mathscr{D}_X(A+\nu(A)))=\sum_{A\in\mathfrak{g}}\mathscr{D}_X(A+\nu(A)).$$

Hence (6.9.1) induces a $\mathscr{D}_X$-isomorphism

(6.9.2) $$\varphi=\varphi_\gamma:\gamma^*\mathscr{D}_X u_\nu\to\mathscr{D}_X u_\nu$$

such that $\varphi(\gamma^* u_\nu)=u_\nu$.

6.10. Some *D*-modules. Put

$$\delta_{\tilde{\omega}^\vee}:=\frac{\tilde{\omega}^\vee\otimes 1_{\tilde{O}_1^\vee\to\tilde{\Omega}^\vee}}{\pi^*(dy_1\wedge\cdots\wedge dy_n)}\in\Gamma(\tilde{\Omega}^\vee,\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee}).$$

(See (6.6) for $\tilde{\omega}^\vee$. Note that $\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee}=\int_{\tilde{i}^\vee}^0\mathcal{O}_{\tilde{O}_1^\vee}$.) Then we can consider the global sections

$$\tilde{F}^*\delta_{\tilde{\omega}^\vee}\in\Gamma(\tilde{\Omega},\tilde{F}^*\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee}),$$

$$\delta_{\omega^\vee}:=\delta_{\tilde{\omega}^\vee}\in\Gamma(\Omega^\vee,\pi_*^\vee\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee})\ (=\Gamma(\tilde{\Omega}^\vee,\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee})),$$

$$F^*\delta_{\omega^\vee}:=\tilde{F}^*\delta_{\tilde{\omega}^\vee}\in\Gamma(\Omega,\pi_*\tilde{F}^*\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee}).$$

(In (3.17), $\delta_{\tilde{\omega}^\vee}$ and $\delta_{\omega^\vee}$ were denoted by $\tilde{h}$ and h, but here we change the notation.) Since $\pi_*\tilde{F}^*\int_{\tilde{i}^\vee}\mathcal{O}_{\tilde{O}_1^\vee}$ is a regular holonomic $\mathcal{D}_\Omega$-module, we can apply the results of §5 to the $\mathcal{D}_V$-modules $\mathcal{D}_V[s](f^s\cdot F^*\delta_{\omega^\vee})$ and $\mathcal{D}_V(f^\alpha\cdot F^*\delta_{\omega^\vee})$ ($\alpha\in\mathbf{C}$). We define $\delta_{\tilde{\omega}}$, $\tilde{F}^{\vee *}\delta_{\tilde{\omega}}$, δ_ω, $F^{\vee *}\delta_\omega$, $\mathcal{D}_{V^\vee}[s](f^{\vee s}\cdot F^{\vee *}\delta_\omega)$ and $\mathcal{D}_{V^\vee}(f^{\vee\alpha}\cdot F^{\vee *}\delta_\omega)$, similarly, interchanging the roles of V and $V^\vee$. For $A\in\mathrm{Lie}(G)$, put $\phi_0(A):=\mathrm{trace}\,\rho(A)$. Then

(6.10.1) $\quad -A\delta_{\omega^\vee}=\phi_0(A)\delta_{\omega^\vee}$ by (3.17),

(6.10.2) $\quad -A(F^*\delta_{\omega^\vee})=\phi_0(A)F^*\delta_{\omega^\vee}$ by (6.8.2), and

(6.10.3) $\quad -A(f^s\cdot F^*\delta_{\omega^\vee})=(s\phi(A)+\phi_0(A))f^s\cdot F^*\delta_{\omega^\vee}$.

6.11. Locally constant sheaves on Ω and $O_1^\vee$. Consider

(6.11.1) *a regular holonomic $\mathcal{D}_\Omega$-module $\mathcal{D}_\Omega u_0$ which is $\mathcal{O}_\Omega$-free of rank one, and such that $-Au_0=\chi(A)u_0$ ($A\in\mathfrak{g}$) with some Lie algebra character χ of $\mathfrak{g}$.*

Define a locally constant sheaf on Ω of rank one by $L:=\mathrm{DR}_\Omega(\mathcal{D}_\Omega u_0)$. By (6.5), F induces an isomorphism $\pi_1(\Omega)\to\pi_1(O_1^\vee)$. Hence L can be uniquely expressed as $L=F^*L^\vee$ with a locally constant sheaf $L^\vee$ on $O_1^\vee$ of rank one.

Example 6.11.2. Let $f_1,\cdots,f_l \in \mathbf{C}[V]\setminus\{0\}$ be relative invariants with characters $\phi_1,\cdots,\phi_l$ such that $f_i(\Omega)\not\ni 0$. Let $f_i^\vee \in \mathbf{C}[V^\vee]\setminus\{0\}$ be a relative invariant with the character ϕ_i^{-1}. Then $f_i^\vee$ (resp. f_i^{-1}) is a non-zero constant multiple of $F^{\vee *}f_i^{-1}$, (resp. $F^*f_i^\vee$), and $u_0 := \Pi_{i=1}^l f_i^{\alpha_i}$ ($\alpha_i \in \mathbf{C}$) satisfies (6.11.1) with $\chi = \Sigma\alpha_i\phi_i$. The locally constant sheaves L on Ω, and $L^\vee$ on $O_1^\vee$ are given by

$$L = \mathbf{C}\cdot\prod_i f_i^{-\alpha_i}, \text{ and}$$

$$L^\vee = \mathbf{C}\cdot\prod_i f_i^{\vee\alpha_i} \mid O_1^\vee .$$

Example 6.11.3. Let ϕ_i, f_i and $f_i^\vee$ $(1\leq i\leq l)$ be as in (6.11.2). Then $u_0 := \Pi_{i=1}^l f_i^{\alpha_i}\cdot F^*\delta_{\omega^\vee}$ ($\alpha_i \in \mathbf{C}$) satisfies the above condition with $\chi = \Sigma_{i=1}^l \alpha_i\phi_i + \phi_0$. The locally constant sheaves L and $L^\vee$ are given by

$$L = \left(\mathbf{C}\cdot\prod_i f_i^{-\alpha_i}\right)\otimes F^*(\mathbf{C}\omega^\vee), \text{ and}$$

$$L^\vee = \left(\mathbf{C}\cdot\prod_i f_i^{\vee\alpha_i} \mid O_1^\vee\right)\otimes \mathbf{C}\omega^\vee .$$

Note that $(\mathbf{C}\omega^\vee)\otimes(\mathbf{C}\omega^\vee) = \mathbf{C}_{O_1^\vee}$ and hence $\mathbf{D}(\mathbf{C}\omega^\vee[m]) = \mathbf{C}\omega^\vee[m]$, where $\mathbf{D}(\ )$ denotes the Verdier dual.

