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Second Order Perturbation Bounds

By

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Abstract

With a view to studying perturbation bounds, the class of functions f for which $\|df(A)\| = \|f^{(1)}(A)\|$ and $\|d^{(2)}f(A)\| = \|f^{(2)}(A)\|$, where $d^k f(A)$ (respectively $f^{(k)}(A)$) denotes the k-th Frechet (ordinary) derivative, $k = 1,2$, has been investigated.

§1. Intoroduction

Let $\mathscr{S}$ be the real space of self adjoint operators defined on a separable Hilbert space $\mathscr{H}$ and $\mathscr{S}_+$ be the subset of $\mathscr{S}$ consisting of positive operators. If I is an open interval in $\boldsymbol{R}$, let $\mathscr{S}_I$ be the set of elements of $\mathscr{S}$ with spectrum in I. Observe that $\mathscr{S}_I$ is an open convex subset of $\mathscr{S}$.

Let f be a real valued measurable function defined on I. If A in $\mathscr{S}_I$ has spectral decomposition $A = \int \lambda dE_\lambda$, where E_λ is the left continuous spectral resolution corresponding to A [6], then denote by $f(A)$ an operator in $\mathscr{S}$ defined as $f(A) = \int f(\lambda) dE_\lambda$. Thus every real measurable function f defined on I induces an operator mapping $f: \mathscr{S}_I \to \mathscr{S}$. In this note we are interested in the mappings of positive operators, so we restrict ourselves to the interval $(0, \infty)$ and all functions are from $(0, \infty)$ to itself.

Let X and Y be Banach spaces and Ω be an open subset of X. Then a mapping f from Ω into Y is Frechet differentiable at $x \in \Omega$ if there exists a map $df(x) \in \mathscr{L}(X, Y)$, the space of bounded linear maps from X to Y, such that

$$f(x+y) - f(x) - df(x)(y) = O(\|y\|).$$

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This map is called the Frechet derivative of f at x. As an element of $\mathscr{L}(X,Y)$ this inherits the norm $\|df(x)\|=\sup\{\|df(x)(y)\|:\|y\|=1\}$. If f is differentiable for all $x\in\Omega$, we have a map $x\mapsto df(x)$ from Ω into $\mathscr{L}(X,Y)$. The derivative of this map, if it exists, is called the second derivative of f at x and is denoted by $d^2f(x)$. Clearly $d^2f(x)$ is an element of $\mathscr{L}(X,(\mathscr{L}(X,Y))$ which can be identified naturally with $\mathscr{L}_2(X,Y)$, the space of bounded bilinear maps from XxX into Y equipped with norm $\|\phi\|=\inf\{\alpha:\|\phi(x,y)\|\leq\alpha\|x\|\|y\|\}$. In fact, $d^2f(x)$ is a symmetric bilinear map. The higher order derivatives can now be defined by induction. Reader may refer to [7] for systematic presentation of Frechet differential calculus and for notations we use.

Let $f^{(k)}$ denote the ordinary k-th derivative of f when it is viewed as a function on $(0,\infty)$. Define that $f\in\mathscr{D}_k$ if for any positive operator A, $\|d^kf(A)\|=\|f^{(k)}(A)\|$. Note that if I is the identity operator, then $\|d^kf(A)(I,\cdots,I)\| =\|f^{(k)}(A)\|$. Thus $\|d^kf(A)\|\geq\|f^{(k)}(A)\|$ for every positive operator A. Further it has been shown in [5] that if f is a real valued k times continuously differentiable function defined on I, then the induced operator mapping is k times Frechet differentiable, i.e., for every $A\in\mathscr{S}_I$, $d^kf(A)$ exists.

Bhatia and Sinha [4] studied the class of $\mathscr{D}_1$ functions and showed that a large class of functions are in $\mathscr{D}_1$. However, the characterisation of class $\mathscr{D}_1$ has eluded the authors. Bhatia [3] showed that operator monotone functions are in $\bigcap_{n=1}^{\infty}\mathscr{D}_n$. He points out that the problem of characterising the class $\mathscr{D}_n$ is intricate.

