Local Uniqueness in the Cauchy Problem for Second Order Elliptic Equations with Non-Lipschitzian Coefficients

By

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Abstract

We show the local uniqueness of the Cauchy problem for the second order elliptic operators whose coefficients of the principal part are real-valued and continuous with some modulus of continuity. These coefficients are not necessarily lipschitz continuous. The proof is given by drawing the Carleman estimates with a weight attached to the modulus of continuity.

§1. Introduction

For the second order elliptic operator with real coefficients, the local uniqueness in the Cauchy problem holds if the coefficients of second order terms are lipschitz continuous (see L. Hörmander [H1] or [H2]). On the other hand, according to the example of A. Pliś ([P]), if the coefficients are only Hölder continuous with the index strictly smaller than 1, the assertion above dose not hold in general. In this paper we introduce some class of continuous functions which contains some non-lipschitz continuous functions and show that if all the coefficients of second order terms belong to this class, the local uniqueness holds.

In order to define a class of continuous functions, we first introduce a function by which we measure the modulus of continuity. We assume that the positive and nondecreasing continuous function $\chi(s)$ defined on the interval $[1, +\infty)$ satisfies the following: there exists a constant $C > 0$ such that on $[1, +\infty)$

(M.1) $$\chi(2s) \leq C\chi(s),$$

next

(M.2)
$$\int_1^{+\infty} \frac{1}{s\chi(s)}\,ds = +\infty,$$

and finally with some constants $\delta \in (0,1)$ and $C > 0$

(M.3)
$$|\chi(s)| \le Cs^\delta \quad \text{for } s \ge 1.$$

Remark. We see that, by choosing large positive constants C_1, C_2 and C_3, the functions $\log(C_1 s)$ and $\log(C_2 s)\log(\log(C_3 s))$ satisfy (M.1), (M.2) and (M.3).

Using the positive and nondecreasing continuous function $\chi(r)$ that satisfies (M.1), (M.2) and (M.3), we define a function space C^χ. Let Ω be an open set in R^{d+1}. We denote by $C^\chi(\Omega)$ the space of all continuous $f(x)$ on Ω satisfying that for any compact set K in Ω, there exist two constants $\epsilon \in (0,1]$ and $C > 0$ such that for any $x, y \in K$ satisfying $|x - y| \le \epsilon$,

$$|f(x) - f(y)| \le C|x - y|\chi\left(\frac{1}{|x - y|}\right).$$

Let Ω be an open set in R^{d+1}. We consider an elliptic operator $E(x, D_x)$ defined by

$$E(x, D_x)u(x) = \sum_{j,k=0}^d a_{j,k}(x)D_{x_j}D_{x_k}u(x) + \sum_{j=0}^d a_j(x)D_{x_j}u(x) + c(x)u(x)$$

where $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$ and the coefficients satisfy the following;
The coefficients $a_{j,k}(x)$ of second order terms are real-valued and belong to $C^\chi(\Omega)$. Furthermore, for any compact set K of Ω, there exists a positive constant C_K such that

(1.1)
$$\sum_{j,k=0}^d a_{j,k}(x)\xi_j\xi_k \ge C_K|\xi|^2$$

for any $x \in K$ and any $(\xi_0, \cdots, \xi_d) \in R^{d+1}$. And

$$a_j(x), \; c(x) \in L^\infty_{loc}(\Omega),$$

that is to say, they are locally essentially bounded.

Under the assumptions above we have the following;

Theorem 1.1. *Let x^0 be a point of Ω, V a neighborhood of x^0 and N a hypersurface in V defined by $\{x \in V \mid \theta(x) = 0\}$ where $\theta(x) \in C^2(V)$ satisfies $\nabla\theta(x) \neq 0$ and $\theta(x^0) = 0$. Then there exists an open neighborhood W of x^0 such that for any $u(x) \in H^2(V)$ that vanishes on $\{x \in V \mid \theta(x) < 0\}$ and satisfies on V*

$$E(x, D_x)u(x) = 0,$$

we have $u(x) = 0$ on W.

Here we denote by $H^2(V)$, where V is an open set $\subset R_x^{d+1}$, the space that consists of all $u(x)$ in $L^2(V)$ whose first and second order derivatives also belong to $L^2(V)$, where $L^2(V)$ is the space of square integrable functions on V.

By a C^2-change of coordinates, we may assume that $\theta(x) = x_0$ and $x^0 = 0$. Note that the function classes $H^2_{loc}(V)$, that is a space of all $f(x)$ satisfying $\epsilon(x)f(x) \in H^2(V)$ for any compactly supported smooth $\epsilon(x)$ on V, and C^χ are invariant under any C^2-change of coordinates. Furthermore, Holmgren's transformation (see L. Nirenberg [Ni, §7, page 29 and 31] or C. Zuily [Z, page 43]) implies that in order to prove Theorem 1.1 we have only to prove the following Theorem 1.2. Indeed after Holmgren's transformation, we may assume that there exist an open neighborhood V of the origin and $t_0 > 0$ such that $u(x) \in H^2(V)$, $E(x, D_x)u(x) = 0$ in V and $u(x) = 0$ on $V \cap \{(x_0, x') \in R^{d+1} \mid x_0 < t_0$ and $x_0 - |x'|^2 < 0\}$ where $\{(x_0, x') \in R^{d+1} \mid |x_0| < t_0, |x'|^2 < t_0\}$ is relatively compact in V. Thus, since $h(x_0)u(x)$, where $h(x_0) \in C^\infty(R)$ verifies $h(x_0) = 0$ for $[\frac{2}{3}t_0, +\infty)$ and $h(x_0) = 1$ on $(-\infty, \frac{1}{2}t_0]$, satisfies, by setting $t = x_0$ and $x = x'$, the assumption of Theorem 1.2, we see $u(x) = 0$ for $x_0 < \frac{1}{3}t_0$.

Theorem 1.2. *Let $E(t, x, D_t, D_x)$ be a second order elliptic operator on $R_t \times R_x^d$ defined by*

$$(1.2) \quad E(t, x, D_t, D_x) = D_t^2 + 2\sum_{j=1}^d a_{0,j}(t, x)D_t D_{x_j} + \sum_{j,k=1}^d a_{j,k}(t, x)D_{x_j}D_{x_k}$$

$$+ a_0(t, x)D_t + \sum_{j=1}^d a_j(t, x)D_{x_j} + c(t, x)$$

where the coefficients satisfy the following;

(1) *All the coefficients are constant outside some compact set K in R^{d+1}.*

(2) *For $1 \leq j$, $k \leq n$, $a_{0,j}(t, x)$ and $a_{j,k}(t, x)$ are real-valued and belong to $C^\chi(R^{d+1})$.*

(3) $a_0(t,x)$, $a_j(t,x)$ for $j=1,\cdots,d$ and $c(t,x)$ are in $L^\infty(\mathbf{R}^{d+1})$

(4)   There exists a constant $C>0$ such that for any $\tau \in \mathbf{R}$, any $(\xi_1,\cdots,\xi_d)\in \mathbf{R}^d$ and any $(t,x)\in \mathbf{R}^{d+1}$

$$\tau^2 + 2\sum_{j=1}^{d} a_{0,j}(t,x)\tau\xi_j + \sum_{j,k=1}^{d} a_{j,k}(t,x)\xi_j\xi_k \geq C(|\tau|^2 + |\xi|^2).$$

Let t_0 be a positive number.   Assume that $u\in H^2(\mathbf{R}^{d+1})$ whose support is contained in $[0,\frac{2}{3}t_0]\times \mathbf{R}^d_x$ satisfies

$$E(t,x,D_t,D_x)u(t,x)=f(t,x)$$

where $f(t,z)\in L^2(\mathbf{R}^{d+1})$ vanishes for $t<\frac{1}{2}t_0$.   Then we have $u(t,x)=0$ for $t<\frac{1}{3}t_0$.

