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A Note on the Approximation Entropies of Certain Shifts

By

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Abstract

Voiculescu has proposed several routes to quantum entropies. Among them, the notion of the "approximation entropies" is a group of four entropies with similar definitions, based on two kinds of approximations. The C*-cases are extensions of the classical topological entropy and the W*-cases are those of the measure-theoretic one. In this paper, we will focus on the approximation entropies and investigate the entropies of Powers' binary shifts with some condition and the Jones shifts.

§1. Preliminaries

Voiculescu [15] has introduced quantum entropies of automorphisms of operator algebras, called approximation entropies. In this section, we will review the definitions and fix the notations. Using two kinds of approximation for the W*-case and the C*-case, Voiculescu has defined four approximation entropies which have similar definitions to each other. As Voiculescu said, one may think of approximation entropies as "growth"-entropies and the key concept is the "δ-rank". The four approximation entropies are defined in the same way, as a matter of form, except for the "δ-rank". So we will only state the definition by subalgebra approximation for the W*-case in detail. (see [15] for the other cases.)

Let $\mathscr{M}$ be a hyperfinite von Neuman algebra with a faithful normal tracial state τ and $\mathscr{F}(\mathscr{M})$ be the set of the unital finite-dimensional C*-subalgebras of $\mathscr{M}$. By $\mathscr{P}f(\mathscr{M})$ we denote the set of finite subsets of $\mathscr{M}$ and by Aut($\mathscr{M}$) the automorphism group of $\mathscr{M}$. For a normal faithful state φ on $\mathscr{M}$, set $\mathrm{Aut}(\mathscr{M},\varphi)=\{\alpha\in\mathrm{Aut}(\mathscr{M})\mid\varphi\circ\alpha=\varphi\}$. For $\omega\in\mathscr{P}f(\mathscr{M})$ and $\mathscr{X}\subset\mathscr{M}$, we shall write

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$\omega \subset_{\delta} \mathscr{X}$ if for every $a \in \omega$ there exists $x \in \mathscr{X}$ such that $\|a-x\|_2 < \delta$ where $\|y\|_2 = \tau(y^*y)^{1/2}$ for $y \in \mathscr{M}$. If $A \in \mathscr{F}(\mathscr{M})$ we denote by $r(A)$ its rank, i.e. the dimension of a maximal abelian self-adjoint subalgebra of A. For $\omega \in \mathscr{P}f(\mathscr{M})$ and $\delta > 0$, we define the *δ-rank with respect to τ* of ω as follows.

$$r_\tau(\omega \mid \delta) = \inf\{r(B) \mid \omega \subset_{\delta} B \in \mathscr{F}(\mathscr{M})\}.$$

For $\alpha \in \mathrm{Aut}(\mathscr{M}, \tau)$, $\delta > 0$ and $\omega \in \mathscr{P}f(\mathscr{M})$, we define

$$ha_\tau(\alpha, \omega \mid \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r_\tau(\omega \cup \alpha(\omega) \cup \cdots \cup \alpha^{n-1}(\omega) \mid \delta),$$

$$ha_\tau(\alpha, \omega) = \sup_{\delta > 0} ha_\tau(\alpha, \omega \mid \delta),$$

$$ha_\tau(\alpha) = \sup\{ha_\tau(\alpha, \omega) \mid \omega \in \mathscr{P}f(\mathscr{M})\}.$$

This $ha_\tau(\alpha)$ is called the *approximation entropy* of α.
One can easily see that $r_\tau(\omega \mid \delta)$ is increasing in ω and decreasing in δ and $r_\tau(\alpha(\omega) \mid \delta) = r_\tau(\omega \mid \delta)$ for any $\alpha \in \mathrm{Aut}(\mathscr{M}, \tau)$.

In the C*-case, we replace hyperfinite von Neumann algebras by unital AF-algebras and $\subset_\delta$ with respect to the 2-norm $\|\cdot\|_2$ by the C*-norm $\|\cdot\|$. Almost all the definitions are repetitions of the W*-case, so they are omitted. No state specified, we write $r(\omega \mid \delta)$, $hat(\alpha, \omega \mid \delta)$, $hat(\alpha, \omega)$ and $hat(\alpha)$ for the C*-versions of $r_\tau(\cdot \mid \delta)$, $ha_\tau(\alpha, \omega \mid \delta)$, $ha_\tau(\alpha, \omega)$ and $ha_\tau(\alpha)$, respectively. We call $hat(\alpha)$ the *topological approximation entropy* of α.