In this paper, we extend the techniques of [4] to study the class of functions which are in $\mathscr{D}_1\cap\mathscr{D}_2$ and show that the function $f(t)=t^p$, $t>0$ is in $\mathscr{D}_1\cap\mathscr{D}_2$ if $p\geq 4$ or if $-\infty<p<1$. Moreover, $f(t)=t^p$, $1<p<\sqrt{2}$, is in $\mathscr{D}_2$ but not in $\mathscr{D}_1$ and for $2<p<\sqrt{2}+1$, f is in D_1 but not in D_2.

Perturbation bounds for functions of positive Hilbert space operators are of immense interest to numerical analysts and operator theorists. Physicists too have evinced keen interest in the problem especially when $f(t)$ is either $t^{1/2}$ or $|t|$ or t^k, where k is a positive integer. In view of Taylor expansion which takes the form

$$f(A+H)=f(A)+df(A)(H)+\frac{1}{2}d^2f(A)(H,H)+\cdots,$$

it is evident that the estimates of $\|df(A)\|$ and $\|d^2f(A)\|$ would lead to second order perturbation bounds for the function f. Indeed, for $f\in\mathscr{D}_1\cap\mathscr{D}_2$, one has

$$\|f(A+H)-f(A)\| \leq \|f^{(1)}(A)\| \, \|H\| + \frac{1}{2}\|f^{(2)}(A)\| \, \|H\|^2 + O(\|H\|^3),$$

where A and H are positive operators. In particular, if $f(t)=t^p$, then for $p \geq 4$, we have

$$\|B^p - A^p\| \leq p\|A\|^{p-1}\|B-A\| + \frac{1}{2}p(p-1)\|A\|^{p-2}\|B-A\|^2 + O(\|B-A\|^3)$$

and if $-\infty < p < 1$, then

$$\|B^p - A^p\| \leq \|pA^{-1}\|^{1-p}\|B-A\| + \frac{1}{2}\|p(1-p)A^{-1}\|^{2-p}\|B-A\|^2 + O(\|B-A\|^3).$$

Following are members of $\mathscr{D}_1 \cap \mathscr{D}_2$ ([2], [4])

(i) The functions $f(t)=t^n$, $n=1,2,\cdots$.

(ii) The functions $f(t)=\sum_{n=0}^{\infty} a_n t^n$, $a_n \geq 0$ for all n. In particular the exponential function is in this class.

(iii) Operator monotone functions [3].

§2. The Main Results

It has been remarked that the exponential function is in $\mathscr{D}_1 \cap \mathscr{D}_2$. In what follows, we compute $d\exp(A)$ and $d^2\exp(A)$ and show that $\|d\exp(A)\| = \|\exp(A)\| = \|d^2\exp(A)\|$ ([1], [8]). Indeed,

$$d\exp(A)(B) = \int_0^1 \exp(sA)\, B \exp((1-s)A) ds$$

and

$$\|d\exp(A)\| = \|\exp(A)\| = \|\exp^{(1)}(A)\|.$$

Since

$$d^2\exp(A)(B_1, B_2) = d(d\exp(A)(B_1))(B_2)$$

$$= \lim_{t \to 0} \frac{1}{t}(d\exp(A+tB_2)(B_1) - d\exp(A)(B_1))$$

$$= \int_0^1 \int_0^1 [\alpha \exp(\alpha\beta A) B_2 \exp(\alpha(1-\beta)A) B_1 \exp((1-\alpha)A)$$
$$+ (1-\alpha)\exp(\alpha A) B_1 \exp(\beta(1-\alpha)A) B_2 \exp((1-\alpha)(1-\beta)A)] d\beta d\alpha,$$

it follows that

$$\|d^2\exp(A)(B_1, B_2)\| \leq \int_0^1 \int_0^1 [\alpha \|B_2\| \|B_1\| \|\exp(A)\| + (1-\alpha)\|B_1\| \|B_2\| \|\exp(A)\|] d\beta d\alpha$$

Consequently,

$$\|d^2\exp(A)\| \leq \|\exp(A)\| = \|\exp^{(2)}(A)\|.$$

Theorem 2.1. *Let f be a function on $(0, \infty)$ which can be written as*

$$f(t) = \int_0^\infty e^{-\lambda t} d\mu(\lambda),$$

where μ is a positive measure on $(0, \infty)$. Then $f \in \mathscr{D}_1 \cap \mathscr{D}_2$.

