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On the Relation between Tautly Imbedded Space Modulo an Analytic Subset S and Hyperbolically Imbedded Space Modulo S

By

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Introduction

Let X be a complex manifold, M a relatively compact domain of X and S an analytic subset of X. We denote the open unit disk in the complex plane $\boldsymbol{C}$ by Δ, the polydisk $\Delta \times \cdots \times \Delta$ in $\boldsymbol{C}^k$ by Δ^k and the Kobayashi pseudodistance of M by d_M (see [Ko] for its definition and basic properties). The space of holomorphic maps from a manifold N to a manifold M with compact-open topology will be denoted by Hol (N, M).

Following Kiernan-Kobayashi [K-K] and [L], we use the following terminologies.

M is tautly imbedded modulo S in X if for each positive integer k and each sequence $\{f_j\}$ in $\mathrm{Hol}(\Delta^k, M)$ we have one of the following:

(a) $\{f_j\}$ has a subsequence which converges in $\mathrm{Hol}(\Delta^k, X)$;

(b) for each compact set $K \subset \Delta^k$ and each compact set $L \subset X \backslash S$, there exists an integer N such that $f_j(K) \cap L = \phi$ for $j \geqq N$.

M is hyperbolically imbedded modulo S in X if, for every pair of distinct points p, q of $\bar{M}$, closure of M, not both contained in S, there exist neighborhoods V_p and V_q of p and q respectively in X such that $d_M(V_p \cap M,\ V_q \cap M) > 0$.

In [K-K], it was proved that if M is tautly imbedded modulo S in X, then M is hyperbolically imbedded modulo S in X and brought up the inverse problem. But we believe there is no results except for the case $S = \phi$ (see Kiernan [Ki2] in case $S = \phi$). In this note, we deal with the inverse problem

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for the case S is a curve and M and X are special manifolds and give an affirmative answer (Theorem 4.4).

§1. Degeneracy Locus of the Kobayashi Pseudodistance

Throughout the sections $1 \sim 2$, let X be a complex manifold, M a relatively compact domain of X and d_M the Kobayashi pseudodistance of M. In [A-S2] we extended d_M to $\bar{M}$, the closure of M in X, as follows:
For $p, q \in \bar{M}$, we define

$$\bar{d}_M(p,q) = \varliminf_{p' \to p, q' \to q} d_M(p', q'), \quad p', q' \in M.$$

It is clear that $0 \leqq \bar{d}_M(p,q) \leqq \infty$. The function $\bar{d}_M$ does not satisfy the triangle inequality. For example, let $M = \{\mathbf{C} \setminus \{0,1\}\} \times \mathbf{C}$, $X = \mathbf{P}^2$, where $\mathbf{P}^2$ is the two-dimensional complex projective space, $p = [0:0:1]$, $q = [1:2:1]$ and $r = [1:3:1]$. It is obvious that $\bar{d}_M(p,q) = \bar{d}_M(p,r) = 0$. And $\bar{d}_M(q,r) = d_M(q,r) \geqq d_{\mathbf{C}\setminus\{0,1\}}(\pi(q), \pi(r)) > 0$, where π is the projection of M to $\mathbf{C}\setminus\{0,1\}$. So $\bar{d}_M$ is not a pseudodistance on $\bar{M}$.

Definition 1.1. *We call $p \in \bar{M}$ a degeneracy point of $\bar{d}_M$ if there exists a point $q \in \bar{M} \setminus \{p\}$ such that $\bar{d}_M(p,q) = 0$. By $S_M(X)$ we denote the set of the degeneracy points of $\bar{d}_M$ on $\bar{M}$ and call it the degeneracy locus of $\bar{d}_M$ in X.*

Definition 1.2. (cf. [T] and [F]). *A closed subset E of X will be called a pseudoconcave subset of order* 1, *if for any coordinate neighborhood*

$$U: |z_1| < 1, \cdots, |z_n| < 1$$

of X and positive numbers r, s with $0 < r < 1$, $0 < s < 1$ such that $U^* \cap E = \phi$, one obtains $U \cap E = \phi$, where

$$U^* = \{p \in U;\ |z_1(p)| \leqq r\} \cup \{p \in U;\ s \leqq \max_{2 \leqq i \leqq n} |z_i(p)|\}.$$

In [A-S2], we proved the following

Theorem 1.3. *The set $S_M(X)$ is a pseudoconcave subset of order* 1 *in X.*

Let S be an analytic subset of X. Using the extended function $\bar{d}_M$, we can show that M is hyperbolically imbedded modulo S in X if and only if for every pair of distinct points $p, q \in \bar{M}$ not both contained in S, $\bar{d}_M(p, q) > 0$ and M is hyperbolically imbedded space in X if and only if for every pair of distinct points $p, q \in \bar{M}$ $\bar{d}_M(p, q) > 0$, that is $S_M(X) = \phi$.