Now we define another δ-rank by an approximation based on completely positive maps instead of subalgebras. Let $\mathscr{M}$ be a hyperfinite von Neumann algebra with a faithful normal state φ. By $CPA(\mathscr{M}, \varphi)$ we denote the set of triples (ψ, ρ, B), where B is a finite-dimensional C*-algebra, $\psi: \mathscr{M} \to B$ and $\rho: B \to \mathscr{M}$ are unital completely positive maps such that $\varphi \circ \psi \circ \rho = \varphi$. For $\omega \in \mathscr{P}f(\mathscr{M})$ and $\delta > 0$, the *completely positive δ-rank with respect to φ* of ω is defined as

$$r_\varphi^{cp}(\omega \mid \delta) = \inf\{r(B) \mid (\psi, \rho, B) \in CPA(\mathscr{M}, \varphi),\ \|(\rho \circ \psi)(a) - a\|_\varphi < \delta \text{ for } a \in \omega\},$$

where $\|y\|_\varphi = \varphi(y^*y)^{1/2}$ for $y \in \mathscr{M}$. We can define $ha_\varphi^{cp}(\alpha, \omega \mid \delta)$, $ha_\varphi^{cp}(\alpha, \omega)$ and $ha_\varphi^{cp}(\alpha)$ as in the subalgebra approximation, and $ha_\varphi^{cp}(\alpha)$ is called the *completely positive approximation entropy* of α. In the C*-case of the completely positive map approximation, the norm $\|\cdot\|_\varphi$ is replaced by the C*-norm and $\mathscr{M}$ by a

nuclear unital C*-algebra with no state specified. As in the above cases, we can define $r^{cp}(\cdot\,|\,\delta)$, $ht(\alpha,\omega\,|\,\delta)$, $ht(\alpha,\omega)$ and $ht(\alpha)$ wich is called the *topological entropy* of α.
Of cource, the approximation entropies are extensions of the classical entropy to the noncommutative framework (i.e. the W*-cases are those of the measure-theoretic Kolmogorov-Sinai entropy and the C*-cases are those of the topological entropy), that is, they coincide with classical ones in commutative cases. Furthermore, Voiculescu has shown that all the approximation entropies have the Kolmogorov-Sinai type theorem.

It is natural to ask what kinds of relations to other entropies they have. In general, one knows that Voiculescu's approximation entropies are larger than or equal to the Connes-Narnhofer-Thirring entropy (CNT-entropy, for short) and in special cases (for example, for noncommutative Bernoulli shifts), they are equal to each other. On the other hand, relations among approximation entropies are, roughly speaking, $ha_\tau^{cp} \leq ha_\tau \leq hat \geq ht$. All the above comparisons were done by Voiculescu in [15].

§2. Approximation Entropies of the Binary Shifts and the Jones Shifts

In this section, we will estimate the approximation entropies of the binary shifts and the Jones shifts and determine their values.

2.1. The binary shifts

First, we will review the fundamental results of the binary shifts.

Let $(a)=\{a_0,a_1,\cdots\}$ be a sequence of 0's and 1's with $a_0=0$. For such a bitstream (a), we consider a sequence of hermitian unitary operators, $\{u_i\,|\,i\in \boldsymbol{Z}\}$ satisfying the following "commutation relations"

$$u_j u_k = (-1)^{a_{|j-k|}}\, u_k u_j.$$

Price [12] has shown that the von Neumann algebra R generated by $\{u_i\,|\,i\in \boldsymbol{Z}\}$ is isomorphic to the hyperfinite II_1-factor if and only if the corresponding bitstream (a) is not mirror-periodic (i.e. $\cdots, a_2,a_1,a_0,a_1,a_2,\cdots$ is not periodic.) From now we shall always assume that (a) is a non-mirror-periodic bitstream. Then, R has the unique faithful normal tracial state τ, which satisfies the following.

