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Normal Algebraic Surfaces with Trivial Tricanonical Divisors

By

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Abstract

We prove that upto isomorphisms there are at least one and at most three rational log Enriques surfaces of index 3 and Type A_{17}.

Introduction

Let T be a normal projective algebraic surface over the complex number field $\mathbf{C}$ with at worst quotient singular points (=Kawamata log terminal singular points in the sense of Kawamata and Kollár [Ka, Ko]). T is called a *log Enriques surface* if the irregularity $\dim H^1(T, \mathcal{O}_T)=0$ and if a positive multiple IK_T of the canonical Weil divisor K_T is linearly equivalent to zero. Without loss of generality, we assume from now on that a log Enriques surface has no Du Val singular points (see the comments after [Z1, Proposition 1.3]).

The smallest I such that $IK_T \sim 0$ is called the (global) *index* of T. It can be proved that $I \leq 66$ (cf. [Z1]). Recently, R. Blache [B1] has shown that $I \leq 21$. He also studied the "generalized" log Enriques surfaces where log canonical singular points are allowed.

Rational log Enriques surfaces T can be regarded as degenerations of K3 or Enriques surfaces, which in turn played important roles in Enriques-Kodaira's classification theory for surfaces. Recently, V. A. Alexeev [A] has proved the boundedness of families of these T. In 3-dimensional case, the base surfaces

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W of elliptically fibred Calabi-Yau threefolds $\Phi_{|D|}: X \to W$ with $D \cdot c_2(X) = 0$ are rational log Enriques surfaces [O1-O5].

Let T be a log Enriques surface of index I. The Galois $\mathbf{Z}/I\mathbf{Z}$-cover

$$\pi: Y := \mathcal{S}pec_{\mathcal{O}_T} \oplus_{i=0}^{I-1} \mathcal{O}_T(-iK_T) \to T$$

is called the *canonical cover*. Clearly, Y is either an abelian surface or a K3 surface with at worst Du Val singular points. We note also that π is unramified over the smooth part T-Sing T.

We say that T is of *Type* A_m or D_n if Y has a singular point of Dynkin type A_m or D_n; T is of *actual Type* $(\oplus A_m) \oplus (\oplus D_n) \oplus (\oplus E_k)$ if Sing $Y = (\oplus A_m) \oplus (\oplus D_n) \oplus (\oplus E_k)$.

Around 1989, M. Reid and I. Naruki asked the author about the uniqueness of rational log Enriques surface of Type D_{19}. The determinations of all isomorphism classes of rational log Enriques surfaces of Type A_{19}, D_{19}, A_{18} and D_{18} have been done in [OZ1, OZ2].

In this series of three papers, we consider the cases A_{17} and D_{17}. Actually, there is no rational log Enriques surface of Type D_{17} (Theorem 4). In the Type A_{17} case, the index I is equal to one of 2, 3, 4, 5, 6, 12 by virtue of [Z3, Theorem 1; OZ5, the proof of Theorem 1]. Our main results are as follows:

Theorem 1. *There is no rational log Enriques surface of Type* A_{17} *and index* $6p$ *for any positive integer* p.

Remark 1. Consequently, a rational log Enriques surface of Type A_{17} has index 2, 3, 4 or 5. The determinations of all isomorphism classes for the cases of index $I = 2$, 4, 5, are done in [Z3, OZ5], while the case $I = 3$ is treated in this note.

Theorem 2. *Upto isomorphisms there is at least one and at most three rational log Enriques surfaces of index* 3 *and Type* A_{17}. *They are all of actual Type* A_{17} *and isomorphic to one of* T_i $(i = 1, 2, 3)$ *in Example* 2.1.

Theorem 3. *Let* T_i *be as in Theorem* 2, $Y_i \to T_i$ *the canonical cover and* $g: X_i \to Y_i$ *the minimal resolution. Write* $\mathrm{Gal}(Y_i/T_i) = \langle \sigma_i \rangle$ *where* σ_i *is an automorphism of order* 3, *and* $\Gamma := g^{-1}(\mathrm{Sing}\ Y_i)$ *which is of Dynkin type* A_{17}.

Then there are two smooth rational curves F, H *on* X_i *such that* $F + H + \Gamma$ *is of Dynkin type* D_{19} *and that the triplet* $(X_i, \langle \sigma_i \rangle, F + H + \Gamma)$ *is isomorphic to Shioda-Inose's unique triplet* $(S_3, \langle g_3 \rangle, \Delta_3)$ *in* [*OZ*1, *Example* 1] (*see also*

Example 2.1 *below*), *where* S_3 *is the unique K*3 *surface of Picard number* 20 *and discriminant* 3.

Question 1. *Determine whether* T_i *and* T_j *are not isomorphic to each other when* $i \neq j$.

Remark 2. The answer to this question may not be easy if one looks at the long arguments in [OZ2, Theorem 1.6] in order to differentiate between two very symmetrically-constructed isomorphism classes of rational log Enriques surfaces of Type A_{18}.

Theorem 4. *There is no rational log Enriques surface of Type* D_{17}.

The organization of the paper is as follows. In §1, we consider automorphisms σ of order 3 or 6 on K3 surfaces, and describe in detail the action of σ around points lying on linear chains of smooth rational curves as well as the action of σ on elliptic fibers. A precise relation between the numbers of σ-fixed isolated points and curves is obtained in Lemma 1.6 by applying the fixed point theorem for holomorphic bundles, which was proved by Atiyah, Segal and Singer in [AS1,2].

In §2, we construct precisely three rational log Enriques surfaces T_i of index 3 and actual Type A_{17}. §3 and §4 are devoted to the proofs of the theorems.

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§1. Preliminaries

In this section, we shall fix the following notation:

T is a rational log Enriques surface of index I and $\pi: Y \to T$ is the canonical cover. $f: S \to T$ and $g: X \to Y$ are minimal resolutions. $\Sigma := f^{-1}(\mathrm{Sing}\, T)$ and $\Gamma := g^{-1}(\mathrm{Sing}\, Y)$ are reduced f-exceptional and g-exceptional divisors, respectively.