It is easy to see the following proposition.

Proposition 1.4. *M is hyperbolically imbedded modulo S in X if and only if $S \supset S_M(X)$.*

§2. Normality and Cluster Sets of a Sequence of Holomorphic Maps

In [A-S1] we defined cluster sets of a sequence of holomorphic maps. Let a sequence $F = \{f_j\}$ in $\mathrm{Hol}(\Delta^k, M)$.

Definition 2.1. *We define the cluster set $F(a:X)$ of F at a point a of Δ^k by*

$$F(a:X) = \bigcap_{\varepsilon>0} \bigcap_{N=1}^{\infty} \overline{\bigcup_{j \geqq N} f_j(U_\varepsilon(a))},$$

where $U_\varepsilon(a) = \{z \in \Delta^k; \|z - a\| < \varepsilon\}$.

Let $F(\Delta^k : X) = \bigcup_{a \in \Delta^k} F(a:X)$.

Definition 2.2. *A sequence $F = \{f_j\}$ in* $\mathrm{Hol}(\Delta^k, M)$ *is normal at $a \in \Delta^k$ if there exists a neighborhood U of a such that every subsequence of F has a convergent subsequence in $Hol(U, X)$.*

Clearly, we have

Proposition 2.3. *If the cluster set $F(a:X)$ of a sequence F in* $\mathrm{Hol}(\Delta^k, M)$ *consists of finite number of points of X, F is normal at a.*

Proposition 2.4. *If there exist a point a and a sequence of points a_j in Δ^k such that $a_j \to a$ and $f_j(a_j) \to p \notin S_M(X)$, then F is normal at a.*

Proof. Since $S_M(X)$ is a closed set, there exists a closed neighborhood V of p biholomorphic to a closed unit ball in X such that $V \cap S_M(X) = \phi$. If we define $\bar{d}_M(p, q) = \infty$ for $q \in X \backslash \bar{M}$, for some $\varepsilon > 0$, $\bar{d}_M(p, \partial V) \geqq \varepsilon$ where ∂V

denotes the boundary of V in X. Then $U(a)=\{z;\ d_{\Delta^k}(a,z)<\frac{\varepsilon}{2}\}$ is relatively compact in Δ^k. We shall prove $f_j(U(a))\subset V$ for $j\geqq N$ where N is a sufficiently large integer. If it is not true, we may assume $f_{j_\lambda}(b_{j_\lambda})\in\partial V$ where $b_{j_\lambda}\in U(a)$, $b_{j_\lambda}\to b\in\overline{U(a)}$ and $f_{j_\lambda}(b_{j_\lambda})\to q\in\partial V$ since $f_j(a_j)\in V$ for sufficiently large j. This is absurd, because

$$\bar{d}_M(p,q)\leqq\lim_{j_\lambda\to\infty} d_M(f_{j_\lambda}(a_{j_\lambda}),\ f_{j_\lambda}(b_{j_\lambda}))$$

$$\leqq\lim_{j_\lambda\to\infty} d_{\Delta^k}(a_{j_\lambda}, b_{j_\lambda})=d_{\Delta^k}(a,b)\leqq\frac{\varepsilon}{2}. \qquad \square$$

Corollary 2.5. (Theorem 1 in [Ki2]). *If $S_M(X)=\phi$, then F has a subsequence which converges in* $\mathrm{Hol}(\Delta^k, X)$ *and consequentry M is tautly imbedded in X.*

Corollary 2.6. *If $F(a:X)\ni p$ and $p\notin S_M(X)$, then F has a subsequence which converges in a neighborhood of a.*

It is easy to see the following

Proposition 2.7. *Let S be a closed subset of X. Let F be a sequence $\{f_j\}$ in* $\mathrm{Hol}(\Delta^k, M)$. *For each compact set K and each compact set $L\subset X\backslash S$, there exists an integer N such that $f_j(K)\cap L=\phi$ for $j\geqq N$ if and only if $F(\Delta^k:X)\subset S$.*

§3. An Auxiliary Theorem (Theorem 3.4)

Lemma 3.1. *Let X be a complex manifold, M be a relatively compact domain of X and F be a sequence $\{f_j\}$ in* $\mathrm{Hol}(\Delta^k, M)$. *Let D be a convergence domain of F with limit $f\in\mathrm{Hol}(D,X)$ and $D\subsetneqq\Delta^k$. If $a\in E=\partial D\backslash\partial\Delta^k$, then $F(a:X)\subset S_M(X)$.*

Proof. We prove the lemma in 3 steps.

