Publ. RIMS, Kyoto Univ.
33 (1997), 509–526

Solvability of Equations for Motion of a Vortex Filament with or without Axial Flow

By

Atusi TANI* and Takahiro NISHIYAMA*

Abstract

It is known that the motion of a vortex filament with axial flow in a perfect fluid is approximately described by a generalization of the localized induction equation. The unique solvability of the initial value problem for it is first established by parabolic regularization. As the axial flow effect vanishes, its solution converges to that of the localized induction equation. Analogous results are obtained in the spatially periodic case.

§1. Introduction

In [13], we proved the weak solvability of some initial and initial-boundary value problems for the localized induction equation

$$x_t = x_s \times x_{ss}, \tag{1.1}$$

which approximately describes the deformation of a vortex filament in a perfect fluid. Here $x = x(s, t)$ represents the position of a point on the filament in $\boldsymbol{R}^3$ as a vector-valued function of arclength $s \in \boldsymbol{R}$ and time t. But the uniqueness and smoothness of the solution were not found.

According to [15], Da Rios [2] firstly formulated (1.1) in 1906. Since then, it has been studied by many authors. Fukumoto and Miyazaki [3] proposed a generalization of (1.1):

$$x_t = x_s \times x_{ss} + a\{x_{sss} + (3/2)x_{ss} \times (x_s \times x_{ss})\}, \tag{1.2}$$

when the vortex filament has an axial flow within its thin vortex core. Here a is a real constant representing the magnitude of the axial flow effect.

Communicated by H. Okamoto, October 23, 1995. Revised October 2, 1996 and May 22, 1997.
1991 Mathematics Subject Classifications: 35Q35, 76C05
* Department of Mathematics, Keio University, Yokohama 223, Japan

By Hasimoto's method in [4], (1.2) can be transformed into the Hirota equation (or the nonlinear Schrödinger equation if $a=0$)

$$\mathrm{i}\Psi_t+\Psi_{ss}+(1/2)|\Psi|^2\Psi-\mathrm{i}a\{\Psi_{sss}+(3/2)|\Psi|^2\Psi_s\}=0 \tag{1.3}$$

for $\Psi=\kappa(s,t)\exp(\mathrm{i}\int_0^s\tau(s,t)\mathrm{d}s+\mathrm{i}\,\eta(t))$ (see [3]). Here $\kappa(s,t)$ and $\tau(s,t)$ are the curvature and the torsion of the filament, respectively, and $\eta(t)$ is a real function of t. Hirota [5] showed that (1.3) has soliton solutions. In addition, the well-posedness of the initial value problem for (1.3) can be proved by the theory of evolution equations ([7], [10], [11]).

However, in investigating the solvability of (1.2) with an arbitrary (non-soliton) initial data, which is the theme of this paper, we do not use the well-posedness of (1.3) for the following reasons.

1. We consider s and t to be independent. When $a\neq 0$, it is nontrivial to prove that $|\boldsymbol{x}_s|=1$ is really derived from (1.2) and $|\boldsymbol{x}_s(s,0)|=1$. Without this proof, Hasimoto's method cannot be applied to (1.2).
2. If the filament is not an infinitely straight line and it has a segment where κ vanishes (or τ is indefinite), then $\operatorname{Arg}\Psi$ is not well-defined even outside this segment. In this case, Hasimoto's transformation from (1.2) into (1.3) is not possible.

Recently, it was shown in [8] that (1.1) can be transformed into the nonlinear Schrödinger equation by not Hasimoto's method but a geometrical one which is valid even if $|\boldsymbol{x}_{ss}|$ vanishes. Then unique and classical solvability of the initial value problem for (1.1) was obtained, although the class of the solution was C^∞.

Differentiating (1.2) with respect to s and setting $\boldsymbol{v}=\boldsymbol{x}_s$ yield

$$\boldsymbol{v}_t=\boldsymbol{v}\times\boldsymbol{v}_{ss}+a[\boldsymbol{v}_{sss}+(3/2)\{\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_s)\}_s]. \tag{1.4}$$

We impose the initial condition

$$\boldsymbol{v}(s,0)=\boldsymbol{v}_0(s),\quad |\boldsymbol{v}_0|=1 \tag{1.5}$$

for $s\in\boldsymbol{R}$. In [14], we proved classical solvability of (1.4) and (1.5) on any finite time interval. The solution for (1.2) and

$$\boldsymbol{x}(s,0)=\boldsymbol{x}_0(s),\quad |\boldsymbol{x}_{0s}|=1 \tag{1.6}$$

is reconstructed from (1.4) and (1.5) with $v_0 = x_{0s}$ by

$$\text{(1.7)} \qquad x = x_0 + \int_0^t [v \times v_s + a\{v_{ss} + (3/2)v_s \times (v \times v_s)\}]dt.$$

The main goal in this paper is to establish the existence of a unique solution to (1.2) and (1.6) for $a \neq 0$ or $a = 0$ such that x_s has unit length and its derivatives belong to the Sobolev spaces bounded-continuously in $t \in [0, \infty)$. To this end, we first investigate a parabolic regularization for (1.4) which has a solution with unit length. At this time, the estimate (2.17) below, which was used without proof in [14], plays a crucial role. To obtain it, a conservation law for (1.3) is suggestive, although we stated above that we do not use the solvability of (1.3) in this paper.

Moreover, analogous results are obtained even if the spatially periodic condition

$$\text{(1.8)} \qquad x_s(s, t) = x_s(s+1, t)$$

is added.

In [9], the system (1.4) with $a = 0$ was derived as a continuum model of the classical Heisenberg spin system. The initial value problem for it was considered and the existence of a unique solution was obtained by Sulem et al. [17]. Their method was applied to the problem (1.1) and (1.6) with $|x_{0ss}(\pm\infty)| \neq 0$ in [12], while $|x_{0ss}(\pm\infty)| = 0$ in [14] and this paper.

We discuss the parabolic regularization for (1.4) in §2. Then the solvability of (1.2) for $a \neq 0$ is obtained from that of (1.4) in §3. In this section, the limit $a \to 0$, or the vanishing axial flow, is also investigated. It is shown in §4 that a spatially periodic case is analogously discussed.