Remark 6.11.4. For a fixed $\lambda \in \mathbf{C}^\times$, define $a \in GL(V)$ by $a(v) = \lambda v$. By (6.8.2), there exists a $\mathscr{D}_{O_0}$-isomorphism $\varphi : a^*\mathscr{D}_{O_0}u_\chi \overset{\sim}{\to} \mathscr{D}_{O_0}u_\chi$ such that $a^*(u_\chi) := 1\otimes u_\chi \mapsto u_\chi$. (See χ for (6.11.1) and u_χ for (6.8.1).) Since $\mathscr{D}_{O_0}u_\chi \simeq \mathscr{D}_{O_0}u_0$ $(u_\chi \mapsto u_0)$, we have $\varphi : a^*\mathscr{D}_{O_0}u_0 \overset{\sim}{\to} \mathscr{D}_{O_0}u_0$ $(a^*u_0 \mapsto u_0)$. Let $\varphi : a^*\mathscr{D}_\Omega u_0 \overset{\sim}{\to} \mathscr{D}_\Omega u_0$ be its extension. (Note that $\mathscr{D}_\Omega u_0$ is the minimal extension of $\mathscr{D}_{O_0}u_0$.) Since $\varphi(a^*u_0) = u_0$ on O_0, and since they are global sections of the free $\mathscr{O}_\Omega$-module $\mathscr{D}_\Omega u_0$,

$$\varphi^*(a^*u_0) = u_0 \quad \text{on the whole } \Omega.$$

The global isomorphism φ on Ω induces $a^*L \simeq L$ and $a^*L^\vee \simeq L^\vee$.

We can consider an infinitesimal version of the above argument. Put $\mathrm{Euler} := \Sigma_{i=1}^n x_i\frac{\partial}{\partial x_i}$. For any C^∞-function ψ on V, we have $A(\psi(cx)) = (A\psi)(cx)$ ($A \in \mathfrak{g}$, $c \in \mathbf{C}^\times$). Differentiating by c and then letting $c \to 1$, we can see that the Euler operator commutes with the $\mathfrak{g}$-action. Hence we can define a morphism $\varphi \in \mathrm{Hom}_{\mathscr{D}} := \mathrm{Hom}_{\mathscr{D}}(\mathscr{D}u_0, \mathscr{D}u_0)$ by $\varphi(u_0) = (\mathrm{Euler})u_0$. Here $\mathscr{D}$ means $\mathscr{D}_{O_0}$, but we may read $\mathscr{D}$ also as $\mathscr{D}_{\Omega}$. (Note that $\mathscr{D}_{\Omega}u_0$ is the minimal extension of $\mathscr{D}_{O_0}u_0$.) Since $\mathscr{D}_{O_0}u_0$ is a locally free $\mathscr{O}_{O_0}$-module of rank one, $\mathrm{Hom}_{\mathscr{D}} = \mathbf{C}$. Hence

$$(6.11.5) \qquad (\mathrm{Euler})u_0 \in \mathbf{C}u_0 .$$

This remark is useful when we need (2.7.2).

Remark 6.11.6. If (G, ρ, V) is a regular prehomogeneous vector space, (6.11.2) and (6.11.3) are essentially the same. See [S, Proposition 11]. Let us show that, in general, if we do not assume the regularity for (G, ρ, V), then the locally constant sheaves L of (6.11.3) can not be obtained as in (6.11.2). Let $f_1, \cdots, f_l$ be the totality (up to $\mathbf{C}^{\times}$) of mutually distinct irreducible relative invariants. Let ϕ_i be the character of f_i. Put $u := F^*\delta_{\omega^\vee}$ (cf. (6.10)), $\underline{f}^{\underline{\alpha}} := f_1^{\alpha_1} \cdots f_l^{\alpha_l}$ $(\alpha_i \in \mathbf{C})$, $L := \mathrm{DR}(\mathscr{D}_{\Omega}u)$, and assume that

$$(6.11.7) \qquad L \simeq \mathbf{C}\underline{f}^{-\underline{\alpha}} (= \bigotimes_i \mathbf{C}f_i^{-\alpha_i}) \text{ on } O_0 .$$

Then $\mathscr{D}_{O_0}u \simeq \mathscr{D}_{O_0}\underline{f}^{\underline{\alpha}}$. Since L is locally constant on Ω, $\alpha_i \in \mathbf{Z}$ whenever $f_i(\Omega) \ni 0$. Removing the factor $\mathbf{C}f_i^{-\alpha_i}$ for such i from the right hand side of (6.11.7), we may assume from the beginning that

$$(6.11.8) \qquad \alpha_i = 0 \quad \text{whenever } i \notin I, \text{ where } I := \{i \mid f_i(\Omega) \not\ni 0\}.$$