Proof of this Theorem follows closely on the lines of (Theorem 2.1,[4]) and the definition of the second order Frechet derivative.

Remark. For $d\mu(\lambda) = (\lambda^{p-1}/\Gamma(p))d\lambda$, $p > 0$, it is well known that

$$t^{-p} = \int_0^\infty e^{-\lambda t} d\mu(\lambda)$$

(Laplace transform of λ^{p-1}). Now it follows from Theorem 2.1 that $t^{-p} \in \mathscr{D}_1 \cap \mathscr{D}_2$ for $p > 0$.

Theorem 2.2. *Let f be a function on $(0, \infty)$ which can be written as*

$$f(t) = t^4 \int_0^\infty e^{-\lambda/t} d\mu(\lambda),$$

where μ is a positive measure on $(0,\infty)$. *Then* $f\in\mathcal{D}_1\cap\mathcal{D}_2$.

Proof. Let $g(t)=\int_0^\infty e^{-\lambda/t}d\mu(\lambda)$. As in the proof of the Theorem 2.1 we can write

$$dg(A)(B)=\int_0^\infty\int_0^\lambda A^{-1}e^{-sA^{-1}}Be^{-(\lambda-s)A^{-1}}A^{-1}ds\,d\mu(\lambda)$$

and

$$
\begin{aligned}
&d^2g(A)(B_1,B_2)=\\
&\int_0^\infty\int_0^\lambda\int_0^\alpha A^{-1}e^{-\beta A^{-1}}A^{-1}B_2A^{-1}e^{-(\alpha-\beta)A^{-1}}B_1e^{-(\lambda-\alpha)A^{-1}}A^{-1}d\beta\, d\alpha\, d\mu(\lambda)\\
&+\int_0^\infty\int_0^\lambda\int_0^{\lambda-\alpha} A^{-1}e^{-\alpha A^{-1}}B_1e^{-\beta A^{-1}}A^{-1}B_2A^{-1}e^{-(\lambda-\alpha-\beta)A^{-1}}A^{-1}d\beta\, d\alpha\, d\mu(\lambda)\\
&-\int_0^\infty\int_0^\lambda A^{-1}e^{-\alpha A^{-1}}B_1e^{-(\lambda-\alpha)A^{-1}}A^{-1}B_2A^{-1}d\alpha\, d\mu(\lambda)\\
&-\int_0^\infty\int_0^\lambda A^{-1}B_2A^{-1}e^{-\alpha A^{-1}}B_1e^{-(\lambda-\alpha)A^{-1}}A^{-1}d\alpha\, d\mu(\lambda).
\end{aligned}
$$

Since $f(A)=A^2g(A)A^2$, we have by the rule for differentiating a product

$$df(A)(B)=(AB+BA)g(A)A^2+A^2g(A)(AB+BA)+A^2dg(A)(B)A^2$$

and

$$
\begin{aligned}
d^2f(A)(B_1,B_2)=&(AB_1+B_1A)g(A)(AB_2+B_2A)+(AB_2+B_2A)g(A)(AB_1+B_1A)\\
&+(B_1B_2+B_2B_1)g(A)A^2+A^2g(A)(B_1B_2+B_2B_1)\\
&+(AB_1+B_1A)dg(A)(B_2)A^2+A^2dg(A)(B_1)(AB_2+B_2A)
\end{aligned}
$$

$$+(AB_2+B_2A)dg(A)(B_1)A^2+A^2dg(A)(B_2)(AB_1+B_1A)$$
$$+A^2d^2g(A)(B_1,B_2)A^2.$$

Hence

$$\|df(A)\|\leq 4\|A\|^3\int_0^\infty \|e^{-\lambda A^{-1}}\|d\mu(\lambda)+\|A\|^2\int_0^\infty \lambda\|e^{-\lambda A^{-1}}\|d\mu(\lambda).$$

Observe that

$$f^{(1)}(t)=4t^3g(t)+\int_0^\infty \lambda t^2e^{-\lambda/t}d\mu(\lambda)$$

is an increasing function of t. Consequently, $\|f^{(1)}(A)\|\geq\|df(A)\|$. Hence $f\in\mathscr{D}_1$.

