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Normal Quintic Surfaces which are Birationally Enriques Surfaces

By

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§0. Introduction

Let S be an Enriques surface over an algebraically closed field k of characteristic $\neq 2$. Then, equivalently, S is a non-singular projective surface with $q(S)=p_g(S)=0$ and $2K_S \sim 0$. It is known (cf. Cossec [Co]) that every Enriques surface admits a morphism of degree one onto a surface of degree 10 in $\mathbf{P}^5$ with isolated rational double points, and also that every Enriques surface is birationally equivalent to a (non-normal) sextic surface in $\mathbf{P}^3$. Then there arises the following problem:

Problem. *Can we birationally embed S in $\mathbf{P}^3$ as a normal hypersurface? If yes, give the lower bound of the degree of the image.*

If X is a normal hypersurface of degree d in $\mathbf{P}^3$, then $\omega_X \cong \mathcal{O}_X(d-4)$. Hence, if $\pi : \tilde{X} \to X$ is the minimal resolution of X, then $\omega_{\tilde{X}} \cong \pi^*\mathcal{O}_{\tilde{X}}(d-4) \otimes \mathcal{O}_{\tilde{X}}(-\tilde{D})$, where $\tilde{D}$ is an effective divisor on $\tilde{X}$, whose support coincides with the sum of the exceptional sets for non-rational singularities of X. So, for X to be birationally an Enriques surface, it is necessary that $d \geq 5$ and $\tilde{D} \neq 0$. Castelnuovo [Ca] and Stagnaro [S] found normal quintic surfaces which are birationally equivalent to Enriques surfaces. In this paper we show (Theorem 2.1) that Enriques surfaces with certain conditions on their elliptic fibrations are birationally equivalent to normal quintic surfaces. Moreover we show (cf. Corollary 2.3) that generic Enriques surfaces and also all Enriques surfaces which are known to us now satisfy these conditions, and, conjecturally, so

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does every Enriques surface. §3 is devoted to proving Theorem 2.1 by constructing birational maps concretely. In §4, we will study the singularity of the normal quintic surfaces constructed in §3, and find their defining equations. It turns out that our quintic surfaces are those in Stagnaro [S]. Then we will prove, under a milder assumption than his, that the surfaces defined by the equations of this type are birationally equivalent to Enriques surfaces and are obtained by the construction of §3 (Theorem 4.3). Consequently the unirationality of the moduli space of Enriques surfaces is shown explicitly (cf. Cossec-Dolgachev [Co-D2] and Kondō [Ko]). In §1, we prove some properties of normal quintic surfaces, which are birationally Enriques surfaces. The author has found the construction in §3 from these observations.

For terminology and results on Enriques surfaces, we refer the reader to Barth-Peters-Van de Ven [B-P-V] and Cossec-Dolgachev [Co-D2]. For example, we call an effective divisor E on an Enriques surface S a *halfpencil* if $|2E|$ is base point free and defines an elliptic fibration on S. Then there exists on S a unique halfpencil E' adjoint with E: $E' \sim E+K_S$. Note that a halfpencil is reduced, and is either a non-singular elliptic curve, a rational curve with one node, or a cycle of non-singular rational curves. For an irreducible curve C, $g(C)$ stands for the genus of the normalization of C, whereas $p_a(C)$ the arithmetic genus of C. If Y_1 and Y_2 are cycles on a variety, we shall denote their intersection by $Y_1 Y_2$ or $Y_1 . Y_2$. However, if it represents a 0-cycle, then the intersection number of Y_1 and Y_2 is also denoted by $Y_1 Y_2$.

After writing up the first version of this paper, the author received Yonggu Kim's paper [Ki], in which he claims that every Enriques surface is birationally equivalent to a normal quintic surface. But actually his argument is incomplete in proving the existence of a divisor which defines the birational map.

The author would like to express her thanks to Professor I.V. Dolgachev for discussions and informing her of the work of Kim and Castelnuovo.

The main results of this paper were announced in [U].

§1. Birational Maps between Enriques Surfaces and Normal Quintic Surfaces

Let X be a normal quintic surface in $\mathbf{P}^3$. Assume that X is birationally equivalent to an Enriques surface S. In this section, we study the birational map between X and S.

Let $\pi: \tilde{S} \to X$ be the minimal resolution of all singularities on X.

Proposition 1.1. $\dim R^1\pi_*\mathcal{O}_{\tilde{S}}=4$.

Proof. Consider the exact sequence:

$$0\to H^1(X,\mathcal{O}_X)\to H^1(\tilde{S},\mathcal{O}_{\tilde{S}})\to R^1\pi_*\mathcal{O}_{\tilde{S}}\to H^2(X,\mathcal{O}_X)\to H^2(\tilde{S},\mathcal{O}_{\tilde{S}}).$$

Since X is a quintic surface, $\dim H^2(X,\mathcal{O}_X)=\dim H^0(X,\mathcal{O}_X(1))=4$. Since $\tilde{S}$ is birationally an Enriques surface, $H^1(\tilde{S},\mathcal{O}_{\tilde{S}})=0$ and $H^2(\tilde{S},\mathcal{O}_{\tilde{S}})=0$. Hence $\dim R^1\pi_*\mathcal{O}_{\tilde{S}}=4$. □

Let $H\subset X$ be a general hyperplane section of X and set $\tilde{H}=\pi^*H$. Then H is a non-singular curve of genus 6 and $\tilde{H}\cong H$.

Proposition 1.2. $\dim H^0(\tilde{S},\mathcal{O}_{\tilde{S}}(\tilde{H}))=4$, *i.e.* π *is defined by the complete linear system* $|\tilde{H}|$.

Proof. From the exact sequence:

$$0\to\mathcal{O}_{\tilde{S}}\to\mathcal{O}_{\tilde{S}}(\tilde{H})\to\mathcal{O}_{\tilde{H}}(\tilde{H})\to 0,$$

we have the long exact sequence:

$$0\to H^0(\tilde{S},\mathcal{O}_{\tilde{S}})\to H^0(\tilde{S},\mathcal{O}_{\tilde{S}}(\tilde{H}))\to H^0(\tilde{H},\mathcal{O}_{\tilde{H}}(\tilde{H}))\to H^1(\tilde{S},\mathcal{O}_{\tilde{S}})\to\cdots.$$

Since $H^1(\tilde{S},\mathcal{O}_{\tilde{S}})=0$, we obtain

$$\dim H^0(\tilde{S},\mathcal{O}_{\tilde{S}}(\tilde{H}))=\dim H^0(\tilde{H},\mathcal{O}_{\tilde{H}}(\tilde{H}))+1.$$

The Riemann-Roch Theorem for $\tilde{H}$ says that

$$\dim H^0(\tilde{H},\mathcal{O}_{\tilde{H}}(\tilde{H}))-\dim H^1(\tilde{H},\mathcal{O}_{\tilde{H}}(\tilde{H}))=\tilde{H}^2-6+1=0.$$

Therefore, by Clifford's Theorem,

$$\dim H^0(\tilde{H},\mathcal{O}_{\tilde{H}}(\tilde{H}))\leq 3.$$

Hence $\dim H^0(\tilde{S},\mathcal{O}_{\tilde{S}}(\tilde{H}))\leq 4$. Here the equality holds since X is not contained in any hyperplane in $\mathbf{P}^3$. □

Let $\mu:\tilde{S}\to S$ be the birational morphism from $\tilde{S}$ to S. Put $n=-K_{\tilde{S}}^2$. Then μ consists of n blowing-downs:

$$\tilde{S}=S_0\xrightarrow{\mu_1}S_1\xrightarrow{\mu_2}S_2\to\cdots\xrightarrow{\mu_n}S_n=S.$$

There exists on $\tilde{S}$ an effective divisor $\tilde{D}$ such that $\tilde{H}-\tilde{D}\sim K_{\tilde{S}}$. $\tilde{D}$ is supported by the sum of the exceptional sets of π, which correspond to non-rational singularities on X. Set $H_i=(\mu_i\circ\cdots\circ\mu_1)_*(\tilde{H})$, $D_i=(\mu_i\circ\cdots\circ\mu_1)_*(\tilde{D})$. We have $K_{S_i}\sim H_i-D_i$. Let $P_i\in S_i$ be the center of the blowing-up μ_i, and let $A_i=\mu_i^{-1}(P_i)$. Moreover we denote by $\tilde{D}_n$ and $\tilde{A}_i$ the proper transform of D_n and A_i to $\tilde{S}$ respectively. Set $m_i=\mathrm{mult}_{P_i}H_i$ and $d_i=\mathrm{mult}_{P_i}D_i$.

Proposition 1.3. (i) $\tilde{D}^2=-5-n$.

(ii) $1\leq n\leq 5$.

(iii) $m_i>0$ *and* $\Sigma_{i=1}^n m_i=5$.

(iv) $d_i>m_i$ *and* $\tilde{D}=\tilde{D}_n+\Sigma_{i=1}^n(d_i-m_i-1)\tilde{A}_i$.

Proof. (i) follows from $-n=K_{\tilde{S}}^2=\tilde{H}^2+\tilde{D}^2=5+\tilde{D}^2$. Let A_i^* be the total transform of A_i to $\tilde{S}$. Then we have first

$$m_i=H_{i-1}A_i=\tilde{H}A_i^*.$$

Since π is the minimal resolution of X and A_i^* has at least one (-1)-curve as a component, $\tilde{H}A_i^*>0$ for every i. So $m_i>0$. Second, from

$$K_{\tilde{S}}\sim\mu^*K_S+\sum_{i=1}^n A_i^*\equiv\sum_{i=1}^n A_i^*,$$

we have

$$5=\tilde{H}^2=\tilde{H}K_{\tilde{S}}=\sum_{i=1}^n\tilde{H}A_i^*,$$

and hence (ii) and (iii) are proved. For (iv), let e_i be the integers such that $\tilde{D}=\tilde{D}_n+\Sigma_{i=1}^n e_i\tilde{A}_i$. Then

$$D_{i-1}=\mu_i^*D_i-d_iA_i+e_iA_i.$$

Hence, with $H_{i-1}=\mu_i^*H_i-m_iA_i$, we have

$$K_{S_{i-1}}\sim H_{i-1}-D_{i-1}\sim\mu_i^*(H_i-D_i)+(d_i-e_i-m_i)A_i\sim\mu_i^*K_{S_i}+(d_i-e_i-m_i)A_i.$$

Thus we obtain $d_i-e_i-m_i=1$, and so $e_i=d_i-m_i-1$. Since $e_i\geq 0$, $d_i\geq m_i+1$. □

§2. A Sufficient Condition for an Enriques Surface to be Birationally a Normal Quintic Surface

Theorem 2.1. *Let S be an Enriques surface. Suppose there exist on S three halfpencils E_1, E_2, E_3 such that*:

(i) $E_1E_2 = E_2E_3 = E_3E_1 = 1$;

(ii) $(E_1 + E_1') \cap E_2 \cap E_3 = \emptyset$, *where E_1' denotes the halfpencil adjoint with E_1. Then S is birationally equivalent to a normal quintic surface in $\mathbf{P}^3$.*

We note, as we will see below, that there is no known example of an Enriques surface which does not satisfy the hypothesis in Theorem 2.1. The proof of Theorem 2.1 is given in the next section.