Let us introduce the notation which we use. The norms in $L^2(\boldsymbol{R})$ and in the Sobolev space $W_2^n(\boldsymbol{R})$ $(n = 1, 2, \cdots)$ are denoted by $\|\cdot\|$ and $\|\cdot\|^{(n)}$, respectively. The set of all continuous (resp. once continuously differentiable) functions in a Hilbert space X on a finite time interval $[0, T]$ is denoted by $C(0, T; X)$ (resp. $C^1(0, T; X)$). The spaces $L^2(0, T; X)$ and $L^\infty(0, T; X)$ are defined in the same way. We represent the supremum over $[0, T]$ by $\langle\cdot\rangle_T$ and the $L^2(\boldsymbol{R}_T)$-norm, $\boldsymbol{R}_T \equiv \boldsymbol{R} \times (0, T)$, by $\|\cdot\|_T$. Positive constants, denoted by c, c_a and c_*, change from line to line. Here c_a is a monotone increasing function in $|a|$ and independent of ε, and c_* is independent of ε and a. The operator

D stands for $\partial/\partial s$.

§2. Parabolic Regularization

We first consider the initial value problem

$$u_t = -\varepsilon u_{ssss} + f(s, t), \tag{2.1}$$

$$u(s, 0) = u_0(s). \tag{2.2}$$

Let $C_b(Q)$ ($Q \subset R$ or R^2) be the set of bounded continuous functions on Q. Then we get

Lemma 2.1. *For given* $\varepsilon > 0$, $T > 0$, $u_0 \in C_b(R)$ *with* $u_{0s} \in W_2^2(R)$ *and* $f \in C(0, T; L^2(R)) \cap L^2(0, T; W_2^2(R))$, *there exists a unique solution* u *of* (2.1) *and* (2.2) *such that* $u \in C_b(R_T)$, $u_s \in C(0, T; W_2^2(R))$, $u_{ssss} \in L^2(0, T; W_2^1(R))$ and $u_t \in L^2(0, T; W_2^1(R))$. *Moreover, the estimate*

$$\sup_{\substack{s \in R \\ 0 < t < T}} |u| + \langle \|u_s\|^{(2)} \rangle_T + \|u_{ssss}\|_T + \|D^5 u\|_T + \|u_t\|_T + \|u_{st}\|_T \tag{2.3}$$

$$\leqq c(\sup_{s \in R} |u_0| + \|u_{0s}\|^{(2)}) + b(T)(\langle \|f\| \rangle_T + \|f_{ss}\|_T)$$

holds, where c *is independent of* T *and* $b(T)$ *is a positive continuous function depending on* T *such that* $b(T) \downarrow 0$ *as* $T \downarrow 0$.

Proof. According to [16], the solution is formally written as

$$u = \int_R E(s - s', t)\, u_0(s') \mathrm{d}s' + \int_0^t \mathrm{d}t' \int_R E(s - s', t - t') f(s', t') \mathrm{d}s'$$

$$\equiv E *_1 u_0 + E * f,$$

where $E(s, t)$ is a fundamental solution (matrix) of (2.1) and is estimated as $|D^n E| \leqq c t^{-(1+n)/4} \exp(-c|s|^{4/3} t^{-1/3})$. Then we have

$$|E *_1 u_0| \leqq c \sup_s |u_0|,$$

$$\|D^p (E *_1 u_0)\| \leqq c \|D^p u_0\| \quad (p = 1, 2, 3),$$

$$|E*f|\leqq ct^{7/8}\langle\|f\|\rangle_t,$$
$$\|D^p(E*f)\|\leqq ct^{1-p/4}\langle\|f\|\rangle_t \ (p=1,2,3).$$

Here we used the formula $\int_0^\infty \exp(-s^{4/3}t^{-1/3})\mathrm{d}s=(3/4)\Gamma(3/4)t^{1/4}$.

On the other hand, the Fourier transformation yields

$$\|D^p(E*_1\boldsymbol{u}_0)\|_t\leqq c\|D^{p-2}\boldsymbol{u}_0\| \ (p=4,5),$$
$$\|(E*f)_{ssss}\|_t\leqq c\|f\|_t\leqq ct^{1/2}\langle\|f\|\rangle_t,$$
$$\|D^5(E*f)\|_t\leqq c\|f_s\|_t\leqq c\|f\|_t^{1/2}\|f_{ss}\|_t^{1/2}\leqq ct^{1/4}\langle\|f\|\rangle_t^{1/2}\|f_{ss}\|_t^{1/2}.$$

The coefficients c's in these estimates depend on ε. However, they are independent of t. Hence the statement of the lemma follows. □

Now, let us investigate a parabolic regularization of (1.4):

$$(2.4)\quad \boldsymbol{v}_t=\boldsymbol{v}\times\boldsymbol{v}_{ss}+a[\boldsymbol{v}_{sss}+(3/2)\{\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_s)\}_s]-\varepsilon\{\boldsymbol{v}_{ssss}+4(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})\boldsymbol{v}+3|\boldsymbol{v}_{ss}|^2\boldsymbol{v}\}$$

for $\varepsilon>0$. Then we obtain

Proposition 2.1. *Let* $\varepsilon>0$, $a\in\boldsymbol{R}$ *and* $\boldsymbol{v}_{0s}\in W_2^2(\boldsymbol{R})$. *Then there exists a positive constant* T_0 *such that* (2.4) *with* (1.5) *has a unique solution* $\boldsymbol{v}$ *satisfying* $\boldsymbol{v}\in C_b(\boldsymbol{R}_{T_0})$, $\boldsymbol{v}_s\in C(0,T_0;W_2^2(\boldsymbol{R}))$, $\boldsymbol{v}_{ssss}\in L^2(0,T_0;W_2^1(\boldsymbol{R}))$ *and* $\boldsymbol{v}_t\in L^2(0,T_0;W_2^1(\boldsymbol{R}))$.