Note that π is a Galois covering such that $\mathrm{Gal}(Y/T) = \mathbf{Z}/I\mathbf{Z}$ and

$Y/(\mathbb{Z}/I\mathbb{Z})=T$. Clearly, there is a natural action of $\mathbb{Z}/I\mathbb{Z}$ on X such that the minimal resolution $g: X \to Y$ is $(\mathbb{Z}/I\mathbb{Z})$-equivariant. We will need the following lemmas for the later use.

Lemma 1.1. *Let T be a rational log Enriques surface of index I with Y the canonical cover. Then $\sigma^*\omega=\zeta_I\omega$ for exactly one generator σ of $(\mathbb{Z}/I\mathbb{Z})$, where $\zeta_I=\exp(2\pi\sqrt{-1}/I)$ and ω is a non-zero holomorphic 2-form on Y or on X.*

Proof. The result follows from the definition of I.

Lemma 1.2. *With the notations and assumptions in Lemma 1.1, we have:*

(1) *The g-exceptional divisor Γ is σ-stable.*

(2) *Every singular point on Y has a non-trivial stabilizer subgroup of $\langle\sigma\rangle=\mathbb{Z}/I\mathbb{Z}$. In particular, every connected component of Γ is σ-stable provided that I is prime.*

(3) *Every σ^i-fixed curve on X where $\sigma^i\neq \mathrm{id}$, is contained in Γ and hence a rational curve.*

Proof. (1) is true because the singular locus Sing Y is σ-stable.

(2) follows from our additional assumption that $T=Y/\sigma$ has no Du Val singular points. (3) is true because $\pi: Y\to T$ is unramified outside the finite set Sing T.

Lemma 1.3. *With the assumption and notation in Lemma 1.1, assume further that $I=pq$ for positive integers p,q. Then $Y_1:=Y/\langle\sigma^q\rangle$ is a rational log Enriques surface of index p with the quotient morphism $Y\to Y_1$ as the canonical cover. Here we assume that $p>1$, $q>1$.*

Proof. Ths follows from the fact that the (global) canonical index is equal to the l.c.m. of local canonical indices.

The following result is proved in [OZ1, Lemmas 2.1 and 2.2].

Lemma 1.4. *Let X be a (smooth) $K3$ surface with an order-three automorphism σ such that $\sigma^*\omega=\zeta\omega$ for a non-zero holomorphic 2-form ω on X and a cubic root ζ of unity. Then the following statements are true.*

(1) *The fixed locus (point wise) X^σ is a disjoint union of smooth curves*

and several isolated points.

(2) *Suppose that* $C_1+C_2+C_3$ *is a linear chain of* σ*-stable smooth rational curves with* C_2 *as the middle component. Then exactly one of* C_i *is* σ*-fixed.*

(3) *Let* C *be a* σ*-stable but not* σ*-fixed smooth rational curve. Then there is a* σ*-fixed curve* D *such that* $C.D=1$.

Lemma 1.5 below is a consequence of Lemma 1.4 and Kodaira's classification of singular elliptic fibers. The condition $n\leq 18$ (resp. $n\leq 17$) in the type(2) (resp. the type(3)) comes from the fact that rank $\operatorname{Pic} X<21$ (cf. [S, Cor. 1.5]).

Lemma 1.5. *Let* X, σ *be as in Lemma* 1.4. *Suppose that* η *is a singular fiber of an elliptic fibration* $\psi: X\to \mathbf{P}^1$ *such that* η *consists of* σ*-stable curves and contains at least one* σ*-fixed curve.* (*We note that every smooth rational curve on* X *is* σ*-stable provided that* $\sigma^*|\operatorname{Pic} X=\mathrm{id}$.) *Then* η *has one of the following types*:

(1) $\eta=H_1+H_2+H_3$ *is of Kodaira type IV, where* H_i*'s share one and the same point.* H_1 *is the only* σ*-fixed curve in* η.

(2) $\eta=H_1+H_2+\cdots+H_n$ *is of Kodaira type* I_n *with* $H_i\cdot H_{i+1}=H_n\cdot H_1=1$ $(1\leq i\leq n-1)$. n *is either one of* 3, 6, 9, 12, 15, 18. *The curves* H_1, H_4, H_7, $\cdots$, H_{n-2} *are the only* σ*-fixed curves in* η.

(3) $\eta=H_1+H_2+2(H_3+H_4+\cdots+H_{n-2})+H_{n-1}+H_n$ *is of Kodaira type* I^*_{n-5} *where* $H_1\cdot H_3=H_i\cdot H_{i+1}=H_{n-2}\cdot H_n=1$ $(2\leq i\leq n-2)$. n *is either one of* 5, 8, 11, 14, 17. H_3, H_6, H_9, $\cdots$, H_{n-2} *are the only* σ*-fixed curves in* η.

(4) $\eta=3H_1+2H_2+H_3+2H_4+H_5+2H_6+H_7$ *is of Kodaira type IV* where* $H_1\cdot H_i=H_i\cdot H_{i+1}=1$ $(i=2, 4, 6)$. H_1 *is the only* σ*-fixed curve in* η.

(5) $\eta=4H_1+2H_2+3H_3+2H_4+H_5+3H_6+2H_7+H_8$ *is of Kodaira type III* where* $H_1\cdot H_i=H_{j-1}\cdot H_j=H_j\cdot H_{j+1}=1$ $(i=2, 3, 6; j=4, 7)$. H_1, H_5, H_8 *are the only* σ*-fixed curves in* η.

(6) $\eta=6H_1+3H_2+4H_3+2H_4+5H_5+4H_6+3H_7+2H_8+H_9$ *is of Kodaira type II* where* $H_1\cdot H_i=H_3\cdot H_4=H_j\cdot H_{j+1}=1$ $(i=2, 3, 5;\ j=5, 6, 7, 8)$. H_1, H_7 *are the only* σ*-fixed curves in* η.