Now

$$B_2Adg(A)(B_1)A^2=B_2\int_0^\infty\int_0^\lambda e^{-\alpha A^{-1}}B_1e^{-(\lambda-\alpha)A^{-1}}Ad\alpha d\mu(\lambda),$$

implies that

$$\|B_2Adg(A)(B_1)A^2\|\leq\|B_2\|\,\|B_1\|\int_0^\infty \lambda\|A\|\,\|e^{-\lambda A^{-1}}\|d\mu(\lambda).$$

Also

$$A^2d^2g(A)(B_1,B_2)A^2+AB_2dg(A)(B_1)A^2+A^2dg(A)(B_1)B_2A$$
$$=\int_0^\infty\int_0^\lambda\int_0^\alpha Ae^{-\beta A^{-1}}A^{-1}B_2A^{-1}e^{-(\alpha-\beta)A^{-1}}B_1e^{-(\lambda-\alpha)A^{-1}}Ad\beta d\alpha d\mu(\lambda)$$
$$+\int_0^\infty\int_0^\lambda\int_0^{\lambda-\alpha} Ae^{-\alpha A^{-1}}B_1e^{-\beta A^{-1}}A^{-1}B_2A^{-1}e^{-(\lambda-\alpha-\beta)A^{-1}}Ad\beta d\alpha d\mu(\lambda)$$

and

$$\left\| \int_0^\infty \int_0^\lambda \int_0^\alpha A e^{-\beta A^{-1}} B_2 A^{-1} e^{-(\alpha-\beta)A^{-1}} B_1 e^{-(\lambda-\alpha)A^{-1}} A d\beta d\alpha d\mu(\lambda) \right\|$$

$$\leq \|B_1\| \|B_2\| \int_0^\infty \int_0^\lambda \int_0^\alpha \|e^{-\lambda A^{-1}}\| d\beta d\alpha d\mu(\lambda)$$

$$= \|B_1\| \|B_2\| \int_0^\infty (\lambda^2/2) \|e^{-\lambda A^{-1}}\| d\mu(\lambda).$$

Thus we have

$$\|d^2 f(A)\| \leq 12 \|A\|^2 \int_0^\infty \|e^{-\lambda A^{-1}}\| d\mu(\lambda) + 6\|A\| \int_0^\infty \lambda \|e^{-\lambda A^{-1}}\| d\mu(\lambda)$$

$$+ \int_0^\infty \lambda^2 \|e^{-\lambda A^{-1}}\| d\mu(\lambda).$$

Now $f^{(2)}(t) = 12t^2 g(t) + 6t \int_0^\infty \lambda e^{-\lambda/t} d\mu(\lambda) + \int_0^\infty \lambda^2 e^{-\lambda/t} d\mu(\lambda)$ is an increasing function of t. Hence $\|f^{(2)}(A)\| \geq \|d^2 f(A)\|$ and this implies that $f \in \mathscr{D}_2$. This completes the proof.

Remarks. (i) The function $f(t) = t^p$, $p \geq 4$, is in $\mathscr{D}_1 \cap \mathscr{D}_2$. This follows from the fact that for $p > 0$,

$$t^p = \frac{1}{\Gamma(p)} \int_0^\infty \lambda^{p-1} e^{-\lambda/t} d\lambda$$

and from Theorem 2.2.

(ii) The function $f(t) = t^p$, $0 < p < 1$, being operator monotone is in $\mathscr{D}_1 \cap \mathscr{D}_2$ Our next proposition shows that $f(t) = t^{p+1} \in \mathscr{D}_2$.

Proposition 2.3. *If* $0 < p < 1$ *and* $f(t) = t^{1+p}$, $t \in (0, \infty)$, *then* $f \in \mathscr{D}_2$.