Proof. Set $\boldsymbol{u}^{(0)}=0$ and let $\boldsymbol{u}^{(n)}$ $(n=1,2,\cdots)$ be a solution in Lemma 2.1 on a time interval $[0,t]$ with $\boldsymbol{u}_0=\boldsymbol{v}_0$ and

$$f=\boldsymbol{u}^{(n-1)}\times\boldsymbol{u}_{ss}^{(n-1)}+a[\boldsymbol{u}_{sss}^{(n-1)}+(3/2)\{\boldsymbol{u}_s^{(n-1)}\times(\boldsymbol{u}^{(n-1)}\times\boldsymbol{u}_s^{(n-1)})\}_s]$$
$$-\varepsilon\{4(\boldsymbol{u}_s^{(n-1)}\cdot\boldsymbol{u}_{sss}^{(n-1)})\boldsymbol{u}^{(n-1)}+3|\boldsymbol{u}_{ss}^{(n-1)}|^2\boldsymbol{u}^{(n-1)}\}.$$

Then $\boldsymbol{u}^{(n)}$ is well-defined for each n. Indeed, it follows from (2.3) and the Sobolev imbedding theorem that

$$A_n\equiv\sup_{s,t}|\boldsymbol{u}^{(n)}|+\langle\|\boldsymbol{u}_s^{(n)}\|^{(2)}\rangle_t+\|\boldsymbol{u}_{ssss}^{(n)}\|_t+\|D^5\boldsymbol{u}^{(n)}\|_t+\|\boldsymbol{u}_t^{(n)}\|+\|\boldsymbol{u}_{st}^{(n)}\|_t$$

is bounded from above by $c_1\{1+b(t)(1+t^{1/2})A_{n-1}(1+A_{n-1})^2\}$ for $n\geqq 2$ and by c_1 for $n=1$. Here c_1 (>0) is a constant depending only on $\|\boldsymbol{v}_{0s}\|^{(2)}$ (note that $|\boldsymbol{v}_0|=1$).

Let M be a constant satisfying $M>c_1$. Then, choosing $t=T_1$ so small that $c_1\{1+b(T_1)(1+T_1^{1/2})M(1+M)^2\}\leqq M$, we have, by induction, $A_n\leqq M$ $(n=1,2,\cdots)$ on $[0,T_1]$.

Setting $w^{(n)}=u^{(n)}-u^{(n-1)}$, we estimate $w^{(n+1)}$ on $[0,t]$, $0<t\leqq T_1$, by (2.3) as

$$\begin{aligned} B_{n+1}&=\sup_{s,t}|w^{(n+1)}|+\langle\|w_s^{(n+1)}\|^{(2)}\rangle_t+\|w_{ssss}^{(n+1)}\|_t \\ &\quad+\|D^5w^{(n+1)}\|_t+\|w_t^{(n+1)}\|+\|w_{st}^{(n+1)}\|_t \\ &\leqq cb(t)(1+t^{1/2})(1+M)^2B_n, \end{aligned}$$

while $B_1=A_1$. If we choose $T_0\in(0,T_1]$ so small that $cb(T_0)(1+T_0^{1/2})(1+M)^2<1$, then B_n converges to zero as $n\uparrow\infty$. Hence $u^{(n)}$ converges to a solution v of (2.4) and (1.5) belonging to the class in the proposition.

It is easy to prove the uniqueness of the solution. □

Next, we prove the following lemma, which implies that the length of the vortex filament is conserved.

Lemma 2.2. *Let v be a solution of* (2.4) *and* (1.5) *belonging to the class in Proposition* 2.1 *with T_0 replaced by an arbitrary $T>0$. Then*

$$|v|=1 \tag{2.5}$$

holds for any $(s,t)\in\mathbf{R}\times[0,T]$.

Proof. Define the function $h(s,t)$ by

$$h(s,t)=|v|^2-1$$

for $s\in\mathbf{R}$, $0\leqq t\leqq T$. Then, from (2.4) and (1.5), we obtain

$$h_t=a\{h_{sss}-3(v\cdot v_{ss})h_s+6(v_s\cdot v_{ss})h\}-\varepsilon\{h_{ssss}+8(v_s\cdot v_{sss})h+6|v_{ss}|^2h\},$$
$$h(s,0)=0.$$

Since $(\mathrm{d}/\mathrm{d}t)\|h\|^2\leqq c(\sup_{s,t}|v|+\|v_s\|^{(3)})\|v_s\|^{(3)}\|h\|^2$ follows, Gronwall's inequality yields that $\|h\|=0$. Hence we have $h=0$, which leads to (2.5). □

Utilizing Lemma 2.2, we derive an a priori estimate for (2.4).

Lemma 2.3. *Let $\boldsymbol{v}$ be as in Lemma 2.2. Then there exists a positive constant ε_0 depending only on T and $\|\boldsymbol{v}_{0s}\|$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the solution $\boldsymbol{v}$ satisfies the estimate*

$$\langle \|\boldsymbol{v}_s\|^{(2)} \rangle_T + \|\boldsymbol{v}_t\|_T + \varepsilon^{1/4}(\|\boldsymbol{v}_{ssss}\|_T + \|\boldsymbol{v}_{st}\|_T) + \varepsilon^{1/2}\|D^5\boldsymbol{v}\|_T \leqq c_a, \tag{2.6}$$

where c_a depends only on $|a|$, $\|\boldsymbol{v}_{0s}\|^{(2)}$, ε_0 and T.

Proof. From (2.5), we have

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{v}_s &= 0, \\ \boldsymbol{v} \cdot \boldsymbol{v}_{ss} &= -|\boldsymbol{v}_s|^2, \\ \boldsymbol{v} \cdot \boldsymbol{v}_{sss} &= -3\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss}, \\ \boldsymbol{v} \cdot \boldsymbol{v}_{ssss} &= -4\boldsymbol{v}_s \cdot \boldsymbol{v}_{sss} - 3|\boldsymbol{v}_{ss}|^2. \end{aligned} \tag{2.7}$$