Cossec and Dolgachev [Co-D2] defined the non-degeneracy invariant $d(S)$ of an Enriques surface S, which is reformulated as follows:

$$d(S) = \max\left\{ r \;\middle|\; \begin{array}{l} \text{There exists on } S \text{ halfpencils } E_1, \dots, E_r \\ \text{such that } E_iE_j = 1 \ (1 \le i < j \le r) \end{array} \right\}.$$

Obviously the divisors $E_1, \dots, E_r$ as above are numerically independent, hence $d(S) \le 10$. Cossec [Co] showed that $d(S) = 10$ if S contains no (-2)-curve, which happens, for example, if S is generic (Barth-Peters [B-P], Cossec-Dolgachev [Co-D1]). On the other hand Cossec [Co] proved that $d(S) \ge 3$ for any Enriques surface S. But according to Cossec and Dolgachev [Co-D2], no Enriques surface with $d(S) = 3$ is known.

As for the condition (ii) of the Theorem 2.1, we have the following:

Proposition 2.2. *Every Enriques surface S with $d(S) \ge 4$ admits halfpencils E_1, E_2, E_3 satisfying the hypothesis* (i) *and* (ii) *of Theorem* 2.1.

Hence we obtain:

Corollary 2.3. *Every Enriques surface S with $d(S) \ge 4$ is birationally equivalent to a normal quintic surface in $\mathbf{P}^3$.*

To prove Proposition 2.2, we will begin with the following

Lemma 2.4. *Let E_1, E_2, E_3 be halfpencils on an Enriques surface S such that $E_1E_2 = E_2E_3 = E_3E_1 = 1$. Let E_i' denote the halfpencil adjoint with E_i $(i = 1, 2, 3)$.*

If E_1, E_2 *and* E_3 *meet at one common point, then so do* E_i, E'_j *and* E'_k, $\{i,j,k\}=\{1,2,3\}$.

Proof. First we note that E_i and E_j $(i\neq j)$ have no common components. (This can be seen easier in our situation, where E_i is at most a cycle of reduced (-2)-curves. A proof for a general situation is given in the Appendix.) Then we can define four different points $P_1=E_1\cap E_2$, $P_2=E_1\cap E'_2$, $P_3=E'_1\cap E_2$, $P_4=E'_1\cap E'_2$. By assumption, $P_1\in E_3$. Then, since $\mathcal{O}_{E_1}(E_2-E'_2)\cong\mathcal{O}_{E_1}(E_3-E'_3)$ and $p_a(E_1)=1$, we have $E_1\cap E'_3=P_2$. Similarly $E_2\cap E'_3=P_3$ by $\mathcal{O}_{E_2}(E_1-E'_1)\cong\mathcal{O}_{E_2}(E_3-E'_3)$ and $p_a(E_2)=1$. Hence, by $\mathcal{O}_{E'_1}(E_2-E'_2)\cong\mathcal{O}_{E'_1}(E'_3-E_3)$ and $p_a(E'_1)=1$, we have $E'_1\cap E_3=P_4$. □

Proof of Proposition 2.2. Let E_1, E_2, E_3, E_4 be any halfpencils on S with $E_iE_j=1$ $(1\leq i\neq j\leq 4)$. Let E'_i denote the halfpencil adjoint with E_i $(1\leq i\leq 4)$. Let $P_1,\dots,P_4$ be the same as in the proof of Lemma 2.4. If E_3 [resp. E_4] passes through neither P_1 nor P_3, then E_1, E_2, E_3 [resp. E_1, E_2, E_4] satisfy the conditions (i) and (ii) of Theorem 2.1.

Let us suppose that E_3 and E_4 pass through either P_1 or P_3. Then, by Lemma 2.4, we can assume that P_1, $P_4\in E_3$, E_4, by exchanging E_3 for E'_3 or E_4 for E'_4 if necessary. Recall that E_3 and E_4 have no common components. Then we obtain $E_3E_4\geq 2$, which contradicts our hypothesis. □

§3. Construction of Birational Maps

In this section we prove Theorem 2.1. For that, we shall explicitly

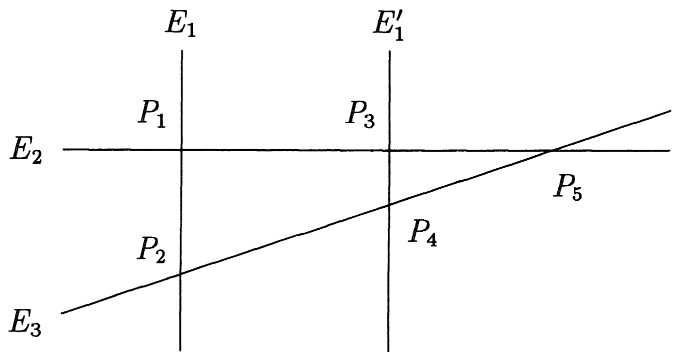

Figure 1

construct birational maps from Enriques surfaces, which satisfy the hypothesis of the Theorem, to normal quintic surfaces.

Let S, E_1, E_1', E_2, E_3 be as in Theorem 2.1. Set $\bar{D}=E_1+E_1'+E_2+E_3$ and $P_1=E_1\cap E_2$, $P_2=E_1\cap E_3$, $P_3=E_1'\cap E_2$, $P_4=E_1'\cap E_3$, $P_5=E_2\cap E_3$, so that $P_1,\dots,P_5$ are the set of double points of $\bar{D}$ other than singular points of E_1, E_1', E_2 and E_3 (see Figure 1).

Let $\bar{H}$ be a general member of $|\bar{D}+K_S|=|2E_1+E_2+E_3|$.

Lemma 3.1. (i) $\bar{H}^2=10$.
(ii) $\dim H^0(S,\mathcal{O}_S(\bar{H}))=6$.

Proof. (i) is clear since $E_iE_j=1-\delta_{ij}$ $(1\leq i,j\leq 3)$. For (ii), we have by the Riemann-Roch Theorem and (i),

$$\dim H^0(S,\mathcal{O}_S(\bar{H}))-\dim H^1(S,\mathcal{O}_S(\bar{H}))+\dim H^2(S,\mathcal{O}_S(\bar{H}))$$
$$=\frac{1}{2}\bar{H}(\bar{H}-K_S)+1=6.$$

$H^2(S,\mathcal{O}_S(\bar{H}))$ is the dual space of $H^0(S,\mathcal{O}_S(K_S-\bar{H}))=H^0(S,\mathcal{O}_S(-\bar{D}))=0$, and $H^1(S,\mathcal{O}_S(\bar{H}))$ is that of $H^1(S,\mathcal{O}_S(-\bar{D}))$. We will prove that $H^1(S,\mathcal{O}_S(-\bar{D}))=0$. Considering the exact sequence

$$0\to H^0(S,\mathcal{O}_S(-\bar{D}))\to H^0(S,\mathcal{O}_S)\to H^0(\bar{D},\mathcal{O}_{\bar{D}})$$
$$\to H^1(S,\mathcal{O}_S(-\bar{D}))\to H^1(S,\mathcal{O}_S)\to\cdots,$$

it is enough to show $H^0(-D,\mathcal{O}_{\bar{D}})\cong k$. In fact, since $\bar{D}$ is a union of different halfpencils, $\bar{D}$ is connected and reduced. Hence $H^0(\bar{D},\mathcal{O}_{\bar{D}})\cong k$. □

For $i=2,3$, let E_i' denote the halfpencil adjoint with E_i. By Lemma 2.4 and our assumption on E_1, E_1', E_2, E_3, we see that E_2' and E_3' do not pass through any of $P_1,\dots,P_5$.

Lemma 3.2. *Let* $s_0,\dots,s_5$ *be sections of* $H^0(S,\mathcal{O}_S(\bar{H}))$ *such that* $(s_0)_0=2E_1'+E_2+E_3$, $(s_1)_0=2E_1+E_2+E_3$, $(s_2)_0=E_1+E_1'+E_2'+E_3$, $(s_3)_0=E_1+E_1'+E_2+E_3'$, $(s_4)_0=2E_1+E_2'+E_3'$, $(s_5)_0=2E_1'+E_2'+E_3'$. *Then* $s_0,\dots,s_5$ *form a basis of* $H^0(S,\mathcal{O}_S(\bar{H}))$.

Proof. By the definition of E_i and E_i', it is easy to check that $s_0,\dots,s_5$

belong to $H^0(S, \mathcal{O}_S(\bar{H}))$. From Lemma 3.1 (ii), it suffices to prove that $s_0, \dots, s_5$ are linearly independent. Put $\bar{H}_i = (s_i)_0$ $(0 \le i \le 5)$. Then $\bar{H}_0$ and $\bar{H}_1$ have $E_2 + E_3$ as fixed components, but $\bar{H}_2$ does not; $\bar{H}_0$, $\bar{H}_1$ and $\bar{H}_2$ have E_3 as a fixed component, but $\bar{H}_3$ does not; $\bar{H}_0, \dots, \bar{H}_3$ have $P_1, \dots, P_5$ as base points, but $\bar{H}_4$ does not pass through P_3, P_4, P_5; $\bar{H}_0, \dots, \bar{H}_4$ have P_1 and P_2 as base points, but $\bar{H}_5$ does not pass through neither. Therefore $s_0, \dots, s_5$ are linearly independent. □

Let $\mu: \tilde{S} \to S$ be the blowing-up at $P_1, \dots, P_5$ and $\tilde{l}_i$ the (-1)-curve over P_i $(1 \le i \le 5)$. Let $\tilde{H}$ be the proper transform of a general member of the linear subspace $\Lambda = |\bar{H} - P_1 - \cdots - P_5|$ of $|\bar{H}|$.

Corollary 3.3. (i) $\dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{H})) = 4$.
(ii) $\tilde{H} \in |\mu^*(\bar{H}) - \tilde{l}_i - \cdots - \tilde{l}_5|$.
(iii) $\tilde{H}^2 = 5$.
(iv) $\mathrm{Bs}|\tilde{H}| = \emptyset$.
(v) *$\tilde{H}$ is an irreducible curve with $p_a(\tilde{H}) = 6$.*

Proof. (i): In the proof of Lemma 3.2, we saw that $s_0, \dots, s_3$ form homogeneous coordinates of the space Λ, and hence $4 = \dim \Lambda + 1 = \dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{H}))$. Moreover we have that $\mathrm{Bs}\Lambda = \{P_1, \dots, P_5\}$.