Proof. For $t>0$ and $0<p<1$, we have $t^{p-1}=\int_0^\infty \frac{d\mu(\lambda)}{\lambda+t}$, where $d\mu(\lambda)=(\sin p\pi/\pi)\lambda^{p-1}d\lambda$. Then we have $df(A)(B)=\int_0^\infty (B-C_\lambda BC_\lambda)d\mu(\lambda)$, where $C_\lambda=\lambda(\lambda I+A)^{-1}$ (See Proposition 2.5 [4]). Now it follows that

$$\begin{aligned} d^2f(A)(B_1,B_2) \\ =\int_0^\infty \lambda^2[(\lambda+A)^{-1}B_2(\lambda+A)^{-1}B_1(\lambda+A)^{-1} \\ +(\lambda+A)^{-1}B_1(\lambda+A)^{-1}B_2(\lambda+A)^{-1}]d\mu(\lambda). \end{aligned}$$

This implies that $\|d^2f(A)\|\leq \int_0^\infty 2\lambda^2\|(\lambda+A)^{-1}\|^3 d\mu(\lambda)$.

If $g(t)=1/(\lambda+t)$, then for $t>0$, it is an increasing function of t. Thus $\|(\lambda+A)^{-1}\|=1/(\lambda+\alpha)$, where $\alpha=\inf\{\langle Ax,x\rangle : \|x\|=1\}$.
Consequently,

$$\|d^2f(A)\|\leq \int_0^\infty 2\lambda^2(\lambda+\alpha)^{-3}d\mu(\lambda)=p(p+1)\alpha^{p-1}$$

(See Step III, Example 4, Section 3).

Now $f^{(2)}(t)=p(p+1)t^{p-1}$ is a decreasing function of t, it follows that $\|f^{(2)}(A)\|=f^{(2)}(\alpha)\geq\|d^2f(A)\|$. Hence $f\in\mathscr{D}_2$.

Our next result is useful in generating examples of functions which are in $\mathscr{D}_1\cap\mathscr{D}_2$.

Proposition 2.4. *Let f and g be functions defined on* $(0,\infty)$ *to itself. Assume that f and g are three times differentiable and all derivatives upto third order are positive. If f and g are in* $\mathscr{D}_1\cap\mathscr{D}_2$, *then* $f+g$, fg *and* $g\circ f$ *(the composite function) are in* $\mathscr{D}_1\cap\mathscr{D}_2$.

Proof. That $f+g$ is in $\mathscr{D}_1\cap\mathscr{D}_2$ is clear.

Differentiaing fg, we have

$$d(fg)(A)(B) = df(A)(B)g(A) + f(A)dg(A)(B)$$

and

$$d^2(fg)(A)(B_1,B_2) = d^2f(A)(B_1,B_2)g(A) + f(A)d^2g(A)(B_1,B_2)$$
$$+ df(A)(B_1)dg(A)(B_2) + df(A)(B_2)dg(A)(B_1).$$

This gives that

$$\|d(fg)(A)\| \le \|df(A)\|\,\|g(A)\| + \|f(A)\|\,\|dg(A)\|$$

and

$$\|d^2(fg)(A)\| \le \|d^2f(A)\|\,\|g(A)\| + \|f(A)\|\,\|d^2g(A)\| + 2\|df(A)\|\,\|dg(A)\|.$$

Now by the hypothesis f and g are in $\mathscr{D}_1 \cap \mathscr{D}_2$ and f, g, $f^{(1)}$, $g^{(1)}$, $f^{(2)}$ and $g^{(2)}$ are increasing functions. If $s = \sup\{\langle Ax,x\rangle : \|x\| = 1\}$, then

$$\|d(fg)(A)\| \le (fg)^{(1)}(s) = \|(fg)^{(1)}(A)\|$$

and

$$\|d^2(fg)(A)\| \le (fg)^{(2)}(s) = \|(fg)^{(2)}(A)\|.$$

Hence $fg \in \mathscr{D}_1 \cap \mathscr{D}_2$.

That $g \circ f \in \mathscr{D}_1 \cap \mathscr{D}_2$ also follows similarly, noticing that for $h = g \circ f$,

$$dh(A)(B) = dg(f(A)).df(A)(B)$$

and

$$d^2h(A)(B_1,B_2) = d^2g(f(A)(df(A)(B_1), df(A)(B_2))$$
$$+ dg(f(A))(d^2f(A)(B_1,B_2)).$$

§3. Examples

1. If p and q are polynomials with positive coefficients, then $p(t)e^{q(t)}$, $t \in (0,\infty)$ is in $\mathscr{D}_1 \cap \mathscr{D}_2$ (Proposition 2.4).