Considering the minimal extension, we get $D_{\Omega}u \simeq D_{\Omega}\underline{f}^{\underline{\alpha}}$. Let $P\underline{f}^{\underline{\alpha}}$ $(P \in D_{\Omega})$ be the element of $\mathscr{D}_{\Omega}\underline{f}^{\underline{\alpha}}$ corresponding to u. Since the G-action on $D_{\Omega} = \Sigma_{m \geq 0} f^{-m} D_V = \Sigma_{m \geq 0} f^{-m} \mathbf{C}[V] \otimes \mathbf{C}[V^\vee]$ is locally finite, there exists a finite dimensional G-submodule W containing P. Let $W = \oplus_\lambda W_\lambda$, $P = \Sigma_\lambda P_\lambda$ be the G-isotypic decomposition. Since $-A\underline{f}^{\underline{\alpha}} = (\Sigma_i \alpha_i \phi_i(A))\underline{f}^{\underline{\alpha}}$ and $-Au = \phi_0(A)u$ $(\phi_0 = \mathrm{trace}\,\rho)$ for all $A \in \mathfrak{g}$, there exists W_λ associated to $\Sigma \alpha_i \phi_i - \phi_0$. For such λ, $P\underline{f}^{\underline{\alpha}} = P_\lambda \underline{f}^{\underline{\alpha}}$, and hence we may assume from the beginning that P is relatively G-invariant with the character $\Sigma \alpha_i \phi_i - \phi_0$, i.e., $(g^*)^{-1} P g^* = (\Sigma \alpha_i \phi_i - \phi_0)(g) \cdot P$. Take $m \geq 0$ so that $W_\lambda \subset f^{-m} D_V$. Let $f = c f_1^{a_1} \cdots f_l^{a_l}$ $(a_i \in \mathbf{Z}_{\geq 0}$, $c \in \mathbf{C}^{\times})$. Then $a_i = 0$ $(i \notin I)$, and $f^m P \in D_V$ is relatively G-invariant with the character $\Sigma_{i \in I}(\alpha_i - m a_i)\phi_i - \phi_0$. Put $\underline{f}^{\underline{s}} = \Pi_{i \in I} f_i^{s_i}$, where s_i's are independent indeterminates. Consider $\underline{f}^{\underline{s}}$ in a simply connected neighbourhood of a point of O_0. Then $f^m(P\underline{f}^{\underline{s}}) \neq 0$, since it does not vanish for $\underline{s} = \underline{\alpha}$. If s_i's are specialized to non-negative integers, then $f^{\underline{s}}$ and $f^m(Pf^{\underline{s}})$ become relatively invariant polynomials. Hence considering the character of $f^m(Pf^{\underline{s}})$, we can show that for $\underline{s}\in(\mathbf{Z}_{\geq 0})^I$, there exists $\underline{c}=\underline{c}(\underline{s})\in(\mathbf{Z}_{\geq 0})^l$ such that

$$\sum_{i\in I}(-\alpha_i+ma_i+s_i)\phi_i+\phi_0=\sum_{i=1}^{l}c_i(s)\phi_i,$$

whenever $Pf^{\underline{s}}\neq 0$. We can easily see that this relation implies that each $c_i(s)$ is a $\mathbf{Z}$-linear combination of s_i $(i\in I)$ and 1. Put $\underline{d}:=\underline{c}(0)$. Then

$$\sum_{i\in I}(-\alpha_i+ma_i)\phi_i+\phi_0=\sum_{i=1}^{l}d_i\phi_i \quad \text{and} \quad d_i\in\mathbf{Z},$$

and hence

$$\phi_0\in\sum_{i\in I}\mathbf{C}\phi_i+\sum_{i\notin I}\mathbf{Z}\phi_i. \tag{6.11.9}$$

Inspecting [Ki, §3, Table B], we can see that (6.11.9) is not satisfied by none of prehomogeneous vector spaces which are listed in the table and satisfy

$$\begin{aligned}&\text{card}\{\text{irreducible components of }(\rho,V)\}\\&>\text{card}\{\text{irreducible relative invariants}\}/\mathbf{C}^\times>0.\end{aligned}$$

Hence for these prehomogeneous vector spaces, (6.11.2) and (6.11.3) are essentially different. (On the other hand, it is easy to see that (6.11.2) and (6.11.3) are essentially the same if these two cardinalities coincide.)

Remark 6.11.10. Consider the $\mathscr{D}_X$-module $\mathscr{D}_X u_\nu$ $(\neq 0)$ as in (6.8) with $X=O_1^\vee$. By the same argument as in (6.11.6), we obtain the relation $\nu\in\Sigma_{i\in I}\mathbf{C}\phi_i+\Sigma_{i\notin I}\mathbf{Z}\phi_i$ in place of (6.11.9). In other words if this relation does not hold, then $\mathscr{D}_X u_\nu=0$.

6.12. *b*-Functions. In the subsequent paragraphs (6.12)–(6.19), we study the '*b*-function' of $f^s u_0$, taking up and fixing some $\mathscr{D}_\Omega u_0$ as (6.11.1).

Let $\mathscr{D}_{O_0}[s]u_{s\phi+\chi}$ denote the $\mathscr{D}_{O_0}[s]$-submodule of $\mathscr{D}_{O_0,\mathbf{K}}u_{s\phi+\chi}$ (cf. (6.8.1)) generated by $u_{s\phi+\chi}$. For a commutative $\mathbf{C}[s]$-algebra C, provisionally put

$$\mathscr{D}_C u_{s\phi+\chi} := C\otimes_{\mathbf{C}[s]}\mathscr{D}_{O_0}[s]u_{s\phi+\chi},$$

(6.12.1) $$\mathscr{D}_C(f^s u_0) := C\otimes_{\mathbf{C}[s]}\mathscr{D}_{O_0}[s](f^s u_0), \text{ and}$$

$$\mathrm{Hom}_{\mathscr{D}_C} := \mathrm{Hom}_{\mathscr{D}_C}(\mathscr{D}_C u_{s\phi+\chi}, \mathscr{D}_C(f^s u_0)).$$

We assume that $\mathbf{C}[s]\subset C$. Since $\mathscr{D}_{O_0}[s]u_{s\phi+\chi}$ and $\mathscr{D}_{O_0}[s](f^s u_0)$ are $\mathbf{C}[s]$-flat (i.e., $\mathbf{C}[s]$-torsion free),

(6.12.2)
$$\mathscr{D}[s]u_{s\phi+\chi}\subset \mathscr{D}_C u_{s\phi+\chi},\ \mathscr{D}[s](f^s u_0)\subset \mathscr{D}_C(f^s u_0), \quad\text{and}\quad \mathrm{Hom}_{\mathscr{D}[s]}\subset \mathrm{Hom}_{\mathscr{D}_C},$$

where $\mathscr{D}=\mathscr{D}_{O_0}$. In the following lemma, we use the notation of (5.7) together with the provisional notation given here.