Lemma 1.6. *Let* X *be a* (*smooth*) *K*3 *surface with an order-six automorphism* σ *such that* $\sigma^*\omega=\zeta\omega$ *for a non-zero holomorphic 2-form* ω *on* X *and a 6-th primitive roof* ζ *of unity. Let* P_1, P_2, $\cdots$, P_M (*resp.* C_1, C_2, $\cdots$, C_N) *be all isolated points* (*resp. all irreducible curves*) *in* X^σ.

Assume that each C_i *is rational. Then* C_i *is smooth and disjoint from* C_j $(i\neq j)$, *and $M_1+M_2/2=3(N+1)$ where M_1, M_2 are non-negative integers to be defined in the proof below satisfying $M_1+M_2=M$.*

Proof. Since $\sigma^*\omega=\zeta\omega$, one has the diagonalization $\sigma^*=\mathrm{diag}(\zeta^2,\zeta^{-1})$ or $\sigma^*=\mathrm{diag}(\zeta^3,\zeta^{-2})$, with suitable local coordinates around P_i. Let P_i for $1\le i\le M_1$ (resp. P_j for $M_1+1\le j\le M_1+M_2=M$) be all isolated points in X^σ such that $\sigma^*=\mathrm{diag}(\zeta^2,\zeta^{-1})$ (resp. $\sigma^*=\mathrm{diag}(\zeta^3,\zeta^{-2})$) around P_i (resp. P_j).

Taking a point $P\in C_k$, we see that $\sigma^*=\mathrm{diag}(1,\zeta)$, with suitable local coordinates (x,y) around P. Thus, around P, X^σ is equal to $\{y=0\}$ and hence smooth. So X^σ is a disjoint union of smooth curves C_k's and points P_l's.

We now calculate the holomorphic Lefschetz number $L(\sigma)$ in two ways as in [AS1, 2, pages 542 and 567]:

$$L(\sigma)=\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\sigma^*\mid H^i(X,\mathcal{O}_X)),$$
$$L(\sigma)=\sum_{i=1}^{M_1}a(P_i)+\sum_{j=M_1+1}^{M}a(P_j)+\sum_{k=1}^{N}b(C_k).$$

Here

$$a(P_i)=1/\det(1-\sigma^*\mid T_{P_i})=1/(1-\zeta^2)(1-\zeta^{-1}),\ a(P_j)=1/(1-\zeta^3)(1-\zeta^{-2}),$$
$$b(C_k)=(1-g(C_k))/(1-\zeta^{-5})-(\zeta^{-5}C_k^2)/(1-\zeta^{-5})^2,$$

where T_{P_l} is the tangent space to X at P_l, $g(C_k)$ the genus of C_k and ζ^5 the eigenvalue of the action σ_* on the normal bundle of C_k.

The first formula yields $L(\sigma)=1+\zeta^{-1}$ by the Serre duality $H^2(X,\mathcal{O}_X)\cong H^0(X,\mathcal{O}(K_X))^\vee$. Plugging this into the second formula for $L(\sigma)$, we get:

$$1+\zeta^{-1}=M_1/(1-\zeta^2)(1-\zeta^{-1})+M_2/(1-\zeta^3)(1-\zeta^{-2})+N(1+\zeta)/(1-\zeta)^2.$$

Multiplying this equality by denominators and simplifying it by the facts that $\zeta^{-1}=1-\zeta$, $\zeta^3=-1$, $\zeta^2=\zeta-1$, we obtain the following one which implies Lemma 1.6:

$$3(1-\zeta)=(M_1+M_2/2-3N)(1-\zeta).$$

Lemma 1.7. *Let X, σ, ζ be as in Lemma 1.6. Assume that $\Sigma_{i=1}^6 C_i$ is a linear chain of σ-stable smooth rational curves C_i with $C_i\cdot C_{i+1}=1$. Set $P_i:=C_i\cap C_{i+1}$.*

Then exactly one of C_i is σ-fixed, say C_r, and the quintuplet $\sigma^*|P_1$, $\sigma^*|P_2,\cdots,\sigma^*|P_5$ of diagonalized local σ^*-actions, is equal to the unique portion of the following recursive sequence such that $\sigma^*|P_r=(1,\zeta)$:

$$(\zeta,1),(1,\zeta),(\zeta^{-1},\zeta^2),(\zeta^{-2},\zeta^3),(\zeta^3,\zeta^{-2}),(\zeta^2,\zeta^{-1}),$$
$$(\zeta,1),(1,\zeta),(\zeta^{-1},\zeta^2),(\zeta^{-2},\zeta^3),(\zeta^3,\zeta^{-2}),(\zeta^2,\zeta^{-1}),\cdots.$$

Proof. Set $P_i:=C_i\cap C_{i+1}$ $(1\le i\le 5)$. If C_i is σ-fixed, then $\sigma^*|P_i=(1,\zeta)$ with suitable local coordinates; otherwise C_i contains exactly two σ-fixed points P_{i-1}, P_i because C_i is smooth rational and σ-stable, and $\sigma^*|P_i=(\zeta^s,\zeta^{1-s})$ for some s because $\sigma^*\omega=\zeta\omega$, where for $i=1,2,\cdots,5$ (resp. for $i=0,1,\cdots,4$), ζ^s (resp. ζ^{1-s}) is the eigenvalue of the action σ^* on the tangent to C_i (resp. C_{i+1}) at P_i and where s, $1-s\ne 1$ (mod 6) because C_i is not σ-fixed.

If C_{i+1} is not σ-fixed, then P_i and P_{i+1} are the only two σ-fixed points on the smooth rational curve C_{i+1} and hence $\sigma^*|P_{i+1}=(\zeta^{s-1},\zeta^{2-s})$. Now Lemma 1.7 is clear.

