

Topological Tensor Products of Unbounded Operator Algebras on Fréchet Domains

By

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Abstract

The aim of this paper is to investigate topological properties of unbounded operator algebras $\mathcal{A} \subset L^+(D)$ and its stability under the formation of topological tensor products $\mathcal{A}_1 \otimes_{\alpha} \mathcal{A}_2$. It is used the method to define topological properties by operator.

§1. Introduction

The present paper is concerned with the study of the formation of topological tensor products $\mathcal{A}_1 \otimes_{\alpha} \mathcal{A}_2$ of unbounded operator algebras $\mathcal{A}_i \subset L^+(D_i)$, where $D_i \subset H_i$ are Fréchet domains of definition, $L^+(D_i)$ are the maximal operator *-algebras on D_i and α is an arbitrary tensor norm. Recall that tensor products of operators play an important role in the quantum field theory.

It is well-known that an operator algebra \mathcal{A} with $\mathcal{C} \subset \mathcal{A} \subset L^+(D)$ is a (DF)-space, if D is a Fréchet space and \mathcal{C} is the space of very continuous operators, see [11]. But the injective tensor product of (DF)-spaces is not a (DF)-space in general. Therefore we will use the concept of (DF)-spaces by operator (DFO). By this condition we obtain good stability properties by taking topological tensor products. The method to define topological properties by operator was introduced by A. Peris for quasinormable spaces and for (DF)-spaces, see [3] and the method was studied for (DF)-spaces with strong dual density condition by the author, see [8]. In Section 3 we will prove that a lot of operator algebras \mathcal{A} satisfy the (DFO)-condition. For such algebras the tensor product $\mathcal{A}_1 \otimes_{\alpha} \mathcal{A}_2$ is a (DF)-space.

In Section 4 we will investigate conditions such that \mathcal{A} and $\mathcal{A}_1 \otimes_{\alpha} \mathcal{A}_2$ are bornological (DF)-spaces. For this purpose we need the concept of the density condition (DC) and the strong dual density condition (SDDC). The density

Communicated by Y. Yamasaki, June 19, 1996.

1991 Mathematics Subject Classification(s) : 47D40, 47A67

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condition was introduced by S. Heinrich [5] in the context of ultrapowers of locally convex spaces. The (SDDC) was introduced and investigated by K.D. Bierstedt and J. Bonet in [1] and [2].

It is known that if $L^+(D)$ or a dense operator algebra \mathcal{A} is bornological, then D has to satisfy (DC), see [6]. The converse assertion is a tough problem and we need in addition that D has partitions of I . Then we get that all ideals $\mathcal{A} \subset L^+(D)$ satisfy the strong dual density condition by operator (SDDCO) and have good stability properties by taking tensor products. This implies that \mathcal{A} and $\mathcal{A}_1 \otimes_\alpha \mathcal{A}_2$, respectively, are bornological.

§2. Notations and Preliminaries

The notation for locally convex spaces is standard. If E is a locally convex space, $\mathcal{U}(E)$ stands for the system of all absolutely convex closed 0-neighbourhoods in E and $\mathcal{B}(E)$ stands for the system of all absolutely convex bounded sets in E . If V is an absolutely convex set, we denote by p_V the Minkowski functional of V , by $p_V^{-1}(0)$ the kernel of p_V and if V is in addition absorbent, we denote by E_V the quotient $E/p_V^{-1}(0)$. Further if E and F are locally convex spaces, then $\mathcal{L}_b(E, F)$ denotes the space of all continuous linear mappings from E into F endowed with the topology of uniform convergence on the bounded sets of E . We write $\mathcal{L}_b(E)$ for $\mathcal{L}_b(E, E)$. If $K \subset E, L \subset F$ and M is a linear subspace of $\mathcal{L}_b(E, F)$, then we write $W(K, L) := \{T \in M : T(K) \subset L\}$. Let C_2 denote the space of Johnson. C_2 is the l_2 -direct sum of the spaces F_k, F'_k with $k \in \mathbb{N}$, where $(F_k)_{k=1}^\infty$ is a sequence of finite-dimensional Banach spaces which is dense in the set of all finite-dimensional Banach spaces endowed with the Banach-Mazur distance. If α is a tensor norm, the topology of $E \otimes_\alpha F$ is given by the system of seminorms

$$(p_U \otimes_\alpha q_V)(z) := \alpha((K_U \otimes K_V)(z); E_U, F_V) \quad z \in E \otimes F, U \in \mathcal{U}(E), V \in \mathcal{U}(F),$$

where $K_U : E \rightarrow E_U$ and $K_V : F \rightarrow F_V$ are the canonical maps.

Throughout the paper, D denotes a linear subspace of a Hilbert space H . We denote the norm and the scalar product of H by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. For a linear operator T on H let $D(T)$ be the domain of definition. The set of linear operators

$$L^+(D) := \{T \in \text{End}(D) : D \subset D(T^*) \text{ and } T^*(D) \subset D\}$$

is the **maximal Op*-algebra** on the domain D with the involution $T^+ := T^*|_D$. The domain D will be endowed with the weakest locally convex topology such that $D \ni \varphi \mapsto \|T_\varphi\|$ are continuous seminorms for all $T \in L^+(D)$. This topology is called the **graph topology** t . We assume that $D[t]$ is a Fréchet space and in

this case we say that D is an **(F)-domain**. Throughout this paper, we fix sequences $(A_k)_{k=1}^\infty \subset L^+(D)$ and $(U_k)_{k=1}^\infty \subset \mathcal{U}(D)$ for each (F)-domain D such that, see [10] :

1. $A_1 := I_D, A_k = A_k^+, \|A_k^2 \varphi\| \leq \|A_{k+1}^2 \varphi\|.$
2. $U_k := \{\varphi \in D : \|A_k^2 \varphi\| \leq 1\}$ and $(U_k)_{k=1}^\infty$ is a basis of 0-neighbourhoods in D .

Let D_b^+ denote the space of all continuous conjugate linear functionals on D , endowed with the topology of uniform convergence on bounded subsets of D . We fix subsets $N_k := U_k^\circ \subset D^+$ for each (F)-domain D . Then $(N_k)_{k=1}^\infty$ is an increasing fundamental sequence of bounded sets of the (DF)-space D_b^+ . We always identify $f \in H$ with a linear functional on D . Then we have dense continuous inclusions $D \subset H \subset D_b^+$.