Lemma 6.13. *Define* $\tilde{\varphi}\in\mathrm{Hom}_{\mathscr{D}_{\mathbf{K}}}$ *by* $\tilde{\varphi}(u_{s\phi+\chi})=f^s u_0$. *Then* (1) $\tilde{\varphi}$ *induces* $\varphi\in\mathrm{Hom}_{\mathscr{D}[s]}$, (2) $\dim_{\mathbf{K}}\mathrm{Hom}_{\mathscr{D}_{\mathbf{K}}}=1$, *and* (3) $\mathrm{Hom}_{\mathscr{D}[s]}=\mathbf{C}[s]\varphi$.

Proof. (1) is obvious. (2) Since $D_{O_0,\mathbf{K}}u_{s\phi+\chi}$ and $\mathscr{D}_{O_0,\mathbf{K}}(f^s u_0)$ are locally free $\mathscr{O}_{O_0,\mathbf{K}}$-modules of rank one, $\dim\mathrm{Hom}_{\mathscr{D}_{\mathbf{K}}}\le\dim\mathrm{Hom}_{\mathscr{O}_{\mathbf{K}}}\le 1$. (Here $\mathrm{Hom}_{\mathscr{O}_{\mathbf{K}}}$ is defined as (6.12.1) replacing $\mathrm{Hom}_{\mathscr{D}_{\mathbf{K}}}$ with $\mathrm{Hom}_{\mathscr{O}_{\mathbf{K}}}$.) Since $0\ne\tilde{\varphi}\in\mathrm{Hom}_{\mathscr{D}_{\mathbf{K}}}$, we get the result. (3) For $\alpha\in\mathbf{C}$, put $C:=\{\xi(s)/\eta(s)\mid \xi,\eta\in\mathbf{C}[s],\ \eta(\alpha)\ne 0\}$. Then C is a discrete valuation ring. Assume that $\mathrm{Hom}_{\mathscr{D}_C}\supsetneq C\varphi$ for some α. Since $\mathrm{Hom}_{\mathscr{D}_C}$ is a torsion free C-module of rank one by (2), it follows that $\varphi=(s-\alpha)\varphi'$ with some $\varphi'\in\mathrm{Hom}_{\mathscr{D}_C}$, and hence that there exists a surjection

$$0=\frac{\mathscr{D}_C(f^s u_0)}{\varphi(\mathscr{D}_C u_{s\phi+\chi})}\twoheadrightarrow\frac{\mathscr{D}_C(f^s u_0)}{(s-\alpha)\mathscr{D}_C(f^s u_0)}=\mathscr{D}_{O_0}(f^\alpha u_0)\ne 0.$$

This is absurd. Therefore $\mathrm{Hom}_{\mathscr{D}_C}=C\varphi$ for all α. Take $\psi\in\mathrm{Hom}_{\mathscr{D}[s]}$. Then there uniquely exists $a\in\mathbf{K}$ such that $\psi=a\varphi$. By what we have porved, $a\in C$ for all α. Hence $a\in\mathbf{C}[s]$. □

Lemma 6.14. *Let* u_0 *be as in* (6.11.1). *Then with some polynomial* $b(s,u_0)\in\mathbf{C}[s]$,

$$f^\vee(\mathrm{grad}_x)(f^{s+1}u_0)=b(s,u_0)f^s u_0$$

in $D_V[s](f^s u_0)$.

Proof. Keep the notation of (6.13). Define $\varphi' \in \mathrm{Hom}_{\mathscr{D}[s]}$ by $\varphi'(u_{s\phi+\chi}) = f^{\vee}(\mathrm{grad}_x)(f^{s+1}u_0)$. (The well-definedness follows from that of $\varphi' \in \mathrm{Hom}_{\mathscr{D}_{\mathbf{K}}}$.) Then $\varphi' = b(s,u_0)\varphi$ with some $b(s,u_0) \in \mathbf{C}[s]$ by (6.13, (3)). Put $u := f^{\vee}(\mathrm{grad}_x)(f^{s+1}u_0) - b(s,u_0)(f^s u_0)$. Then $u \mid O_0 = 0$ in $\mathscr{D}_{O_0}[s](f^s u_0)$, and hence $u \mid O_0 = 0$ in $\mathscr{D}_{O_0,\mathbf{K}}(f^s u_0)$. Since $\mathscr{D}_{V,\mathbf{K}}(f^s u_0)$ is a simple $\mathscr{D}_{V,\mathbf{K}}$-module by (6.7.3), $u \mid O_0 = 0$ implies $u = 0$ in $\mathscr{D}_{V,\mathbf{K}}(f^s u_0)$ by the next sublemma. Hence we get the result by (6.12.2). □

Sublemma 6.14.1. *Let $\mathscr{M}$ be a simple $\mathscr{D}$-module, i.e., a non-zero coherent $\mathscr{D}$-module without proper coherent $\mathscr{D}$-submodule, and $U \subset X$ an open subset. If $\mathscr{M}|_U \neq 0$, then the restriction map $\Gamma(X,\mathscr{M}) \to \Gamma(U,\mathscr{M})$ is injective.*

Proof. Assume that $0 \neq u \in \Gamma(X,\mathscr{M})$ and $u|_U = 0$. Since $\mathscr{M}$ is simple, $\mathscr{M} = \mathscr{D}u$ and hence $\mathscr{M}|_U = 0$. □

6.15. Orbit O^0. Put $O^0 := \{v - F^{\vee}F(v) \mid v \in O_0\}$. Then O^0 is a G-orbit of V, since it is an image of G-orbit by an equivariant mapping.

Example 6.15.1. In the example (3.24), $O^0 = O_8$, the orbit appearing at the bottom of the right half of the holonomy diagram.