§2. Examples of Index 3 and Actual Type A_{17}

In the present section, we shall construct rational log Enriques surfaces of index 3 and actual Type A_{17} (cf. Theorem 2).

Example 2.1 (index 3 and actual Type A_{17}). Let (S_3,g_3,Δ_3) be the Shioda-Inose's triplet in [OZ1, Example 1], where S_3 is the unique algebraic K3 surface of Picard number 20 and discriminant 3 (cf. [SI] or [V]), g_3 is an order-three automorphism on S_3 such that $g_3^*\omega=\zeta_3\omega$ with a non-zero holomorphic 2-form ω and $\zeta_3=exp(2\pi\sqrt{-1}/3)$, and Δ_3 is a reduced simple normal crossing divisor on S_3 of Dynkin type D_{19} as follows:

$$\begin{array}{ccc} & & \Gamma_{18} \\ & & | \\ \Gamma_1-\Gamma_2-\cdots-\Gamma_{16}- & & \Gamma_{17}\,. \\ & & | \\ & & \Gamma_{19} \end{array}$$

Note that g_3 acts trivially on $\operatorname{Pic} S_3$ such that the fixed locus (point wise)

$$(S_3)^{g_3}=\operatorname{Supp}\left(\sum_{i=1}^{6}\Gamma_{3i-1}\right)\coprod\{q_0,q_1,q_{3,4},q_{6,7},q_{9,10},q_{12,13},q_{15,16},q_{18},q_{19}\}.$$

Here $q_{i,i+1}=\Gamma_i\cap\Gamma_{i+1}$, $q_j\in\Gamma_j$ and q_0 is a point not on Δ_3.

For $i=1$ (resp. 2, or 3), let $\gamma_i: S_3 \to S_{\gamma_i}$ be the contraction of $\Delta_3-(\Gamma_{18}+\Gamma_{19})$ (resp. $\Delta_3-(\Gamma_1+\Gamma_{18})$, or $\Delta_3-(\Gamma_1+\Gamma_{19})$). Then S_{γ_i} is a K3 surface with a Du Val singular point of type A_{17}. The point $\gamma_i(q_0)$ and the only singular point on S_{γ_i}, together with two points $\gamma_1(q_{18})$, $\gamma_1(q_{19})$ (resp. $\gamma_2(q_1)$ and $\gamma_2(q_{18})$, or $\gamma_3(q_1)$, $\gamma_3(q_{19})$) are the only fixed points of the induced action of g_3 on S_{γ_i}. Clearly the quotient surface $S_{\gamma_i}/\langle g_3\rangle$ is a rational log Enriques surface of index 3 and actual Type A_{17}.

§3. Extend A_{17} to A_{18} or D_{18}

Let T be a rational log Enriques surface of index 3 and Type A_{17}. We employ the notation at the beginning of §1 and in Lemma 1.1:

$\pi : Y \to T$, $f : S \to T$, $g : X \to Y$, $\Sigma = f^{-1}(\mathrm{Sing}\, T)$, $\Gamma = g^{-1}(\mathrm{Sing}\, Y)$, $\langle\sigma\rangle = \mathrm{Gal}(Y/T)$, $\sigma^*\omega = \zeta_3\omega$.

We also denote by $\Gamma(1)=\Sigma_{i=1}^{17}\Gamma_i$ where $\Gamma_i\cdot\Gamma_{i+1}=1$, the unique connected component of Γ of Dynkin type A_{17}.

Lemma 3.1. *Let T be a rational log Enriques surface of index* 3 *and Type A_{17}. Then we have*:

(1) *T is of actual Type A_{17}, i.e., $\Gamma=\Gamma(1)$.*

(2) *After relabelling Γ_i as Γ_{18-i} if necessary, the σ-fixed locus* (*point wise*) *X^σ is a disjoint union of six curves Γ_{3i-1} $(1\le i\le 6)$ and nine isolated points.*

(3) *The pair (X,σ) is isomorphic to Shioda-Inose's pair (S_3,g_3) defined in* [*OZ*1, *Example* 1] (*see also Example* 2.1). *In particular, X is isomorphic to the unique K*3 *surface of Picard number* 20 *and discriminant* 3, *and σ acts trivially on* Pic X.

Proof. Let $\Gamma(i)$ $(1\le i\le k)$ be all of connected components of Γ. Since $\rho(X)\le 20$, one has $k\le 3$ and each Γ_j $(j\ge 2)$ is of Dynkin type A_1 or A_2. By Lemma 1.2, each $\Gamma(i)$ is σ-stable. Since $3=\mathrm{ord}(\sigma)$ is coprime to the order of the graph-automorphism group $\mathbf{Z}/2\mathbf{Z}$ of a Dynkin diagram $A_n (n\ge 2)$, each irreducible component of Γ is σ-stable. Hence each $\Gamma(i)$ contains at least one σ-fixed curve (cf. Lemmas 1.2 and 1.4).

By Lemmas 1.2 and 1.4, the set of σ-fixed curves contained in $\Gamma(1)$, is equal to $\Sigma_{i=1}^{6}\Gamma_{3i-1}$ after relabelling if necessary. Since X^σ can not contain more than six (smooth rational) curves [OZ1, Theorem 3, Example 1 and Remark

3], one has $\Gamma=\Gamma(1)$. Now, (2) and (3) follow from [ibid.]. This proves Lemma 3.1.

Remark 3.2. (1) Since $\sigma^*|\operatorname{Pic} X=\mathrm{id}$, each smooth rational curve, especially each component of Γ_i, is σ-stable. Hence one has the following (cf. Lemma 1.4(3)):

$$X^\sigma=\operatorname{Supp}\left(\sum_{i=1}^{6}\Gamma_{3i-1}\right)\coprod\{p_1,p_{3,4},p_{6,7},p_{9,10},p_{12,13},p_{15,16},q_1,q_2,q_3\}$$

where $p_{i,i+1}=\Gamma_i\cap\Gamma_{i+1}$, $p_1\in\Gamma_1$ and q_j's are points not on Γ.