Here $L^\infty(\mathbf{R}^{d+1})$ is the space of all essentially bounded functions on $\mathbf{R}^{d+1}$. The proof of Theorem 1.2 is given in Section 4 after two preparations.   First, in the next section, we define the regularization of functions in C^χ and show their properties.   As for the second one we draw the Carleman estimates for some elliptic first order equations in Section 3.

Remark.   The function class C^χ is studied by H. Bahouri and J.-Y. Chemin ([B-C]) in the context of fluid dynamics.   Furthermore F. Colombini and N. Lerner ([C-L]) studied the Cauchy problem for second order strictly hyperbolic operators with log-lipschitz continuous coefficients, that is to say belonging to C^χ with $\chi(r)=\log(2r)$.

In the following sections, we use the notation of the multi-index α which is a d-tuple of non-negative integers $(\alpha_1,\cdots,\alpha_d)$.   We set $|\alpha|=\Sigma_{j=1}^d \alpha_j$.   The space $C_0^\infty(\mathbf{R}^d)$ is the space of all compactly supported smooth functions on $\mathbf{R}^d$.   We denote by $\|\cdot\|$ [resp. $\|\|\cdot\|\|$] the L^2-norm in $\mathbf{R}^d_x$ [resp. $\mathbf{R}_t\times \mathbf{R}^d_x$] that is to say;

$$\|f(x)\|^2 = \int_{\mathbf{R}^d} |f(x)|^2 dx,$$

[resp.

$$\|\|f(t,x)\|\|^2 = \int_{\mathbf{R}^{d+1}} |f(t,x)|^2 dt\, dx.]$$

Set $\langle\xi\rangle=\sqrt{|\xi|^2+1}$.

For a symbol $a(t,x,\xi)$, which does not necessarily depend on t or x, we

denote by $a(t,x,D_x)$ the pseudodifferential operator defined by

$$a(t,x,D_x)f(x)=\frac{1}{(2\pi)^d}\int_{R^d}e^{i\langle x,\xi\rangle}a(t,x,\xi)\hat{f}(\xi)d\xi$$

where $\hat{f}(\xi)$ is the Fourier transform of $f(x)$, that is to say

$$\hat{f}(\xi)=\int_{R^d}e^{-i\langle\xi,x\rangle}f(x)dx.$$

Furthermore we denote by C or suffixed C_* some positive constant which may be different line by line.

§2. Preliminaries

In this section, we define the regularization of functions in C^χ and show its properties.

Let a nonnegative and nonincreasing function $\Psi(y)\in C^\infty(R)$ satisfy

$$0\leq\Psi(y)\leq 1,$$

$$\Psi(y)=\begin{cases}1, & \text{for } y\leq 1 \\ 0, & \text{for } y\leq 2.\end{cases}$$

We define a sequence of functions $\{\psi_j(y)\}$ in the following way:

$$\psi_0(y)=\Psi(y)$$

$$\psi_j(y)=\Psi(2^{-j}y)-\Psi(2^{-j+1}y) \quad \text{for } j\geq 1.$$

Then we see that, for $j\geq 1$, the support of $\psi_j(y)$ is contained in $\{y\,|\,2^{j-1}\leq y\leq 2^{j+1}\}$ and that

(2.1) $$|\frac{d^k}{dy^k}\psi_j(y)|\leq C2^{-jk}$$

where the constant C may depend on k.

For a positive and nondecreasing continuous function $\chi(s)$ satisfying (M.1), (M.2) and (M.3), we define a nondecreasing function $\tilde{\chi}(s)$ by

$$\tilde{\chi}(s)=\sum_{j=0}^{+\infty}\psi_j(s)\chi(2^j).$$

The property (M.1) implies that

$$C^{-1} \le \frac{\tilde{\chi}(s)}{\chi(s)} \le C \quad \text{on } [1, +\infty),$$

which shows that $C^\chi = C^{\tilde{\chi}}$. We see also that $\tilde{\chi}(s)$ satisfies (M.2) and (M.3). Furthermore from (2.1) we obtain

(M.4) $$|\frac{d^n}{ds^n}\tilde{\chi}(s)| \le C(1+|s|)^{-n}\tilde{\chi}(s)$$

where the constant C may depend on n.

Therefore from now on we assume that the function $\chi(s)$ is a positive and nondecreasing smooth function enjoying the properties (M.1), (M.2), (M.3) and (M.4).

Next we define the regularization $\tilde{a}(t,x,\xi)$ of a function $a(t,x) \in C^\chi(R^{d+1})$ by using the functions above $\psi_j(y)$ and a nonnegative function $\gamma(w) \in C_0^\infty(R^{d+1})$ satisfying

$$\int_{R^{d+1}} \gamma(w)dw = 1$$

and

$$\gamma(w) = 0 \quad \text{if } |w| < 1 \text{ or } |w| > 2.$$

Set

(2.2) $$\rho_j = (2^j \chi(2^j))^{\frac{1}{2}}.$$

We define $\tilde{a}(t,x,\xi)$ by

$$\tilde{a}(t,x,\xi) = \sum_{j=0}^{+\infty} \psi_j(|\xi|)\rho_j^{d+1} \int_{R^{d+1}} \gamma(\rho_j(t-s), \rho_j(x-y))a(s,y)ds\,dy.$$

Then we see the following:

Proposition 2.1. *For any $a(t,x) \in C^\chi(R^{d+1})$ that is constant out of a compact set, we have the following:*

(2.3) $$|\tilde{a}(t,x,\xi) - a(t,x)| \le C\langle\xi\rangle^{-\frac{1}{2}} (\chi(\langle\xi\rangle))^{\frac{1}{2}},$$

and for $\alpha_0 + |\alpha| + |\beta| > 0$

(2.4)
$$|\frac{\partial^{\alpha_0}}{\partial t^{\alpha_0}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\beta}}{\partial \xi^{\beta}} \tilde{a}(t, x, \xi)|$$

$$\leq C\langle\xi\rangle^{\frac{|\alpha_0| + |\alpha| - 1}{2}} (\chi(<\xi>))^{\frac{|\alpha_0| + |\alpha| + 1}{2}} \langle\xi\rangle^{-|\beta|}$$

And the operator

$$(\tilde{a}(t, x, D_x) - a(t, x))\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{-\frac{1}{2}}$$

is an L^2 bounded operator and strongly continuous with respect to t.