Proving (ii), (iii) and (iv) is then equivalent to showing, by Lemma 3.1 (i), that for each P_i there exist i_1 and i_2 $(0 \le i_1 \ne i_2 \le 3)$ such that $\bar{H}_{i_1}$ and $\bar{H}_{i_2}$ are smooth at P_i and have different tangents at P_i, where $\bar{H}_j = (s_j)_0$. Indeed we find such $\bar{H}_{i_1}$ and $\bar{H}_{i_2}$ as follows:

$$
\begin{aligned}
P_1 &: \quad \bar{H}_0,\ \bar{H}_2 \\
P_2 &: \quad \bar{H}_0,\ \bar{H}_3 \\
P_3 &: \quad \bar{H}_1,\ \bar{H}_2 \\
P_4 &: \quad \bar{H}_1,\ \bar{H}_3 \\
P_5 &: \quad \bar{H}_2,\ \bar{H}_3 .
\end{aligned}
$$

(v): By (iv), $\tilde{H}$ is irreducible. Moreover we have proved that $\mu_*\tilde{H}$ is also non-singular at $P_1, \dots, P_5$. Hence $p_a(\tilde{H}) = p_a(\bar{H}) = 1/2\bar{H}^2 + 1 = 6$ by Lemma 3.1, (i). □

This Corollary says that $|\tilde{H}|$ defines a morphism from $\tilde{S}$ to $\mathbf{P}^3$ and its image is a surface. Let X denote the image and $\pi:\tilde{S}\to X$ the induced morphism.

Lemma 3.4. *π is a birational morphism and* $\deg X=5$.

Proof. Let H be a general hyperplane section of $X\subset\mathbf{P}^3$. Then we may assume $\tilde{H}=\pi^*H$, and hence $\tilde{H}^2=\deg\pi\cdot H^2$. By Corollary 3.3 (iii), we obtain $\deg\pi=1$ and $H^2=5$. □

Lemma 3.5. *X is normal.*

Proof. The restriction morphism $\pi_{|\tilde{H}}:\tilde{H}\to H$ is a birational morphism onto a plane quintic curve H. From Corollary 3.3, (v), we get $p_a(\tilde{H})=6=p_a(H)$, which implies that the surface X has only isolated singularities, and hence that X is normal. □

Therefore $\pi\circ\mu^{-1}$ is a birational map from S to a normal quintic surface X, and hence Theorem 2.1 is proved.

§4. Singularity of X and Defining Equations

Let S be an Enriques surface which satisfies the hypothesis of Theorem 2.1, and X the quintic surface birationally equivalent to S as constructed from S in §3. We use also the other notations such as E_i, E_i', $\bar{H}$, $\bar{D}$, s_i, μ, $\tilde{H}$, $\tilde{l}_i$, π as in §2 and §3.

Let $\tilde{E}_i$ and $\tilde{E}_i'$ be the proper transform of E_i and E_i' to $\tilde{S}$ respectively $(1\le i\le 3)$. Then $\tilde{E}_1$, $\tilde{E}_1'$, $\tilde{E}_2$, $\tilde{E}_3$ are disjoint from each other, $\tilde{E}_1^2=\tilde{E}_1'^2=-2$ and $\tilde{E}_2^2=\tilde{E}_3^2=-3$. On E_i and E_i', μ is at most blowing-up at non-singular points for any i. Hence $p_a(\tilde{E}_i)=p_a(\tilde{E}_i')=1$. Moreover $p_a(A)\le 0$ for any non-zero effective divisor A such that $A<\tilde{E}_i$ or $A<\tilde{E}_i'$, and $\tilde{E}_i$ and $\tilde{E}_i'$ has no (-1)-curve as a component. Since $\tilde{E}_1$, $\tilde{E}_1'$, $\tilde{E}_2$, $\tilde{E}_3$ are disjoint from $\tilde{H}$, the image on X of these divisors are singular points.

Let $\bar{H}_i$ be as in the proof of Lemma 3.2. Set $\tilde{H}_i=\mu^*\bar{H}_i-\tilde{l}_1-\cdots-\tilde{l}_5$ $(0\le i\le 3)$. Then $\tilde{H}_i\in|\tilde{H}|$ by Corollary 3.3 (ii), and

$$\tilde{H}_0=2\tilde{E}_1'+\tilde{E}_2+\tilde{E}_3+2\tilde{l}_3+2\tilde{l}_4+\tilde{l}_5$$

$$\tilde{H}_1=2\tilde{E}_1+\tilde{E}_2+\tilde{E}_3+2\tilde{l}_1+2\tilde{l}_2+\tilde{l}_5$$

$$\tilde{H}_2=\tilde{E}_1+\tilde{E}_1'+\tilde{E}_2'+\tilde{E}_3+\tilde{l}_2+\tilde{l}_4$$

$$\tilde{H}_3 = \tilde{E}_1 + \tilde{E}'_1 + \tilde{E}_2 + \tilde{E}'_3 + \tilde{l}_1 + \tilde{l}_3 .$$

Since $\tilde{H}\tilde{l}_i = 1$, the image of $\tilde{l}_i$ on X is a line for $1 \leq i \leq 5$. Therefore we have first that $\tilde{E}_1$, $\tilde{E}'_1$, $\tilde{E}_2$, $\tilde{E}_3$ are contracted by π to distinct singular points, which we call Q_0, Q_1, Q_2, Q_3 respectively. We know (Laufer [L]) that each of them is either a simple elliptic or a cusp singularity. The other singularities, if they exist, are rational double points (cf. Proposition 1.1). Second, if $\tilde{C}$ is a curve on $\tilde{S}$ which is contracted by π to a rational double point, then $\tilde{H}_i\tilde{C} = 0$ for every i. Hence $\tilde{C}$ is disjoint from $\tilde{E}_1 + \tilde{E}'_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{l}_1 + \cdots + \tilde{l}_5$, and therefore $C = \mu_* \tilde{C}$ is disjoint from $\bar{D}$. Conversely, if an irreducible curve C on S is disjoint from $\bar{D}$, then C is a (-2)-curve and is also disjoint from $\bar{H}$, and so $\tilde{C} = \mu^* C$ is a (-2)-curve, which is disjoint from $\tilde{H}$ and $\tilde{E}_1 + \tilde{E}'_1 + \tilde{E}_2 + \tilde{E}_3$. Hence the image of $\tilde{C}$ on X is a rational double point.

For a singular point, let Z be the fundamental cycle of its minimal resolution. Then to sum up we have

Proposition 4.1. *The singularity of X consists of*

1) *two singularities with $Z^2 = -2$, each of which is either a simple elliptic or a cusp singularity; these correspond to E_1 and E'_1 on S;*

2) *two singularities with $Z^2 = -3$, each of which is either a simple elliptic or a cusp singularity; these correspond to E_2 and E_3 on S;*

and possibly

3) *rational double points; each of them corresponds to a connected component of the sum of all curves on S, which are disjoint from $\bar{D}$.*

Moreover, $\tilde{S}$ is the minimal resolution of all singularities on X.

Now we are going to find the defining equation of X in $\mathbf{P}^3$. Lemma 3.2 and Corollary 3.3 (ii) show that we can choose homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$ of $\mathbf{P}^3$ so that $\pi^* H_i = \tilde{H}_i$ $(0 \leq i \leq 3)$, where H_i denotes the section of X by the hyperplane $\{X_i = 0\}$. We fix such coordinates. $\tilde{H}_0$ is disjoint from $\tilde{E}_1$, while $\tilde{H}_i \geq \tilde{E}_1$ for $i \neq 0$. Therefore $Q_0 = (1:0:0:0)$. Similarly we have $Q_1 = (0:1:0:0)$, $Q_2 = (0:0:1:0)$ and $Q_3 = (0:0:0:1)$. By $\tilde{E}_1^2 = \tilde{E}_1'^2 = -2$, Q_0 and Q_1 are double points; by $\tilde{E}_2^2 = \tilde{E}_3^2 = -3$, Q_2 and Q_3 are triple points (Laufer [L]).

Let l_i denote the line $\pi_* \tilde{l}_i$ $(1 \leq i \leq 5)$. Then their defining equations are as follows (see Figure 2):

$l_1: X_1 = X_3 = 0$

$l_2: X_1 = X_2 = 0$

$l_3: X_0 = X_3 = 0$

$l_4: X_0 = X_2 = 0$

$l_5: X_0 = X_1 = 0$

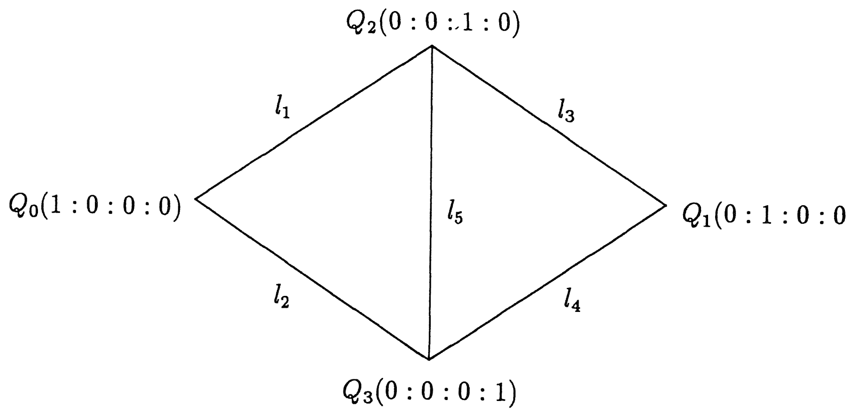

Figure 2

The defining equation of X in $\mathbf{P}^3$ corresponds to a non-trivial linear equation relating the monomials $\{s_0^{n_0}s_1^{n_1}s_2^{n_2}s_3^{n_3} \mid n_0+n_1+n_2+n_3=5\}$ in $H^0(S, \mathcal{O}_S(5\bar{H}))$. We find 14 out of these monomials, whose divisors of zeros are greater than $4E_1+4E_1'+3E_2+3E_3$, as follows:

$$s_0^3s_1^2,\ s_0^2s_1^3,\ s_0^2s_1s_2^2,\ s_0^2s_1s_3^2,\ s_0^2s_1^2s_2,\ s_0^2s_1^2s_3,\ s_0^2s_1s_2s_3,$$
$$s_0s_1^2s_2^2,\ s_0s_1^2s_3^2,\ s_0s_2^2s_3^2,\ s_0s_1^2s_2s_3,\ s_0s_1s_2^2s_3,\ s_0s_1s_2s_3^2,\ s_1s_2^2s_3^2.$$