6.16. Microlocal b-function. Let u_0 be as in (6.11.1), and $b^{\mathrm{loc}}(s,u_0)$ (resp. $b^{\mathrm{loc}}(s,\mathscr{F}(u_0))$) be the microlocal b-function of $f^s u_0$ at $\Psi(O_0) \subset T^*V$ (resp. $f^{\vee s}\mathscr{F}(u_0)$ at $\{0\} \times O_0 \subset T^*V^{\vee}$, i.e., the open orbit in the conormal bundle of $\{0\} \subset V^{\vee}$). (See (6.5) for Ψ.) We normalize b^{loc} to be monic. Thus we get the functional equation, $f^{s+1}u_0 = b^{\mathrm{loc}}(s,u_0)P(s)(f^s u_0)$ with some microdifferential operator $P(s) \in \mathscr{E}[s]$ whose principal symbol is invertible on $\Psi(O_0)$, and similarly for $f^{\vee s}\mathscr{F}(u_0)$. See [G3, (0.5) and (6.1)].

Remark 6.16.1. In [G3], we have exclusively studied u_0 as in (6.11.2). However the argument in [G3] is based on the relative invariance of $f^{\alpha}u_0$, and works for general u_0 as in (6.11.1).

Lemma 6.17. *Let u_0 be as in* (6.11.1). *Take $v \in O^0$ and $v' \in \overline{O^0}$. Let $a(u_0)$ be the leading coefficient of $b(s,u_0)$. Then*

$$a(u_0)^{-1}b(s,u_0) = \tilde{B}_0(s,u_0) = B_0(s,u_0) = B_{v'}(s,u_0) = B_v(s,u_0) = b^{\mathrm{loc}}(s,u_0).$$

(See (6.14), (5.10) and (6.16) for the various b-functions.)

Proof. (1) We have

$$\{0\}\times O_1^\vee \subset (TO_1^\vee)^\perp \subset \mathrm{ch}(Df^\alpha) \text{ by (3.3)}$$
$$=\mathrm{ch}(D(f^\alpha u_0)) \text{ by (6.7.4)}.$$

Since $f^\vee(\mathrm{grad}_x)$ is invertible as a microdifferential operator at any point of $\{0\}\times O_1^\vee$, we get the first equality by (5.15). (Note that $\{0\}\times O_1^\vee$ projects to 0.) The second equality is a part of (5.12).

(2) Let us show that B_v divides $B_{v'}$, using (5.11.5) with $\Gamma=G$ and $X=V$. It suffices to prove (5.11.2). Since $\mathscr{D}_{O_0}u_\chi \overset{\sim}{\to} \mathscr{D}_{O_0}u_0$ $(u_\chi \mapsto u_0)$, we have for $\gamma\in G$, a $\mathscr{D}_{O_0}$-isomorphism $\varphi=\varphi_\gamma:\gamma^*\mathscr{D}_{O_0}u_0\to\mathscr{D}_{O_0}u_0$ such that $\varphi(\gamma^*u_0)=u_0$ by (6.9.2). This φ uniquely extends to an isomorphism $\varphi:\gamma^*\mathscr{D}_\Omega u_0\to\mathscr{D}_\Omega u_0$. Since $(\varphi(\gamma^*u_0)-u_0)\,|\,O_0=0$, $\varphi(\gamma^*u_0)=u_0$ on Ω. Thus we get (5.11.2).

(3) By the same argument as in (2), using the $\mathbf{C}^\times$-action instead of the G-action, and using (6.11.4), we can show that $B_{v'}$ divides B_0.

(4) Let us show that $B_v(s,u_0)=b^{\mathrm{loc}}(s,u_0)=a(u_0)^{-1}b(s,u_0)$. Take $v_0\in O_0$ so that $v_0-F^\vee F(v_0)=v$. By (3.3), (6.5), and (6.7.4), $\Psi(v_0)=(v,F(v_0))\in(TO_1^\vee)^\perp \subset \mathrm{ch}\mathscr{D}f^\alpha=\mathrm{ch}\mathscr{D}(f^\alpha u_0)$, which is a point lying over v of the cotangent bundle T^*V, and whose G-orbit $\Psi(O_0)$ is of dimension n. Since $f^\vee(\mathrm{grad}_x)$ is invertible in $\mathscr{E}_{\Psi(v_0)}$, we get the equality. (Cf. 5.15).) □

Lemma 6.18. *Let u_0 be as in* (6.11.1). *Then* $\deg b(s,u_0)=d$ $(=\deg f)$.

Proof. By (6.17), it suffices to calculate $\deg b^{\mathrm{loc}}(s,u_0)$. By [G3, (0.5, (4))], this degree does not depend on u_0. Since we have already proved in (1.7) that $\deg b(s,1)$ $(=\deg b^{\mathrm{loc}}(s,1))=d$, where 1 denotes the identity element of $\mathbf{C}[\Omega]$, we get the result. □

Lemma 6.19. *Let u_0 be as in* (6.11.1). (1) $f(\mathrm{grad}_y)(f^{\vee s+1}\mathscr{F}(f^\alpha u_0))$ $=(-1)^d b(\alpha-s-1,u_0)f^{\vee s}\mathscr{F}(f^\alpha u_0)$, *where $\mathscr{F}(-)$ denotes the Fourier transformation of D_V-modules.* (*See* (2.7).) (2) $a(u_0)^{-1}(-1)^d b(-s-1,u_0)=\tilde{B}_0(s,\mathscr{F}u_0))$ $=B_0(s,\mathscr{F}(u_0))=b^{\mathrm{loc}}(s,\mathscr{F}(u_0))$.

Proof. (1) Read the proof of (3.1) replacing $b(s)\to b(s,u_0)$ and $f^\alpha \to f^\alpha u_0$.

(2) We have

$$
\begin{aligned}
&\operatorname{ch} D(f^{\vee\beta}\cdot\mathscr{F}(f^{\alpha}u_0))\\
&=\operatorname{ch} D(\mathscr{F}(f^{\alpha}u_0)[f^{\vee -1}]) \text{ by (5.6.1)}\\
&=\operatorname{ch} D(\mathscr{F}(f^{\alpha}u_0)) \text{ (cf. the proof of (3.2))}\\
&=\operatorname{ch} D(f^{\alpha}u_0) \text{ by (2.7.2) and (6.11.5)}\\
&=\operatorname{ch} Df^{\alpha} \text{ by (6.7.4)}\\
&\supset \{0\}\times O_0.
\end{aligned}
$$

Since $f(\mathrm{grad}_y)$ is invertible as a microdifferential operator on $\{0\}\times O_0$, and since $\{0\}\times O_0$ is an open dense G-orbit of the irreducible component $\{0\}\times V$ of $\operatorname{ch} Df^{\alpha}$, we get the assertion. (The second equality is a part of (5.12).) □

Now we can prove the following theorem in the same way as (3.11), using the results of this section so far.