Proof. In this proof C represents an arbitrary constant which is independent of j. Since $a(t,x) \in C^\chi(R^{d+1})$, $\chi(s)$ is nondecreasing and $\rho_j \leq 2^j$ for large j, which follows from (2.2) and (M.3), then we see that, if $(\rho_j(s-t), \rho_j(y-x)) \in \text{supp } \gamma(w)$, then

(2.5)
$$|a(s,y) - a(t,x)| \leq C\rho_j^{-1}\chi(\rho_j)$$
$$\leq C(2^{-j}\chi(2^j))^{\frac{1}{2}},$$

where we used the assumption that $a(t,x)$ is constant out of a compact set and $\chi(\rho_j) \leq C\chi(2^j)$ which follows from (M.1) and from $\rho_j \leq 2^j$ for large j. Thus we see (2.3).

Next we remark that the estimate (2.1) implies, for $j \geq 1$,

(2.6)
$$|\frac{\partial^{\beta}}{\partial \xi^{\beta}} \psi_j(|\xi|)| \begin{cases} \leq C\langle\xi\rangle^{-|\beta|} & \text{on } \{\xi \mid 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \\ = 0 & \text{otherwise.} \end{cases}$$

If $|\alpha_0| + |\alpha| > 0$, since $\int \frac{\partial^{\alpha_0}}{\partial s^{\alpha_0}} \frac{\partial^{\alpha}}{\partial y^{\alpha}} \gamma(s,y) ds\, dy = 0$, we have

$$\int_{R^{d+1}} \left(\frac{\partial^{\alpha_0}}{\partial s^{\alpha_0}} \frac{\partial^{\alpha}}{\partial y^{\alpha}} \gamma(\rho_j(t-s), \rho_j(x-y)) \right) a(s,y) ds\, dy$$

$$= \int_{R^{d+1}} \left(\frac{\partial^{\alpha_0}}{\partial s^{\alpha_0}} \frac{\partial^{\alpha}}{\partial y^{\alpha}} \gamma(\rho_j(t-s), \rho_j(x-y)) \right) (a(s,y) - a(t,x)) ds\, dy.$$

From (2.5) we see that the absolute value of the right hand side of the equation above is equal or inferior to $C\rho_j^{-(d+2)}\chi(\rho_j)$. Since $\chi(\rho_j) \leq C\chi(2^j)$ and $|\alpha_0| + |\alpha| > 0$,

using (2.2), we get

$$\rho_j^{|\alpha_0|+|\alpha|}\rho_j^{-1}\chi(\rho_j) \le C(2^{\frac{1}{2}j})^{|\alpha_0|+|\alpha|-1}(\chi(2^j))^{\frac{|\alpha_0|+|\alpha|+1}{2}},$$

from which and from (2.6) follows the estimate (2.4) for $|\alpha_0|+|\alpha|>0$. In the case where $|\beta|>0$ and $|\alpha_0|+|\alpha|=0$, since $\sum_{j=0}^{+\infty}\psi_j(|\xi|)=1$ and $\rho_j^{d+1}\int\gamma(\rho_j(t-s),$ $\rho_j(x-y))ds\,dy=1$, we see

$$\frac{\partial^\beta}{\partial\xi^\beta}\tilde{a}(t,x,\xi) = \sum_{j=0}^{+\infty}\frac{\partial^\beta}{\partial\xi^\beta}\psi_j(|\xi|)\rho_j^{d+1}\int_{\mathbf{R}^{d+1}}\gamma(\rho_j(t-s),\rho_j(x-y))(a(s,y)-a(t,x))ds\,dy,$$

from which and (2.5) we see the estimate of (2.4) in this case.

Finally we show the L^2-boundedness of the operator

(2.7) $$(\tilde{a}(t,x,D_x)-a(t,x))\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{-\frac{1}{2}}$$

by using the method of M. Nagase ([Na]).

Put

$$A(t,x,\xi)=(\tilde{a}(t,x,\xi)-a(t,x))\langle\xi\rangle^{\frac{1}{2}}(\chi(\langle\xi\rangle))^{-\frac{1}{2}}.$$

Then we see from (2.3), (2.4) and (M.4)

$$\left|\frac{\partial^\beta}{\partial\xi^\beta}A(t,x,\xi)\right| \le C\langle\xi\rangle^{-|\beta|}.$$

On the other hand we obtain from (M.4), for any β,

$$\left|\frac{\partial^\beta}{\partial\xi^\beta}(A(t,x_1,\xi)-A(t,x_2,\xi))\right|$$

$$\le C|x_1-x_2|\chi(|x_1-x_2|^{-1})\langle\xi\rangle^{\frac{1}{2}}(\chi(\langle\xi\rangle))^{-\frac{1}{2}}\langle\xi\rangle^{-|\beta|}.$$

Therefore for the symbol $\tilde{A}(t,x,\xi)$ defined by

$$\tilde{A}(t,x,\xi)=\langle\xi\rangle^{d\mu}\int_{\mathbf{R}^d}\gamma_1(\langle\xi\rangle^\mu(x-y))A(t,y,\xi)dy$$

where $\mu=\frac{\delta+2}{3}$ with the constant δ in (M.3) and a function $\gamma_1(y)\in C_0^\infty(\mathbf{R}^d)$ satisfies $\gamma_1(y)=0$ out of $\{y\,|\,1\le|y|\le2\}$ and

$$\int_{\mathbf{R}^d}\gamma_1(y)dy=1,$$

we have the following; since we see, from $0 < \mu < 1$, $\chi(\langle\xi\rangle^\mu) \le \chi(\langle\xi\rangle))$, then for any β

$$(2.8) \qquad |\frac{\partial^\beta}{\partial\xi^\beta}(\tilde{A}(t,x,\xi) - A(t,x,\xi))| \le C\langle\xi\rangle^{-\mu+\frac{1}{2}}(\chi(\langle\xi\rangle))^{1-\frac{1}{2}}\langle\xi\rangle^{-|\beta|},$$

since $\chi(\langle\xi\rangle) \le C\langle\xi\rangle^\delta$ and $\mu = \frac{\delta+2}{3}$,

$$\le C\langle\xi\rangle^{\frac{\delta-1}{6}}\langle\xi\rangle^{-|\beta|},$$

and

$$\tilde{A}(t,x,\xi) \in S_{1,\mu}^0.$$

Here we denote by $S_{\rho,\kappa}^m$ the set of all continuous $s(t,x,\xi)$ on $\mathbf{R}_t \times \mathbf{R}^{2d}$ satisfying for any α and β,

$$|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial\xi^\beta}s(t,x,\xi)| \le C\langle\xi\rangle^{m-\rho|\beta|+\kappa|\alpha|}$$

on $\mathbf{R}_t \times \mathbf{R}^{2d}$. Since $\delta < 1$, we have $\mu < 1$. Thus the operator $\tilde{A}(t,x,D_x)$ is an L^2 bounded operator with the continuous parameter t. Since $\delta - 1 < 0$, using the result of M. Nagase [Na, Theorem 2 and 3], we see that the estimate (2.8) implies that $\tilde{A}(t,x,D_x) - A(t,x,D_x)$ is an L^2 bounded operator with the continuous parameter t. Hence the operator $A(t,x,D_x)$ is an L^2 bounded operator with the continuous parameter t. The proof of Proposition 2.1 is completed.