We shall show that these are linearly dependent. Set $G=5\bar{H}-(4E_1+4E_1'+3E_2+3E_3)\sim 2E_1+2E_2+2E_3$ and $G'=E_1+E_1'+F_2+F_3$, where F_2 and F_3 are general members of $|2E_2|$ and $|2E_3|$ respectively. Then G' is connected and reduced, so $\dim H^0(G', \mathcal{O}_{G'})=1$. Moreover $G'\sim G-K_S$. By the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(-G')) \to H^0(S, \mathcal{O}_S) \to H^0(G', \mathcal{O}_{G'})$$
$$\to H^1(S, \mathcal{O}_S(-G')) \to H^1(S, \mathcal{O}_S) \to \cdots,$$

we have $H^1(S, \mathcal{O}_S(-G'))=0$ and so $H^1(S, \mathcal{O}_S(G))=0$. Therefore, with $H^2(S, \mathcal{O}_S(G)) \cong H^0(S, \mathcal{O}_S(-G'))^\vee = 0$, we obtain by the Riemann-Roch Theorem

$$\begin{aligned}\dim H^0(S, \mathcal{O}_S(G)) &= 1/2G(G-K_S)+1 \\ &= 13.\end{aligned}$$

Thus there exists a non-trivial linear relation among the 14 monomials above, and hence X is defined by an equation of the form:

$$\begin{aligned}(*)\quad F= {}& a_1X_0^3X_1^2 + a_1X_0^2X_1^3 + a_3X_0^2X_1X_2^2 + a_4X_0^2X_1X_3^2 \\ & + a_5X_0^2X_1^2X_2 + a_6X_0^2X_1^2X_3 + a_7X_0^2X_1X_2X_3 + a_8X_0X_1^2X_2^2 \\ & + a_9X_0X_1^2X_3^2 + a_{10}X_0X_2^2X_3^2 + a_{11}X_0X_1^2X_2X_3 \\ & + a_{12}X_0X_1X_2^2X_3 + a_{13}X_0X_1X_2X_3^2 + a_{14}X_1X_2^2X_3^2 \\ = {}& 0 \qquad\qquad (a_1, \ldots, a_{14} \in k).\end{aligned}$$

Since X has double points at $Q_0=(1:0:0:0)$ and $Q_1=(0:1:0:0)$, $a_1, a_2 \neq 0$. If $a_{10}=0$, then $X_1 \mid F$; if $a_{14}=0$, then $X_0 \mid F$. Hence $a_{10}, a_{14} \neq 0$. If $a_3=0$, then each monomial of F is divisible by X_1^2, X_1X_3 or X_3^2, and so X has singularity along l_1. Similarly X has singularity along l_2 if $a_4=0$, along l_3 if $a_8=0$, along l_4 if $a_9=0$. Since X is normal, we obtain $a_3, a_4, a_8, a_9 \neq 0$.

Stagnaro [S] has given these equations as one type of examples of quintics, whose resolutions are (blowing-up of) Enriques surfaces. Here we study them with regard to our construction in §3.

By multiplying some coordinates X_i by non-zero constants, we can assume, for example, $a_1=a_2=a_3=a_4=1$. Thus we have proved the following:

Theorem 4.2. *Let X be a normal quintic surface in $\mathbf{P}^3$, which is constructed in the way of §3 from an Enriques surface S satisfying the hypothesis of Theorem 2.1. Then, with suitable homogeneous coordinates $(X_0:X_1:X_2:X_3)$ of $\mathbf{P}^3$, the defining equation of X is of the following form*:

$$\begin{aligned}(**)\quad F= {}& X_0^3X_1^2 + X_0^2X_1^3 + X_0^2X_1X_2^2 + X_0^2X_1X_3^2 \\ & + c_1X_0^2X_1^2X_2 + c_2X_0^2X_1^2X_3 + c_3X_0^2X_1X_2X_3 + c_4X_0X_1^2X_2^2 \\ & + c_5X_0X_1^2X_3^2 + c_6X_0X_2^2X_3^2 + c_7X_0X_1^2X_2X_3 \\ & + c_8X_0X_1X_2^2X_3 + c_9X_0X_1X_2X_3^2 + c_{10}X_1X_2^2X_3^2\end{aligned}$$

$=0$

$$(c_4, c_5, c_6, c_{10} \neq 0,\ c_1, \dots, c_{10} \in k).$$

Remark. The condition $c_4, c_5, c_6, c_{10} \neq 0$ in Theorem 4.2 is not a sufficient condition for $X = \{F=0\}$ to be a normal surface.

In what follows we shall prove the converse of our construction, namely:

Theorem 4.3. *Let* $F=0$ *be the equation of the form* (**). *Set* $X = \{F=0\} \subset \mathbf{P}^3$, *and suppose that* X *has at worst isolated rational singularities outside* $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$ *and* $(0:0:0:1)$. *Then* X *is birationally equivalent to an Enriques surface* S, *which satisfies the hypothesis in Theorem* 2.1, *and* X *is constructed from* S *in the way described in* §3.

Remark. Stagnaro [S] proved that the surface defined by the equation (*) is birationally equivalent to an Enriques surface under the assumption that the singularity of the surface is the same as what may arise from an Enriques surface, which is the case, for example, if the coefficients a_i's are general.

Corollary 4.4. *Let* M *be the moduli space of the quadruple* (S, E_1, E_2, E_3) *as in Theorem* 2.1. *Then* M *is isomorphic to an open subset of the affine space* $\mathbf{A}^{10}$. *In particular, the moduli space of the Enriques surfaces is unirational.*

Before proving the Theorem, we note the following elementary facts.

Lemma 4.5. *Let* Y *be a non-singular surface and* $\sigma: \tilde{Y} \to Y$ *the blowing-up at a point* $P \in Y$. *Let* $\tilde{D}$ *be an effective divisor on* $\tilde{Y}$. *Then we have* $p_a(\tilde{D}) \leq p_a(\sigma_* \tilde{D})$.

Proof. Let E denote the exceptional curve for σ. Set $D = \sigma_* \tilde{D}$. Then $\tilde{D} = \sigma^* D + pE$ for some $p \in \mathbf{Z}$. If $p=0$, then

$$p_a(\tilde{D}) = p_a(\sigma^* D) = 1/2(\sigma^* D)(\sigma^* D + \sigma^* K_Y + E) + 1$$
$$= 1/2 D(D + K_Y) + 1 = p_a(D).$$

For $m > 0$, we have

$$p_a(mE)=1/2(mE)(mE+\sigma^*K_Y+E)+1=1-1/2m(m+1)\leq 0.$$

Hence if $p>0$, then

$$p_a(\tilde{D})=p_a(\sigma^*D)+p_a(pE)+(\sigma^*D)(pE)-1<p_a(\sigma^*D)=p_a(D).$$

If $p<0$, put $m=-p$. Then

$$\begin{aligned} p_a(D)&=p_a(\sigma^*D)=p_a(\sigma^*D-mE)+p_a(mE)+(\sigma^*D-mE)(mE)-1\\ &=p_a(\tilde{D})+1-1/2m(m+1)+m^2-1\geq p_a(\tilde{D}). \end{aligned}$$

□

Corollary 4.6. *Let $\sigma:\tilde{Y}\to Y$ be a birational morphism between non-singular surfaces. Let $\tilde{A}$ be a curve on $\tilde{Y}$ and set $A=\sigma(\tilde{A})$. Then*

$$\sup_{\substack{\tilde{D}>0\\ \mathrm{Supp}\tilde{D}\subseteq\tilde{A}}} p_a(\tilde{D})\leq \sup_{\substack{D>0\\ \mathrm{Supp}D\subseteq A}} p_a(D).$$

Proof. This follows from Lemma 4.5, since σ consists of blowing-ups. □

Proof of Theorem 4.3. The original form (∗) of F with the condition $a_1,a_2,a_3,a_4,a_8,a_9,a_{10},a_{14}\neq 0$ is symmetric with respect to X_0 and X_1; and X_2 and X_3. Hence $X=\{F=0\}$ has the same symmetry property, unless we assume some additional condition, which is incompatible with it. The reduction procedure from (∗) to (∗∗) and the assumptions on the singularity of X in our Theorem are compatible with this symmetry.

Set $Q_0=(1:0:0:0)$, $Q_1=(0:1:0:0)$, $Q_2=(0:0:1:0)$ and $Q_3=(0:0:0:1)$. Then Q_0 and Q_1 are double points, and Q_2 and Q_3 are triple points. Let $\pi:\tilde{S}\to X$ be the minimal resolution of X and $\tilde{H}=\pi^*H$, where H is a hyperplane section of X. Then there exists an effective divisor $\tilde{D}$ on $\tilde{S}$ such that (1) $\mathrm{Supp}\,\tilde{D}$ is included in the union of the exceptional sets over Q_0,Q_1,Q_2 and Q_3, and (2) $K_{\tilde{S}}\sim\tilde{H}-\tilde{D}$. Let H_0,H_1,H_2 denote the hyperplane sections of X through $Q_0,Q_2,Q_3;Q_1,Q_2,Q_3$; and Q_0,Q_1,Q_2 respectively, i.e., $H_0=X\cap\{X_1=0\}$, $H_1=X\cap\{X_0=0\}$ and $H_2=X\cap\{X_3=0\}$.

On X there are at least five lines (cf. Figure 2):

$$l_1=\overline{Q_0Q_2}\colon\ X_1=X_3=0$$

$$l_2=\overline{Q_0Q_3}\colon\ X_1=X_2=0$$

$$l_3=\overline{Q_1Q_2}:\ X_0=X_3=0$$

$$l_4=\overline{Q_1Q_3}:\ X_0=X_2=0$$

$$l_5=\overline{Q_2Q_3}:\ X_0=X_1=0$$

Let $\tilde{l}_i$ denote the proper transform of l_i to $\tilde{S}$. Let X_{smooth} be the smooth part of X. Then

Lemma 4.7. $l_i\backslash\{Q_0,\dots,Q_3\}\subset X_{\text{smooth}}$ $(1\le i\le 5)$.

Proof. Let $\mathscr{H}_i$ be the hyperplane $\{X_i=0\}$ in $\mathbf{P}^3$ with homogeneous coordinate $(X_0:\cdots:\check{X}_i:\cdots:X_3)$. Then we have $X.\mathscr{H}_3=l_1+l_3+C$, where C denote the curve defined in $\mathscr{H}_3$ by

$$X_0^2X_1+X_0X_1^2+X_0X_2^2+c_1X_0X_1X_2+c_4X_1X_2^2=0.$$

Moreover $C\cap l_1=\{Q_0,Q_2\}$ and $C\cap l_3=\{Q_1,Q_2\}$. Therefore $(l_1+l_3)\backslash\{Q_0,Q_1,Q_2\}\subset X_{\text{smooth}}$. From the symmetry, we have also $(l_2+l_4)\backslash\{Q_0,Q_1,Q_3\}\subset X_{\text{smooth}}$. Since $X.\mathscr{H}_0=2l_3+2l_4+l_5$, $l_5\backslash\{Q_2,Q_3\}\subset X_{\text{smooth}}$.