We define

$$\mathcal{C}(D_b^+, D) := \{T \in \mathcal{L}_b(D, D_b^+) : \exists S \in \mathcal{L}(D_b^+, D) \ \forall \varphi \in D \ T\varphi = S\varphi\}$$

and abbreviate $\mathcal{C}(D_b^+, D)$ by \mathcal{C} . The elements of \mathcal{C} are called **very continuous operators** and \mathcal{C} is a *-ideal in $L^+(D)$. \mathcal{F} stands for the space of all finite rank operators belonging to $L^+(D)$. The spaces $\mathcal{F} \subset \mathcal{C} \subset L^+(D) \subset \mathcal{L}_b(D, D_b^+) =: \mathcal{L}$ are endowed with the topology of uniform convergence on bounded sets. The closure of \mathcal{F} in $L^+(D)$ is denoted by \mathcal{V} .

Let T be an operator in $L^+(D) \subset \mathcal{L}$. Then there is an extension \tilde{T} of the operator T which belongs to $\mathcal{L}(D_b^+, D_b^+)$ by $\tilde{T} := (T^+)$. Clearly $(\tilde{T}\varphi)(\psi) = \varphi(T^+\psi) = \langle \varphi, T^+\psi \rangle = \langle T\varphi, \psi \rangle = (T\varphi)(\psi)$ for $\varphi \in D \subset D^+$ and $\psi \in D$. By [11], Proposition 2.4.c) the sequence $(\tilde{A}_k^2(U_H))_{k=1}^\infty$ is a fundamental sequence of bounded subsets of D_b^+ and we have $N_k = U_k^\circ = \tilde{A}_k^2(U_H)$.

For more information about unbounded operator algebras see [10], [11] and [13].

§3. The (DF)-Property by Operator for Unbounded Operator Algebras

In this section we will prove several propositions about permanence of the (DF)-property by taking topological tensor products of unbounded operator algebras on (F)-domains. K.-D. Kürsten proved in [11] that the very continuous operators \mathcal{C} and the *-algebra $L^+(D)$ are (DF)-spaces. We will obtain that these spaces are (DF)-spaces by operator and have good stability properties by taking tensor products. (DF)-spaces by operator were investigated in [8] and [12].

Definition 3.1. *A locally convex space E with an increasing fundamental sequence $(M_k)_{k=1}^\infty$ of bounded sets is said to be **(DF)-space by operator**,*

abbreviated by (DFO)-space, if

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \forall (U_n)_{n=1}^\infty \subset \mathcal{U}(E) \exists U \in \mathcal{U}(E) \exists (S_k)_{k=1}^\infty \subset \mathcal{L}(E):$$

$$S_k(U) \subset \lambda_k M_k \text{ and } (I_E - \sum_{k=1}^n S_k)(U) \subset U_n \quad k, n \in \mathbb{N}. \tag{1}$$

Remark. From the characterization of (DF)-spaces given by Bierstedt and Bonet in [2], Lemma 5.A., we deduce that a (DFO)-space is always a (DF)-space :

A locally convex space is a (DF)-space if and only if it has an increasing fundamental sequence $(M_k)_{k=1}^\infty$ of bounded sets such that for every positive sequence $(\varepsilon_k)_{k=1}^\infty$ and for every sequence $(U_n)_{n=1}^\infty$ of 0-neighbourhoods in E , there is $U \in \mathcal{U}(E)$ such that

$$U \subset \bigcap_{n=1}^\infty \left(\sum_{k=1}^n \varepsilon_k M_k + U_n \right).$$

Now we give a characterization of the (DFO)-property for reflexive spaces, see [8], Theorem 2.3.

Proposition 3.2. *Let E be a reflexive (DF)-space. Then the following assertions are equivalent :*

1. E is a (DFO)-space.
2. $X \otimes_\varepsilon E$ (and $L_b(X, E)$, resp.) is a (DF)-space for each Banach space X .

It is clear that a Banach space is always a (DFO)-space. For examples and counter-examples see [8]. By Lemma 5 and Proposition 7 in [12] we have the following

Proposition 3.3. *If α is a tensor norm and E, F are (DFO)-spaces, then $E \otimes_\alpha F$ is a (DF)-space.*

The next theorem states that D_b^+ is a (DFO)-space and it gives more information about the operators $(S_k)_{k=1}^\infty$ if D is an (F)-domain.

Theorem 3.4. *Let D be an (F)-domain. Then D_b^+ is a (DFO)-space and the corresponding operators are continuous extensions of very continuous operators, i.e. :*

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \forall (V_n)_{n=1}^\infty \subset \mathcal{U}(D_b^+) \exists V \in \mathcal{U}(D_b^+) \exists (S_k)_{k=1}^\infty \subset \mathcal{C} :$$

$$\tilde{S}_k(V) \subset \lambda_k N_k \text{ and } \left(I_{D^+} - \sum_{k=1}^n \tilde{S}_k \right) (V) \subset V_n \quad k, n \in \mathbb{N}.$$

Proof. Let us be given a positive sequence $(\lambda_k)_{k=1}^\infty$ and a sequence $(V_n)_{n=1}^\infty \subset \mathcal{U}(D_b^+)$. Since D_b^+ is a (DF)-space, there is some sequence of numbers $\rho_n > 0$ such that $V_0 := \bigcap_{n=1}^\infty \rho_n V_n$ belongs to $\mathcal{U}(D_b^+)$. Then the polar V_0° is bounded in D and there exists a positive sequence $(\lambda'_k)_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty \lambda'_k \sup\{\|A_k \varphi\|^2 : \varphi \in V_0^\circ\} < 1.$$

On the linear space

$$D_{t_0} := \left\{ \varphi \in D : \sum_{k=1}^\infty \lambda'_k \|A_k \varphi\|^2 < \infty \right\}$$

we define a hermitian sesquilinear form t_0 by

$$t_0(\varphi, \psi) := \sum_{k=1}^\infty \lambda'_k \langle A_k \varphi, A_k \psi \rangle.$$

By K.-D. Kürsten [10], proof of Theorem 4.1, it follows that there exists a positive selfadjoint operator S_0 acting on the Hilbert space $\bar{D}_{t_0} \subset H$ with domain D_{t_0} and $\langle S_0 \varphi, S_0 \psi \rangle = t_0(\varphi, \psi)$ for $\varphi, \psi \in D_{t_0}$. We define $B_0 := S_0^{-2} \oplus 0|_D \in \mathcal{C}$ with respect to the orthogonal direct sum $\bar{D}_{t_0} \oplus (H \ominus \bar{D}_{t_0})$ and one sees immediately that $V_0^\circ \subset \bar{B}_0(U_H)$.