§3. Carleman Estimates for First Order Operators

We introduce two classes of symbols. We say that a smooth function $p(t,x,\xi)$ [resp. $p(x,\xi)$] belongs to $S_{1,*}^m$ [resp. $S_{*,*}^0$] if we have, for any α_0, α and β,

$$|\frac{\partial^{\alpha_0}}{\partial t^{\alpha_0}}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial\xi^\beta}p(t,x,\xi)| \le C(\langle\xi\rangle^{\frac{1}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}})^{|\alpha_0|+|\alpha|}\langle\xi\rangle^{m-|\beta|}$$

on $\mathbf{R}_t \times \mathbf{R}^{2d}$ [resp. if we have, for any α and β,

$$|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial\xi^\beta}p(x,\xi)| \le C_{\alpha,\beta}(\langle\xi\rangle\chi(\langle\xi\rangle))^{\frac{|\alpha|-|\beta|}{2}}$$

on $\mathbf{R}^{2d}$.]

From (M.3) we see $S^m_{1,*} \subset S^m_{1,1+\delta}$. Since (M.4) implies that there exists two positive constant c and C such that

$$C^{-1} \leq \frac{\chi(\xi)}{\chi(\eta)} \leq C \quad \text{if } |\xi - \eta| \leq c|\xi|,$$

we see that $\Phi(\xi) = \langle \xi \rangle$ or $\Phi(\xi) = (\langle \xi \rangle \chi(\langle \xi \rangle))^{\frac{1}{2}}$ and $\varphi(\xi) = (\langle \xi \rangle \chi(\langle \xi \rangle))^{-\frac{1}{2}}$ are weights in a sense of R. Beals and C. Fefferman ([B-F] or see R. Beals [B] or H. Kumano-go [K, Ch. 7]) and then we can apply the results on the calculus of pseudodifferential operators given by R. Beals and C. Fefferman ([B-F], see also R. Beals [B] and H. Kumano-go [K. Ch. 7] for the calculus of the operators associated to $S^m_{1,*}$ or $S^0_{*,*}$.) Especially the pseudodifferential operators whose symbols are in $S^0_{1,*}$ or $S^0_{*,*}$ are L^2-bounded operators that are strongly continuous with respect to the parameter t. Next we remark that, since the estimate (M.3) implies $\chi(\langle \xi \rangle) \leq C\langle \xi \rangle^{\frac{1}{2}} \chi(\langle \xi \rangle)^{\frac{1}{2}}$, the symbol $\tilde{a}(t,x,\xi)$ defined in Section 2 belongs to $S^0_{1,*}$.

Now we define the weight function $\phi(s)$ with which we draw the Carleman estimate.

Since $\chi(s)$ is continuous and positive, the function $r(s)$ on $[1, +\infty)$ defined by

$$r(s) = \int_1^s \frac{1}{s\chi(s)} ds$$

is strictly increasing and (M.2) means $r(s) \to +\infty$ as $s \to +\infty$. Hence $r(s)$ has the C^1 inverse function $s(r)$ defined on $[0, +\infty)$. And we define $\phi(r)$ by

$$\phi(r) = \int_0^r s(r) dr.$$

By the definition we see that

(3.1) $\phi''(r) = s\chi(s) \quad \text{if } \phi'(r) = s.$

Since $\chi(s)$ is nondecreasing and $\phi'(0) = 1$, we see that

(3.2) $\phi''(r) \geq \chi(1).$

Proposition 3.1. *For real-valued symbols* $p(t,x,\xi)$ *and* $q(t,x,\xi)$ *in* $S^1_{1,*}$ *satisfying*

(3.3) $|q(t,x,\xi)| \geq C\langle \xi \rangle,$

$$(3.4) \qquad \chi(\langle\xi\rangle)^{-1}\frac{\partial}{\partial t}p(t,x,\xi), \ \chi(\langle\xi\rangle)^{-1}\frac{\partial}{\partial t}q(t,x,\xi)\in S_{1,*}^{1},$$

and for $1\leq j\leq d$

$$(3.5) \qquad \chi(\langle\xi\rangle)^{-1}\frac{\partial}{\partial x_{j}}p(t,x,\xi), \ \chi(\langle\xi\rangle)^{-1}\frac{\partial}{\partial x_{j}}q(t,x,\xi)\in S_{1,*}^{1},$$

we have the following. For any $t_{0}>0$ there exists a positive integer n_{0} such that for any $v(t,x)$ in $C_{0}^{\infty}(\mathbf{R}^{d+1})$ whose support is a compact subset of $[0,t_{0}]\times\mathbf{R}^{d}$, and for any $n\geq n_{0}$

$$C_{0}n^{\frac{1}{2}(1-\delta)}\||e^{\frac{1}{n}\phi(n(2t_{0}-t))}(\chi(\langle D_{x}\rangle)\langle D_{x}\rangle)^{\frac{1}{2}}v(t,x)\||^{2}$$

$$\leq\||e^{\frac{1}{n}\phi(n(2t_{0}-t))}(D_{t}+p(t,x,D_{x})-iq(t,x,D_{x}))v(t,x)\||^{2},$$

where the constant C_{0} is independent of n and, of course, of $v(t,x)$.

Proof. In this proof any positive constant C is assumed to be independent of n. Since $p(t,x,\xi)$ and $q(t,x,\xi)$ are real-valued symbols in $S_{1,*}^{1}$,

$$p_{a}(t,x,D_{x})=\frac{1}{2}(p(t,x,D_{x})-p^{*}(t,x,D_{x}))$$

and

$$q_{a}(t,x,D_{x})=\frac{1}{2}(q(t,x,D_{x})-q^{*}(t,x,D_{x})),$$

are pseudodifferential operators whose symbols $p_{a}(t,x,\xi)$ and $q_{a}(t,x,\xi)$ satisfy

$$(3.6) \qquad p_{a}(t,x,\xi)(\chi(\langle\xi\rangle)\langle\xi\rangle)^{-\frac{1}{2}}\in S_{1,*}^{0}$$
$$q_{a}(t,x,\xi)(\chi(\langle\xi\rangle)\langle\xi\rangle)^{-\frac{1}{2}}\in S_{1,*}^{0},$$

where $p^{*}(t,x,D_{x})$ [resp. $q^{*}(t,x,D_{x})$] is the formal adjoint of $p(t,x,D_{x})$ [resp. $q(t,x,D_{x})$] with respect to the inner product on $L^{2}(\mathbf{R}_{x}^{d})$.