Theorem 6.20. *Let u_0 be as in* (6.11.1) *and put*

$$A_+ := \{\alpha\in\mathbf{C} \mid b(\alpha+j, u_0)\neq 0 \textit{ for } j=0,1,2,\cdots\}, \textit{ and}$$

$$A_- := \{\alpha\in\mathbf{C} \mid b(\alpha-j, u_0)\neq 0 \textit{ for } j=1,2,\cdots\}.$$

(1) $D(f^{\alpha}u_0)=D(f^{\alpha}u_0)[f^{-1}]$ *if* $\alpha\in A_-$.
(2) $D(f^{\alpha}u_0)=((D(f^{\alpha}u_0))^*[f^{-1}])^*$ *if* $\alpha\in A_+$.
(3) $\mathscr{F}(D(f^{\alpha}u_0))=\mathscr{F}(D(f^{\alpha}u_0))[f^{\vee -1}]$ if $\alpha\in A_+$.
(4) $\mathscr{F}(D(f^{\alpha}u_0))=(\mathscr{F}(D(f^{\alpha}u_0))^*[f^{\vee -1}])^*$ *if* $\alpha\in A_-$.
(5) *Let I be the defining ideal of $\overline{O_1^{\vee}}$ in* $\mathbf{C}[V^{\vee}]$, $\phi_0 := \operatorname{trace}\rho(A)$, *and let $D_{V^{\vee}}u''_{\alpha}$ be the D-module defined by*

$$-Au''_{\alpha}=(\alpha\phi+\chi+\phi_0)(A)u''_{\alpha} \textit{ for } A\in\mathrm{Lie}(G), \textit{ and}$$

$$au''_{\alpha}=0 \textit{ for } a\in I.$$

Then for any $\alpha\in\mathbf{C}$ and for any $k\in\mathbf{Z}$,

$$\mathscr{F}(D(f^{\alpha}u_0))[f^{\vee -1}]=(Du''_{\alpha+k})[f^{\vee -1}],$$

where $\mathscr{F}(f^{\alpha}u_0)$ is identified with $f^{\vee -k}u''_{\alpha+k}$.

Corollary 6.21. (*Cf.* (3.23).) *Let* u_0, L *and* $L^\vee$ *be as in* (6.11).

(1) $\mathrm{DR}(D_V(f^\alpha u_0))=Rj_*(\mathbf{C}f^{-\alpha}\otimes L)$ *if* $\alpha\in A_-$.

(2) $\mathrm{DR}(D_V(f^\alpha u_0))=j_!(\mathbf{C}f^{-\alpha}\otimes L)$ *if* $\alpha\in A_+$.

(3) $\mathrm{DR}(\mathscr{F}(D_V(f^\alpha u_0)))[n]=Rj_*^\vee i_*^\vee(\mathbf{C}f^{\vee\,-\alpha}\otimes L^\vee\otimes\mathbf{C}\omega^\vee[m])$ *if* $\alpha\in A_+$.

(4) $\mathrm{DR}(\mathscr{F}(D_V(f^\alpha u_0)))[n]=j_!^\vee i_*^\vee(\mathbf{C}f^{\vee\,-\alpha}\otimes L^\vee\otimes\mathbf{C}\omega^\vee[m])$ *if* $\alpha\in A_-$.

Corollary 6.22. (*Cf.* [G2, Theorem 4].) *Let* L *and* $L^\vee$ *be as in* (6.11), *and* $\mathscr{F}$ *denote the Sato-Fourier transformation.*

(1) $\mathscr{F}(Rj_*L[n])=j_!^\vee i_*^\vee(L^\vee\otimes\mathbf{C}\omega^\vee)[m]$.

(2) $\mathscr{F}(j_!L[n])=Rj_*^\vee i_*^\vee(L^\vee\otimes\mathbf{C}\omega^\vee)[m]$.

(3) $\mathscr{F}(Rj_*(L\otimes F^*\mathbf{C}\omega^\vee)[n])=j_!^\vee i_*^\vee L^\vee[m]$.

(4) $\mathscr{F}(j_!(L\otimes F^*\mathbf{C}\omega^\vee)[n])=Rj_*^\vee i_*^\vee L^\vee[m]$.

Moreover, for these perverse sheaves, $\mathscr{F}=\mathscr{F}^{-1}$.

Example 6.23. Let f and $f^\vee$ be as in (6.3). Let $\{f_i\}_{1\le i\le k}$ (resp. $\{f_i^\vee\}_{1\le i\le k}$) be relative invariants on V (resp. $V^\vee$) such that $f_i(v)f_i^\vee(v^\vee)$ ($(v,v^\vee)\in V\times V^\vee$) are absolutely invariant, and such that $0\notin f_i(\Omega)$ and $0\notin f_i^\vee(\Omega^\vee)$ for all i. Then (6.22) holds for $L=\mathbf{C}f_1^{\alpha_1}\cdots f_k^{\alpha_k}$ and $L^\vee=\mathbf{C}f_1^{\vee\,-\alpha_1}\cdots f_k^{\vee\,-\alpha_k}\,|\,O_1^\vee$.

6.24. Remark on $\mathbf{C}\omega^\vee$. Let $\pi^\vee:\tilde{O}_1^\vee\to O_1^\vee$ be the double covering defined by $\omega^\vee$ (cf. (6.6)), and let $L(\omega^\vee)$ denote the isotypic part of $\pi_*^\vee\mathbf{C}_{\tilde{O}_1^\vee}$ corresponding to the non-trivial character of $\mathrm{Gal}(\tilde{O}_1^\vee/O_1^\vee)$. Then $L(\omega^\vee)=\mathbf{C}\omega^\vee$. This description of $\mathbf{C}\omega^\vee$ enables us to consider an analogue of (6.22) in the category of étale $\overline{\mathbf{Q}}_l$-sheaves, which will be used in a study of character sums associated to prehomogeneous vector spaces over a finite field.