Now we choose a sequence $(\varepsilon_k)_{k=1}^\infty$ with $0 < \varepsilon_k < \min\{\lambda_k, \lambda'_k\}$ and $\varepsilon_k N_k \subset 2^{-k} V_k$ for all $k \in \mathbb{N}$. On the linear space

$$D_t := \left\{ \varphi \in D : \sum_{k=1}^\infty \varepsilon_k \|A_k \varphi\|^2 < \infty \right\} \supset D_{t_0}$$

we define a hermitian sesquilinear form t once more by

$$t(\varphi, \psi) := \sum_{k=1}^\infty \varepsilon_k \langle A_k \varphi, A_k \psi \rangle. \tag{2}$$

There exists a positive selfadjoint operator S acting on $\bar{D}_t \subset H$ with domain D_t and $\langle S\varphi, S\psi \rangle = t(\varphi, \psi)$ for $\varphi, \psi \in D_t$. We set again $B := S^{-2} \oplus 0|_D \in \mathcal{C}$ with respect to the orthogonal direct sum $\bar{D}_t \oplus (H \ominus \bar{D}_t)$. Note that $S^2 \bar{B}|_{\bar{D}_t} = I_{\bar{D}_t}$. For more details relative to this construction see [10]. Since $\bar{D}_{t_0} \subset \bar{D}_t$, it follows $\ker \bar{B} \subset \ker \bar{B}_0$.

Define $V := \{f \in D_b^+ : \|\tilde{B}f\| \leq 1\} \in \mathcal{U}(D_b^+)$ and $S_k := \varepsilon_k A_k^2 B \in \mathcal{C}$. We conclude

$$\tilde{S}_k(V) \subset \varepsilon_k A_k^2 (U_H \cap D) \subset \varepsilon_k N_k \subset \lambda_k N_k$$

and the first relation of Theorem 3.4 is satisfied. Fix some $n \in \mathbb{N}$. Let $\vartheta \in V \cap \bar{D}_t$, $\psi \in V_n^\circ \subset D$ and set $\varphi := \tilde{B}\vartheta \in U_H \cap D(S^2) \subset D$. Since $\varepsilon_k A_k^2 \varphi \in \varepsilon_k N_k$, it follows $|\langle \varepsilon_k A_k^2 \varphi, \psi \rangle| \leq 2^{-k}$ for all $k \geq n$ by choice of the sequence $(\varepsilon_k)_{k=1}^\infty$. By (2) we get

$$\begin{aligned} \left| \left\langle \left(S^2 - \sum_{k=1}^n \varepsilon_k A_k^2 \right) \varphi, \psi \right\rangle \right| &= \left| \sum_{k=1}^n \varepsilon_k \langle A_k^2 \varphi, \psi \rangle - \sum_{k=1}^n \varepsilon_k \langle A_k^2 \varphi, \psi \rangle \right| \\ &\leq \sum_{k=n+1}^{\infty} \varepsilon_k |\langle A_k^2 \varphi, \psi \rangle| \leq \sum_{k=n+1}^{\infty} 2^{-k} \leq 1. \end{aligned}$$

Further, we have

$$\begin{aligned} \left| \left\langle \vartheta - \sum_{k=1}^n \tilde{S}_k \vartheta, \psi \right\rangle \right| &= \left| \left\langle S^2 \tilde{B} \vartheta - \sum_{k=1}^n \varepsilon_k A_k^2 \tilde{B} \vartheta, \psi \right\rangle \right| \\ &= \left| \left\langle \left(S^2 - \sum_{k=1}^n \varepsilon_k A_k^2 \right) \varphi, \psi \right\rangle \right| \leq 1 \end{aligned}$$

and this implies

$$\left(I - \sum_{k=1}^n \tilde{S}_k \right) (V \cap \bar{D}_t) \subset V_n^{\circ\circ} = V_n. \tag{3}$$

Now let ϑ be an element in $H \ominus \bar{D}_t$. Then $\vartheta \in \ker \bar{B} \subset \ker \bar{B}_0$ and this shows $\vartheta \in p_{V_0}^{-1}(0)$. By construction of V_0 we get $\vartheta \in p_{V_k}^{-1}(0)$ and $\vartheta \in V_k$ for all $k \in \mathbb{N}$. Therefore

$$\left(I_{D^+} - \sum_{k=1}^n \tilde{S}_k \right) \vartheta = \vartheta - \sum_{k=1}^n \varepsilon_k A_k^2 \tilde{B} \vartheta = \vartheta \in V_n$$

and by (3) a short calculation indicates

$$\left(I_{D^+} - \sum_{k=1}^n \tilde{S}_k \right) (V \cap H) \subset V_n.$$

Moreover, since H is dense in D_b^+ , \tilde{S}_k are continuous operators and V_n is a closed set, we have proved the second relation in the theorem, too. \diamond

Theorem 3.5. *Let D be an (F) -domain and let \mathcal{A} be an algebra of operators such that $\mathcal{C} \subset \mathcal{A} \subset L^+(D)$ or let \mathcal{A} be an ideal in $L^+(D)$. Then \mathcal{A} is a (DFO) -space.*

Proof. By Theorem 3.4 the space D_b^+ is a (DFO) -space, i.e.

$$\begin{aligned} \forall (\lambda_k)_{k=1}^{\infty} \text{ pos. } \forall (V_n)_{n=1}^{\infty} \subset \mathcal{U}(D_b^+) \exists V \in \mathcal{U}(D_b^+) \exists (S_k)_{k=1}^{\infty} \subset \mathcal{C} : \\ \tilde{S}_k(V) \subset \lambda_k N_k, \left(I_{D^+} - \sum_{k=1}^n \tilde{S}_k \right) (V) \subset V_n \quad k, n \in \mathbb{N}. \end{aligned} \tag{4}$$