Hence for any $f(x)\in C_{0}^{\infty}(\mathbf{R}^{d})$

$$(3.7) \qquad \|p_{a}(t,x,D_{x})f(x)\|\leq C\|(\chi(\langle D_{x}\rangle)\langle D_{x}\rangle)^{\frac{1}{2}}f(x)\|$$
$$\|q_{a}(t,x,D_{x})f(x)\|\leq C\|(\chi(\langle D_{x}\rangle)\langle D_{x}\rangle)^{\frac{1}{2}}f(x)\|.$$

By using the operators

(3.8)
$$p_s(t,x,D_x) = p(t,x,D_x) - p_a(t,x,D_x)$$
$$q_s(t,x,D_x) = q(t,x,D_x) - q_a(t,x,D_x)$$

we define the operator L by

(3.9)
$$Lv(t,x) = (\frac{\partial}{\partial t} + ip_s(t,x,D_x) + q_s(t,x,D_x))v(t,x).$$

We set

(3.10)
$$w(t,x) = e^{\frac{1}{n}\phi(n(2t_0 - t))}v(t,x).$$

Then, since $e^{\frac{1}{n}\phi(n(2t_0 - t))}Lv(t,x) = (L + \phi'(n(2t_0 - t)))w(t,x)$,

$$\|e^{\frac{1}{n}\phi(n(2t_0 - t))}Lv(t,x)\|^2$$

$$= \|(\frac{\partial}{\partial t} + ip_s(t,x,D_x))w(t,x)\|^2 + \|(\phi'(n(2t_0 - t)) + q_s(t,x,D_x))w(t,x)\|^2$$

$$+ 2\Re((\frac{\partial}{\partial t} + ip_s(t,x,D_x))w(t,x), (\phi'(n(2t_0 - t)) + q_s(t,x,D_x))w(t,x)).$$

Since $(p_s(t,x,D_x))^* = p_s(t,x,D_x)$ and $(q_s(t,x,D_x))^* = q_s(t,x,D_x)$, the third term of the right hand side is equal to

$$\frac{\partial}{\partial t}((\phi'(n(2t_0 - t)) + q_s(t,x,D_x))w(t,x), w(t,x))$$

$$+ ((n\phi''(n(2t_0 - t)) - \frac{\partial}{\partial t}q_s(t,x,D_x))w(t,x), w(t,x))$$

$$+ i([q_s(t,x,D_x),p_s(t,x,D_x)]w(t,x), w(t,x)).$$

From the (3.4), (3.5), (3.6) and (3.8) we see that

$$\frac{\frac{\partial}{\partial t}q_s(t,x,D_x)}{\langle\xi\rangle\chi(\langle\xi\rangle)} \in S^0_{1,*}$$

and

$$\frac{\frac{\partial}{\partial \xi_j} q_s(t,x,\xi) \frac{\partial}{\partial x_j} p_s(t,x,\xi) - \frac{\partial}{\partial \xi_j} p_s(t,x,\xi) \frac{\partial}{\partial x_j} q_s(t,x,\xi)}{\langle \xi \rangle \chi(\langle \xi \rangle)} \in S_{1,*}^0,$$

which imply that

$$|(\frac{\partial}{\partial t} q_s(t,x,D_x))w(t,x), w(t,x))| + |([q_s(t,x,D_x), p_s(t,x,D_x)]w(t,x), w(t,x))|$$

$$\leq C \|(\langle D_x \rangle \chi(\langle D_x \rangle))^{\frac{1}{2}} w(t,x)\|^2.$$

Therefore we have, since $w(t,x)$ vanishes at $t=0$ and $t=t_0$,

$$(3.11) \qquad \int_0^{t_0} \|e^{\frac{1}{n}\phi(n(2t_0-t))} Lv(t,x)\|^2 dt$$

$$\geq \int_0^{t_0} (\|(\phi'(n(2t_0-t)) + q_s(t,x,D_x))w(t,x)\|^2$$

$$+ n\phi''(n(2t_0-t))\|w(t,x)\|^2) dt$$

$$- C \int_0^{t_0} \|(\langle D_x \rangle \chi(\langle D_x \rangle))^{\frac{1}{2}} w(t,x)\|^2 dt$$

Thus in order to finish the proof of Proposition 3.1 we have only to show the following estimate: for $t \in [0, t_0]$

$$(3.12) \qquad (Vw(t,x), w(t,x)) \geq (C_1 n^{\frac{1}{2}(1-\delta)} - C_2)\|(\langle D_x \rangle \chi(\langle D_x \rangle))^{\frac{1}{2}} w(t,x)\|^2,$$

where $V = (\phi'(n(2t_0-t)) + q_s(t,x,D_x))^2 + n\phi''(n(2t_0-t))$.

Indeed it follows from (3.11) and (3.12) that

$$\int_0^{t_0} \|e^{\frac{1}{n}\phi(n(2t_0-t))} Lv(t,x)\|^2 dt$$

$$\geq (C_1 n^{\frac{1}{2}(1-\delta)} - C_2) \int_0^{t_0} \|(\langle D_x \rangle \chi(\langle D_x \rangle))^{\frac{1}{2}} w(t,x)\|^2 dt,$$

from which, noting (3.7) and

$$D_t + p(t,x,D_x) - iq(t,x,D_x) = \frac{1}{i}\{L + ip_a(t,x,D_x) + q_a(t,x,D_x)\},$$

we obtain the assertion of Proposition 3.1 for some n_0.

Set

$$Q(t,x,\xi) = \sqrt{n\phi''(n(2t_0 - t))} - i(\phi'(n(2t_0 - t)) + q(t,x,\xi)).$$

Noting (3.7), (3.8) and

$$(Vw(t,x), w(t,x)) = \|(\sqrt{n\phi''(n(2t_0 - t))} - i(\phi'(n(2t_0 - t)) + q_s(t,x,D_x)))w(t,x)\|^2,$$

we see that (3.12) is equivalent to

(3.13) $\|Q(t,x,D_x)w(t,x)\| \geq (C_1 n^{\frac{1}{4}(1-\delta)} - C_2)\|(\langle D_x\rangle\chi(\langle D_x\rangle))^{\frac{1}{2}}w(t,x)\|.$

In order to obtain the estimate (3.13), first we show that for $t \in [0, t_0]$

(3.14) $|Q(t,x,\xi)| \geq Cn^{\frac{1}{4}(1-\delta)}(\chi(\langle\xi\rangle)\langle\xi\rangle)^{\frac{1}{2}}$

and that, setting

$$E_n(t,x,\xi) = \frac{n^{\frac{1}{4}(1-\delta)}(\chi(\langle\xi\rangle)\langle\xi\rangle)^{\frac{1}{2}}}{Q(t,x,\xi)},$$

(3.15) $\{E_n(t,x,\xi)\}_{n\geq 1, t\in[0,t_0]}$ is a bounded family in $S^0_{*,*}$.