$$\tau(w)=0, \quad \text{for any nontrivial word } w \text{ of } u_i\text{'s}.$$

Powers' binary shift is the unique extension of the mapping defined by $\sigma(u_i)=u_{i+1}$ to an automorphism on R, which we also denote by σ. And we call the extension on the C*-algebra generated by $\{u_i \mid i\in \boldsymbol{Z}\}$ the *topological binary shift*. For computations of the entropy of binary shifts, we review the structure of the subalgebra of R generated by $u_0,\cdots,u_n$, which we denote by B_n. We can easily see that $\dim B_n=2^{n+1}$. Furthermore, it is known [11, 13] that the dimension of the center of B_n is 2^{c_n} for some $c_n\in\{0,1,2,\cdots\}$, and the algebra B_n decomposes as the direct sum of 2^{c_n} copies of $2^{m_n}\times 2^{m_n}$ matrix algebras, where $m_n=(1/2)(n+1-c_n)$. We call the sequence $\{c_0,c_1,\cdots\}$ the *center sequence* for the shift corresponding to the bitstream (a). Powers and Price [11] determined the form of center sequences. The center sequence consists of a disjoint union of infinitely many finite strings of the form $1\ 2\ \cdots\ m-1\ m$ $m-1\ \cdots\ 2\ 1\ 0$. The value of m may vary in the sequence. Furthermore, they have shown in [11] that if a bitstream (a) is eventually periodic, that is, for some $p\in \boldsymbol{N}$ the subsequence $a_pa_{p+1}\cdots$ of (a) is periodic, then the center sequence associated with (a) is also eventually periodic.

Now, we have prepared for calculation of the entropy of the binary shifts. By using the tensor product inequality and the monotonicity of ha_τ^{cp}, Voiculescu has shown in [15] that for any Powers' binary shift σ,

$$\frac{1}{2}\log 2\leq ha_\tau^{cp}(\sigma)\leq ha_\tau(\sigma)\leq \log 2.$$

Similarly to the above result, for any topological binary shift σ, we have

$$\frac{1}{2}\log 2\leq ht(\sigma)\leq hat(\sigma)\leq \log 2.$$

Proposition 2.1. *Let σ be the Powers' binary shift automorphism with the corresponding bitstream eventually periodic. Then, all the approximation entropies are* $(1/2)\log 2$.

Proof. By virtue of the Voiculescu's result, it suffices to show that $hat(\sigma)\leq(1/2)\log 2$, since $ha_\tau(\sigma)\leq hat(\sigma)$ by Proposition 2.4. in [15].

Let B_n be the algebra generated by $\{u_0,u_1,\cdots,u_n\}$ and c_n be its center sequence. Then, B_n is the direct sum of 2^{c_n} copies of $2^{m_n}\times 2^{m_n}$ matrix algebras, where $m_n=(1/2)(n+1-c_n)$. Hence, $r(B_n)=2^{(1/2)(n+1+c_n)}$. Since $r(\{u_0,u_1,\cdots,u_n\}|\delta)\leq r(B_n)$ for any $\delta>0$,

$$hat(\sigma,u_0)\leq \limsup_n \frac{1}{n+1}\log r(B_n)$$

$$=\limsup_n \frac{n+1+c_n}{2(n+1)}\log 2.$$

Since $\{c_n\}$ is eventually periodic, $\{c_n\}$ is bounded, and by Proposition 2.3. in [15], we get

$$hat(\sigma)=hat(\sigma,u_0)\leq \frac{1}{2}\log 2.$$

■

2.2. The Jones shifts

Let $\{e_i \mid i\in \boldsymbol{Z}\}$ be a sequence of projections satisfying the following "Temperly-Lieb relation".

(1) $e_ie_j=e_je_i$, for $|i-j|\geq 2$,

(2) $e_ie_{i\pm 1}e_i=\lambda e_i$,

where $\lambda\in(0,\frac{1}{4}]\cup\left\{\left(4\cos^2\frac{\pi}{m}\right)^{-1}\,\middle|\, m\in \boldsymbol{N},\ m\geq 3\right\}$.

As shown by Jones [8], the von Neumann algebra generated by $\{e_i \mid i\in \boldsymbol{Z}\}$ and the unit 1, which we denote by R, is the hyperfinite II_1-factor with the canonical tracial state τ. The *Jones shift* is the automorphism on R defined by $\theta_\lambda(e_i)=e_{i+1}$. We will also use the same notation when we consider the Jones shifts on the C*-algebra generated by $\{e_i \mid i\in \boldsymbol{Z}\}$.

Various entropies of the Jones shifts have been already calculated, and we will recall some of the results here. (In connection with the index theory for type II_1 subfactors, one can find deeper results on the entropies in [3], [4] and [6], viewing the Jones shifts as the square roots of the canonical shifts.)