Let us be given a positive sequence $(\lambda_k)_{k=1}^{\infty}$ and a sequence $(M_k)_{k=1}^{\infty} \subset D$ of bounded subsets. Define $V_k := M_k^{\circ} \in \mathcal{U}(D_b^+)$ and choose V and $(S_k)_{k=1}^{\infty}$ such that (4) is true. Since $U_k = N_k^{\circ}$, $M := V^{\circ} \in \mathcal{B}(D)$, $(\tilde{S}_k)' \in \mathcal{L}(D, D)$ and $f((S_k)' \varphi) = (\tilde{S}_k f)(\varphi) = f(S_k^+ \varphi)$ for $\varphi \in D$, $f \in D_b^+$, we see with $R_k := S_k^+$ the following

$$\begin{aligned} \forall (\lambda_k)_{k=1}^{\infty} \text{ pos. } \forall (M_n)_{n=1}^{\infty} \subset \mathcal{B}(D) \exists M \in \mathcal{B}(D) \exists (R_k)_{k=1}^{\infty} \subset \mathcal{C} : \\ R_k(U_k) \subset \lambda_k M, \left(I_D - \sum_{k=1}^n R_k \right) (M_n) \subset M \quad k, n \in \mathbb{N}. \end{aligned} \tag{5}$$

By (4) and (5) the conditions (j) and (jj) are satisfied in the proof of [12], Theorem 13. Therefore, the space $\mathcal{L} = \mathcal{L}_b(D, D_b^+)$ is a (DFO)-space, where the corresponding continuous linear operators Q_k on $\mathcal{L}_b(D, D_b^+)$ have the structure

$$Q_k(T) := \sum_{(i,j) \in I_k} \tilde{S}_i T R_j \tag{6}$$

where $I_k \subset \mathbb{N} \times \mathbb{N}$ is a finite subset and $S_i, R_j \in \mathcal{C}$. Hence, we get

$$\forall (\varepsilon_k)_{k=1}^\infty \text{ pOS. } \forall (W_n)_{n=1}^\infty \subset \mathcal{U}(\mathcal{L}) \exists W \in \mathcal{U}(\mathcal{L}) \exists (Q_k)_{k=1}^\infty \text{ with struct. (6):} \tag{6}$$

$$Q_k(W) \subset \varepsilon_k \mathcal{M}_k \text{ and } \left(I_{\mathcal{L}} - \sum_{k=1}^n Q_k \right) (W) \subset W_n \quad k, n \in \mathbb{N}. \tag{7}$$

where $(\mathcal{M}_k)_{k=1}^\infty$ is an increasing fundamental sequence of bounded sets in \mathcal{L} . By construction the linear operators $Q_k(A) = \sum \tilde{S}_i A R_j$ leave \mathcal{A} invariant. This implies $Q_k|_{\mathcal{A}} \in \mathcal{L}(\mathcal{A}, \mathcal{A})$ and it is readily seen that \mathcal{A} satisfies an analogous condition to (7) with the increasing fundamental sequence $(\mathcal{M}_k \cap \mathcal{A})_{k=1}^\infty$ of bounded sets. Therefore \mathcal{A} is a (DFO)-space. \diamond

Corollary 3.6. *Let D, D_1 and D_2 be (F)-domains, then :*

1. *The spaces $\mathcal{F}, \mathcal{V}, \mathcal{C}$ and $L^+(D)$ are (DFO)-spaces.*
2. *The Calkin algebra $\mathcal{A}_c := L^+(D)/\mathcal{V}$ is a (DFO)-space.*
3. *If α is an arbitrary tensor norm and the algebras $\mathcal{A}_i \subset L^+(D_i) \quad i=1, 2$ satisfy the assumption in Theorem 3.5, then $\mathcal{A}_1 \otimes_\alpha \mathcal{A}_2$ and $\mathcal{A}_1 \tilde{\otimes}_\alpha \mathcal{A}_2$ are (DF)-spaces.*

Proof. We are going to prove (2). Given a positive sequence $(\lambda_k)_{k=1}^\infty$ and a sequence for $(V_k)_{k=1}^\infty \subset \mathcal{U}(\mathcal{A}_c)$, there is $(W_k)_{k=1}^\infty \subset \mathcal{U}(L^+(D))$ such that $W_k + \mathcal{V} \subset V_k$ for $k \in \mathbb{N}$. By Theorem 3.5 we find a $W \subset \mathcal{U}(L^+(D))$ and $(Q_k)_{k=1}^\infty \subset \mathcal{L}(L^+(D))$ with the structure (6) such that

$$Q_k(W) \subset \varepsilon_k \mathcal{M}_k \text{ and } \left(I_{\mathcal{L}} - \sum_{k=1}^n Q_k \right) (W) \subset W_n \quad k, n \in \mathbb{N}$$

where $(\mathcal{M}_k)_{k=1}^\infty$ is an increasing fundamental sequence of bounded sets in \mathcal{L} . We set $V := W + \mathcal{V}$ and $\mathcal{N}_k := \mathcal{M}_k + \mathcal{V} \in \mathcal{B}(\mathcal{A}_c)$. Since $L^+(D)$ is a (DF)-space, the canonical quotient map lifts bounded sets and \mathcal{N}_k is a fundamental sequence of bounded sets in \mathcal{A}_c . We can infer from $Q_k(\mathcal{V}) \subset \mathcal{V}$ that the linear operators $\tilde{Q}_k \in \mathcal{L}(\mathcal{A}_c, \mathcal{A}_c)$ are well-defined by $\tilde{Q}_k(T + \mathcal{V}) := Q_k(T) + \mathcal{V}$. Thus

$$\begin{aligned} \tilde{Q}_k(V) &= \tilde{Q}_k(W + \mathcal{V}) \subset Q_k(W) + \mathcal{V} \subset \varepsilon_k \mathcal{M}_k + \mathcal{V} = \varepsilon_k \mathcal{N}_k, \\ \left(I_{\mathcal{A}_c} - \sum_{k=1}^n \tilde{Q}_k \right) (V) &\subset \left(I_{L^+(D)} - \sum_{k=1}^n Q_k \right) (W) + \mathcal{V} \subset W_n + \mathcal{V} \subset V_n, \end{aligned}$$

and (2) is proven. The assertion (3) follows by Proposition 3.3. \diamond

§4. The Strong Dual Density Condition by Operator for Unbounded Operator Algebras

The density condition plays an important role in the theory of the maximal Op*-algebras $L^+(D)$ on (F)-domains D , see [6] and [7]. For example if D is a commutatively dominated (F)-domain, then $L^+(D)$ is a bornological space if and only if D satisfies the density condition. In this section we will study the situation for subalgebras $\mathcal{A} \subset L^+(D)$ and tensor products of subalgebras. We start with some definitions, see [5] and [1].