2. The function $f(t) = t + t^{-1}$, $t \in (0,\infty)$ is in $\mathscr{D}_2$ but not in $\mathscr{D}_1$. This follows from (Example 2.B, [4]) and that

$$d^2f(A)(B_1,B_2)=A^{-1}B_2A^{-1}B_1A^{-1}+A^{-1}B_1A^{-1}B_2A^{-1}.$$

3. Let $f(t)=t^{p+1}$, $t\in(0,\infty)$ and $0<p<\sqrt{2}-1$. Then it is shown in (Example 2.A, [4]) that $f(t)$ is not in $\mathscr{D}_1$. Further Proposition 2.3 shows that it is in $\mathscr{D}_2$.

4. Let $0<p<1$. It is shown in [4] that $f(t)=t^{2+p}$, $t\in(0,\infty)$ is in $\mathscr{D}_1$. Now we show that for some values of p it is not in $\mathscr{D}_2$. This we shall accomplish in several steps.

Step I. Writing $f(A)=\frac{1}{2}(Ah(A)+h(A)A)$, where $h(t)=t^{p+1}$ and using (Proposition 2.5 [4]) we have

$$df(A)(B)=\frac{1}{2}[Bh(A)+h(A)B+dh(A)(B)A+Adh(A)(B)]$$

and

$$d^2f(A)(B_1,B_2)=\frac{1}{2}[B_1dh(A)(B_2)+B_2dh(A)(B_1)+dh(A)(B_2)B_1+dh(A)(B_1)B_2$$
$$+Ad^2h(A)(B_1,B_2)+d^2h(A)(B_1,B_2)A],$$

where

$$h(A)=A\int_0^\infty(\lambda+A)^{-1}d\mu(\lambda)A,\quad dh(A)(B)=\int_0^\infty(B-C_\lambda BC_\lambda)d\mu(\lambda),$$

$$d^2h(A)(B_1,B_2)=\int_0^\infty\frac{1}{\lambda}[C_\lambda B_2C_\lambda B_1C_\lambda+C_\lambda B_1C_\lambda B_2C_\lambda]d\mu(\lambda),$$

$$d\mu(\lambda)=(\sin p\pi/\pi)\lambda^{p-1}d\lambda \quad\text{and}\quad C_\lambda=\lambda(\lambda I+A)^{-1}.$$

Step II. Now we compute $d^2f(A)(B,I)$, where $A=P+\varepsilon Q$, P and Q are complementary projections and ε is positive. From Step I we have

$$(C_\lambda A)/\lambda=A(A+\lambda)^{-1}=I-C_\lambda$$

and

$$d^2f(A)(B,I)=\int_0^\infty(2B-C_\lambda BC_\lambda^2-C_\lambda^2BC_\lambda)d\mu(\lambda).$$

Using that $C_\lambda = \frac{\lambda}{\lambda+1}P + \frac{\lambda}{\lambda+\varepsilon}Q$, we write

$d^2f(A)(B,I) =$

$$\int_0^\infty 2\left[1 - \frac{\lambda^3}{(\lambda+1)^3}\right]d\mu(\lambda)PBP + \int_0^\infty \left[1 - \frac{\lambda^3}{(\lambda+1)(\lambda+\varepsilon)^2}\right]d\mu(\lambda)(PBQ+QBP)$$

$$\int_0^\infty \left[1 - \frac{\lambda^3}{(\lambda+\varepsilon)(\lambda+1)^2}\right]d\mu(\lambda)(PBQ+QBP) + \int_0^\infty 2\left[1 - \frac{\lambda^3}{(\lambda+\varepsilon)^3}\right]d\mu(\lambda)QBQ.$$

Step III. We next compute the above integrals. Using the representation

$t^{p-1} = \int_0^\infty \frac{d\mu(\lambda)}{\lambda+t}$, $0<p<1$, $t>0$, it follows that $\int_0^\infty \frac{d\mu(\lambda)}{(\lambda+t)^2} = (1-p)t^{p-2}$ and