(2) Since the discriminant of X is 3, for any 20 curves C_i on X one has $\det(C_i\cdot C_j)=-3n^2$ for some non-negative integer n. Here n is the index of the sublattice $\Sigma_{i=1}^{20}\mathbf{Z}C_i$ in the lattice $\operatorname{Pic} X$ when C_i's are linearly independent, and zero otherwise.

The rest of this section is devoted to the proof of the following:

Proposition 3.3. *Let T, S, Σ, Y, X, Γ be as in Lemma* 3.1. *Then there is a smooth rational curve H on X such that $H+\Gamma$ is of Dynkin type A_{18} or D_{18}.*

By Remark 3.2, Σ consists of four connected components $\Sigma(1):=\Pi_1+\Sigma_2+\Sigma_5+\Sigma_8+\Sigma_{11}+\Sigma_{14}+\Sigma_{17}$, Λ_i $(i=1, 2, 3)$ with the following dual graph:

$$\Pi_1-\Sigma_2-\Sigma_5-\Sigma_8-\Sigma_{11}-\Sigma_{14}-\Sigma_{17},\ \Lambda_1,\ \Lambda_2,\ \Lambda_3.$$

Here $\Pi_1^2=-2$, $\Sigma_2^2=-3$, $\Sigma_i^2=-2$ $(i=5, 8, 11, 14)$, $\Sigma_{17}^2=-4$, $\Lambda_j^2=-3$ $(j=1, 2, 3)$.

The canonical cover $\pi: Y\to T$ induces a rational map $\pi: X\cdots\to S$ such that after relabelling if necessary, Γ_{3i-1} $(1\le i\le 6)$ on X is the strict transform of Σ_{3i-1} and that Λ_j is mapped to by π^{-1}, the three σ-fixed points q_j which do not lie on Γ.

Lemma 3.4. *Let T, S, Σ be as in Lemma* 3.1.

(1) *One has* $3(K_S+\Sigma^*)\sim 0$ *where* $\Sigma^*:=(\Pi_1+2\Sigma_{i=1}^{6}\Sigma_{3i-1}+\Sigma_{j=1}^{3}\Lambda_j)/3$.

(2) $K_S^2=-3$ and $\rho(S)=13$.

(3) *For any (-1)-curve E on X one has $E\cdot\Sigma^*=1$. If H is an irreducible curve on X with $H^2<0$, then H is either a component of Γ or a (-1)-curve.*

Proof. (1) follows from the fact that $0\sim f^*(3K_T)=3(K_S+\Sigma^*)$ while (2) follows from (1). (1) and the genus formula imply the first half of (3) and that a curve H with $H^2<0$ either satisfies (3), or is a (-2)-curve disjoint from Γ. The latter case is impossible because $\sigma^* \mid \mathrm{Pic}\, X = \mathrm{id}$ by Lemma 3.1. This proves Lemma 3.4.

Now our Proposition 3.3 will follow from the following Lemmas 3.5–3.9.

Lemma 3.5. *Let T, S, Σ be as in Lemma* 3.1. *Then there is a (-1)-curve E, or two disjoint (-1)-curves E_1, E_2, or three disjoint (-1)-curves E_1, E_2, E_3, on S such that one of the following cases occurs* (*after relabelling if necessary*):

Case(1). $E\cdot\Lambda_1 = E\cdot\Gamma_{3r-1} = 1$ *for either one of* $r=1, 2, \cdots, 6$,

Case(2). $E\cdot\Lambda_1 = E\cdot\Lambda_2 = E\cdot\Pi_1 = 1$,

Case(3). $E_2\cdot\Lambda_j = 1$ $(j=1, 2, 3)$, *and* $(E_1\cdot\Pi_1, E_1\cdot\Lambda_1) = (1, 2)$ *or* $(2, 1)$,

Case(4). $(E_j\cdot\Pi_1, E_j\cdot\Lambda_j) = (1,2)$ *or* $(2,1)$ $(j=1,2,3)$,

Case(5). *each of* $(E_j\cdot\Pi_1, E_j\cdot\Lambda_j)$ $(j=1, 2)$ *and* $(E_3\cdot\Lambda_3, E_3\cdot(\Lambda_1+\Lambda_2))$ *equals* $(1, 2)$ *or* $(2, 1)$,

Case(6). *each of* $(E_1\cdot\Pi_1, E_1\cdot\Lambda_1)$, $(E_2\cdot\Lambda_1, E_2\cdot\Lambda_2)$ *and* $(E_3\cdot\Lambda_1, E_3\cdot\Lambda_3)$ *equals* $(1, 2)$ *or* $(2, 1)$,

Case(7). *each of* $(E_1\cdot\Pi_1, E_1\cdot\Lambda_1)$, $(E_2\cdot\Lambda_1, E_2\cdot\Lambda_2)$ *and* $(E_3\cdot\Lambda_2, E_3\cdot\Lambda_3)$ *equals* $(1, 2)$ *or* $(2, 1)$.

Proof. Let $\nu: S \to \Sigma_m$ be a smooth blowing-down of smooth rational curves to points on some Hirzebruch surface F_m of degree m. Since $K_{F_m} + \nu_*\Sigma^* \equiv 0$ (Lemma 3.4), $\nu_*\Sigma$ contains at least one horizontal component and is hence connected.

Claim(1). $\mathrm{Supp}\,\nu(\Gamma) = \mathrm{Supp}\,\nu_*\Gamma$, that is, no connected component of Σ is contracted to a point not lying on $\nu_*\Sigma$.

Suppose to the contrary that a maximum union Σ' of connected components of Σ is contracted to a point p not lying on $\nu_*\Sigma$ so that $\nu(\Sigma') \cap \nu(\Sigma - \Sigma') = \phi$. Factorize $\nu = \nu_3 \circ \nu_2 \circ \nu_1$ so that $\nu_1(\Sigma')$ is a (-1)-curve and ν_2 is the blowing down of $\nu_1(\Sigma')$. Then we have $0 = \nu_1(\Sigma')\cdot \nu_{1*}(K_S + \Sigma^*) = -1-\alpha < 0$, where α is the coefficient in Σ^* of the proper transform $\nu_1'(\nu_1(\Sigma'))$. This is a contradiction. So Claim (1) is true.