Definition 4.1. (1) Let F denote a metrizable space and $(U_k)_{k=1}^\infty$ a countable basis of closed absolutely convex 0-neighbourhoods in F .

F is said to satisfy the **density condition**, abbreviated by (DC), if the following holds

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists B \in \mathcal{B}(F) : \bigcap_{k=1}^m \lambda_k U_k \subset U_n + B.$$

(2) Let E denote a locally convex space with an increasing fundamental sequence $(M_k)_{k=1}^\infty$ of bounded sets. E is said to satisfy the **strong dual density condition**, abbreviated by (SDDC), if the following holds

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists U \in \mathcal{U}(E) : M_n \cap U \subset \Gamma \bigcup_{k=1}^m \lambda_k M_k. \tag{8}$$

E is said to satisfy the **strong dual density condition by operator**, abbreviated by (SDDCO), if the following holds

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists U \in \mathcal{U}(E) \exists (Q_k)_{k=1}^m \text{ lin. op. on } E : \sum_{k=1}^m Q_k = I_E \text{ and } Q_k(M_n \cap U) \subset \lambda_k M_k, k=1, \dots, m. \tag{9}$$

Quasinormable Fréchet spaces and Fréchet-Montel spaces are examples for spaces satisfying (DC), see [5]. For examples and counter-examples of (F)-domains with (DC) see [6] and [7]. By taking polars, it follows that the (F)-space E satisfies (DC) if and only if the strong dual E'_b satisfies (SDDC). Remark that U_k° is $\sigma(E', E)$ -compact, such that $\Gamma \bigcup_{k=1}^m \lambda_k^{-1} U_k^\circ$ is closed. It is readily seen, that (SDDCO) implies (SDDC). We give a characterization of the (SDDCO) property analogous to the (DFO)-property in Section 3. For details see [8], Theorem 1.6.

Proposition 4.2. Let E be a reflexive (DF)-space. The following assertions are equivalent :

1. E satisfies the strong dual density condition by operator (SDDCO).
2. $X \otimes_\varepsilon E$ (and $L_b(X, E)$, resp.) satisfies the strong dual density condition

(SDDC) for each Banach space X .

Remark. The direction (1) \Rightarrow (2) is valid for arbitrary (DF)-spaces.

If E is a reflexive (DFO)-space, then (SDDC) implies (SDDCO), see [8], Theorem 2.4. Thus (SDDC) and (SDDCO) are equivalent properties for D_b^+ . But we know nothing about the linear operators Q_k . We are going to study this operators for a large class of (F)-domains. The next definition substitutes the properties (1)-(3) in [13], Proposition 4.1.4.

Definition 4.3. Let D denote an (F)-domain. D is said to have **partitions of I** if the following holds

$$\forall (\varepsilon_k)_{k=1}^\infty \text{ pos. } \exists (\varepsilon'_k)_{k=1}^\infty \text{ pos. } \forall p \in \mathbb{N} \exists m \in \mathbb{N} \exists (Q_k)_{k=1}^m \subset L^+(D):$$

$$\sum_{i=1}^m Q_i = I_D \text{ and } Q_k(\varepsilon_n U_k) \subset \bigcap_{i=1}^p \varepsilon'_i U_i \quad k=1, \dots, m.$$

Proposition 4.4. Let D be a commutatively dominated (F)-domain, i.e.

$$D := \bigcap_{k=1}^\infty D(h_k(T)),$$

where T is a self-adjoint operator on a Hilbert space H and h_k is a sequence of real measurable functions on the spectrum $\sigma(T)$ of T such that $1 = h_1(t)$ and $h_k(t)^2 \leq h_{(k+1)}(t)$ a.e. for all $k \in \mathbb{N}$, see [13], 4.3. Then D has partitions of I .

Proof. Let us be given a positive sequence $(\varepsilon_k)_{k=1}^\infty$ and $p \in \mathbb{N}$. We set $\varepsilon'_i := \varepsilon_i$ and define bounded measurable functions q_k on $\sigma(T)$ by

$$q_k(t) := \frac{\varepsilon_k^{-1} h_k^2(t)}{\sum_{i=1}^p \varepsilon_i^{-1} h_i^2(t)} \quad t \in \sigma(T), \quad k=1, \dots, p,$$

and operators by $Q_k := q_k(T)|_D$. Since the domain of $h_n(T)$ is invariant for $q_k(T)$, it follows that $Q_k \in L^+(D)$ for $k=1, \dots, p$. Fix some k and let $\psi \in \varepsilon_k U_k$, where $U_j := \{\varphi \in D : \|h_j^2(T)\varphi\| \leq 1\}$ for $j \in \mathbb{N}$. We set $\vartheta := \varepsilon_k^{-1} h_k^2(T)\psi \in U_H \cap D$. For an arbitrary $l=1, \dots, p$ the estimate

$$h_l^2(T)q_k(T)\psi = \frac{\varepsilon_k^{-1} h_l^2(T)h_k^2(T)}{\sum_{i=1}^p \varepsilon_i^{-1} h_i^2(T)}\psi = \varepsilon_l \frac{\varepsilon_l^{-1} h_l^2(T)}{\sum_{i=1}^p \varepsilon_i^{-1} h_i^2(T)}\vartheta \in \varepsilon_l U_H$$

shows $Q_k\psi \in \varepsilon_l U_l$. This means $Q_k(\varepsilon_k U_k) \subset \bigcap_{i=1}^p \varepsilon_i U_i$ for $k=1, \dots, p$. By construction of the operators we get $\sum_{i=1}^p Q_i = I_D$. \diamond

Proposition 4.5. Let D be a quasinormable (F)-domain, i.e.

$$\forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall \varepsilon > 0 \exists M \in \mathcal{B}(D): U_m \subset \varepsilon U_n + M.$$

Then D has partitions of I .

We note that a quasinormable (F)-space always satisfies (DC). Thus a quasinormable (F)-domain is a domain satisfying (DC) with partitions of I .