$\int_0^\infty \frac{d\mu(\lambda)}{(\lambda+t)^3} = \frac{1}{2}(1-p)(2-p)t^{p-3}$. Elementary calculations show that:

$$\int_0^\infty \left[1 - \frac{\lambda^3}{(\lambda+1)^3}\right]d\mu(\lambda) = \frac{(p+1)(p+2)}{2}$$

$$\int_0^\infty \left[1 - \frac{\lambda^3}{(\lambda+1)(\lambda+\varepsilon)^2}\right]d\mu(\lambda) = \frac{1-\varepsilon^{p+1}((p+1)(1-\varepsilon)+1)}{(1-\varepsilon)^2}$$

$$\int_0^\infty \left[1 - \frac{\lambda^3}{(\lambda+\varepsilon)(\lambda+1)^3}\right]d\mu(\lambda) = \frac{\varepsilon^{p+2}+(p+2)(1-\varepsilon)-1}{(1-\varepsilon)^2}$$

$$\int_0^\infty \left[1 - \frac{\lambda^3}{(\lambda+\varepsilon)^3}\right]d\mu(\lambda) = \varepsilon^p\frac{(p+1)(p+2)}{2}$$

Step IV. On the space $H = C^2$, consider the following matrices:

$A=\begin{bmatrix}1 & 0\\ 0 & \varepsilon\end{bmatrix}$, $0<\varepsilon<1$, $B=\begin{bmatrix}1 & b\\ b & -1\end{bmatrix}$, $b>0$. Then we have that $d^2f(A)(B,I)=\begin{bmatrix}\alpha & \beta\\ \beta & \chi\end{bmatrix}$, where $\alpha=(p+1)(p+2)$, $\chi=-\varepsilon^p(p+1)(p+2)$ and $\beta=\dfrac{\varepsilon^{p+2}+(p+2)(1-\varepsilon)-\varepsilon^{p+1}((p+1)(1-\varepsilon)+1)}{(1-\varepsilon)^2}b.$

If we call this matrix as $X(\varepsilon)$, then

$$X(0)=\begin{bmatrix}(p+1)(p+2) & (p+2)b\\ (p+2)b & 0\end{bmatrix}.$$

It follws that

$$\|X(0)\|=(p+2)((p+1)+\sqrt{((p+1)^2+4b^2)}).$$

Also

$$\|f^{(2)}(A)\|\,\|B\|\,\|I\|=(p+1)(p+2)\|A\|^{p+1}(1+b^2)^{1/2}.$$

After some algebraic manipulation, we have that

$$\|X(0)\|>\|f^{(2)}(A)\|\,\|B\|\,\|I\| \text{ if and only if } p^2+2p<(1+b^2)^{-1/2}.$$

When b is near zero, the right hand side of the above inequality is near 1. So, in this case, the inequality is true if $0<p<\sqrt{2}-1$. Thus for $0<p<\sqrt{2}-1$, t^{2+p} does not belong to $\mathscr{D}_2$.

5. Let

$$f(\lambda)=\begin{cases}0 & 0<\lambda<\alpha\\ e^{-\beta(\lambda-\alpha)} & \lambda>\alpha\end{cases}$$

where $\alpha,\beta>0$. Then

$$\int_0^\infty e^{-\lambda t}f(\lambda)d\lambda=(t+\beta)^{-1}e^{-\alpha t}.$$

Using Theorem 1, Section 2, we get $(t+\beta)^{-1}e^{-\alpha t}\in\mathscr{D}_1\cap\mathscr{D}_2$.

§4. Concluding Remark

Let $f(t)=t^p$, $t>0$. Then $f(t)\in\mathscr{D}_1\cap\mathscr{D}_2$ if $p\geq 4$ or $-\infty<p<1$. If $1<p<\sqrt{2}$, then f is in $\mathscr{D}_2$ but not in $\mathscr{D}_1$ and if $2<p<\sqrt{2}+1$, then f is in $\mathscr{D}_1$ but not in $\mathscr{D}_2$. Here our results may be compared with those obtained by Bhatia and Sinha [4]. It would be interesting to know the status of the functions involved for remaining values of p.

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