Now, let us look into properties near Q_0 and Q_1.

Lemma 4.8. (i) *Q_0 and Q_1 are not rational double points.*

(ii) *If Q_i ($i=0$ or 1) is minimally elliptic, then the fundamental cycle Z of its resolution satisfies $Z^2=-2$.*

Proof. It suffices to prove for Q_0. Let $\tilde{\sigma}:\tilde{\mathbf{P}}\to\mathbf{P}^3$ be the blowing-up at Q_0, $\mathbf{E}=\tilde{\sigma}^{-1}(Q_0)$, X' the proper transform of X and $\sigma:X'\to X$ the restriction of $\tilde{\sigma}$ to X'. Set $U=\{X_0\neq 0\}$, and $x=X_1/X_0$, $y=X_2/X_0$, $z=X_3/X_0$, so that (x,y,z) form coordinates of U with $Q_0=(0,0,0)$. Then

$$X\cap U=\left\{\begin{array}{l}x^2+x^3+xy^2+xz^2+c_1x^2y+c_2x^2z+c_3xyz+c_4x^2y^2\\ \qquad +c_5x^2z^2+c_6y^2z^2+c_7x^2yz+c_8xy^2z+c_9xyz^2+c_{10}xy^2z^2=0\end{array}\right\}.$$

$\tilde{U}:=\tilde{\sigma}^{-1}(U)$ is covered by three coordinate neighbourhoods $\tilde{U}_x$, $\tilde{U}_y$ and $\tilde{U}_z$. Let us consider $\tilde{U}_z$ with coordinates $u=x/z$, $v=y/z$, $w=z$. On $\tilde{U}_z$, $\mathbf{E}$ is defined by $w=0$, and X' by

$$u^2+u^3w+uv^2w+uw+c_1u^2vw+c_2u^2w+c_3uvw+c_4u^2v^2w^2$$

$$+c_5u^2w^2+c_6v^2w^2+c_7u^2vw^2+c_8uv^2w^2+c_9uvw^2+c_{10}uv^2w^3=0.$$

Hence $\{u=w=0\}=X'\cap \mathrm{E}\cap \tilde{U}_z$ is the one-dimensional singular locus of $X'\cap \tilde{U}_z$. By the symmetry, we conclude that $X'\cap \mathrm{E}=\sigma^{-1}(Q_0)$ is the singular locus of X' over Q_0. The blowing-up of a rational double point or a minimally elliptic singularity such that the fundamental cycle Z of its resolution satisfies $Z^2=-1$ has only isolated singularity (cf. Brieskorn [B], Laufer [L]). □

Corollary 4.9. (i) $p_g(\tilde{S})=0$.
(ii) $\dim R^1\pi_*\mathcal{O}_{\tilde{S}}=4+q(\tilde{S})$.

Proof. By the Lemma and since Q_2 and Q_3 are triple points, none of $Q_0,\dots,Q_3$ are rational double points. Hence $\pi^{-1}(Q_i)\subset \operatorname{Supp}\tilde{D}$ $(0\leq i\leq 3)$. But there are no hyperplane which passes through all Q_i, and so $p_g(\tilde{S})=\dim H^0(\tilde{S},\mathcal{O}_{\tilde{S}}(\tilde{H}-\tilde{D}))=0$. (ii) follows from (i) and the exact sequence

$$0\to H^1(X,\mathcal{O}_X)\to H^1(\tilde{S},\mathcal{O}_{\tilde{S}})\to R^1\pi_*\mathcal{O}_{\tilde{S}}\to H^2(X,\mathcal{O}_X)\to H^2(\tilde{S},\mathcal{O}_{\tilde{S}}).$$

□

Lemma 4.10. *For $i=0,1$, the multiplicity of any component of $\pi^{-1}(Q_i)$ in π^*H_i is greater than one.*

Proof. Again it suffices to prove for $i=0$. We use the notations in the proof of Lemma 4.8. If $\mathscr{H}_1$ is the hyperplane defined by $X_1=0$, then $\tilde{\sigma}^*\mathscr{H}_1$ is defined by $uw=0$ on $\tilde{U}_z$. And both $\{u=0\}$ and $\{w=0\}$ contain $\sigma^{-1}(Q_0)\cap\tilde{U}_z$. Therefore, if $\varphi:\tilde{X}'\to X$ is a resolution of X, which factors through X', then the multiplicity of any component of $\varphi^{-1}(Q_0)$ in φ^*H_0 is greater than 1. Hence the Lemma follows because any resolution of X factors through $\tilde{S}$. □

Next, we examine the blow-up of X at Q_2 and Q_3. Let $\pi':\tilde{S}'\to X$ be a resolution of X, which factors through the blowing-up X'' of X at Q_2 and Q_3. We assume furthermore that $\tilde{S}'$ is the minimal resolution of X'' if we restrict the morphism $\tilde{S}'\to X''$ to the normal part of X''. Let Z_i' denote the fundamental cycle of $\pi'^{-1}(Q_i)$ $(0\leq i\leq 3)$, and $\tilde{l}_i'$ [resp. l_i'] the proper transform of l_i to $\tilde{S}'$ [resp. X''] $(1\leq i\leq 5)$. Then

Lemma 4.11. *$\tilde{S}'$ has an elliptic fibration $\tilde{f}:\tilde{S}'\to\mathbf{P}^1$ such that*

(i) $\tilde{l}'_i$ *is contained in a fibre of* $\tilde{f}$ $(1 \leq i \leq 5)$;

(ii) *every multiple fibre of* $\tilde{f}$ *has multiplicity two*;

(iii) *the number of multiple fibres of* $\tilde{f}$ *is at most four*;

(iv) $2(Z'_0 + \tilde{l}'_1 + \tilde{l}'_2)$ *and* $2(Z'_1 + \tilde{l}'_3 + \tilde{l}'_4)$ *are multiple fibres of* $\tilde{f}$.

Moreover, Z'_0 *and* Z'_1 *are reduced.*

Proof. Let $\mathscr{H}_\alpha \cong \mathbf{P}^2$ be the hyperplane $\{X_1 = \alpha X_0\}$ in $\mathbf{P}^3$ with homogeneous coordinates $(X_0 : X_2 : X_3)$. *Then* $\mathscr{H}_\alpha . X$ is of the form $l_5 + C_\alpha$, where C_α is a curve of degree 4 on $\mathscr{H}_\alpha$ defined by

$$\begin{aligned}G_\alpha = &(\alpha^2 + \alpha^3)X_0^4 + \alpha^2 c_1 X_0^3 X_2 + \alpha^2 c_2 X_0^3 X_3 \\ &+ (\alpha + \alpha^2 c_4)X_0^2 X_2^2 + (\alpha + \alpha^2 c_5)X_0^2 X_3^2 + (\alpha c_3 + \alpha^2 c_7)X_0^2 X_2 X_3 \\ &+ \alpha c_8 X_0 X_2^2 X_3 + \alpha c_9 X_0 X_2 X_3^2 + (c_6 + \alpha c_{10})X_2^2 X_3^2 \\ = &0,\end{aligned}$$

while l_5 is defined by $X_0 = 0$ on $\mathscr{H}_\alpha$. Therefore, if α is general, then $C_\alpha \cap l_5 = \{Q_2, Q_3\}$ since $c_{10} \neq 0$. Let $x = X_0/X_3$ and $y = X_2/X_3$. Then the defining equation of C_α near Q_3 on $\mathscr{H}_\alpha$ is

$$\begin{aligned}g_\alpha = &(\alpha + \alpha^2 c_5)x^2 + \alpha c_9 xy + (c_6 + \alpha c_{10})y^2 + (\text{higher terms}) \\ = &0.\end{aligned}$$

From this and the symmetry we see that C_α has two nodes at Q_2 and Q_3, and that their branches are transversal to l_5 at Q_2 and Q_3, provided that α is general. Therefore the proper transform $\tilde{C}_\alpha$ of C_α to $\tilde{S}'$ is a curve of $p_a(\tilde{C}_\alpha) = 1$ and is disjoint from $\tilde{l}'_5$. Moreover $\tilde{C}_\alpha \cap \tilde{C}_{\alpha'} = \emptyset$ for general α, α'. Hence we deduce that $|\tilde{C}_\alpha|$ defines a (quasi-)elliptic fibration $\tilde{f} : \tilde{S}' \to \mathbf{P}^1$. For $\alpha = \infty$, we set $\mathscr{H}_\infty = \{X_0 = 0\}$ and $\mathscr{H}_\infty . X = l_5 + C_\infty$. Fix a general $\alpha_0 \in \mathbf{P}^1$ and let $L = \pi'^*(C_{\alpha_0} + l_5) - \tilde{C}_{\alpha_0} - \tilde{l}'_5$. Then $\operatorname{Supp} L = \pi'^{-1}(Q_2) \cup \pi'^{-1}(Q_3)$ by Lemma 4.7. We set $\tilde{C}_\alpha = \pi'^*(C_\alpha + l_5) - L - \tilde{l}'_5$ for every $\alpha \in \mathbf{P}^1$, then $\tilde{C}_\alpha$ is a fibre of $\tilde{f}$. If we take α satisfying $c_6 + \alpha c_{10} = 0$, so that $X_0 \mid G_\alpha$, we have $l_5 \subset C_\alpha$ and so $\tilde{l}'_5 \subset \tilde{C}_\alpha$. For $\alpha = 0$ and ∞, we have $C_0 = 2l_1 + 2l_2$ and $C_\infty = 2l_3 + 2l_4$, and so $2\tilde{l}'_1 + 2\tilde{l}'_2 \subset \tilde{C}_0$ and $2\tilde{l}'_3 + 2\tilde{l}'_4 \subset \tilde{C}_\infty$. Hence (i) is proved.

Let m_α denote the multiplicity of the fibre $\tilde{C}_\alpha$, then $g_\alpha = h_\alpha^{m_\alpha}$ for some h_α. Suppose $m_\alpha \geq 2$. Then the part of degree 2 of g_α is a square of a linear form. Therefore $(\alpha c_9)^2 - 4(\alpha + \alpha^2 c_5)(c_6 + \alpha c_{10}) = 0$. This equation has at most three solution in k, and hence there exist at most four $\alpha \in \mathbf{P}^1$ such that $m_\alpha \geq 2$. This proves (iii).