Proof. Let us be given a positive sequence $(\varepsilon_k)_{k=1}^\infty$ and the fundamental system $(N_k)_{k=1}^\infty$ of bounded sets in D_b^+ . We define bounded sets $B_k := N_k + \sum_{i=1}^k \varepsilon_i^{-1} N_i$. Since D is a quasinormable (F)-domain, its strong dual satisfies the strict Mackey condition. Thus, we get

$$\forall n \in \mathbb{N} \exists m(n) \in \mathbb{N} \ m(n) > n \exists V_n \in \mathcal{U}(D_b^+): B_n \cap V_n \subset \varepsilon_{m(n)}^{-1} N_{m(n)}. \tag{10}$$

By Theorem 3.4 D_b^+ is a (DFO)-space and we can choose $V \in \mathcal{U}(D_b^+)$ and $(S_k)_{k=1}^\infty \subset \mathcal{C}$ such that

$$\tilde{S}_k(V) \subset \varepsilon_k^{-1} N_k \text{ and } \left(I_{D^+} - \sum_{k=1}^n \tilde{S}_k \right) (V) \subset V_n \ k, n \in \mathbb{N}. \tag{11}$$

Let $V^\circ \subset \bigcap_{i=1}^\infty \delta_i U_i$ and define $\varepsilon'_i := 1 + |\delta_i|$ for all $i \in \mathbb{N}$. We fix some $p \in \mathbb{N}$ and set $m := m(p)$. By (10) and (11) we get

$$\begin{aligned} \left(I_{D^+} - \sum_{k=1}^p \tilde{S}_k \right) (N_p \cap V) &\subset \left(N_p - \sum_{k=1}^p \tilde{S}_k(V) \right) \cap V_p \\ &\subset B_p \cap V_p \subset \varepsilon_m^{-1} N_m. \end{aligned}$$

By taking polars we find

$$\begin{aligned} S_k^+(\varepsilon_k U_k) &\subset V^\circ \subset \bigcap_{i=1}^p \varepsilon'_i U_i \ k \in \mathbb{N} \text{ and} \\ \left(I_D - \sum_{k=1}^p S_k^+ \right) (\varepsilon_m U_m) &\subset U_p + V^\circ \subset \bigcap_{i=1}^p \varepsilon'_i U_i. \end{aligned}$$

Now we define linear operators by $Q_k := S_k^+$ for $k=1, \dots, p$; $Q_k := 0$ for $k = p+1, \dots, m-1$ and $Q_m := I_D - \sum_{i=1}^p S_i^+$. Then we have $\sum_{i=1}^m Q_i = I_D$ and $Q_k(\varepsilon_k U_k) \subset \bigcap_{i=1}^p \varepsilon'_i U_i$ for $k=1, \dots, m$. \diamond

Proposition 4.6. *Let D be an (F)-domain with partitions of I and density condition (DC). Then D_b^+ (SDDCO) and the corresponding operators are continuous extensions of operators in $L^+(D)$, more exactly*

$$\begin{aligned} \forall (\lambda_k)_{k=1}^\infty \text{ pos. } \exists V \in \mathcal{U}(D_b^+) \forall n \subset \mathbb{N} \exists m \in \mathbb{N} \exists (Q_k)_{k=1}^m \subset L^+(D): \\ \sum_{k=1}^m \tilde{Q}_k = I_{D^+} \text{ and } \tilde{Q}_k(N_n \cap V) \subset \lambda_k N_k \ k=1, \dots, m. \end{aligned} \tag{12}$$

Remark. Assertion (12) is a little bit stronger than (SDDCO), because the quantors “ $\forall n$ ” and “ $\exists V$ ” are swapped. It is known that Assertion (12) with

arbitrary linear operators and (SDDCO) are equivalent properties for (DFO)-spaces, [8], Theorem 2.4, but the concrete operators can differ.

Proof. Let us be given a positive sequence $(\lambda_k)_{k=1}^\infty$. We set $\varepsilon_k := \lambda_k^{-1}$. Since D has partitions of I , we can find a positive sequence $(\varepsilon'_k)_{k=1}^\infty$ such that for all p there exist $m \in \mathbb{N}$ and operators $(Q_k)_{k=1}^m \in L^+(D)$ such that

$$\sum_{i=1}^m Q_i = I_D \text{ and } Q_k^+(\varepsilon_k U_k) \subset \bigcap_{i=1}^p \varepsilon'_i U_i \quad k=1, \dots, m. \tag{13}$$

By assumption D satisfies the density condition. Since the bounded set in Definition 4.1.(1) of (DC) can be chosen not depending on $n \in \mathbb{N}$, see [5], 1. 4, we find a bounded set $M \subset D$ such that for all $n \in \mathbb{N}$ there exists $q \in \mathbb{N}$ with

$$\bigcap_{i=1}^q 2\varepsilon'_i U_i \subset U_n + M. \tag{14}$$

Choose m and the operators in (13) for $p := q$. Then we infer

$$\sum_{i=1}^m Q_i = I_D \text{ and } Q_k^+(\varepsilon_k U_k) \subset \frac{1}{2}(U_n + M) \quad k=1, \dots, m. \tag{15}$$

By taking polars we get $U_n^\circ \cap M^\circ \subset 2(U_n + M)^\circ \subset ((Q_k^+)')^{-1}(\varepsilon_k^{-1} U_k^\circ)$ and with $V := M^\circ \in \mathcal{U}(D_b^+)$ and $N_n = U_n^\circ$ we conclude that

$$(Q_k^+)'(N_n \cap V) \subset \lambda_k N_k \quad k=1, \dots, m.$$

Since $\tilde{Q}_k = (Q_k^+)'$ and $\sum_{i=1}^m \tilde{Q}_i = I_{D^+}$, the statement of the proposition follows. \diamond

Theorem 4.7. *Let E and G be (DFO)-spaces with increasing fundamental sequences of bounded sets $(M_k)_{k=1}^\infty$ and $(C_k)_{k=1}^\infty$ respectively. Let E satisfy the property*

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \exists U \in \mathcal{U}(E) \forall n \in \mathbb{N} \exists m \in \mathbb{N} \exists (S_k)_{k=1}^m \subset \mathcal{L}(E):$$

$$\sum_{k=1}^m S_k = I_E \text{ and } S_k(M_n \cap U) \subset \lambda_k M_k \quad k=1, \dots, m \tag{16}$$

and let G satisfy a similar relation

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \exists U' \in \mathcal{U}(G) \forall n' \in \mathbb{N} \exists m' \in \mathbb{N} \exists (R_k)_{k=1}^{m'} \subset \mathcal{L}(G):$$

$$\sum_{k=1}^{m'} R_k = I_G \text{ and } R_k(C_{n'} \cap U') \subset \lambda_k C_k \quad k=1, \dots, m'. \tag{17}$$

Then $\mathcal{L}_b(G'_b, E)$ is a (DF)-space satisfying the strong dual density condition by operator (SDDCO) and the corresponding continuous linear operators Q_k on $\mathcal{L}_b(G'_b, E)$ have the structure

$$Q_k(T) = \sum_{(i,j) \in I_k} S_i T R'_j \quad T \in \mathcal{L}_b(G'_b, E) \tag{18}$$

where $I_k \subset \mathbb{N} \times \mathbb{N}$ is a finite subset.