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Let us list some symbols used in [G1] (=Part I) and the present paper (=Part II). Some of them are included in both lists if the symbol is appeared in Part I and is reviewed in Part II.

List of Symbols

Part I

1.1. $G,\ \rho,\ V,\ O_0,\ v_0$
1.2. $f,\ \phi$
1.3. $n := \dim V,\ d := \deg f,\ \Omega = \Omega(f) := V \backslash f^{-1}(0)$
1.4. $O_1 = O_1(f)$
1.5. $\rho^\vee,\ V^\vee,\ \langle \mid \rangle = \langle\, ,\rangle,\ \{v_1, \cdots, v_n\},\ \{v_1^\vee, \cdots, v_n^\vee\},\ f^\vee,\ \Omega^\vee = \Omega^\vee(f^\vee) := V^\vee \backslash f^{\vee\,-1}(0),\ O_0^\vee,$
$O_1^\vee = O_1^\vee(f^\vee)$
1.6. $b(s) = b_0 s^d + b_1 s^{d-1} + \cdots + b_d$
1.8. $F := \operatorname{grad} \log f,\ F^\vee := \operatorname{grad} \log f^\vee$
1.10. $T_pX,\ (T_pX)^\perp$
1.11. $F(\Omega)_0$
1.16. $B_{v^\vee},\ G_{v^\vee}$
1.18. $(TO_1^\vee)^\perp,\ \Phi(v, v^\vee) := v + F^\vee(v^\vee),\ \Psi(v) = (v - F^\vee F(v), F(v))$
2.1.1. $A(V) = \mathbf{C}[V],\ \{x_1, \cdots, x_n\},\ D(V),\ A = A(U),\ D = D(U),\ \partial_i = \frac{\partial}{\partial x_i},\ \mathrm{ord}(P),\ D_k,\ \mathrm{gr}_k(D),\ \mathrm{gr}(D),\ \sigma = \sigma_k$
2.1.2. $P^*,\ \Omega$
2.1.3. $A_v,\ m_v,\ \tilde{A}_v,\ \tilde{m}_v,\ \hat{A}_v,\ \hat{m}_v,\ D_v,\ \tilde{D}_v,\ E_p$
2.1.4. $D_{V' \to V},\ \tilde{D}_{V' \to V, v},\ 1_{V' \to V}$
2.1.7. $M[f^{-1}]$ for a D-module M
2.2.3. $\mathrm{ch}(M)$
2.2.4. $m(C) = m(C, M)$ (multiplicity of a D-module M), $\underline{\mathrm{ch}}(M) = \mathbf{ch}(M)$
2.3.1. $D[s],\ D[s]f^s,\ Df^\alpha$
2.3.2. $N = D[s](f^2 u),\ N(\alpha) = D(f^\alpha u)$
2.3.5. $\mathbf{C}[s,t],\ D[s,t]$
2.3.6. $A_+ = A_+(c),\ A_- = A_-(c)$
2.4.1. $W,\ W_0$
2.5.1. $B(s),\ B_v(s),\ \tilde{B}_v(s),\ \hat{B}_v(s)$
2.5.7. $b_p(s)$
2.6.3. $M^* := \mathrm{Ext}_D^n(M, D) \otimes_A \Omega^{-1}$ (dual holonomic D-module)
2.7.1. $\mathscr{F}$ (Fourier transformation)
2.8.1. $\mathscr{O} = \mathscr{O}_X,\ \Omega = \Omega_X,\ \mathscr{D} = \mathscr{D}_X,\ \mathscr{O}^{\mathrm{an}} = \mathscr{O}_X^{\mathrm{an}},\ \mathscr{M}^{\mathrm{an}} = \mathscr{O}^{\mathrm{an}} \otimes_{\mathscr{O}} \mathscr{M}$
2.8.3. $\mathrm{Sol}(\mathscr{M}) := R\underline{\mathrm{Hom}}_{\mathscr{D}^{\mathrm{an}}}(\mathscr{M}^{\mathrm{an}}, \mathscr{O}^{\mathrm{an}}),\ \mathrm{DR}(\mathscr{M}) := R\underline{\mathrm{Hom}}_{\mathscr{D}^{\mathrm{an}}}(\mathscr{O}^{\mathrm{an}}, \mathscr{M}^{\mathrm{an}})$, etc.
3.5. $\phi_0 := \mathrm{trace}(\rho(A)),\ Du'_\alpha,\ Du''_\alpha$
3.12. $O_1 \xrightarrow{i} \Omega \xrightarrow{j} V,\ O_1^\vee \xrightarrow{i^\vee} \Omega^\vee \xrightarrow{j^\vee} V^\vee,\ \omega^{\vee 2}$
3.14. $\omega^\vee,\ \pi^\vee : \tilde{O}_1^\vee \to O_1^\vee,\ \tilde{\omega}^\vee$
3.15. $\tilde{F} : \tilde{O}_1 \to \tilde{O}_1^\vee$

3.17. $\delta_{\tilde{\omega}^\vee} = \delta_{\tilde{O}_1^\vee, \tilde{\omega}^\vee} = \dfrac{\tilde{\omega}^\vee \otimes 1_{\tilde{O}_1^\vee \to \tilde{\Omega}^\vee}}{\pi^{\vee *}(dy_1 \wedge \cdots \wedge dy_n)}$, $\delta_{\omega^\vee} = \delta_{O_1^\vee, \omega^\vee} = \dfrac{\omega^\vee \otimes 1_{O_1^\vee \to \Omega^\vee}}{dy_1 \wedge \cdots \wedge dy_n}$
(In [G1], $\delta_{\tilde{\omega}^\vee}$ (resp. $\delta_{\omega^\vee}$) was denoted by $\tilde{h}$ (resp. h), but we change the notation.)