(1) We show if $F(a:X)=Q\cup S$, $Q\neq\phi$, $Q\cap S_M(X)=\phi$ and $S\subset S_M(X)$, then $Q=\{p\}$. If $Q\ni p_1, p_2$, there exist a neighborhood $U(a)$ and subsequences F_1, F_2 of F such that F_i converges to f_i on $U(a)$ and $f_i(a)=p_i$ $(i=1,2)$ from Corollary 2.6. Since f_1 and f_2 are analytic continuations of f, $p_1=p_2$ from the uniqueness of continuation.

(2) We show if $F(a:X)=\{p\}\cup S$ and $S\subset S_M(X)$ then $S=\phi$. Since f has an

analytic continuation at a and $f(a)=p$ from Corollary 2.6, $f(a_j)\to p$ for every $a_j\in D$ such that $a_j\to a$. Let $\{a_j\}_{j=1,2,\dots}$ be some points of D such that $a_j\to a$ and $f(a_j)=p_j$ and ε_j be positive numbers such that $\varepsilon_j\to 0$. There exists an integer N_1 such that $d(f_j(a_1),p_1)<\varepsilon_1$ for $j>N_1$, an integer N_2 such that $d(f_j(a_1),p_1)<\varepsilon_2$ and $d(f_j(a_2),p_2)<\varepsilon_2$ for $j>N_2$, an integer N_3 such that $d(f_j(a_1),p_1)<\varepsilon_3$, $d(f_j(a_2),p_2)<\varepsilon_3$ and $d(f_j(a_3),p_3)<\varepsilon_3$ for $j>N_3$ and so on, where d is a distance on X. We may assume $N_1<N_2<N_3<\cdots$. Set $\tilde{a}_j=a_1$ for $j\leqq N_2$, $\tilde{a}_j=a_2$ for $N_2<j\leqq N_3$, $\tilde{a}_j=a_3$ for $N_3<j\leqq N_4$ and so on. Then $\tilde{a}_j\in D$, $\tilde{a}_j\to a$ and $f_j(\tilde{a}_j)\to p$. Let $q\in S$, then there exist $b_\lambda\in\Delta^k$ such that $b_\lambda\to a$ and $f_{j_\lambda}(b_\lambda)\to q$. Since there exists a sufficiently small closed neighborhood V of p such that $V\cap S_M(X)=\phi$, there exist $c_\mu\in\Delta^k$ such that $c_\mu\to a$ and $f_{j_{\lambda_\mu}}(c_\mu)\to r\in\partial V$. This contradicts $Q=\{p\}$.

(3) If $F(a:X)=\{p\}$, F is normal at a from Proposition 2.3. Then F converges on a neighborhood of a from Vitali's theorem. This is absurd. If $Q=\phi$, there is no problem. □

Lemma 3.2. *Let A and S be curves of $\boldsymbol{P}^2$ and set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\backslash A$. Let M be tautly imbedded modulo S in X and let F be a sequence $\{f_j\}$ in* $\mathrm{Hol}(\Delta^k, M)$ *without any convergent subsequence in* $\mathrm{Hol}(\Delta^k, X)$. *Let $D\neq\phi$ be the convergence domain of F with limit $f\in\mathrm{Hol}(D,X)$. Set $E=\Delta^k\backslash D$. Then either E is contained in an analytic subset of Δ^k or $f(D)\subset S_M(X)$.*