Claim(1) and its preceding argument imply that $\nu(\Sigma)$ is connected. So

$v^{-1}v(\Sigma)$ is also connected. We can write $v^{-1}v(\Sigma)=\Sigma+E_{-1}$ where E_{-1} is a union of (-1)-curves (Lemma 3.4). Now Lemma 3.5 follows from Lemma 3.4(3) and the fact that E_{-1} consists of disjoint (-1)-curves.

Lemma 3.6. *Cases* (4)–(7) *in Lemma* 3.5 *are impossible.*

Proof. Lemma 3.6 can be proved by dividing Cases (4)–(7) into subcases and applying Remark 3.2(2). We illustrate our method by considering the following subcase of Case (6): $E_1\cdot\Pi_1=E_2\cdot\Lambda_2=E_3\cdot\Lambda_3=1$, $E_j\cdot\Lambda_1=2$ $(j=1,2,3)$.

Denote by G_{17+j} the strict transform on X of E_j. Then G_{17+j} has self intersection 2 and, is either an elliptic curve with an ordinary node or a rational curve with a cusp of type (2,5). Moreover, $G_{18}\cdot\Gamma_1=G_{18}\cdot\Gamma=1$, $G_{19}\cdot\Gamma=G_{20}\cdot\Gamma=0$ and $G_i\cdot G_j=4$ for $i,j=18$, 19, 20; $i\neq j$. Set $G_i=\Gamma_{18-i}$ $(i=1,2,\cdots,17)$. Using "Mathematica", we get $\det(G_i\cdot G_j)=-516=-3\times2^2\times43$. This contradicts Remark 3.2(2). This way, one can prove Lemma 3.6.

Lemma 3.7. *Suppose Case*(1) *in Lemma* 3.5 *occurs. Then Proposition* 3.3 *is true.*

Proof. Let E with $E\cdot\Lambda_1=E\cdot\Sigma_{3r-1}=1$ be as in Case(1). Denote by F the strict transform on X of E. Then F is a smooth rational curve such that $F\cdot\Gamma_{3r-1}=F\cdot\Gamma=1$. If $r=1$ (resp. $r=6$), then $F+\Gamma$ is of Dynkin type D_{18} (resp. A_{18}), whence Proposition 3.3 is true.

Therefore, we may assume that $r=2$, 3, 4 or 5. Set $\eta_0:=4\Gamma_{3r-1}+3(\Gamma_{3r-2}+\Gamma_{3r})+2(\Gamma_{3r-3}+\Gamma_{3r+1}+F)+\Gamma_{3r-4}+\Gamma_{3r+2}$. Applying the Riemann-Roch theorem, there is an elliptic fibration $\psi: X\to\mathbf{P}^1$ with η_0 as a fiber.

Case(1.1) $r=2$. Let η_1 be the fiber containing $\Gamma_{10}+\Gamma_{11}+\cdots+\Gamma_{17}$. By Lemma 1.5 and the fact that $\Gamma_9\cdot\eta_i=1$ $(i=0,1)$, η_1 fits either type(2) with $n=9$ or type(3) with $n=11$ there. For type(3), we let H be a tip component in η_1 which meets Γ_{17} but not ψ's cross-section Γ_1. Then $H+\Gamma$ is of Dynkin type A_{18}.

For type(2), the cross-section Γ_1 meets η_1 at a point on the unique component G of η_1 which is not contained in Γ. Thus the smooth rational curve G contains three σ-fixed points $G\cap\Gamma_1$, $G\cap\Gamma_{10}$, $G\cap\Gamma_{17}$, and is hence σ-fixed (Lemma 1.4). This contradicts Remark 3.2(1).

Case(1.2) $r=3$. Let η_1 be the fiber containing $\Gamma_{13}+\Gamma_{14}+\cdots+\Gamma_{17}$. By Lemma 1.5 and the argument in Case(1.1) for type(2) there, η_1 fits type(3) in

Lemma 1.5 and either $\eta_1=F_2+\Gamma_3+2(\Gamma_2+\Gamma_1+F+\Gamma_{17}+\Gamma_{16}+\Gamma_{15}+\Gamma_{14})+\Gamma_{13}+F_{14}$ where $F_j\cdot\Gamma_j=F\cdot\Gamma_1=F\cdot\Gamma_{17}=1$, or $\eta_1=F_{17}'+F_{17}+2(\Gamma_{17}+\Gamma_{16}+\Gamma_{15}+\Gamma_{14})+\Gamma_{13}+F_{14}$ where $F_j\cdot\Gamma_j=F_{17}'\cdot\Gamma_{17}=1$ and the cross-section Γ_4 does not meet F_{17}. In the first (resp. second) subcase $F_2+\Gamma$ (resp. $F_{17}+\Gamma$) is of Dynkin type D_{18} (resp. A_{18}).

Case(1.3) $r=4$. Let η_1 be the fiber containing $\Gamma_1+\Gamma_2+\cdots+\Gamma_6$. By Lemma 1.5, η_1 fits type(3) in Lemma 1.5, and either $\eta_1=F_5+\Gamma_6+2(\Gamma_5+\Gamma_4+\cdots+\Gamma_1+F+\Gamma_{17})+\Gamma_{16}+F_{17}$ where $F_j\cdot\Gamma_j=F\cdot\Gamma_1=F\cdot\Gamma_{17}=1$, or $\eta_1=F_2+\Gamma_1+2(\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5)+\Gamma_6+F_5$ where $F_j\cdot\Gamma_j=1$.