Since $q(t,x,\xi) \in S^1_{1,*}$ is real-valued and satisfies (3.3), we see that there exist two positive constants C_1 and C_2 such that if $q(t,x,\xi) \geq 0$,

(3.16) $C_1\langle\xi\rangle \leq q(t,x,\xi) \leq C_2\langle\xi\rangle$

and if $q(t,x,\xi) < 0$,

(3.17) $C_1\langle\xi\rangle \leq -q(t,x,\xi) \leq C_2\langle\xi\rangle.$

In the case where, using the constants above,

$$\frac{1}{2}C_1\langle\xi\rangle \leq \phi'(n(2t_0 - t)) \leq 2C_2\langle\xi\rangle,$$

we see from (3.1) and the monotonicity of $\chi(s)$, that

$$\phi''(n(2t_0 - t)) \geq \frac{1}{2}C_1\langle\xi\rangle\chi(\frac{1}{2}C_1\langle\xi\rangle)$$

from (M.1)

$$\geq C\langle\xi\rangle\chi(\langle\xi\rangle).$$

Hence

$$|Q(t,x,\xi)| \geq Cn^{\frac{1}{2}}(\langle\xi\rangle\chi(\langle\xi\rangle))^{\frac{1}{2}},$$

which implies (3.14).

In the case where

$$\frac{1}{2}C_1\langle\xi\rangle \geq \phi'(n(2t_0-t)) \quad \text{or} \quad \phi'(n(2t_0-t)) \geq 2C_2\langle\xi\rangle,$$

since $\phi'(r) \geq 1$ for $r \geq 0$, we obtain from (3.16) and (3.17)

$$|(\phi'(n(2t_0-t)) + q(t,x,\xi))| \geq C\langle\xi\rangle.$$

Hence in this case, since $\phi''(r) \geq \chi(1)$ for $r \geq 0$ (see (3.2)),

$$|Q(t,x,\xi)| \geq C(n^{\frac{1}{2}} + \langle\xi\rangle)$$

$$\geq C(n^{\frac{1}{2}})^{1 - \frac{1}{2}(\delta+1)}\langle\xi\rangle^{\frac{1}{2}(\delta+1)},$$

from (M.3)

$$\geq Cn^{\frac{1}{4}(1-\delta)}(\chi(\langle\xi\rangle)\langle\xi\rangle)^{\frac{1}{2}}$$

which shows (3.14).

Next if $|\alpha| > 0$, then

(3.18)
$$|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta}Q(t,x,\xi)| = |\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta}q(t,x,\xi)|$$

$$\leq C(\langle\xi\rangle\chi(\langle\xi\rangle))^{\frac{1}{2}(|\alpha|+1)}\langle\xi\rangle^{-|\beta|}.$$

Hence, taking (3.14) into account, if $|\alpha| > 0$,

(3.19)
$$|\frac{\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta}Q(t,x,\xi)}{Q(t,x,\xi)}| \leq C(\langle\xi\rangle\chi(\langle\xi\rangle))^{\frac{1}{2}|\alpha|}\langle\xi\rangle^{-|\beta|}.$$

Similarly for $|\beta| > 0$, since $|\frac{\partial^\beta}{\partial \xi^\beta}Q(t,x,\xi)| = |\frac{\partial^\beta}{\partial \xi^\beta}q(t,x,\xi)|$,

(3.20)
$$\left| \frac{\dfrac{\partial^{\beta}}{\partial \xi^{\beta}} Q(t,x,\xi)}{Q(t,x,\xi)} \right| \le C(\langle \xi \rangle \chi(\langle \xi \rangle))^{-\frac{1}{2}} \langle \xi \rangle^{1-|\beta|}.$$

The assertion (3.15) follows from (3.19) and (3.20). Thus (3.14) and (3.15) are proved.

It follows from (3.15) that for $t \in [0, t_0]$

(3.21)
$$\|E_n(t,x,D_x)f(x)\| \le C\|f(x)\|.$$

We define the operator $R_n(t,x,D_x)$ by

(3.22)
$$R_n(t,x,D_x) = n^{\frac{1}{4}(1-\delta)} \langle D_x \rangle^{\frac{1}{2}} (\chi(\langle D_x \rangle))^{\frac{1}{2}} - E_n(t,x,D_x)Q(t,x,D_x)$$

and let $R_n(t,x,\xi)$ be its symbol. Since the symbol $R_n(t,x,\xi)$ is given by the following oscillatory integral

$$\sum_{j=1}^{d} \int_0^1 d\theta \frac{1}{(2\pi)^d} \int_{R^{2d}} e^{-i\langle y,\eta \rangle} (-1)^{\frac{1}{2}} \frac{\partial}{\partial \xi_j} E_n(t,x,\xi+\theta\eta) \frac{\partial}{\partial x_j} Q(t,x+y,\xi) d\eta \, dy$$

then it follows from (3.15) and (3.18) that

$$\left\{ \frac{R_n(t,x,\xi)}{(\langle \xi \rangle \chi(\langle \xi \rangle))^{\frac{1}{2}}} \right\}_{n \ge 1, t \in [0,t_0]}$$

is a bounded family in $S^0_{*,*}$ (see [B-F, §2, Theorem 1 and Lemma 2]). Thus we obtain, for $t \in [0, t_0]$,

(3.23)
$$\|R_n(t,x,D_x)f(x)\| \le C\|\langle D_x \rangle^{\frac{1}{2}} (\chi(\langle D_x \rangle))^{\frac{1}{2}} f(x)\|.$$

Since

$$\|n^{\frac{1}{4}(1-\delta)} \langle D_x \rangle^{\frac{1}{2}} (\chi(\langle D_x \rangle))^{\frac{1}{2}} w(t,x)\|$$

$$\le \|E_n(t,x,D_x)Q(t,x,D_x)w(t,x)\| + \|R_n(t,x,D_x)w(t,x)\|,$$

the desired estimate (3.13) follows from (3.21) and (3.23). This completes the proof of Proposition 3.1

§4. Proof of Theorem 1.2.