Pimsner and Popa [10] computed the Connes-Størmer entropy of the Jones shifts, except for the case $\lambda=1/4$ which was settled by Yin [16] and Choda [2]. One who wants to know the whole treatment of the CNT-version of the calculation may refer to Section 17 in [9]. It is known that

$$h_\tau(\theta_\lambda)=\begin{cases} -\frac{1}{2}\log\lambda, & \text{when } \frac{1}{4}<\lambda\leq 1,\\[1ex] \eta(t)+\eta(1-t), & \text{when } 0<\lambda\leq\frac{1}{4}, \quad \text{where } t(1-t)=\lambda, \end{cases}$$

where $h_\tau(\cdot)$ is the CNT-entropy with respect to τ and $\eta(t) = -t\log t$.

Let A_n be the C*-subalgebra of R generated by $1, e_0, \cdots, e_n$ and $\mathscr{A}_{loc} = \cup_n A_n$, which is a local C*-algebra with $\theta_\lambda(\mathscr{A}_{loc}) \subset \mathscr{A}_{loc}$. (In this case, we think of θ_λ as a unital $*$-endomorphism on $\mathscr{A}_{loc}$). It is known [14] that the Thomsen's topological entropy $\hbar$ of the Jones sifts has the following formula and one can compute the value with it.

$$\hbar(\theta_\lambda \mid_{\mathscr{A}_{loc}}) = \limsup_n \frac{\log r(A_n)}{n} = \begin{cases} -\frac{1}{2}\log\lambda, & \text{when } \frac{1}{4} < \lambda \leq 1, \\ \log 2, & \text{when } 0 < \lambda \leq \frac{1}{4}. \end{cases}$$

Of cource, we can think of θ_λ as an automorphism for which there is a canonical local C*-subalgebra A of R such that $\theta_\lambda(A) = A$. For this automorphic version, we obtain exactly the same result as above.

Proposition 2.2. *If* $\frac{1}{4} \leq \lambda < 1$, *then all the approximation entropies are* $-\frac{1}{2}\log\lambda$.

Proof. By Propositions 3.6, 3.7, 2.4, 4.6 and 4.5. in [15], we have $h_\tau(\theta_\lambda) \leq ha_\tau^{cp}(\theta_\lambda) \leq ha_\tau(\theta_\lambda) \leq hat(\theta_\lambda)$ and $h_\tau(\theta_\lambda) \leq ht(\theta_\lambda) \leq hat(\theta_\lambda)$. Since $h_\tau(\theta_\lambda) = -\frac{1}{2}\log\lambda$, it is sufficient to show that $hat(\theta_\lambda) \leq -\frac{1}{2}\log\lambda$.

Let A_n be the finite-dimensional C^*-subalgebra generated by $\{e_i \mid 0 \leq i \leq n-1\}$. For any $\delta > 0$, we have $r(\{e_0, \cdots, e_{n-1}\} \mid \delta) \leq r(A_n)$. By Proposition 2.3. in [15],

$$hat(\theta_\lambda) = hat(\theta_\lambda, e_0) \leq \limsup_n \log \frac{r(A_n)}{n} = \hbar(\theta_\lambda) = -\frac{1}{2}\log\lambda.$$

■

Remark. Here we return to Powers' binary shifts and consider the Powers' binary shift σ_0 corresponding to a bitstream 0100 $\cdots$. This shift is very important, for all the Powers' binary shifts with commutant index 2 are cocycle conjugate to it, though there are countably many non-conjugate binary shifts of commutant index 2 (see [13]). And one knows that σ_0 is the Jones shift with $\lambda = 1/2$, which is easily seen as follows.

Let $\{u_i \mid i \in \boldsymbol{Z}\}$ be the hermitian unitary generators with respect to σ_0. Putting $e_i = (1/2)(1 + u_i)$, one can easily see that e_i's are projections and satisfy the Temperly-Lieb relation with $\lambda = 1/2$ and that $\sigma_0(e_i) = e_{i+1}$. Furthermore, of cource, the approximation entropies of σ_0 coincide with those of $\theta_{1/2}$ by Propositions 2.1 and 2.2.

Proposition 2.3. *If* $0 < \lambda \leq \frac{1}{4}$, *then* $hat(\theta_\lambda) = ht(\theta_\lambda) = \log 2$.