Then, in (2.4),

$$\{\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_s)\}_s = (|\boldsymbol{v}_s|^2 \boldsymbol{v})_s = 2(\boldsymbol{v}_s \cdot \boldsymbol{v}_{ss})\boldsymbol{v} + |\boldsymbol{v}_s|^2 \boldsymbol{v}_s \tag{2.8}$$

holds. It also follows from (2.5) that, at each point where $|\boldsymbol{v}_s|$ is nonzero, the vectors $\boldsymbol{v}$, $\boldsymbol{v}_s/|\boldsymbol{v}_s|$ and $\boldsymbol{v} \times \boldsymbol{v}_s/|\boldsymbol{v}_s|$ are the unit tangent, the unit principal normal and the unit binormal vectors in $\boldsymbol{R}^3$, respectively.
Then

$$\boldsymbol{v}_s \times D^n\boldsymbol{v} = \boldsymbol{v}_s \times [(\boldsymbol{v} \cdot D^n\boldsymbol{v})\boldsymbol{v} + \{(\boldsymbol{v} \times \boldsymbol{v}_s) \cdot D^n\boldsymbol{v}\}\boldsymbol{v} \times \boldsymbol{v}_s/|\boldsymbol{v}_s|^2]$$

holds for $n = 2, 3, 4, \cdots$ and leads to

$$\boldsymbol{v}_s \times D^n\boldsymbol{v} = -(\boldsymbol{v} \cdot D^n\boldsymbol{v})\boldsymbol{v} \times \boldsymbol{v}_s + \{(\boldsymbol{v} \times \boldsymbol{v}_s) \cdot D^n\boldsymbol{v}\}\boldsymbol{v}. \tag{2.9}$$

Clearly, (2.9) is also valid where $|\boldsymbol{v}_s| = 0$.

Multiplying (2.4) by $\boldsymbol{v}_{ss}$, integrating over $\boldsymbol{R}$ and using (2.7) and (2.8), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s(\cdot, t)\|^2 &= -2\varepsilon\Big(\|\boldsymbol{v}_{sss}\|^2 + 4\int_{\boldsymbol{R}} |\boldsymbol{v}_s|^2 \boldsymbol{v}_s \cdot \boldsymbol{v}_{sss} \mathrm{d}s + 3\| |\boldsymbol{v}_s| |\boldsymbol{v}_{ss}| \|^2\Big) \\ &\leqq -\varepsilon\|\boldsymbol{v}_{sss}\|^2 + \varepsilon c_0 \|\boldsymbol{v}_s\|^{10}. \end{aligned} \tag{2.10}$$

Here c_0 is a positive constant given by use of the multiplicative and Young's inequalities. Note that the scalar equation $\mathrm{d}r/\mathrm{d}t=\varepsilon c_0 r^5$ with $r(0)=\|v_{0s}\|^2$ has the solution $r(t)=(\|v_{0s}\|^{-8}-4\varepsilon c_0 t)^{-1/4}$ when $4\varepsilon c_0 t<\|v_{0s}\|^{-8}$. Then, choosing ε_0 so small that

$$0<\varepsilon_0<(4c_0T\|v_{0s}\|^8)^{-1}, \tag{2.11}$$

we have

$$\|v_s(\cdot,t)\|\leqq r(t)^{1/2}\leqq c_* \tag{2.12}$$

on $[0,T]$ for all $\varepsilon\in(0,\varepsilon_0]$.

To obtain further a priori estimates for (2.4), a conservation law for (1.3) is suggestive, as was mentioned in Sect. 1 (see also [12]). According to the soliton theory ([1]), Zakharov-Shabat's recurrence formula ([18]) yields

$$\|\Psi_{ss}\|^2-2\||\Psi||\Psi_s|\|^2+\frac{1}{8}\||\Psi|^3\|^2-\frac{1}{4}\int_{\mathbf{R}}(\Psi_s^2\bar{\Psi}^2+\Psi^2\bar{\Psi}_s^2)\mathrm{d}s$$

(the bar denotes the complex conjugate) as an invariant for (1.3) independently of a (see also [10], [11]). This invariant represents the same quantity as

$$\|v_{sss}\|^2-\frac{7}{2}\||v_s||v_{ss}|\|^2-14\|v_s\cdot v_{ss}\|^2+\frac{21}{8}\||v_s|^3\|^2 \tag{2.13}$$

if Hasimoto's transformation is possible. Thus we anticipate that, for (2.4), the time derivative of (2.13) is bounded from above by a constant proportional to ε. The correctness of this anticipation will be verified in (2.17) below.

Set

$$\begin{aligned}&-2v_{ssss}\cdot v_{sst}-7(v_s\cdot v_{st})|v_{ss}|^2-7|v_s|^2(v_{ss}\cdot v_{sst})\\&-28(v_s\cdot v_{ss})(v_s\cdot v_{ss})_t+(63/4)|v_s|^4(v_s\cdot v_{st})\end{aligned}$$

equal to $I_1+aI_2+\varepsilon I_3$, where I_1, I_2 and I_3 are independent of a and ε. It is clear that the time derivative of (2.13) is equal to $\int_{\mathbf{R}}(I_1+aI_2+\varepsilon I_3)\mathrm{d}s$. Then, noting (2.8), we have

$$\begin{aligned}I_1=&-4(v_s\times v_{sss})\cdot v_{ssss}+7|v_s|^2(v\times v_{ss})\cdot v_{ssss}+14|v_s|^2(v_s\times v_{ss})\cdot v_{sss}\\&+7|v_{ss}|^2(v\times v_s)\cdot v_{sss}+28(v_s\cdot v_{ss})(v\times v_s)\cdot v_{ssss}\end{aligned}$$

$$+28(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{sss}-(63/4)|\boldsymbol{v}_s|^4(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss},$$

$$\begin{aligned}I_2=&-2\boldsymbol{v}_{ssss}\cdot D^5\boldsymbol{v}+(945/8)|\boldsymbol{v}_s|^6(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})+72(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})(\boldsymbol{v}_{ss}\cdot\boldsymbol{v}_{sss})\\&+54|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}_{ss}\cdot\boldsymbol{v}_{sss})-7|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ssss})+6(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ssss})\\&-46(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}_{ss}\cdot\boldsymbol{v}_{ssss})-3|\boldsymbol{v}_s|^2(\boldsymbol{v}_{sss}\cdot\boldsymbol{v}_{ssss})-7|\boldsymbol{v}_s|^2(\boldsymbol{v}_{ss}\cdot D^5\boldsymbol{v})\\&-28(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}_s\cdot D^5\boldsymbol{v})+(105/2)|\boldsymbol{v}_s|^4(\boldsymbol{v}_{ss}\cdot\boldsymbol{v}_{sss})+(147/4)|\boldsymbol{v}_s|^4(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ssss})\\&-(777/2)|\boldsymbol{v}_s|^2|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})-273|\boldsymbol{v}_s|^2(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})-420(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})^3.\end{aligned}$$