In the first subcase $F_{17}+\Gamma$ is of Dynkin type A_{18}. In the second subcase, if the cross-section Γ_{15} does not meet F_2 (resp. F_5) then $F_2+\Gamma$ is of Dynkin type D_{18} (resp. then we are reduced to Case(1.1) with F replaced by F_5). Hence Proposition 3.3 is true by the arguments in Case(1.1).

Case(1.4) $r=5$. Let η_1 be the fiber containing $\Gamma_1+\Gamma_2+\cdots+\Gamma_9$. By Lemma 1.5, η_1 fits type(3) with $n=11$ there. Let H ($\neq\Gamma_1$) be the tip component of η_1 meeting Γ_2. Then $H+\Gamma$ is of Dynkin type D_{18}. This proves Lemma 3.7.

Lemma 3.8. *Suppose Case*(2) *in Lemma* 3.5 *occurs. Then Proposition* 3.3 *is true.*

Proof. Let E be as in Case(2). Denote by F the strict transform on X of E. Then F is a smooth elliptic curve with $F\cdot\Gamma_1=F\cdot\Gamma=1$. Let η_1 be the fiber of the elliptic fibration $\Phi_{|F|}:X\to\mathbb{P}^1$ containing $\Gamma_2+\Gamma_3+\cdots+\Gamma_{17}$. By Lemma 1.5, η_1 fits either type(3) with $n=20$, or type(2) with $n=18$. The first subcase is impossible for $n\leq 17$ (cf. Lemma 1.5). In the second subcase, if we let H be one of two tip components in η_1 which meets Γ_{17}, then $H+\Gamma$ is of Dynkin type A_{18}. This proves Lemma 3.8.

Lemma 3.9. *Suppose Case*(3) *in Lemma* 3.5 *occurs. Then Proposition* 3.3 *is true.*

Proof. Let E_1, E_2 be as in Case(3). Denote by F_j the strict transform on X of E_j. Then F_2 is a smooth elliptic curve, while F_1 is a curve of self intersection 2 such that $(F_1\cdot\Gamma_1,\ F_1\cdot F_2)=(1,2)$ or $(2,1)$. Applying Lemma 1.5 to the elliptic fibration $\psi:=\Phi_{|F_2|}:X\to\mathbb{P}^1$ we see that there is a smooth rational curve F_3 on X such that $F_3\cdot\Gamma_1=F_3\cdot\Gamma_{17}=1$ and $\eta_1:=F_3+\Gamma$ is a fiber of ψ of Kodaira type I_{18}. Since $F_1\cdot F_2=F_1\cdot\eta_1$, we see that $(F_1\cdot\Gamma_1,\ F_1\cdot F_2)=(1,2)$ and $(E_1\cdot\Pi_1,$

$E_1 \cdot \Lambda_1)=(1, 2)$.

$F_2 \sim \eta_1$ implies that $\xi_0 \sim \xi_1$ where $\xi_0 := 3E_2 + \Sigma_{i=1}^3 \Lambda_i$, $\xi_1 := 3E_3 + 2\Pi_1 + \Sigma_{i=1}^6 \Sigma_{3i-1}$ are "π-direct images" of F_2, η_1 and where E_3 is the π-image of F_3. (We note that the six isolated σ-fixed points form the set of the indeterminant or fundamental points of the rational map $\pi: X \cdots \to S$.) Hence there is an elliptic fibration $\varphi: S \to \mathbf{P}^1$ with ξ_i as fibers.

Claim(1). φ is multiple fiber free.

Since the fibration ψ on the K3 surface X is multiple fiber free, it suffices to show that the inverse on X of each fiber ($\neq \xi_0$, ξ_1) of φ splits into three distinct fibers of ψ.

We note that both F_2 and η_1 are σ-stable because $\sigma^* | \operatorname{Pic} X = \mathrm{id}$ and hence σ^* permutes fibers of ψ and induces an automorphism σ on the base curve $\mathbf{P}^1$ of ψ. So it suffices to show that the action of σ on $\mathbf{P}^1$ is non-trivial because then $\psi(F_2)$, $\psi(\eta_1)$ are the only σ-fixed points on $\mathbf{P}^1$ and σ acts freely on the set of all fibers of ψ minus F_2, η_1.

If the action of σ on $\mathbf{P}^1$ is trivial then $\pi_* \eta = 3\xi$ for a general fiber η of ψ where $\xi = \pi(\eta)$. So 3ξ is linearly equivalent to the "π-direct image" ξ_i ($i=0$, 1). This is impossible because there are infinitely many such 3ξ but the φ can have at most one multiple fiber by noting that the Kodaira dimension of S is $-\infty$ and applying the canonical divisor formula for elliptic surfaces. This proves Claim(1).

By Claim(1) and by the canonical divisor formula, one has $K_S + \xi_i \sim 0$ ($i=0$, 1). Let E be a (-1)-curve on S. Then $E \cdot \xi_i = 1$ and hence $E \cdot \Sigma_{3i-1} = E \cdot \Lambda_j = 1$ for some $1 \leq i \leq 6$ and $1 \leq j \leq 3$. So we are reduced to Case(1) in Lemma 3.5 after relabelling Λ_j as Λ_1. Thus Proposition 3.3 is true by Lemma 3.7. This completes the proof of Lemma 3.9 and also that of Proposition 3.3.

§4. Proofs of Theorems

First, we prove Theorem 2. Let T be a rational log Enriques surface of index 3 and Type A_{17}. We shall use the notations T, S, X, Γ in Lemma 3.1. By Proposition 3.3, there is a smooth rational curve H on X such that $H + \Gamma$ is of Dynkin type A_{18} or D_{18}. By Lemma 3.1(3) and [OZII, Theorems 3 and 4], there is a smooth rational curve F on X such that $(X, \langle \sigma \rangle, F + H + \Gamma\rangle$ is isomorphic to Shioda-Inose's triplet (S_3, $\langle g_3\rangle$, Δ_3) in Example 2.1. Thus (X, $\langle\sigma\rangle$, Γ) is isomorphic to (S_3, $\langle g_3\rangle$, $\Delta_3-(\Gamma_{18}+\Gamma_{19})$), ($S_3$, $\langle g_3\rangle$, $\Delta_3-(\Gamma_1+\Gamma_{18})$) or ($S_3$, $\langle g_3\rangle$, $\Delta_3-(\Gamma_1+\Gamma_{19})$). Now Theorem 2 follows.