Proof. Since E and G are (DFO)-spaces, E and G'_b satisfy the conditions (j) and (jj) in the proof of [12], Theorem 13. Thus $\mathcal{L}_b(G'_b, E)$ is a (DF)-space. By taking polars of (17) we get for the space G'_b the following property

$$\forall (\lambda_k)_{k=1}^\infty \text{ pos. } \exists B \in \mathcal{B}(G'_b) \forall n' \in \mathbb{N} \exists m' \in \mathbb{N} \exists (R_k)_{k=1}^{m'} \in \mathcal{L}(G):$$

$$\sum_{k=1}^{m'} R'_k = I_{G'} \text{ and } R'_k(\lambda_{k'}^{-1} V_k) \subset V_{n'} + B \quad k=1, \dots, m', \tag{19}$$

where $(V_k)_{k=1}^\infty$ is a decreasing basis of 0-neighbourhoods in G'_b . Now given a decreasing positive sequence $(\varepsilon_k)_{k=1}^\infty$. By induction we define a new decreasing positive sequence $(\lambda_k)_{k=1}^\infty$ such that

$$\lambda_j \sum_{i=1}^j \lambda_i \leq \frac{1}{4} \varepsilon_j \quad j \in \mathbb{N}. \tag{20}$$

Take $U \in \mathcal{U}(E)$ and $B \in \mathcal{B}(G'_b)$ such that (16) and (19) are satisfied for $(\lambda_k)_{k=1}^\infty$. We fix some $n \in \mathbb{N}$. Choose $\rho_1, \rho_2 \geq 1$ with $M_n \subset \rho_1 U$ and $B \subset \rho_2 V_n$. Because of (16), there exist $m \in \mathbb{N}$ and linear operators $(S_k)_{k=1}^m \subset \mathcal{L}(E)$ such that

$$\sum_{k=1}^m S_k = I_E \text{ and } S_k(\rho_2 M_n \cap U) \subset \lambda_k M_k \quad k=1, \dots, m \tag{21}$$

and by (19) there exist m' and linear operators $(R_k)_{k=1}^{m'} \subset \mathcal{L}(G)$ such that

$$\sum_{k=1}^{m'} R'_k = I_{G'} \text{ and } R'_k(\lambda_{k'}^{-1} V_k) \subset \rho_1^{-1} V_n + B \quad k=1, \dots, m'. \tag{22}$$

Without loss of generality, we can assume that $m = m'$ and we choose

$$T \in W(V_n, M_n) \cap W(B, U). \tag{23}$$

By (22), (23) and the choice of ρ_1, ρ_2 it follows

$$TR'_l(\lambda_l^{-1} V_l) \subset T(\rho_1^{-1} V_n) + T(B)$$

$$\subset \rho_1^{-1} M_n + U \subset 2U \quad l=1, \dots, m \tag{24}$$

and

$$TR'_l(\lambda_l^{-1} V_l) \subset T(\rho_1^{-1} V_n) + T(B)$$

$$\subset T(V_n) + T(\rho_2 V_n) \subset 2\rho_2 M_n \quad l=1, \dots, m. \tag{25}$$

The relations (21), (24) and (25) imply

$$S_k TR'_l(\lambda_l^{-1} V_l) \subset 2S_k(\rho_2 M_n \cap U)$$

$$\subset 2\lambda_k M_k \quad k, l=1, \dots, m. \tag{26}$$

We define continuous linear operators Q_k on $\mathcal{L}(G'_b, E)$ by

$$Q_k(T) := \sum_{i=1}^{k-1} S_i TR'_k + \sum_{i=1}^k S_k TR'_i. \tag{27}$$

Consequently

$$\sum_{k=1}^m Q_k(T) = \sum_{k=1}^m \sum_{i=1}^{k-1} S_i TR'_k + \sum_{k=1}^m \sum_{i=1}^k S_k TR'_i = T$$

and because of (26) we infer that

$$\begin{aligned} Q_k(T)(V_k) &\subset \sum_{i=1}^{k-1} S_i TR'_k(V_k) + \sum_{i=1}^k S_k TR'_i(V_i) \\ &\subset \sum_{i=1}^{k-1} 2\lambda_k \lambda_i M_i + \sum_{i=1}^k 2\lambda_k \lambda_i M_k \\ &\subset 4 \sum_{i=1}^k \lambda_k \lambda_i M_k \subset \varepsilon_k M_k. \end{aligned}$$

We proved the following: For each decreasing positive sequence $(\varepsilon_k)_{k=1}^\infty$ there exists a 0-neighbourhood $W(B, U)$ such that for each bounded set $W(V_n, M_n)$, we find an $m \in \mathbb{N}$ and linear operators Q_k with the structure (18) such that

$$\sum_{i=1}^m Q_i = I_{\mathcal{L}(G'_b, E)} \text{ and } Q_k(W(V_n, M_n) \cap W(B, U)) \subset \varepsilon_k W(V_k, M_k)$$

for $k=1, \dots, m$. This means that $\mathcal{L}(G'_b, E)$ satisfies (SDDCO) (even a bit stronger, because the quantors are swapped). \diamond

If D is an (F)-domain with (DC) and partitions of I and $E = G := D_b^\dagger$, then the spaces E and G satisfy the assumption of Theorem 4.7 and the corresponding linear operators S_i, R_j are elements of $L^+(D)$ by Proposition 4.6. It follows that the linear operators $Q_k(T) = \sum S_i TR_j$ leave ideals of $L^+(D)$ invariant. This proves the following theorem.

Theorem 4.8. *Let D be an (F)-domain with partitions of I and \mathcal{A} an ideal of $L^+(D)$ containing \mathcal{C} . Then are equivalent :*

1. D satisfies the density condition (DC).
2. \mathcal{A} satisfies the strong dual density condition by operator (SDDCO).
3. \mathcal{A} is a bornological (DFO)-space.