Proof. Since M is hyperbolically imbedded modulo S in X from Theorem 1 in [K-K], $S\supset S_M(X)$ from Proposition 1.4. Then $S_M(X)=\phi$ or a curve from Theorem 1.3. Since F has not any convergent subsequence in $\mathrm{Hol}(\Delta^k, X)$, $S_M(X)$ is a curve from Corollary 2.5. Since $F(\Delta^k:X)\subset S$ from Proposition 2.7, $f(D)\subset S$. Suppose $f(D)\subset S_0$ and $f(D)\not\subset S_M(X)$ where S_0 is an irreducible component of S which is not contained in $S_M(X)$. If $f(a_j)\to p$ for a point $a\in\partial E\backslash\partial\Delta^k$, a sequence of points $a_j\in D$ and $a_j\to a$, $p\in S_0\cap S_M(X)$ from Lemma 3.1. There is a rational function g on $\boldsymbol{P}^2$ which takes zero only on $S_M(X)$ and takes pole only on a line L such that $L\cap S_M(X)\cap S_0=\phi$. Then the points of indeterminancy of g are contained in $\boldsymbol{P}^2\backslash S_0$. So $\Phi=g\circ f$ is meromorphic in D and has not a point of indeterminancy. And $\lim_{z\to a}\Phi(z)=0$ for $z\in D$ and $a\in\partial E\backslash\partial\Delta^k$. Therefore $\bar{P}\cap(\partial E\backslash\partial\Delta^k)=\phi$ where P denotes the pole divisor of Φ.

If we set $\Phi\equiv 0$ on E, then Φ is continuous on $Y=\Delta^k\backslash P$, E is contained in the zeros of Φ and Φ is holomorphic in $Y\backslash E$. So Φ is holomorphic in Y from Rado's theorem. Therefore either $\Phi\equiv 0$ on Δ^k or $\Phi\not\equiv 0$ and E is contained in an analytic subset of Δ^k. The former case contradicts to the assumption since $f(D)\not\subset S_M(X)$. □

Lemma 3.3. *Let $A_1,\cdots,A_l$ be l ($l\geqq 1$) irreducible hypersurfaces of P^n and set $X=P^n$, $M=P^n\backslash(A_1\cup\cdots\cup A_l)$. Let $\{f_j\}$ be in* $\mathrm{Hol}(\Delta^k, M)$, *$E$ be an analytic subset of Δ^k and $\{f_j\}$ converge to f in* $\mathrm{Hol}(\Delta^k\backslash E, X)$. *Then either $\{f_j\}$ converges to f in* $\mathrm{Hol}(\Delta^k, X)$ *or $f(\Delta^k\backslash E)\subset\bigcap_{i=1}^{l} A_i$.*

Proof. Since $f_j(\Delta^k)\cap A_i=\phi$ for all j, we have, from Hurwitz's theorem, either $f(\Delta^k\backslash E)\cap A_i=\phi$ or $f(\Delta^k\backslash E)\subset A_i$ for each $i=1,\cdots,l$. Therefore, if $f(\Delta^k\backslash E)\not\subset\bigcap_{i=1}^{l} A_i$, then $f(\Delta^k\backslash E)\cap A_i=\phi$ for some i. Since $P^n\backslash A_i$ is a Stein manifold, it is imbedded into C^N by Φ. Recalling the maximum principle, $\Phi\circ f_j$ converges in $\mathrm{Hol}(\Delta^k, C^N)$. Therefore $\{f_j\}$ converges in $\mathrm{Hol}(\Delta^k, P^n\backslash A_i)$. □

Theorem 3.4. *Let A be a curve of P^2 whose number of irreducible components is greater than 1 and S be a curve of P^2. Set $X=P^2$ and $M=P^2\backslash A$. If M is tautly imbedded modulo S in X, then M is tautly imbedded modulo $S_M(X)$ in X.*

Proof. Since $S\supset S_M(X)$, $S_M(X)=\phi$ or a curve. If $S_M(X)=\phi$, above theorem is correct from Corollary 2.5. So we assume $S_M(X)$ is a curve and show that $F(\Delta^k\colon X)\subset S_M(X)$ if F be a sequence $\{f_j\}$ in $\mathrm{Hol}(\Delta^k, M)$ which has not any convergent subsequence in $\mathrm{Hol}(\Delta^k, X)$.

Suppose there exists a point $a\in\Delta^k$ such that $F(a\colon X)\ni p\notin S_M(X)$. Then there are a subsequence F' of F and a neighborhood $U(a)$ of a such that F' converges to f in $\mathrm{Hol}(U(a), X)$ from Corollary 2.6. Let D be a convergence domain of F' which contains $U(a)$. From the assumption $D\subsetneqq\Delta^k$. Set $E=\Delta^k\backslash D$. From Lemma 3.2 E is contained in an analytic subset of Δ^k. From Lemma 3.3 $f(\Delta^k\backslash E)\subset\bigcap_{i=1}^{l} A_i$, where $A_1,\cdots,A_l$ are irreducible components of A. If $\bigcap_{i=1}^{l} A_i=\phi$, it is a contradiction since $f(\Delta^k\backslash E)\neq\phi$. If $\bigcap_{i=1}^{l} A_i=\{q_1\}\cup\cdots\cup\{q_t\}$, $f(\Delta^k\backslash E)=\{q_s\}=\{p\}$ $(1\leqq s\leqq t)$. So $F'(E\colon X)\ni p$. This is a contradiction since $F'(E\colon X)\subset S_M(X)$ from Lemma 3.1. □