To prove (ii) and (iv), we look at the morphism $X'' \to X$ near Q_3. Let $\tilde{\tau}: \tilde{\mathbf{P}}' \to \mathbf{P}^3$ be the blowing-up at Q_2 and Q_3, $\mathrm{E}' = \tilde{\tau}^{-1}(Q_3)$ and $\tau = \tilde{\tau}_{|X''}: X'' \to X$. Set $V = \{X_3 \neq 0\}$ and $x = X_0/X_3$, $y = X_1/X_3$, $z = X_2/X_3$, so that (x, y, z) are coordinates of V with $Q_3 = (0, 0, 0)$. Then

$$X \cap V = \left\{ \begin{array}{l} x^3y^2 + x^2y^3 + x^2yz^2 + x^2y + c_1x^2y^2z \\ \qquad + c_2x^2y^2 + c_3x^2yz + c_4xy^2z^2 + c_5xy^2 \\ \qquad + c_6xz^2 + c_7xy^2z + c_8xyz^2 + c_9xyz + c_{10}yz^2 = 0 \end{array} \right\}.$$

$\tilde{V} := \tilde{\tau}^{-1}(V)$ is covered by three coordinate neighbourhoods $\tilde{V}_x$, $\tilde{V}_y$, and $\tilde{V}_z$. Let us consider $\tilde{V}_x$ with coordinates $u = x$, $v = y/x$, $w = z/x$. On $\tilde{V}_x$, E' is defined by $u = 0$ and X'' by

$$u^2v^2 + u^2v^3 + u^2vw^2 + v + c_1u^2v^2w + c_2uv^2 + c_3uvw + c_4u^2vw^2$$
$$+ c_5v^2 + c_6w^2 + c_7uv^2w + c_8uvw^2 + c_9vw + c_{10}vw^2 = 0.$$

So we see first that X'' is non-singular at $Q = (0, 0, 0)$. Since $l_2 = \{y = z = 0\}$ on V, $l_2' \cap \tilde{V} \subset \tilde{l}_2' \cap \tilde{V}_x$ and $\tilde{l}_2'$ is defined by $v = w = 0$ on $\tilde{V}_x$. On the other hand, $l_5' \cap \tilde{V}_x = \emptyset$ and $l_1' \cap \tilde{V}_x = \emptyset$. Let $E = \mathrm{E}'.X''$. Then E is non-singular at Q. Moreover we have that $\tau^*(2l_1 + 2l_2 + l_5) = X''.\tilde{\tau}^*\mathscr{H}_0 = 2l_2' + E$ on $\tilde{V}_x$ and that l_2' and E meet only at Q transversally. From these we have that $\pi'^*(2l_1 + 2l_2 + l_5) = 2\tilde{l}_2' + D_3'$ over $\tilde{V}_x$ for some effective divisor D_3' with $\mathrm{Supp}\, D_3' = \pi'^{-1}(Q_3)$ and $\tilde{l}_2' D_3' = 1$. Hence $(D_3' - L)\tilde{l}_2' = 0$. Since $\tilde{C}_0$ is connected, it follows from Lemma 4.7 and the symmetry that $\tilde{C}_0 = D_0' + 2\tilde{l}_1' + 2\tilde{l}_2'$ for some effective divisor D_0' such that $\mathrm{Supp}\, D_0' = \pi'^{-1}(Q_0)$. In particular we have $\tilde{C}_0 D_3' = 2$, which implies (ii). On the other hand, $\pi'^*(C_0 + l_5) = \pi'^* H_0$. So, by Lemma 4.10, all components of D_0' has multiplicity greater than one in D_0'. Hence $\tilde{C}_0$ is a multiple fibre of multiplicity two. Hence we can write $\tilde{C}_0 = 2\tilde{F}_0$. Let $\mu': \tilde{S}' \to S$ be a successive blowing-downs of (-1)-curves in fibres of $\tilde{f}$ such that the induced fibration $f: S \to \mathbf{P}^1$ is relatively minimal. Let $F_0 = \mu'_* \tilde{F}_0$ and $\bar{C}_0 = 2F_0$. Then $\bar{C}_0$ is a fibre of f and $\tilde{C}_0 = \mu'^* \bar{C}_0$. Note that F_0 is reduced. Moreover the induced morphism $D_0' \to F_0$ is surjective. In fact, since Q_0 is not a rational singularity (Lemma 4.8 (i)),

$$\sup_{\substack{D > 0 \\ \mathrm{Supp} D \subseteq \pi'^{-1}(Q_0)}} p_a(D) > 0$$

by Artin [A], and hence the surjectivity follows from Corollary 4.6. In particular $\tilde{l}_1'$ and $\tilde{l}_2'$ are exceptional for μ'. Since π' is the minimal resolution with respect to Q_0, $\tilde{l}_1'$ and $\tilde{l}_2'$ are the only possible (-1)-curves in $\tilde{C}_0$, and their multiplicities in $\tilde{F}_0$ is equal to one. This implies that $\tilde{F}_0$ is reduced. Since $\tilde{F}_0\Gamma=0$ for any irreducible component Γ of $\tilde{F}_0$, we see that $\tilde{F}_0-\tilde{l}_1'-\tilde{l}_2'$ coincides with the fundamental cycle Z_0'. Therefore, with the symmetry, (iv) and the last part of the Lemma is proved. □

Let $\mu':\tilde{S}'\to S$ be, as in the proof of Lemma 4.11, successive blowing-downs of (-1)-curves in fibres of $\tilde{f}$ so that we get a relatively minimal (quasi-)elliptic fibration $f:S\to\mathbf{P}^1$.

We consider in two cases.

Case 1: $q(S)=0$. Lemma 4.8 (i) and Corollary 4.9 (ii) say that Q_0, Q_1, Q_2, Q_3 are all minimally elliptic singularities. By Laufer [L] and Lemma 4.8 (ii), the minimal resolution factors the blowing-up of $Q_0,\dots,Q_3$. Hence $\tilde{S}'=\tilde{S}$, and so we write $\mu=\mu':\tilde{S}\to S$. For every i, $\tilde{l}_i$ lies in a fibre of $\tilde{f}$ (Lemma 4.11 (i)) and $\tilde{l}_i\cong\mathbf{P}^1$. Hence $\tilde{l}_i^2<0$. Moreover $\tilde{H}\tilde{l}_i=1$ and $\tilde{D}\tilde{l}_i\geq 2$ because l_i passes through two points from $\{Q_0,\dots,Q_3\}$. So, by $\tilde{H}\tilde{l}_i-\tilde{D}\tilde{l}_i+\tilde{l}_i^2=-2$, we have $\tilde{D}\tilde{l}_i=2$ and $\tilde{l}_i^2=-1$. Hence $\tilde{l}_i$ is contracted to a point by μ. Let $\tilde{D}=\tilde{E}_0+\cdots+\tilde{E}_3$, where $\tilde{E}_i$ is the part of $\tilde{D}$ such that $\operatorname{Supp}\tilde{E}_i=\pi^{-1}(Q_i)$. In our case $\tilde{E}_i$ coincides with the fundamental cycle of $\pi^{-1}(Q_i)$ (Laufer [L]). Therefore, from Lemma 4.7 and 4.11 (and its proof),

$$\pi^*H_0=2\tilde{E}_0+\tilde{E}_2+\tilde{E}_3+2\tilde{l}_1+2\tilde{l}_2+\tilde{l}_5+\Delta_0$$

$$\pi^*H_1=2\tilde{E}_1+\tilde{E}_2+\tilde{E}_3+2\tilde{l}_3+2\tilde{l}_4+\tilde{l}_5+\Delta_1,$$

where Δ_0 and Δ_1 are effective divisors whose support is included in $\operatorname{Supp}(\tilde{E}_2+\tilde{E}_3)$. On the other hand, since $\tilde{D}\tilde{l}_i=2$, we have

$$\begin{aligned}
&\tilde{E}_0\tilde{l}_1=\tilde{E}_2\tilde{l}_1=1, \quad \tilde{E}_1\tilde{l}_1=\tilde{E}_3\tilde{l}_1=0,\\
&\tilde{E}_0\tilde{l}_2=\tilde{E}_3\tilde{l}_2=1, \quad \tilde{E}_1\tilde{l}_2=\tilde{E}_2\tilde{l}_2=0,\\
&\tilde{E}_1\tilde{l}_3=\tilde{E}_2\tilde{l}_3=1, \quad \tilde{E}_0\tilde{l}_3=\tilde{E}_3\tilde{l}_3=0, \qquad (***)\\
&\tilde{E}_1\tilde{l}_4=\tilde{E}_3\tilde{l}_4=1, \quad \tilde{E}_0\tilde{l}_4=\tilde{E}_2\tilde{l}_4=0,\\
&\tilde{E}_2\tilde{l}_5=\tilde{E}_3\tilde{l}_5=1, \quad \tilde{E}_0\tilde{l}_5=\tilde{E}_1\tilde{l}_5=0.
\end{aligned}$$

Moreover, from Lemma 4.8 and Laufer [L], $\tilde{E}_0^2=\tilde{E}_1^2=-2$, $\tilde{E}_2^2=\tilde{E}_3^2=-3$. Hence we can calculate

$$\Delta_0^2 = \{\pi^* H_0 - (2\tilde{E}_0 + \tilde{E}_2 + \tilde{E}_3 + 2\tilde{l}_1 + 2\tilde{l}_2 + \tilde{l}_5)\}^2 = 0,$$

which means $\Delta_0 = 0$. Similarly $\Delta_1 = 0$. Therefore we obtain

$$\begin{aligned} 2K_{\tilde{S}} &\sim \pi^* H_0 + \pi^* H_1 - 2(\tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) \\ &= 2(\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 + \tilde{l}_5). \end{aligned}$$

This implies that the contraction of $\tilde{l}_1, \dots, \tilde{l}_5$ coincides with S and that S is a minimal surface with $2K_S \sim 0$. Together with $p_g(S) = 0$ (Corollary 4.9 (i)) and $q(S) = 0$, we conclude that S is an Enriques surface. Let $E_i = \mu_* \tilde{E}_i$ $(0 \le i \le 3)$. By Lemma 4.11 (iv), E_0 and E_1 are the halfpencils of the elliptic fibration f. Let us show that E_2 and E_3 are also halfpencils. By (***) we see $E_i^2 = 0$ and $p_a(\tilde{E}_i) = p_a(E_i) = 1$ for every i. But $|E_i|$ does not define an elliptic fibration since $E_i E_j = 1$ for some j. Since $Q_0, Q_1, Q_2 \in H_2$ and $Q_3 \notin H_2$, we have that $\pi^* H_2 = \tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}'_3$ for some effective divisor $\tilde{E}'_3$, which is disjoint from $\tilde{E}_3$. Hence

$$K_{\tilde{S}} \sim \pi^* H_2 - \tilde{D} = \tilde{E}'_3 - \tilde{E}_3,$$

and so $K_S \sim E'_3 - E_3$, where $E'_3 = \mu_* \tilde{E}'_3$. E_3 and E'_3 have no common components and $2E_3 - 2E'_3 \sim 2K_S \sim 0$. Thus $2E_3$ defines an elliptic fibration, and so E_3 is a halfpencil. Also E_2 is a halfpencil. Moreover (***) implies that $E_0 E_2 = E_2 E_3 = E_3 E_0 = 1$, E_0, E_1, E_2, E_3 meet at five different points, which we call $P_1, \dots, P_5$ (cf. Figure 1 with E_1 and E'_1 replaced by E_0 and E_1 respectively), and that μ is the blowing-up at $P_1, \dots, P_5$. Moreover

$$\begin{aligned} \mu_* \tilde{H} &\sim \mu_*(\pi^* H_0) = 2E_0 + E_2 + E_3 \\ &\sim E_0 + E_1 + E_2 + E_3 + K_S, \\ \tilde{H} &\sim \pi^* H_0 = \mu^*(\mu_*(\pi^* H_0)) - \tilde{l}_1 - \cdots - \tilde{l}_5 \\ &\sim \mu^*(\mu_* \tilde{H}) - \tilde{l}_1 - \cdots - \tilde{l}_5. \end{aligned}$$

Therefore X can be reconstructed from S in the way described in §3.