Proof. In this case, according to Jones' result [8], the C*-algebra generated by $\{e_i \mid i \in \boldsymbol{Z}\}$ has the Bratteli diagram arising from reflections of Dynkin diagram A_∞. As pointed out in Example 1.3 of [7], θ_λ are independent of λ up to unitary equivalence. Hence, $hat(\theta_\lambda)$ and $ht(\theta_\lambda)$ are also independent of λ. As in the previous proposition, we have

$$h_\tau(\theta_\lambda) \leq ht(\theta_\lambda) \leq hat(\theta_\lambda) \leq \hbar(\theta_\lambda) = \log 2, \qquad \text{for all } \lambda \in (0,\ 1/4].$$

Since $h_\tau(\theta_\lambda) = \eta(t) + \eta(1-t)$, where $t(1-t) = \lambda \leq 1/4$, $h_\tau(\theta_\lambda)$ is maximized at $\lambda = 1/4$ and its maxium is $\log 2$. So we get this proposition. ∎

Proposition 2.4. *If* $0 < \lambda \leq \frac{1}{4}$, *then* $ha_\tau^{cp}(\theta_\lambda) = \eta(t) + \eta(1-t)$, *where* $t = \dfrac{1 + \sqrt{1 - 4\lambda}}{2}$.

Proof. Pimsner and Popa [10] have shown that in the case of $0 < \lambda \leq \frac{1}{4}$, the Jones shift is a kind of the Bernoulli shift. To illustrate it, we consider the infinite tensor product of replicas of 2×2 matrix algebra $M_2(\boldsymbol{C})$. Let $M_i = M_2(\boldsymbol{C})$, $M = \otimes_{i \in \boldsymbol{Z}} M_i$ and $M_{[n,m]} = \otimes_{i=n}^m M_i$. We denote the matrix units in M_n by $w_{ij}^{(n)}$ ($n \in \boldsymbol{Z}$, $1 \leq i, j \leq 2$). Let φ_i be a state with its density $tw_{11}^{(i)} + (1-t)w_{22}^{(i)}$ in M_i and φ be the product state $\otimes_{i \in \boldsymbol{Z}} \varphi_i$ on M. For $n \in \boldsymbol{N}$, we set

$$\begin{aligned} e_n = \cdots 1 \otimes \big((1-t)w_{11}^{(n)} \otimes w_{22}^{(n+1)} + \sqrt{t(1-t)}\,w_{12}^{(n)} \otimes w_{21}^{(n+1)} \\ + \sqrt{t(1-t)}\,w_{21}^{(n)} \otimes w_{12}^{(n+1)} + t w_{22}^{(n+1)} \otimes w_{11}^{(n+1)}\big) \otimes 1 \cdots \end{aligned}$$

$$= \cdots 1 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-t & \sqrt{t(1-t)} & 0 \\ 0 & \sqrt{t(1-t)} & t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \cdots \quad \in M_{[n,n+1]}.$$

It is straightforward to see that e_n's are projections and satisfy the Temperley-Lieb relation. So we can realize the Jones shifts with $\lambda < 1/4$ on the infinite tensor product. Let R be the AF-algebra generated by $\cdots, e_{-1}, e_0, e_1, \cdots$ and $\bar{\varphi}$ be the extension of φ to $\pi_\varphi(M)''$, where π_φ is the GNS-representation with respect to φ. We write γ for the right shift on M and $\bar{\gamma}$ for its extension on $\pi_\varphi(M)''$. Pimsner and Popa have shown that $\pi_\varphi(R)''$ is just the centralizer $\mathscr{X}$ of $\bar{\varphi}$ and $\bar{\gamma}|_{\mathscr{X}}$ is the Jones shift. (It is also well-known that Connes and Størmer [5] have proved that $\mathscr{X}$ is the hyperfinite II_1 factor with the canonical tracial state $\bar{\varphi}|_{\mathscr{X}}$ and $H(\bar{\gamma}|_{\mathscr{X}}) = \eta(t) + \eta(1-t)$. They called this shift the Bernoulli shift defined by $\{1-t, t\}$.) Applying this idea to the approximation entropy, we get

$$h_\tau(\theta_\lambda) \leq ha_\tau^{cp}(\theta_\lambda) = ha_{\bar{\varphi}|\mathscr{X}}^{cp}(\bar{\gamma}|_{\mathscr{X}}) \leq ha_{\bar{\varphi}}^{cp}(\bar{\gamma}) = h_{\bar{\varphi}}(\bar{\gamma}).$$