Theorem 3 follows from the above arguments or Theorem 2.

Next, we prove Theorem 4. Suppose the contrary that T is a rational log Enriques surface of index I and Type D_{17}. We use the same notations as at the beginning of §1. So Γ contains a connected component $\Gamma(1)$ of Dynkin type D_{17} as follows:

$$\begin{array}{ccccccc} & & & & & & \Gamma_{18} \\ & & & & & & | \\ \Gamma_3 & — & \Gamma_4 & —\cdots— & \Gamma_{16} & — & \Gamma_{17}. \\ & & & & & & | \\ & & & & & & \Gamma_{19} \end{array}$$

The existence of such $\Gamma(1)$ on X implies that $\rho(X)\geq 18$. Thus Euler's Phi-function $\varphi(I)\leq \text{rank } T_X=22-\rho(X)\leq 4$ (cf. "added in proof" at the end of [Z1]), and hence $I=2$, 4, 8, 12, 3, 6, 5, 10. By Lemma 1.3, it suffices to consider the cases $I=2$, 3, 5.

If $I=2$ then every singular point on the canonical cover Y is of Dynkin type A_{2n-1} for some $n\geq 1$ (cf. [Z1, Lemma 3.1]). Hence $I\neq 2$. We can also use [OZ1, Lemma 3.2] to rule out the case $I=2$.

Consider the case $I=3$. Then each irreducible component Γ_i in $\Gamma(1)$ is σ-stable because $3=\text{ord}(\sigma)$ is coprime with the order of the graph-automorphism group $\mathbf{Z}/2\mathbf{Z}$ of $\Gamma(1)$ (cf. Lemma 1.2(2)). Now the intersection points of Γ_{17} with Γ_{16}, Γ_{18} and Γ_{19} are σ-fixed. Hence the smooth rational curve Γ_{17} is σ-fixed. Applying Lemma 1.4(2), we see that Γ_5, Γ_8, Γ_{11}, Γ_{14}, Γ_{17} are the only σ-fixed curves in $\Gamma(1)$. Applying Lemma 1.4(3) to $C:=\Gamma_3$, we get a contradiction (cf. Lemma 1.2(3)). So the case $I=3$ is impossible.

Consider the case $I=5$. As in the case $I=3$, each irreducible component Γ_i of $\Gamma(1)$ is σ-stable and Γ_{17} is σ-fixed. Applying [OZ5, Lemma 1.6] which is an analogy of Lemma 1.4 for the case $I=5$, we see that Γ_7, Γ_{12}, Γ_{17} are the only σ-fixed components in $\Gamma(1)$. This contradicts [OZ5, Lemmas 1.2 and 1.6] which are analogies of Lemmas 1.2 and 1.4, applied to the linear chain $\Gamma_3+\Gamma_4+\Gamma_5$. So $I=5$ is impossible. This completes the proof of Theorem 4.

Finally, we prove Theorem 1. Suppose the contrary that T is a rational log Enriques surface of index $6p$ and Type A_{17} for some $p \geq 1$. In view of Lemma 1.3, it suffices to consider the case $p=1$.

We shall employ the notation $\pi: Y \to T$, $\mathrm{Gal}(Y/T)=\langle\sigma\rangle$, $g: X \to Y$, $\Gamma = g^{-1}(\mathrm{Sing}\, Y)$ at the beginning of §1 and in Lemma 1.1. By Lemma 1.3, $T_3 := Y/\langle\sigma^2\rangle$ is a rational log Enriques surface of index 3 and Type A_{17}. In view of Lemma 3.1, T_3 is of actual Type A_{17}, i.e., $\Gamma = \Gamma(1) = \Sigma_{i=1}^{17} \Gamma_i$ where $\Gamma_i \cdot \Gamma_{i+1} = 1$. By Lemma 1.2, the fixed locus X^σ is a subset of Γ.

Now applying Lemma 1.7 and using the fact that each σ-stable but not σ-fixed smooth rational curve has exactly two σ-fixed points, we see that X^σ is equal to one of the following three sets, after relabelling Γ_i as Γ_{18-i} if necessary, where $p_{i,i+1} = \Gamma_i \cap \Gamma_{i+1}$, $p_j \in \Gamma_j$:

$$\mathrm{Supp}(\Gamma_1 + \Gamma_7 + \Gamma_{13})$$
$$\amalg\ \{p_{2,3}, p_{3,4}, p_{4,5}, p_{5,6}, p_{8,9}, p_{9,10}, p_{10,11}, p_{11,12}, p_{14,15}, p_{15,16}, p_{16,17}, p_{17}\},$$
$$\mathrm{Supp}(\Gamma_2 + \Gamma_8 + \Gamma_{14})$$
$$\amalg\ \{p_1, p_{3,4}, p_{4,5}, p_{5,6}, p_{6,7}, p_{9,10}, p_{10,11}, p_{11,12}, p_{12,13}, p_{15,16}, p_{16,17}, p_{17}\},$$
$$\mathrm{Supp}(\Gamma_3 + \Gamma_9 + \Gamma_{15})$$
$$\amalg\ \{p_1, p_{1,2}, p_{4,5}, p_{5,6}, p_{6,7}, p_{7,8}, p_{10,11}, p_{11,12}, p_{12,13}, p_{13,14}, p_{16,17}, p_{17}\}.$$

By Lemma 1.7, in all these three cases, we have $M_1 = M_2 = 6$, $N = 3$ in the notations of Lemma 1.6. This contradicts the equality in Lemma 1.6. Therefore, Theorem 1 is true.

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