Case 2: $q(S) > 0$. From the canonical bundle formula and Lemma 4.11, we have $K_S \sim f^* \mathscr{M} + \Sigma_{i=0}^m F_i$, where $\mathscr{M}$ is a line bundle on $\mathbb{P}^1$ of degree $\chi(\mathcal{O}_S) - 2$ and $\{2F_0, \dots, 2F_m\}$ is the set of the multiple fibres of f with $1 \le m \le 3$. Let F be a general fibre of f. Then $K_S \equiv (\chi(\mathcal{O}_S) - 2 + 1/2(m+1))F$.

Obviously $\kappa(S) \le 1$. If $\kappa(S) = 1$, then $\chi(\mathcal{O}_S) - 2 + 1/2(m+1) > 0$, and hence $\chi(\mathcal{O}_S) > 0$. This is impossible since $q(S) > 0$ and $p_g(S) = 0$ (Corollary 4.9 (i)). If

$\kappa(S)=0$, then S is a (quasi-)hyperelliptic surface. But if $ch(k)=0$, then no normal quintic surface can be birationally a hyperelliptic surface (Nakamura-Umezu [N-U]). We will show that no (quasi-)hyperelliptic surface appears in our situation in any characteristic. Assume that S is a (quasi-)hyperelliptic surface. Then a similar argument as in the proof of Proposition 1.1–3 shows that $\dim R^1\pi_*\mathcal{O}_{\tilde{S}}=5$ and that the natural map $\mu:\tilde{S}\to S$ consists of at most five blow-ups. Moreover any rational curve C on S is, if it exists, a cuspidal curve with $C^2=0$ (Bombieri-Mumford [B-M]). From these we can deduce that the map μ is just the contraction of the disjoint (-1)-curves $\tilde{l}_1,\dots,\tilde{l}_5$ and that there exists a non-singular rational curve Γ in $\tilde{D}$ which is not exceptional for μ. Then we have $\Gamma\tilde{l}_i\geq 2$ for some i. Since $\tilde{l}_i$ meets at least two components of $\tilde{D}$, we obtain $K_{\tilde{S}}\tilde{l}_i=(\tilde{H}-\tilde{D})\tilde{l}_i\leq -2$, and hence a contradiction.

Suppose $\kappa(S)=-\infty$. Then S is birationally a ruled surface over a curve of genus $q(S)$. Since there are curves of arithmetic genus 1 on S, we obtain $q(S)=1$, $\chi(\mathcal{O}_S)=0$ and $m=1$ or 2. Moreover $K_S^2=0$ implies that S itself is minimal. Let $g:S\to E$ (E is an elliptic curve) denote the ruling of S. Since all fibres of f are mapped surjectively onto E, they are non-singular elliptic curves. Corollary 4.9 (ii) with our assumption says that there exists a unique i_0 $(0\leq i_0\leq 3)$ such that Q_{i_0} is a singularity of geometric genus 2 (i.e. $\dim(R^1\pi_*\mathcal{O}_{\tilde{S}})_{Q_{i_0}}=2$ by the definition due to Wagreich [W]), and other Q_i's are minimally elliptic singularities. We may assume $i_0=0$ or 3. If $i_0=0$, then $\tilde{S}'=\tilde{S}$. Let us show that the morphism $\mu':\tilde{S}'\to S$ factors through $\tilde{S}$ if $i_0=3$, too. Assume to the contrary. Let $\tilde{S}'=\tilde{S}_0\overset{\nu_1}{\to}\tilde{S}_1\overset{\nu_2}{\to}\cdots\overset{\nu_h}{\to}\tilde{S}_h=\tilde{S}$ be a sequence of blowing-downs from $\tilde{S}'$ to $\tilde{S}$. Let $A_i\subset\tilde{S}_{i-1}$ denote the exceptional (-1)-curve for ν_i. The proper transform $\tilde{A}_i$ of A_i to $\tilde{S}'$ is a component of $\pi'^{-1}(Q_3)$. Then there exists h_0 $(0\leq h_0\leq h)$ such that μ' factors through $\tilde{S}_{h_0}$ but does not through $\tilde{S}_{h_0+1}$. That is, $\tilde{A}_1,\dots,\tilde{A}_{h_0}$ are contained in fibres of $\tilde{f}$ but $\tilde{A}_{h_0+1}$ is not. Let $\rho:\tilde{S}_{h_0}\to S$ denote the induced morphism. Since $\rho_*A_{h_0+1}$ is a rational curve, it is a fibre of g. On the other hand, $\mu'_*(Z'_i)$ $(i=0,1)$ is not contained in a fibre of g because of Corollary 4.6 and $p_a(Z'_i)=1$. Hence $(\rho_*A_{h_0+1})(\mu'_*Z'_i)>0$ for $i=0,1$. Since A_{h_0+1} is disjoint from $(\nu_{h_0}\circ\cdots\circ\nu_1)_*(Z'_i)$ $(i=0,1)$ and since $\mu'_*(Z'_0)\cap\mu'_*(Z'_1)=\emptyset$ by Lemma 4.11 (iv), there lie on $\rho_*A_{h_0+1}$ at least two centers of blowing-ups in ρ. Therefore we get $A_{h_0+1}^2\leq -2$. This is a contradiction, and hence we conclude that $\mu':\tilde{S}'\to S$ factors through $\tilde{S}$. Let $\mu:\tilde{S}\to S$ be the induced morphism. Then, as in case 1, we see that $\tilde{l}_1,\dots,\tilde{l}_5$ are disjoint (-1)-curves and are exceptional for μ, and that $\tilde{D}\tilde{l}_i=2$ for every i. Let $\tilde{D}=\tilde{D}_0+\cdots+\tilde{D}_3$ such that $\operatorname{Supp}\tilde{D}_i=\pi^{-1}(Q_i)$. Set $D_i=\mu_*\tilde{D}_i$.

Since Q_i is not a rational singularity, D_i is not contained in a fibre of g (Artin [A] and Corollary 4.6), and hence D_i has a component E_i such that $g(E_i)\geq 1$. By Hidaka-Watanabe [H-W], Q_{i_0} is also an elliptic singularity. By definition (Wagreich [W]),

$$\sup_{\substack{D>0 \\ \mathrm{Supp}D\subseteq\pi'^{-1}(Q_{i_0})}} p_a(D)=1.$$

Let $\tilde{E}_i$ denote the proper transform of E_i on $\tilde{S}$. Then $\tilde{E}_i$ is a non-singular elliptic curve $(0\leq i\leq 3)$, $\tilde{D}_i=\tilde{E}_i$ for $i\neq i_0$, and

$$\tilde{D}_{i_0}=\gamma\tilde{E}_{i_0}+(\text{trees of non-singular rational curves}).$$

Hence $\pi:\tilde{S}\to X$ is also the minimal good resolution of singularities of X, and we can apply results on elliptic sequence. What we need here is the following (Yau [Y1], [Y2], Tomari [T]): There exists a decomposition $\tilde{D}_{i_0}=Z_{i_0,0}+Z_{i_0,1}+\cdots+Z_{i_0,l}+\tilde{E}_{i_0}$ $(l\geq 0)$ such that

(i) $Z_{i_0,0}$ is the fundamental cycle of $\pi^{-1}(Q_{i_0})$;

(ii) $Z_{i_0,0}>Z_{i_0,1}>\cdots>Z_{i_0,l}>\tilde{E}_{i_0}$;

(iii) $Z_{i_0,0}^2\leq Z_{i_0,1}^2\leq\cdots\leq Z_{i_0,l}^2\leq\tilde{E}_{i_0}^2$;

(iv) $\tilde{D}_{i_0}^2=\Sigma_{j=0}^{l}Z_{i_0,j}^2+\tilde{E}_{i_0}^2$;

(v) $\mathrm{mult}_{Q_{i_0}}X\geq-\Sigma_{j=0}^{l}Z_{i_0,j}^2$;

(vi) $\tilde{E}_{i_0}Z_{i_0,0}=0$.

Assume $i_0=0$. By Lemma 4.11 we have that $2E_0$ is a fibre of f and that $\mu^*E_0=Z_{0,0}+\tilde{l}_1+\tilde{l}_2$, which is reduced. Moreover we have seen that $\tilde{l}_1$ and $\tilde{l}_2$ are (-1)-curves and are disjoint from each other. Hence we get $Z_{0,0}^2=-2$. Since Q_0 is a double point, we have $l=0$ and so $\tilde{D}_0=Z_{0,0}+\tilde{E}_0$. Therefore

$$\mathcal{O}_{\tilde{E}_0}\cong\mathcal{O}_{\tilde{E}_0}(K_{\tilde{E}_0})\cong\mathcal{O}_{\tilde{E}_0}(\tilde{H}-\tilde{D}+\tilde{E}_0)\cong\mathcal{O}_{\tilde{E}_0}(-Z_{0,0}),$$

and hence $\mathcal{O}_{\tilde{E}_0}(\tilde{E}_0)\cong\mathcal{O}_{\tilde{E}_0}(-(Z_{0,0}-\tilde{E}_0))$. This happens only if every exceptional curve of μ, such that its center of blowing-up lies on E_0 or its proper transform, is a component of $Z_{0,0}$, and $\mathcal{O}_{E_0}(E_0)\cong\mathcal{O}_{E_0}$. But this is impossible since $\mathcal{O}_{E_0}(E_0)$ is of order 2 because $2E_0$ is a multiple fibre.