Since (2.7) and (2.9) yield

$$\begin{aligned}&(\boldsymbol{v}_s\times\boldsymbol{v}_{sss})\cdot\boldsymbol{v}_{ssss}\\&\quad=3(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ssss}-4(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}-3|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss},\end{aligned}$$

we get

$$\begin{aligned}I_1=&\,7|\boldsymbol{v}_s|^2(\boldsymbol{v}\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{ssss}+14|\boldsymbol{v}_s|^2(\boldsymbol{v}_s\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{sss}+19|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}\\&+16(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ssss}+16(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}\\&+28(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{sss}-(63/4)|\boldsymbol{v}_s|^4(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}.\end{aligned}$$

If I_1 can be written in the differentiated form, then it is as

$$\begin{aligned}(2.14)\quad&\{k_1(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}+k_2|\boldsymbol{v}_s|^2(\boldsymbol{v}\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{sss}\\&+k_3(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ss}+k_4|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ss}+k_5|\boldsymbol{v}_s|^4(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ss}\}_s\end{aligned}$$

with constants $k_1,\cdots,k_5$. In fact, ignoring the coefficients, we see that this contains all terms of I_1. Since

$$(2.15)\qquad |\boldsymbol{v}_s|^2(\boldsymbol{v}\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{ssss}=(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ssss}-(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ssss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ss}$$

can be proved in the same way as (2.9) and

$$(2.16)\qquad |\boldsymbol{v}_s|^2(\boldsymbol{v}_s\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{sss}=|\boldsymbol{v}_s|^4(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}-3|\boldsymbol{v}_s|^2(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ss}$$

is verified by (2.7) and (2.9), (2.14) is equal to

$$\begin{aligned}&(k_1+k_4)|\boldsymbol{v}_{ss}|^2(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}+(k_1+k_3)(\boldsymbol{v}_s\cdot\boldsymbol{v}_{sss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{sss}\\&+(k_1+2k_2)(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_{ss})\cdot\boldsymbol{v}_{sss}+(k_1+k_6)(\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss})(\boldsymbol{v}\times\boldsymbol{v}_s)\cdot\boldsymbol{v}_{ssss}\end{aligned}$$

$$+(k_2-k_7)|v_s|^2(v_s\times v_{ss})\cdot v_{sss}+(k_2-k_6)|v_s|^2(v\times v_{ss})\cdot v_{ssss}$$
$$+(k_3+2k_4)(v_{ss}\cdot v_{sss})(v\times v_s)\cdot v_{ss}+(k_3-k_6)(v_s\cdot v_{ssss})(v\times v_s)\cdot v_{ss}$$
$$+(4k_5-3k_7)|v_s|^2(v_s\cdot v_{ss})(v\times v_s)\cdot v_{ss}+(k_5+k_7)|v_s|^4(v\times v_s)\cdot v_{sss}.$$

Here k_6 and k_7 are constants related to (2.15) and (2.16), respectively. Comparing the coefficients in it with those in I_1, we get

$$k_1=18,\quad k_2=5,\quad k_3=-2,\quad k_4=1,\quad k_5=-27/4$$

and

$$k_6=-2,\quad k_7=-9.$$

In the same way, under the assumption that I_2 can be written as

$$\{-|v_{ssss}|^2+(945/64)|v_s|^8+m_1(v_s\cdot v_{ss})(v_{ss}\cdot v_{sss})+m_2(v_s\cdot v_{sss})^2$$
$$+m_3|v_{ss}|^2(v_s\cdot v_{sss})+m_4|v_{ss}|^4+m_5(v_s\cdot v_{ss})(v_s\cdot v_{ssss})$$
$$+m_6|v_s|^2(v_{ss}\cdot v_{ssss})+m_7|v_s|^2|v_{sss}|^2+m_8|v_s|^4|v_{ss}|^2$$
$$+m_9|v_s|^2(v_s\cdot v_{ss})^2+m_{10}|v_s|^4(v_s\cdot v_{sss})\}_s,$$

we obtain

$$m_1=-4,\quad m_2=17,\quad m_3=21,\quad m_4=37/4,\quad m_5=-28,$$
$$m_6=-7,\quad m_7=2,\quad m_8=63/8,\quad m_9=-210,\quad m_{10}=147/4.$$

Thus,

$$\int_{\boldsymbol{R}} I_1 ds=\int_{\boldsymbol{R}} I_2 ds=0$$

holds.

On the other hand, by the multiplicative and Young's inequalities, we have

$$\int_{\boldsymbol{R}} I_3 ds\leqq -2\|D^5v\|^2+c_*(\|v_s\|^{9/4}\|D^5v\|^{7/4}+\|v_s\|^{9/2}\|D^5v\|^{3/2}+\|v_s\|^{27/4}\|D^5v\|^{5/4})$$
$$\leqq -\|D^5v\|^2+c_*\|v_s\|^{18}.$$

Hence

(2.17) $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{v}_{sss}\|^2-\frac{7}{2}\||\boldsymbol{v}_s||\boldsymbol{v}_{ss}|\|^2-14\|\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss}\|^2+\frac{21}{8}\||\boldsymbol{v}_s|^3\|^2\right)$$

$$=\int_{\boldsymbol{R}}(I_1+aI_2+\varepsilon I_3)\mathrm{d}s$$

$$=\varepsilon\int_{\boldsymbol{R}} I_3\mathrm{d}s\leqq-\varepsilon\|D^5\boldsymbol{v}\|^2+c_*$$

follows for $\varepsilon\in(0,\varepsilon_0]$. Then

$$\|\boldsymbol{v}_{sss}\|^2-(7/2)\||\boldsymbol{v}_s|\,|\boldsymbol{v}_{ss}|\|^2-14\|\boldsymbol{v}_s\cdot\boldsymbol{v}_{ss}\|^2$$
$$\leqq\|\boldsymbol{v}_{0sss}\|^2+(21/8)\||\boldsymbol{v}_{0s}|^3\|^2-\varepsilon\|D^5\boldsymbol{v}\|_t^2+c_*t$$

holds. Since $\||\boldsymbol{v}_s|\,|\boldsymbol{v}_{ss}|\|^2\leqq c_*\|\boldsymbol{v}_s\|^{5/2}\|\boldsymbol{v}_{sss}\|^{3/2}\leqq(1/20)\|\boldsymbol{v}_{sss}\|^2+c_*$, we have

$$\langle\|\boldsymbol{v}_{sss}\|\rangle_T+\varepsilon^{1/2}\|D^5\boldsymbol{v}\|_T\leqq c_*.$$