First we regularize the principal part of the operator:

$$e(t,x,\tau,\xi)=\tau^2+\sum_{j=1}^{d}2a_{0,j}(t,x)\tau\xi_j+\sum_{j,k=1}^{d}a_{j,k}(t,x)\xi_j\xi_k.$$

Set

$$b_{k,j}(t,x)=a_{j,k}(t,x)-a_{0,j}(t,x)a_{0,k}(t,x).$$

Then we see that

$$(4.1)\qquad e(t,x,\tau,\xi)=(\tau+\sum_{j=1}^{d}a_{0,j}(t,x)\xi_j)^2+\sum_{j,k=1}^{d}b_{j,k}(t,x)\xi_j\xi_k$$

and from the ellipticity

$$(4.2)\qquad \sum_{j,k=1}^{d}b_{j,k}(t,x)\xi_j\xi_k\geq C|\xi|^2.$$

We denote by $\tilde{a}_{0,j}(t,x,\xi)$ [resp. $\tilde{b}_{j,k}(t,x,\xi)$] the regularization of $a_{0,j}(t,x)$ [resp. $b_{j,k}(t,x)$] defined in Section 2.

Set

$$p(t,x,\xi)=\sum_{j=1}^{d}\tilde{a}_{0,j}(t,x,\xi)\xi_j$$

and

$$b(t,x,\xi)=\sum_{j,k=1}^{d}\tilde{b}_{j,k}(t,x,\xi)\xi_j\xi_k.$$

Then the definition of $b_{j,k}(t,x)$ and (4.2) imply

$$(4.3)\qquad b(t,x,\xi)\geq C|\xi|^2.$$

Here we introduce a notation. Let $m(\xi)$ be a continuous function on $\boldsymbol{R}^d$ with a polynomial growth. We say that an operator $P(t)$ on $C_0^\infty(\boldsymbol{R}^d)$ with a parameter $t\in[0,t_0]$ belongs to $\mathscr{L}\{m(\xi)\}$ if we have for any $f(x)\in C_0^\infty(\boldsymbol{R}^d)$ and any $t\in[0,t_0]$

$$\|P(t)f(x)\|\leq C\|m(D_x)f(x)\|.$$

Lemma 4.1.

$$(D_t + p(t,x,D_x))^2 + b(t,x,D_x)$$

$$= D_t^2 + 2 \sum_{j=1}^{d} a_{0,j}(t,x)D_t D_{x_j} + \sum_{j,k=1}^{d} a_{j,k}(t,x)D_{x_j}D_{x_k} + R_0(t)D_t + R_1(t)$$

where $R_j(t) \in \mathscr{L}\{\langle\xi\rangle^{\frac{1+2j}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$.

Proof. Since $\tilde{a}_{0,j}(t,x,\xi) \in S_{1,*}^0$,

$$[D_t, p(t,x,D_x)]$$

and

$$p(t,x,D_x)^2 - \sum_{j,k=1}^{d} \tilde{a}_{0,j}(t,x,D_x)\tilde{a}_{0,k}(t,x,D_x)D_{x_j}D_{x_k}$$

belong to $\mathscr{L}\{\langle\xi\rangle^{\frac{3}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$. Since, thanks to Proposition 2.1, $\tilde{a}_{0,j}(t,x,D_x)$ $-a_{0,j}(t,x) \in \mathscr{L}\{\langle\xi\rangle^{-\frac{1}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$, we see that

$$\sum_{j=1}^{d} a_{0,j}(t,x)D_{x_j} - p(t,x,D_x) \in \mathscr{L}\{\langle\xi\rangle^{\frac{1}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$$

and that

$$\tilde{a}_{0,j}(t,x,D_x)\tilde{a}_{0,k}(t,x,D_x)D_{x_j}D_{x_k} - a_{0,j}(t,x)a_{0,k}(t,x)D_{x_j}D_{x_k}$$

belongs to $\mathscr{L}\{\langle\xi\rangle^{\frac{3}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$. Therefore we obtain that the following operator can be written as $r_0(t)D_t + r_1(t)$ with $r_j(t) \in \mathscr{L}\{\langle\xi\rangle^{\frac{1+2j}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$

$$D_t^2 + \sum_{j=1}^{d} 2a_{0,j}(t,x)D_t D_{x_j} + \sum_{j,k=1}^{d} a_{0,j}(t,x)a_{0,k}(t,x)D_{x_j}D_{x_k} - (D_t + p(t,x,D_x))^2.$$

Similarly, since

$$\tilde{b}_{j,k}(t,x,D_x) - b_{j,k}(t,x) \in \mathscr{L}\{\langle\xi\rangle^{-\frac{1}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\},$$

we see

$$b(t,x,D_x) - \sum_{j,k=1}^{d} b_{j,k}(t,x)D_{x_j}D_{x_k} \in \mathscr{L}\{\langle\xi\rangle^{\frac{3}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}.$$

Hence we obtain the assertion of Lemma 4.1.   □

Set, with a positive constant $l \geq 1$

$$q_l(t,x,\xi) = \sqrt{b(t,x,\xi) + l^2}.$$

From Proposition 2.1, $b(t,x,\xi) \in S^2_{1,*}$ and (4.3), we see that $q_l(t,x,\xi)$ satisfies the assumption on $q(t,x,\xi)$ of Proposition 3.1 and that

(4.4) $q_l(t,x,D_x)q_l(t,x,D_x) - b(t,x,D_x) \in \mathscr{L}\{\langle\xi\rangle^{\frac{3}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}.$

Furthermore choosing l large enough, we see that there exists $o(t,x,\xi) \in S^{-1}_{1,\delta_0}$ where $\delta_0 = \frac{1+\delta}{2}$ such that

(4.5) $o(t,x,D_x)q_l(t,x,D_x) = 1.$

Indeed, since $q_l(t,x,\xi) \geq C(|\xi|+l)$ and $b(t,x,\xi) \in S^2_{1,\delta_0}$ we obtain for $|\alpha|+|\beta|>0$

$$|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial \xi^\beta} q_l(t,x,\xi)| \leq C_{\alpha,\beta}(|\xi|+l)^{-1}\langle\xi\rangle^{2+\delta_0|\alpha|-|\beta|}$$

with the constant $C_{\alpha,\beta}$ that is independent of $l \geq 1$. Hence $\{\frac{\partial}{\partial x_j}q_l(t,x,\xi)l^{(1-\delta_0)} \,|\, 1 \leq j \leq d, \, l \geq 1\}$ is bounded in S^2_{1,δ_0}. Set $q_l^{-1}(t,x,\xi) = \dfrac{1}{q_l(t,x,\xi)}$ and $r_l(t,x,\xi)$ the symbol of the operator

$$q_l^{-1}(t,x,D_x)q_l(t,x,D_x) - I.$$

Then, since $\{q_l^{-1}(t,x,\xi) \,|\, l \geq 1\}$ is bounded in S^{-1}_{1,δ_0}, $\{r_l(t,x,\xi)l^{1-\delta_0} \,|\, l \geq 1\}$ becomes a bounded set in S^0_{1,δ_0} (see H. Kumano-go ([K, Ch. 3, §3, Theorem 3.1]). Hence for large l, the operator $I + r_l(t,x,D_x)$ has the inverse $I(t,x,D_x)$ whose symbol $I(t,x,\xi)$ belongs to S^0_{1,δ_0} (see H. Kumano-go ([K, Appendix, Theorem I.1]). Therefore $I(t,x,D_x)q_l^{-1}(t,x,D_x)$ is a desired operator satisfying (4.5). Since we consider $q_l(t,x,\xi)$ with one fixed l for which (4.5) is valid, in the following we write $q(t,x,\cdot)$ in the place of $q_l(t,x,\cdot)$.