Proof. (1) \Rightarrow (2): This direction is obvious by the above arguments, Proposition 4.6 and Theorem 4.7.

(2) \Rightarrow (3): The ideal \mathcal{A} is a (DFO)- and (DF)-space by Theorem 3.5 and (SDDCO) implies (SDDC). But a (DF)-space satisfying (SDDC) is always bornological by [1], 1.4.(c).

(3) \Rightarrow (1) : Since $\mathcal{C} \subset \mathcal{A} \subset \mathcal{L}$ and \mathcal{C} is dense in \mathcal{L} , see [10], Theorem 6.1, it follows that the completion of \mathcal{A} is \mathcal{L} and we conclude that \mathcal{L} is barrelled. The assertion follows by [6], Theorem 4.1. \diamond

Corollary 4.9. *Let D be an (F)-domain satisfying (DC) with partitions of I , then :*

1. *The spaces \mathcal{F} , \mathcal{V} , \mathcal{C} and $L^+(D)$ are bornological (DFO)-spaces.*
2. *The Calkin-algebra $\mathcal{A}_C := L^+(D)/\mathcal{V}$ is a bornological (DFO)-space.*

We omit the proof.

Theorem 4.10. *Let D_i be (F)-domains with partitions of I and \mathcal{A}_i ideals in $L^+(D_i)$ containing \mathcal{C}_i for $i=1, 2$. Further, let α be an arbitrary tensor norm. Then are equivalent :*

1. *D_1 and D_2 satisfy the density condition (DC).*
2. *$\mathcal{A}_1 \otimes_\alpha \mathcal{A}_2$ is a bornological (DF)-space.*

Proof. (1) \Rightarrow (2) : Let E, G be (DF)-spaces and $X := C_2$ the space of Johnson. We define a vector valued trace

$$\text{tr} : (E \otimes_\varepsilon X') \otimes_\pi (X \otimes_\alpha G) \rightarrow E \otimes_\alpha G \tag{28}$$

by $(u \otimes x') \otimes (x \otimes v) \mapsto \langle x, x' \rangle u \otimes v$. Due to A. Defant tr is a surjective topological homomorphism, see [4], Sections 29 and 35. Put $E := \mathcal{A}_2$ and $G := X = C_2$. By Propositions 3.3 and 4.2 and the remark after 4.2. $\mathcal{A}_2 \otimes_\varepsilon X'$ is a (DF)-space satisfying (SDDC). By [1], 1.4.(c), $\mathcal{A}_2 \otimes_\varepsilon X'$ is a bornological (DF)-space. Of course, $X \otimes_\alpha X'$ is a bornological (DF)-space, too. Since this property is hereditary under taking projective tensor products, see [9], §41.(7) and (8), and under taking quotients, it follows that $X \otimes_\alpha \mathcal{A}_2$ is a bornological (DF)-space. By the same argument for

$$\text{tr} : (\mathcal{A}_1 \otimes_\varepsilon X') \otimes_\pi (X \otimes_\alpha \mathcal{A}_2) \rightarrow \mathcal{A}_1 \otimes_\alpha \mathcal{A}_2$$

we prove the assertion.

(2) \Rightarrow (1) : Since \mathcal{A}_i is a complementable subspace of $\mathcal{A}_1 \otimes_\alpha \mathcal{A}_2$, the ideal \mathcal{A}_i is a bornological (DF)-space. Now, we can conclude as (3) \Rightarrow (1) in the proof of Theorem 4.8. \diamond

Remark. (1) A computation as in the proof of Theorem 4.7 indicates that $L^+(D_1) \otimes_\varepsilon L^+(D_2)$ satisfies (SDDC) if D_1 and D_2 are (F)-domains with (DC) and partitions of I .

(2) With the present method, it is possible to study the space $\mathcal{L}_b(D_1, L^+(D_2))$, where D_1 and D_2 are (F)-domains. For example, let D_1 be the Schwartz space $\mathbf{S}(\mathbf{R}^4)$ and $D := D_2$ an (F)-domain with partitions of I satisfying (DC), then the space $\mathcal{L}_b(\mathbf{S}(\mathbf{R}^4), L^+(D))$ is a bornological (DF)-space by Proposition 4.6 and Theorem 4.7. We note that the Schwartz space is a commutatively dominated (F)-space satisfying (DC).

References

- [1] Bierstedt, K.D. and Bonet, J., Dual density conditions in (DF)-spaces I, *Results in Math.*, **14** (1988), 242-274.
- [2] _____, Dual density conditions in (DF)-spaces II, *Bull. Soc. Roy. Sci. Liège.*, **57** (1988), 567-589.
- [3] Boyd, C. and Peris, A., A projective description of the Nachbin-ported topology, *J. Math. Anal. Appl.*, **197** (1996), 635-657.
- [4] Defant, A. and Floret, K., *Tensor Norms and Operator Ideals*, North-Holland Math. Studies, North-Holland, Amsterdam, 1993.
- [5] Heinrich, S., Ultrapowers of locally convex spaces and applications I., *Math. Nachr.*, **118** (1984), 285-315.
- [6] Heinrichs, W.-D., The Density Property in Fréchet-Domains of Unbounded Operator *-algebras, *Math. Nachr.*, **165** (1994), 49-60.
- [7] _____, On Unbounded Positive *-Representations on Fréchet-Domains, *Publ. RIMS, Kyoto Univ.*, **30** (1994), 1123-1138.
- [8] _____, The Density Condition and the Strong Dual Density Condition by Operator, *Collect. Math.*, to appear.
- [9] Köthe, G., *Topological Vector Spaces II*, Springer Verlag, New York, Heidelberg, Berlin, 1979.
- [10] Kürsten, K.-D., The Completion of the Maximal Op*-Algebras on Fréchet Domains, *Publ. RIMS, Kyoto Univ.*, **22** (1986), 151-175.
- [11] _____, Duality for Maximal Op*-Algebras on Fréchet Domains, *Publ. RIMS, Kyoto Univ.*, **24** (1988), 585-620.
- [12] Peris, A. and Rivera, M.J., Localization of Bounded Sets in Tensor Products, *Rev. Univ. Complutense Madrid*, to appear.
- [13] Schmüdgen, K., *Unbounded Operator Algebras and Representations*, Akademie-Verlag, Berlin, 1990.