This estimate together with (2.12) yields

$$\langle\|\boldsymbol{v}_s\|^{(2)}\rangle_T+\varepsilon^{1/4}\|\boldsymbol{v}_{ssss}\|_T\leqq c_*.$$

It is easy to derive $\|\boldsymbol{v}_t\|_T+\varepsilon^{1/4}\|\boldsymbol{v}_{st}\|_T\leqq c_a$. □

Similar a priori estimates were obtained in [14], where we used (2.17) without proof.

From Proposition 2.1, Lemmas 2.2 and 2.3, we have

Theorem 2.1. *Let* $T>0$, $\boldsymbol{v}_{0s}\in W_2^2(\boldsymbol{R})$ *and* $a\in\boldsymbol{R}$. *Then, for each* $\varepsilon\in(0,\varepsilon_0]$ *with* ε_0 *satisfying* (2.11), *there exists a unique solution* $\boldsymbol{v}$ *to* (2.4) *with* (1.5) *such that* $\boldsymbol{v}_s\in C(0,T;W_2^2(\boldsymbol{R}))$, $\boldsymbol{v}_{ssss}\in L^2(0,T;W_2^1(\boldsymbol{R}))$, $\boldsymbol{v}_t\in L^2(0,T;W_2^1(\boldsymbol{R}))$, (2.5) *and* (2.6) *hold.*

Furthermore, the following lemma is clear since $\|\boldsymbol{v}-\boldsymbol{v}_0\|\leqq\int_0^t\|\boldsymbol{v}_t\|\mathrm{d}t$.

Lemma 2.4. *The solution in Theorem* 2.1 *satisfies*

$$(2.18)\qquad \langle\|v-v_0\|\rangle_T \leqq c_a.$$

§3. Vortex Filament with or without Axial Flow

In this section, we first consider $\varepsilon\downarrow 0$. Consequently, the following theorem is established. Its proof is mainly based on the method in [6].

Theorem 3.1. *Let* $v_{0s}\in W_2^2(R)$ and $a\neq 0$. Then, for any $T>0$, there exists a unique solution v of (1.4) with (1.5) such that (2.5) is satisfied and $(v-v_0)\in C(0, T; W_2^3(R))\cap C^1(0, T; L^2(R))$.

Proof. Let $v^\varepsilon(s, t)$ be the solution of (2.4) with (1.5), where the existence is guaranteed by Theorem 2.1. Noting (2.8) and taking the difference of (2.4) for $\varepsilon=\varepsilon'$ and that for $\varepsilon=\varepsilon''$ $(0<\varepsilon''<\varepsilon'\leqq\varepsilon_0)$, we have

$$\begin{aligned} z_t = v^{\varepsilon'}\times z_{ss} + z\times v_{ss}^{\varepsilon''} + a[z_{sss} + (3/2)\{|v_s^{\varepsilon'}|^2 z + (v_s^{\varepsilon'}\cdot z_s)v^{\varepsilon''} + (z_s\cdot v_s^{\varepsilon''})v^{\varepsilon''}\}_s] \\ -\varepsilon'\{v_{ssss}^{\varepsilon'} + 4(v_s^{\varepsilon'}\cdot v_{sss}^{\varepsilon'})v^{\varepsilon'} + 3|v_{ss}^{\varepsilon'}|^2 v^{\varepsilon'}\} \\ +\varepsilon''\{v_{ssss}^{\varepsilon''} + 4(v_s^{\varepsilon''}\cdot v_{sss}^{\varepsilon''})v^{\varepsilon''} + 3|v_{ss}^{\varepsilon''}|^2 v^{\varepsilon''}\}, \end{aligned}$$

where $z=v^{\varepsilon'}-v^{\varepsilon''}$. For the function z,

$$v^{\varepsilon''}\cdot z_s = -v_s^{\varepsilon'}\cdot z (= v_s^{\varepsilon'}\cdot v^{\varepsilon''})$$

holds (by (2.7)) and leads to

$$v^{\varepsilon''}\cdot z_{ss} = -(v_s^{\varepsilon'}+v_s^{\varepsilon''})\cdot z_s - v_{ss}^{\varepsilon'}\cdot z.$$

Moreover, (2.6) and (2.18) yield

$$\langle\|z\|^{(3)}\rangle_T \leqq \langle\|v_s^{\varepsilon'}\|^{(2)}\rangle_T + \langle\|v_s^{\varepsilon''}\|^{(2)}\rangle_T + \langle\|v^{\varepsilon'}-v_0\|\rangle_T + \langle\|v^{\varepsilon''}-v_0\|\rangle_T \leqq c_a.$$

Noting these relations, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 &= \int_R z_t\cdot z\,\mathrm{d}s \\ &\leqq \left|\int_R (v_s^{\varepsilon'}\times z_s)\cdot z\,\mathrm{d}s\right| + \frac{3}{2}|a|\left|\int_R \{|v_s^{\varepsilon'}|^2 z + (v_s^{\varepsilon'}\cdot z_s)v^{\varepsilon''} + (z_s\cdot v_s^{\varepsilon''})v^{\varepsilon''}\}\cdot z_s\,\mathrm{d}s\right| + c_a\varepsilon' \end{aligned}$$

$$\leqq c_a(\|z\|\,\|z_s\|+\|z_s\|^2)+c_a\varepsilon',$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_s\|^2=-\int_{\boldsymbol{R}} z_t\cdot z_{ss}\,\mathrm{d}s$$

$$\leqq\left|\int_{\boldsymbol{R}}(z\times v_{ss}^{\varepsilon''})\cdot z_{ss}\mathrm{d}s\right|+\frac{3}{2}|a|\left|\int_{\boldsymbol{R}}[2(v_s^{\varepsilon'}\cdot v_{ss}^{\varepsilon'})(z\cdot z_{ss})+|v_s^{\varepsilon'}|^2(z_s\cdot z_{ss})\right.$$