3.22. $L(\alpha) = \mathbf{C}f^\alpha$, $L^\vee(\alpha) = \mathbf{C}f^{\vee\alpha}$, $H^\vee = L(\omega^\vee)$ ($L(\omega^\vee)$ is a new notation introduced in the part II.)
3.23. $m := \dim O_1 = \dim O_1^\vee$
4.1. $\mathscr{B} = \mathscr{B}_X = \{\text{hyperfunctions}\}$
4.2. V_k, $V(K)$, etc.
4.4. Ω_j, $\Omega_j^\vee$
4.7. $l = l' = l''$
4.8. $G(\mathbf{R})^+$
4.10. $h^\vee = \left| \dfrac{F^{\vee *}\omega \wedge \delta(z_{m+1}, \cdots, z_n) dz_{m+1} \wedge \cdots \wedge dz_n}{dy_1 \wedge \cdots \wedge dy_n} \right|$, $|f^\vee|_j^{-\alpha} \cdot h^\vee$ $(1 \leq j \leq 1)$
4.14. $|f|_j^\alpha$ $(1 \leq j \leq l)$

Part II

5.1. $\mathcal{O} = \mathcal{O}_X$, $\mathscr{D} = \mathscr{D}_X$, $\mathscr{D}_A = \mathscr{D}_{X,A} := \mathscr{D}_X \otimes_{\mathbf{C}} A$, $D_A = D_{X,A} := D_X \otimes_{\mathbf{C}} A$, $\mathscr{D}[s] = \mathscr{D}_X[s] := \mathscr{D}_X \otimes_{\mathbf{C}} \mathbf{C}[s]$, $\mathscr{D}[s,t] = \mathscr{D}_X[s,t] := \mathscr{D} \otimes_{\mathbf{C}} \mathbf{C}[s,t]$
5.2. $X_0 := X \setminus f^{-1}(0)$, $\mathscr{D}_X[s](f^s \underline{u}) = \Sigma_i \mathscr{D}_X[s](f^s \underline{u})_i$, $\mathscr{D}_X(f^\alpha \underline{u}) = \Sigma_i \mathscr{D}_X(f^\alpha \underline{u})_i$, $D_X[s](f^s \underline{u})$, $D_X(f^\alpha \underline{u})$, $f^s \underline{u} \mid V$, $(f^s \underline{u} \mid V)_i$, $f^\alpha \underline{u} \mid V$, $(f^\alpha \underline{u} \mid V)_i$
5.3. $b(s, \mathscr{N})$, $A_+(\mathscr{N})$, $A_-(\mathscr{N})$
5.5. $(f^\alpha, \mathscr{M})_*$, $(f^\alpha, \mathscr{M})_!$
5.7. $\mathbf{K}$ (= algebraic closure of $\mathbf{C}(s)$)
5.10. $B_{x_0}(s, \underline{u})$, $\tilde{B}_{x_0}(s, \underline{u})$, $\hat{B}_{x_0}(s, \underline{u})$, $B(s, \underline{u})$,
6.1. G, ρ, V, O_0, v_0, $\mathfrak{g}$
6.2. ϕ, f
6.3. $\rho^\vee$, $V^\vee$, $f^\vee$, $\langle v^\vee, v \rangle = \langle v, v^\vee \rangle$, $\dim V = \dim V^\vee =: n$, $\deg f = \deg f^\vee =: d$
6.4. $F := \operatorname{grad} \log f$, $F^\vee := \operatorname{grad} \log f^\vee$, $O_1 \xrightarrow{i} \Omega \xrightarrow{j} V$, $O_1^\vee \xrightarrow{i^\vee} \Omega^\vee \xrightarrow{j^\vee} V^\vee$
6.6. ω^2, ω, $\tilde{\omega}$, π, $\tilde{O}_1$, $\tilde{\Omega}$, $\tilde{F}$, $\omega^{\vee 2}$, $\omega^\vee$, $\tilde{\omega}^\vee$, $\pi^\vee$, $\tilde{O}_1^\vee$, $\tilde{\Omega}^\vee$, $\tilde{F}^\vee$,
6.8. u_ν (ν is a Lie algebra character.), $\xi^* u$
6.9. $\varphi = \varphi_y$
6.10. $\delta_{\tilde{\omega}^\vee} = \delta_{\tilde{O}_1^\vee, \tilde{\omega}^\vee}$, $\tilde{F}^* \delta_{\tilde{\omega}^\vee} = \tilde{F}^* \delta_{\tilde{O}_1, \tilde{\omega}^\vee}$, $\delta_{\omega^\vee} = \delta_{O_1^\vee, \omega^\vee}$, $F^* \delta_{\omega^\vee} = F^* \delta_{O_1^\vee, \omega^\vee}$, $\phi_0(A) = \operatorname{trace}(\rho(A))$
6.11. u_0, χ, L, $L^\vee$
6.14. $b(s, u_0)$
6.15. O^0
6.16. $b^{\mathrm{loc}}(s, u_0)$, $b^{\mathrm{loc}}(s, \mathscr{F}(u_0))$
6.17. $a(u_0)$
6.24. $L(\omega^\vee)$

Errata of [G1]

$p.896$, $\uparrow l.12$: $T_{s^{-1}v^\vee} O_1^\vee \to (T_{s^{-1}v^\vee} O_1^\vee)^\perp$.
$p.903$, $\uparrow l.15$: $\det(B_{gv}(\partial'_{i,gv^\vee}, \partial'_{j,gv^\vee})) = \det(c_{ij})^2 \det(B_{v^\vee}(\partial'_{i,v}, \partial'_{j,v^\vee}))$
$\to \det(B_{gv^\vee}(\partial'_{i,gv^\vee}, \partial'_{j,gv^\vee})) = \det(c_{ij})^2 \det(B_{v^\vee}(\partial'_{i,v^\vee}, \partial'_{j,v^\vee}))$

(In the two places, v should be replaced with $v^\vee$.)
p.903, ↑*l*.7: $dz_1^{(v)} \wedge \cdots \wedge z_m^{(v)} \to dz_1^{(v)} \wedge \cdots \wedge dz_m^{(v)}$.