Now using

$$L_1 = D_t + p(t,x,D_x) - iq(t,x,D_x)$$

and

$$L_2 = D_t + p(t,x,D_x) + iq(t,x,D_x),$$

we can rewrite the operator $E(t,x,D_t,D_x)$ in the following way.   Since $p(t,x,\xi)$ and $q(t,x,\xi) \in S_{1,*}^1$,

(4.6) $(D_t + p(t,x,D_x))^2 + q(t,x,D_x)^2 - L_1 L_2 \in \mathscr{L}\{\langle\xi\rangle^{\frac{3}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}.$

Since $D_t + p(t,x,D_x) = \frac{1}{2}(L_1 + L_2)$, $q(t,x,D_x) = \frac{1}{2i}(L_2 - L_1)$ and the coefficients $a_j(t,x)$ $c(t,x)$ of $E(t,x,D_t,D_x)$ are bounded, it follows from Lemma 4.1, (4.4), (4.5) and (4.6) that there exist $Q_{1,1}$ and $Q_{1,2}$ in $\mathscr{L}\{\langle\xi\rangle^{\frac{1}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$ such that

(4.7) $L_1 L_2 = E(t,x,D_t,D_x) + \sum_{j=1,2} Q_{1,j} L_j.$

Similarly we see that there exist $Q_{2,1}$ and $Q_{2,2}$ in $\mathscr{L}\{\langle\xi\rangle^{\frac{1}{2}}(\chi(\langle\xi\rangle))^{\frac{1}{2}}\}$ such that

(4.8) $L_2 L_1 = E(t,x,D_t,D_x) + \sum_{j=1.2} Q_{2,j} L_j.$

Finally let $u(t,x)$ and $f(t,x)$ be those of Theorem 1.2.   Set for $j=1,2$

$$v_j(t,x) = L_{3-j} u(t,x).$$

Then we see that for $j=1,2$ $v_j(t,x) \in H^1(\mathbf{R}^{d+1})$ and the support of $v_j(t,x)$ is in $[0, \frac{2}{3} t_0] \times \mathbf{R}^d$.   Proposition 3.1 and the density argument give the following; there exists a positive n_0 such that for any $n \geq n_0$

(4.9) $n^{\frac{1-\delta}{4}} |\!|\!| e^{\phi_n(t)} \langle D_x \rangle^{\frac{1}{2}}(\chi(\langle D_x \rangle))^{\frac{1}{2}} v_j(t,x) |\!|\!| \leq C |\!|\!| e^{\phi_n(t)} L_j v_j(t,x) |\!|\!|,$

where $\phi_n(t) = \frac{1}{n}\phi(n(2t_0 - t))$.

On the other hand from (4.7) we obtain

$$L_1 v_1(t,x) = f(t,x) + \sum_{j=1,2} Q_{1,3-j} v_j(t,x),$$

which implies

(4.10) $|\!|\!| e^{\phi_n(t)} L_1 v_1(t,x) |\!|\!|$

$$\leq |\!|\!| e^{\phi_n(t)} f(t,x) |\!|\!| + C \sum_{j=1,2} |\!|\!| e^{\phi_n(t)} \langle D_x \rangle^{\frac{1}{2}}(\chi(\langle D_x \rangle))^{\frac{1}{2}} v_j(t,x) |\!|\!|.$$

Similarly from (4.8) we get

(4.11) $\||e^{\phi_n(t)}L_2 v_2(t,x)\||$

$$\leq \||e^{\phi_n(t)}f(t,x)\|| + C \sum_{j=1,2} \||e^{\phi_n(t)}\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{\frac{1}{2}}v_j(t,x)\||.$$

Therefore, by choosing a larger n_0 if necessary, we obtain from (4.9), (4.10) and (4.11)

(4.12) $n^{\frac{1-\delta}{4}}(\||e^{\phi_n(t)}\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{\frac{1}{2}}v_1(t,x)\||$

$$+ \||e^{\phi_n(t)}\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{\frac{1}{2}}v_2(t,x)\||)$$

$$\leq C\||e^{\phi_n(t)}f(t,x)\||$$

for $n \geq n_0$.

Since $\phi_n(t)$ is decreasing and $f(t,x)=0$ for $t \leq \frac{1}{2}t_0$, we get from (4.12)

$$\sum_{j=1,2} n^{\frac{1-\delta}{2}}e^{2\phi_n(\frac{1}{3}t_0)}\int_0^{\frac{1}{3}t_0} \|\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{\frac{1}{2}}v_j(t,x)\|^2 dt$$

$$\leq Ce^{2\phi_n(\frac{1}{2}t_0)}\int_{\frac{1}{2}t_0}^{t_0} \|f(t,x)\|^2 dt.$$

Therefore for $n \geq n_0$

(4.13) $\sum_{j=1,2}\int_0^{\frac{1}{3}t_0} \|\langle D_x\rangle^{\frac{1}{2}}(\chi(\langle D_x\rangle))^{\frac{1}{2}}v_j(t,x)\|^2 dt$

$$\leq Cn^{-\frac{1-\delta}{2}}e^{2(\phi_n(\frac{1}{2}t_0)-\phi_n(\frac{1}{3}t_0))}\int_{\frac{1}{2}t_0}^{t_0} \|f(t,x)\|^2 dt.$$

Since $\phi_n(t)$ is decreasing, $\phi_n(\frac{1}{2}t_0)-\phi_n(\frac{1}{3}t_0)$ is negative. Then, since $\delta < 1$, as $n \to +\infty$, the right hand side of (4.13) tends to zero. Hence the left hand side of (4.13) is equal to zero. Thus we see that $v_1(t,x)=0$ and $v_2(t,x)=0$ for $t < \frac{1}{3}t_0$. Hence, from $u(t,x)=\frac{1}{2i}o(t,x,D_x)(v_1(t,x)-v_2(t,x))$, we get $u(t,x)=0$ for $t < \frac{1}{3}t_0$. The proof of Theorem 1.2 is completed.

Remark. By the definiton of $\phi_n(t)$, we see that

$$\phi_n(\frac{1}{3}t_0) - \phi_n(\frac{1}{2}t_0) = \frac{1}{n}\int_{\frac{3}{2}t_0 n}^{\frac{5}{3}t_0 n} \phi'(r)dr.$$

Since $t_0 > 0$ and $\phi'(r) = s(r) \to +\infty$ as $r \to +\infty$, we see that the right hand side tends to $+\infty$ as $n \to +\infty$.

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