$$\left.+\{(v_s^{\varepsilon'}+v_s^{\varepsilon''})\cdot z_s\}_s(v^{\varepsilon''}\cdot z_{ss})+\{(v_s^{\varepsilon'}+v_s^{\varepsilon''})\cdot z_s\}(v_s^{\varepsilon''}\cdot z_{ss})]\mathrm{d}s\right|+c_a\varepsilon'$$

$$\leqq\left|\int_{\boldsymbol{R}}(z\times v_{sss}^{\varepsilon''})\cdot z_s\mathrm{d}s\right|+\frac{3}{2}|a|\left|\int_{\boldsymbol{R}}[2(v_s^{\varepsilon'}\cdot v_{ss}^{\varepsilon'})_s(z\cdot z_s)+3(v_s^{\varepsilon'}\cdot v_{ss}^{\varepsilon'})|z_s|^2\right.$$

$$+\{(v_s^{\varepsilon'}+v_s^{\varepsilon''})\cdot z_s\}_s\{(v_s^{\varepsilon'}+v_s^{\varepsilon''})\cdot z_s+v_{ss}^{\varepsilon'}\cdot z\}$$

$$\left.-\{(z+2v_s^{\varepsilon''})\cdot z_s\}\{(v_s^{\varepsilon''}\cdot z_s)_s-v_{ss}^{\varepsilon''}\cdot z_s\}]\mathrm{d}s\right|+c_a\varepsilon'$$

$$\leqq c_a(\|z\|^{1/2}\|z_s\|^{3/2}+\|z\|\,\|z_s\|+\|z_s\|^2)+c_a\varepsilon'.$$

They yield

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2+\|z_s\|^2)\leqq c_a(\|z\|^2+\|z_s\|^2)+c_a\varepsilon'. \tag{3.1}$$

Since $z(s,0)=\mathbf{0}$, if follows that $\langle\|z\|^{(1)}\rangle_T\leqq c_a(\varepsilon')^{1/2}$. Thus, $v^\varepsilon-v_0$ converges to some function $(w-v_0)\in C(0,T;\,W_2^1(\boldsymbol{R}))$ in the $W_2^1(\boldsymbol{R})$-norm uniformly on $[0,T]$ as $\varepsilon\downarrow 0$. From this, we see that $|w|=1$.

On the other hand, (2.6) and (2.18) imply that $w-v_0$ belongs to $L^\infty(0,T;\,W_2^3(\boldsymbol{R}))$. Let $\Phi=\Phi(s)$ be an infinitely differentiable vector function with a compact support. Denoting the scalar product in $W_2^n(\boldsymbol{R})$ by $(\cdot,\cdot)^{(n)}$, we have

$$|(v^{\varepsilon'}-v^{\varepsilon''},\Phi)^{(3)}|=\left|\int_{\boldsymbol{R}}(v^{\varepsilon'}-v^{\varepsilon''})\cdot(1-D^2+D^4-D^6)\Phi\mathrm{d}s\right|$$

$$\leqq 2\|\Phi\|^{(6)}\|v^{\varepsilon'}-v^{\varepsilon''}\|.$$

This means that $(v^\varepsilon-v_0,\Phi)^{(3)}$ converges uniformly. Therefore, by the density theorem, $v^{\varepsilon}-v_0$ converges to $w-v_0$ weakly in $W_2^3(R)$ uniformly on $[0,T]$ and $w-v_0$ is weakly continuous in $W_2^3(R)$. It is clear that $w(s,0)=v_0$ and $\|w(\cdot,t)-v_0\|^{(3)}\leqq c_a$.

Let

$$G(v^{\varepsilon})=(v^{\varepsilon}\times v_s^{\varepsilon})_s+\frac{3}{2}a\{v_s^{\varepsilon}\times(v^{\varepsilon}\times v_s^{\varepsilon})\}_s.$$

Then $\langle\|G(v^{\varepsilon})\|^{(1)}\rangle_T\leqq c_a$ follows from (2.5) and (2.6). Moreover, since

$$|(G(v^{\varepsilon'})-G(v^{\varepsilon''}),\Phi)^{(1)}|\leqq c_a\|\Phi\|^{(3)}\|v^{\varepsilon'}-v^{\varepsilon''}\|^{(1)},$$

$G(v^{\varepsilon})$ converges to $G(w)\in L^{\infty}(0,T;W_2^1(R))$ weakly in $W_2^1(R)$ uniformly on $[0,T]$.

Next, integrate (2.4) over $[t',t'']\subset[0,T]$. Then we get

$$v^{\varepsilon}(s,t'')-v^{\varepsilon}(s,t')=\int_{t'}^{t''}[av_{sss}^{\varepsilon}+G(v^{\varepsilon})-\varepsilon\{v_{ssss}^{\varepsilon}+4(v_s^{\varepsilon}\cdot v_{sss}^{\varepsilon})v^{\varepsilon}+3|v_{ss}^{\varepsilon}|^2v^{\varepsilon}\}]\mathrm{d}t.$$

Taking its scalar product in $L^2(R)$ with an arbitrary element of $L^2(R)$ and making use of the above convergence and (2.6), we have

$$w(s,t'')-w(s,t')=\int_{t'}^{t''}(aw_{sss}+G(w))\mathrm{d}t.$$

in $L^2(R)$. Hence w is a solution of (1.4) with (1.5) and $\|w-v_0\|$ is Lipschitz continuous.

Suppose that there exist two solutions w' and w'' for (1.4) with the same initial data and set $z=w'-w''$. Then we obtain the same estimate as (3.1) with $\varepsilon'=0$. Hence $z=0$ and the uniqueness of the solution follows.