The first inequality and the second one are due to Proposition 3.6. and Proposition 3.5. in [15] respectively and the last equality is due to Proposition 3.9. in [15]. One knows that $h_\tau(\theta_\lambda) = h_{\bar{\varphi}}(\bar{\gamma}) = \eta(t) + \eta(1-t)$. So we get this proposition. ■

Remark. By the same method as in Proposition 2.2, we can calculate the approximation entropies of other shifts. Let S be a finite subset of $\boldsymbol{N}$ and $n \in \boldsymbol{N}$. Then there exists a family $\{u_i \mid i \in \boldsymbol{Z}\}$ of unitaries satisfying the following conditions.

(1) $u_i^n = 1$, for any $i \in \boldsymbol{Z}$,

(2) $u_i u_j = \exp(2\pi\sqrt{-1}/n) u_j u_i$, if $|i-j| \in S$,

(3) $u_i u_j = u_j u_i$, if $|i-j| \notin S$.

Let P be the von Neumann algebra generated by the family and θ be the automorphism defined by $\theta(u_i) = u_{i+1}$ on P. (We also write θ for the automorphism given by $u_i \to u_{i+1}$ on the C*-algebra generated by the family.) One knows that P is the hyperfinite II_1-factor and $h_\tau(\theta) = \hbar(\theta) = (1/2)\log n$. (See [1], [2] and [14].) Since $\{u_0, \cdots, u_{k-1}\}$ generates a finite-dimensional C*-subalgebra of P, as in Proposition 2.2, we obtain

$$h_\tau(\theta) \leq ha_\tau^{cp}(\theta) \leq ha_\tau(\theta) \leq hat(\theta) \leq \hbar(\theta) \text{ and } h_\tau(\theta) \leq ht(\theta) \leq hat(\theta) \leq \hbar(\theta).$$

Hence, all the approximation entropies are $\frac{1}{2}\log n$.

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References

[1] Choda, M., Shifts on the Hyperfinite II_1-factor, *J. Operator Theory*, **17** (1987), 223–235.
[2] ———, Entropy for *-Endomorphisms and Relative Entropy for Subalgebras, *J. Operator Theory*, **25** (1991), 125–140.
[3] ———, Entropy for Canonical Shifts, *Trans. Amer. Math. Soc.*, **334** (1992), 827–849.
[4] Choda, M. and Hiai, F., Entropy for Canonical Shifts. II, *Publ. RIMS, Kyoto Univ.*, **27** (1991), 461–489.
[5] Connes, A. and Størmer, E., Entropy of II_1 von Neumann algebras, *Acta Math.*, **134** (1975), 289–306.
[6] Hiai, F., Entropy for Canonical Shifts and Strong Amenability, *Internat. J. Math.*, **6** (1995), 381–396.
[7] Hiai, F. and Petz, D., Quantum Mechanics in AF C*-systems, *Preprint*.
[8] Jones, V. F. R., Index for Subfactors, *Invent. Math.*, **72** (1983), 1–25.
[9] Ohya, M. and Petz, D., *Quantum Entropy and Its Use*, Berlin, Heidelberg, New York: Springer, 1993.
[10] Pimsner, M. and Popa, S., Entropy and Index for Subfactors, *Ann. Sci. École Norm. Sup. Sér.*4, **19** (1986), 57–106.
[11] Powers, R. T. and Price, G. L., Cocycle Conjugacy Classes of Shifts on the Hyperfinite II_1 Factor, *J. Func. Anal.*, **121** (1994), 275–295.
[12] Price, G. L., Shifts on Type II_1 Factors, *Canad. J. Math*, **39** (1987), 492–511.
[13] ———, Cocycle Conjugacy Classes of Shifts on the Hyperfinite II_1 Factor, II, *Preprint*.
[14] Thomsen, K., Topological Entropy for Endomorphisms of Local C*-Algebras, *Comm. Math. Phys.*, **164** (1994), 181–193.
[15] Voiculescu, D., Dynamical Approximation Entropies and Topological Entropy in Operator Algebras, *Comm. Math. Phys.*, **170** (1995), 249–281.
[16] Yin, H. S., Entropy of Certain Noncommutative Shifts, *Rocky Mount. J. Math.*, **20** (1990), 651–656.