In what follows we assume $i_0=3$. Then $\tilde{D}_i=\tilde{E}_i$ for $0\leq i\leq 2$, and

$\tilde{E}_0^2=\tilde{E}_1^2=-2$, $\tilde{E}_2^2=-3$ (Laufer [L] and Lemma 4.8). By Lemma 4.11, the morphism μ is over E_0 [resp. over E_1] blowing-ups at two different points, say P_1 and P_2 [resp. P_3 and P_4], and $\mu^{-1}(P_j)=\tilde{l}_j$ $(1\leq j\leq 4)$. Hence E_0 intersects D_3 transversally only at P_2, and E_0 and D_3 are non-singular at P_2. Therefore $E_0D_3=1$. Let G be a general fibre and C_0 a minimal section of g. Put $e=-C_0^2$. Then $e\geq -1$ by Nagata [N]. From

$$-2C_0-eG\equiv K_S\equiv(-2+1/2(m+1))F,$$

we have $0\leq C_0(-K_S)=-e$, and so $e=0$ or -1.

Let us first suppose $e=0$. Then C_0 is contained in a fibre of f. Since $C_0G=1$ and f has no section, $2C_0$ is a multiple fibre. By Lemma 4.11 (iv), there is another multiple fibre $2C_1$. Then there are no multiple fibres of f other than $2C_0$ and $2C_1$ since f induces a double cover of rational curves $G\to\mathbf{P}^1$. We may assume $C_0=E_0$ and $C_1=E_1$. Hence $C_0D_3=E_0D_3=1$. If D_3 has no rational comoponents, then $D_3=\gamma E_3$ with $\gamma\geq 2$, which contradicts with $C_0D_3=1$. Hence D_3 contains a fibre G_0 of g as a component. Since $C_0(D_3-G_0)=0$, we conclude that $D_3=\gamma E_3+G_0$ with $\gamma\geq 2$ and E_3 is a (non-multiple) fibre of f. Since the restriction of g to E_3 is an unramified morphism of degree 2, E_3 and G_0 intersect transversally at two different points R_1 and R_2. Let $\tilde{G}_0$ denote the proper transform of G_0 to $\tilde{S}$ and $\tilde{R}_i$ the point on $\tilde{E}_3$ over R_i for $i=1,2$. We may assume that the connected component of $\tilde{D}_3-\gamma\tilde{E}_3$ which contains $\tilde{G}_0$ meets $\tilde{E}_3$ at $\tilde{R}_1$ and that R_2 is one of the centers of the blowing-ups in μ. Let n denote the number of blowing-ups in μ. Then

$$\begin{aligned} n&=K_S^2-K_{\tilde{S}}^2=-(\tilde{H}-\tilde{D})^2=-(5+\tilde{D}_0^2+\cdots+\tilde{D}_3^2)\\ &=2-\tilde{D}_3^2=2-\tilde{E}_3^2-\sum_{j=0}^{l}Z_{3,j}^2. \end{aligned}$$

Since Q_3 is a triple point, we have $-\Sigma_{j=0}^{l}Z_{3,j}^2\leq 3$, therefore $n\leq 5-\tilde{E}_3^2$. This implies that the centers of the blowing-ups in μ, other than $P_1,\dots,P_4$, lie on E_3 or its proper transform except for at most one point. Since $\tilde{D}\tilde{l}_5=2$, $\tilde{D}_3\tilde{l}_5=\tilde{E}_2\tilde{l}_5=1$. Therefore $\tilde{l}_5$ meets exactly one component of $\tilde{D}_3$, whose multiplicity in $\tilde{D}_3$ is equal to one. This component is not $\tilde{E}_3$ since $\gamma\geq 2$. Let P be the point obtained by contracting $\tilde{l}_5$. Then P is the center of a blowing-up in μ, other than $P_1,\dots,P_4$, which does not lie on a proper transform of E_3. From this and $\tilde{E}_3^2\geq Z_{3,0}^2\geq -3$, and since $\tilde{D}_3$ has no (-1)-curve as a component, we can deduce that the configuration of $\tilde{D}_3$ is

$$\tilde{E}_3 + (\text{less than three) chains of non-singular rational curves},$$

and that the fundamental cycle $Z_{3,0}$ is reduced. By $\tilde{E}_3 Z_{3,0}=0$, we have that $-\tilde{E}_3^2$ is equal to the number of the chains of rational curves in $\tilde{D}_3$, which is less than three. Furthermore we note that $\mathcal{O}_{E_3}(E_3)\simeq\mathcal{O}_{E_3}$. If $\tilde{E}_3^2=-1$, then we obtain $Z_{3,0}=\tilde{E}_3+\tilde{G}_0$ and $\tilde{E}_3\tilde{G}_0=\tilde{R}_1$, and so $l=0$, $\tilde{D}_3=2\tilde{E}_3+\tilde{G}_0$. Therefore

$$\mathcal{O}_{\tilde{E}_3}\cong\mathcal{O}_{\tilde{E}_3}(K_{\tilde{E}_3})\cong\mathcal{O}_{\tilde{E}_3}(\tilde{H}-\tilde{D}+\tilde{E}_3)\cong\mathcal{O}_{\tilde{E}_3}(-\tilde{E}_3-\tilde{G}_0)\cong\mathcal{O}_{\tilde{E}_3}(\tilde{R}_2-\tilde{R}_1),$$

and hence $R_1=R_2$, which is a contradiction. Assume $\tilde{E}_3^2=-2$. Since $Z_{3,0}^2\le\cdots\le Z_{3,l}^2\le\tilde{E}_3^2=-2$ and $-\Sigma_{j=0}^{l}Z_{3,j}^2\le 3$, we have $l=0$ and $\tilde{D}_3=\tilde{E}_3+Z_{3,0}$. Hence $\mathcal{O}_{\tilde{E}_3}\cong\mathcal{O}_{\tilde{E}_3}(-Z_{3,0})$ as above, and so $\mathcal{O}_{\tilde{E}_3}(Z_{3,0}-\tilde{E}_3)\cong\mathcal{O}_{\tilde{E}_3}(-\tilde{E}_3)$. Let $\tilde{R}$ be the point on $\tilde{E}_3$ such that $(Z_{3,0}-\tilde{E}_3)\tilde{E}_3=\tilde{R}_1+\tilde{R}$. We note that $\tilde{R}\neq\tilde{R}_1$. Moreover let R' be the other center than R_2 of the blowing-ups in μ, which lies on the proper transform of E_3. Then we have $\mathcal{O}_{\tilde{E}_3}(\tilde{R}_1+\tilde{R})\cong\mathcal{O}_{\tilde{E}_3}(\tilde{R}_2+\tilde{R}')$, where $\tilde{R}'$ denotes the point on $\tilde{E}_3$ over R'. If $\tilde{R}=\tilde{R}_2$, then $\tilde{R}'=\tilde{R}_1$, which is impossible because then the exceptional curve of the blowing-up at either R_1 or R_2, which should be a component of $Z_{3,0}$, remains (-1)-curve on $\tilde{S}$. But $\tilde{R}\neq\tilde{R}_2$ implies $\tilde{R}=\tilde{R}'$, and so $R_1=R_2$, again a contradiction.

Suppose $e=-1$. Then $(2-1/2(m+1))C_0F=C_0(-K_S)=1$, and hence $m=2$ and $C_0F=2$ since f has no section. Let s and t be the integers such that $E_0\equiv sC_0+tG$. Then we obtain $s+t=1$ by $C_0E_0=1/2C_0F=1$, and $s+2t=0$ by $E_0^2=0$ and $s=E_0G>0$. Therefore $s=2$ and $t=-1: E_0\equiv 2C_0-G$. Hence, from $E_0D_3=1$, we have

$$E_0(D_3-G)=-1<0.$$

This implies that D_3 has no rational curve as a component. Hence $D_3=\gamma E_3$, which contradicts $E_0D_3=1$ since $\gamma\ge 2$.

Therefore the case of $\kappa=-\infty$ does not occur, and we have completed the proof of Theorem 4.3.

Remark. In the proof above, we use the assumption on the singularity of X to exclude the case that $\kappa=-\infty$ and that X has five simple elliptic singularities. In fact, we can construct normal quintic surfaces X from elliptic ruled surfaces S with $e=-1$ or 0 (in the latter case $S=\mathbf{P}(\mathcal{O}_E\oplus\mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on the elliptic curve E such that $\mathcal{L}\not\cong\mathcal{O}$ and $\mathcal{L}^{\otimes 2}\cong\mathcal{O}_E$), such that X have five simple elliptic singularities at $Q_0(1:0:0:0)$, $Q_1(0:1:0:0)$,

$Q_2(0:0:1:0)$, $Q_3(0:0:0:1)$, and another point Q and satisfy Lemma 4.11. But the author does not know if their defining equations can be reduced to the form (**).

Appendix

Here we assume that the ground field k is an algebraically closed field of arbitrary characteristic.

Proposition. *Let E_1 and E_2 be effective divisors on a non-singular surface S such that:*

(i) *E_1 and E_2 have no (-1)-curve as a component;*

(ii) *$|m_1E_1|$ and $|m_2E_2|$ define respective elliptic fibrations for some $m_1, m_2 \geq 1$;*

(iii) $E_1E_2=1$.

Then E_1 and E_2 have no common components.

Proof. Assumption (ii) implies $E_iC=0$ for every irreducible component C of E_i. Moreover, with (i), any curve, whose support is properly contained in $\operatorname{Supp} E_i$, can be contracted to rational double points.

Set $E_1=Z_1+F_1$ and $E_2=Z_2+F_2$, where $\operatorname{Supp} Z_1=\operatorname{Supp} Z_2$, F_1, F_2, Z_1 have no common components, and Z_1, Z_2, F_1, F_2 are all effective. Put $A=(Z_1)_{\mathrm{red}}=(Z_2)_{\mathrm{red}}$. Assumimg $A\neq 0$, we shall deduce a contradiction. Assumption (iii) implies $F_1\neq 0$ and $F_2\neq 0$. Hence A is the (possibly disconnected) exceptional set of the minimal resolution of rational double points. Let Z_0 denote the fundamental cycle of any connected component of A. It is clear that Z_1C, $Z_2C\leq 0$ for every irreducible component C of A, and so we obtain $Z_0\leq Z_1$, Z_2. Therefore we have

$$Z_1Z_2\leq Z_1Z_0\leq Z_0^2=-2.$$

On the other hand, we have from (iii)

$$1=E_1E_2=(Z_1+F_1)E_2=F_1E_2=F_1Z_2+F_1F_2.$$

Here $F_1Z_2\geq 1$ since E_1 is connected, and $F_1F_2\geq 0$. Therefore $F_1Z_2=1$. But this contradicts $Z_1Z_2\leq -2$, because

$$0=E_1Z_2=Z_1Z_2+F_1Z_2.$$

□

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