Corollary 3.5. *Let A be a curve of $\boldsymbol{P}^2$ whose number of irreducible components is greater than* 1. *Set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\backslash A$.* If $S_M(X)\subset A$, $\boldsymbol{P}^2\backslash A$ *is tautly imbedded modulo $S_M(x)$ in X.*

Proof. Since $\boldsymbol{P}^2\backslash A$ is hyperbolically imbedded modulo $S_M(X)$ in $\boldsymbol{P}^2$ and $S_M(X)\subset A$, $\boldsymbol{P}^2\backslash A$ is complete hyperbolic from Theorem 4 in [K-K]. Then $\boldsymbol{P}^2\backslash A$ is taut from [E] and [Ki1]. By the definition, $\boldsymbol{P}^2\backslash A$ is tautly imbedded modulo A in $\boldsymbol{P}^2$. So $\boldsymbol{P}^2\backslash A$ is tautly imbedded modulo $S_M(X)$ in $\boldsymbol{P}^2$ by Theorem 3.4. □

§4. Theorem 4.4

In [A-S1] we defined a nonhyperbolic curve as follows.

Definition 4.1. *Let A be curve of $\boldsymbol{P}^2$. An irreducible curve C of $\boldsymbol{P}^2$ will be called a nonhyperbolic curve with respect to A if the normalization of $C\backslash A$ is isomorphic to $\boldsymbol{C}$ or $\boldsymbol{C}^*=\boldsymbol{C}\backslash\{0\}$. If C is an irreducible component of A, we shall say that C is a nonhyperbolic curve with respect to A if the normalization of $C\backslash A'$ is isomorphic to $\boldsymbol{C}$, $\boldsymbol{C}^*$, $\boldsymbol{P}^1$ or an elliptic curve, where A' is the union of the components of A except C which may be ϕ.*

Theorem 4.2. (Theorem 2 in [A-S1]). *Let A be a curve with l ($l\geqq 4$) irreducible components of $\boldsymbol{P}^2$. Set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\backslash A$. Suppose that the number of the nonhyperbolic curves of $\boldsymbol{P}^2$ with respect to A is finite, then there is a curve S of $\boldsymbol{P}^2$ such that M is tautly imbedded modulo S in X. Here we may take $S=\phi$ if there is no nonhyperbolic curve of $\boldsymbol{P}^2$ with respect to A.*

Theorem 4.3. (Corollary of Theorem in [A]). *Let A be a curve with l ($l\geqq 4$) irreducible components of $\boldsymbol{P}^2$. Set $X=\boldsymbol{P}^2$ and $M=\boldsymbol{P}^2\backslash A$.*
(1) *If the number of the nonhyperbolic curves of $\boldsymbol{P}^2$ with respect to A is at most finite, $S_M(X)$ is empty or a curve.*
(2) *If the number of the nonhyperbolic curves of $\boldsymbol{P}^2$ with respect to A is infinite, then $S_M(X)=X$.*

Therefore, if $S_M(X)$ is a curve, there is a curve S of $\boldsymbol{P}^2$ such that M is tautly imbedded modulo S in X by Theorem 4.3 and Theorem 4.2. And then, M is tautly imbedded modulo $S_M(X)$ in X from Theorem 3.4. So we have the following

Theorem 4.4. *Let A be a curve with l ($l \geqq 4$) irreducible components of P^2. Set $X=P^2$ and $M=P^2 \backslash A$. If $S_M(X)$ is a curve, M is tautly imbedded modulo $S_M(X)$ in X.*

Remark. Let X and M be the same in Theorem 4.4 and S be a curve of X. If M is hyperbolically imbedded modulo S in X, M is tautly imbedded modulo S in X. Because, $S_M(X) \subset S$ by Proposition 1.4 and $S_M(X)$ is a curve or an empty set by Theorem 1.3. So M is tautly imbedded modulo $S_M(X)$ in X by Theorem 4.4 and Corollary 2.5. Therefore M is tautly imbedded modulo S in X.

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