We can derive $(\mathrm{d}/\mathrm{d}t)\|v_{sss}^{\varepsilon}\|^2\leqq c_a$ from (2.6) and (2.4) for $\varepsilon\in(0,\varepsilon_0]$. Therefore, $\|v_{sss}^{\varepsilon}(\cdot,t)\|\leqq\|v_{0sss}\|+c_at^{1/2}$ follows. Letting $\varepsilon\downarrow 0$, we have $\|w_{sss}(\cdot,t)\|\leqq\|v_{0sss}\|+c_at^{1/2}$, which leads to $\limsup_{t\downarrow 0}\|w_{sss}(\cdot,t)\|\leqq\|v_{0sss}\|=\|w_{sss}(\cdot,0)\|$. Since w_{sss} is weakly continuous in $L^2(R)$, $\lim_{t\downarrow 0}\|w_{sss}(\cdot,t)\|=\|v_{0sss}\|$ is obtained. As in [6], by the uniqueness of w and the reversibility of (1.4) in t, we can show $w_{sss}\in C(0,T;L^2(R))$. Then it yields $(w-v_0)\in C(0,T;W_2^3(R))\cap C^1(0,T;L^2(R))$. □

Now, by (1.7), we establish the unique solvability of (1.2) and (1.6).

Theorem 3.2. *Let* $x_{0ss}\in W_2^2(\boldsymbol{R})$ *and* $a\neq 0$. *Then, for any* $T>0$, *there exists a unique solution* x *of* (1.2) *with* (1.6) *such that* $|x_s|=1$ *and* $(x-x_0)\in C(0,T;W_2^4(\boldsymbol{R}))\cap C^1(0,T;W_2^1(\boldsymbol{R}))$. *Moreover,*

$$\langle\|x-x_0\|^{(4)}\rangle_T\leqq c_a \tag{3.2}$$

is valid.

Next, we consider the limit $a\to 0$.

Theorem 3.3. *Let* x^a *be the solution of* (1.2) *and* (1.6) *for* $a\neq 0$, *which is guaranteed by Theorem* 3.2. *Then* x^a-x_0 *converges strongly in* $W_2^3(\boldsymbol{R})$ *and weakly in* $W_2^4(\boldsymbol{R})$ *uniformly on* $[0,T]$ *as* $a\to 0$. *The limit function* x *is a unique solution of* (1.1) *with* (1.6) *such that* $|x_s|=1$ *and* $(x-x_0)\in C(0,T;W_2^4(\boldsymbol{R}))\cap C^1(0,T;W_2^2(\boldsymbol{R}))$.

Proof. Taking the difference of (1.2) for $a=a'$ and that for $a=a''$ and setting $p=x^{a'}-x^{a''}$, we have

$$p_t=x_s^{a'}\times p_{ss}+p_s\times x_{ss}^{a''}+a'\{x_{sss}^{a'}+(3/2)|x_{ss}^{a'}|^2x_s^{a'}\}-a''\{x_{sss}^{a''}+(3/2)|x_{ss}^{a''}|^2x_s^{a''}\}.$$

Let $0<|a''|<|a'|\leqq 1$. Then, since $c_a\leqq c_*$ holds if $|a|\leqq 1$, it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p\|^2\leqq\|p\|\,\|p_{ss}\|+\sup_s|x_{ss}^{a''}|\,\|p\|\,\|p_s\|+c_*|a'|,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_s\|^2=-\int_{\boldsymbol{R}}p_{ss}\cdot p_t\mathrm{d}s\leqq\sup_s|x_{ss}^{a''}|\,\|p_s\|\,\|p_{ss}\|+c_*|a'|,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_{ss}\|^2=-\int_{\boldsymbol{R}}p_{sss}\cdot p_{st}\mathrm{d}s\leqq\sup_s|p_s|\,\|x_{ssss}^{a''}\|\,\|p_{ss}\|+c_*|a'|.$$

Here we noted that

$$(x_s^{a'}\times p_{ss}+p_s\times x_{ss}^{a''})_s=x_s^{a'}\times p_{sss}+p_s\times x_{sss}^{a''}$$

and

$$\langle\|p\|^{(4)}\rangle_T\leqq\langle\|x^{a'}-x_0\|^{(1)}\rangle_T+\langle\|x^{a''}-x_0\|^{(1)}\rangle_T+\langle\|x_{ss}^{a'}\|^{(2)}\rangle_T+\langle\|x_{ss}^{a''}\|^{(2)}\rangle_T\leqq c_*.$$

The above estimates lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|p\|^{(2)})^2 \leqq c_*(\|p\|^{(2)})^2 + c_*|a'|,$$

which yields $\langle \|p\|^{(2)} \rangle_T \leqq c_*|a'|^{1/2}$ since $p(s,0)=0$. Thus, $x^a - x_0$ converges to some function $(x-x_0) \in C(0,T;W_2^2(\mathbb{R}))$ in the $W_2^2(\mathbb{R})$-norm uniformly on $[0,T]$ as $a \to 0$. Therefore, (3.2) with $|a| \leqq 1$ gives the weak convergence of $x^a - x_0$ to $x - x_0$ in $W_2^4(\mathbb{R})$ uniformly on $[0,T]$ and the strong convergence in $W_2^3(\mathbb{R})$ uniformly on $[0,T]$. In the same way as in the proof of Theorem 3.1, we can prove that x is a unique solution of (1.1) and (1.6) belonging to the class in the theorem. □

From Theorems 3.2 and 3.3, the following theorem is derived.

Theorem 3.4. *Let x be a solution to* (1.2) *and* (1.6) *for $a \in \mathbb{R}$ which is guaranteed by Theorem* 3.2 *or* 3.3. *Then it is extended so that $\|(x-x_0)(\cdot,t)\|^{(1)}$ is continuous in $t \in [0,\infty)$, and $\|x_{ss}(\cdot,t)\|^{(2)}$ and $\|x_t(\cdot,t)\|^{(1)}$ (or $\|x_t(\cdot,t)\|^{(2)}$ if $a=0$) are bounded and continuous in $t \in [0,\infty)$.*

Proof. It follows from (2.10) and (2.17) that every solution to (1.2) satisfies

$$\|x_{ss}\| = \|x_{0ss}\|,$$

$$\|x_{ssss}\|^2 - (7/2)\||x_{ss}|\,|x_{sss}|\|^2 - 14\|x_{ss}\cdot x_{sss}\|^2 \leqq \|x_{0ssss}\|^2 + (21/8)\||x_{0ss}|^3\|^2$$

whether $a \neq 0$ or $a=0$. Then x_{ss} is bounded in $W_2^2(\mathbb{R})$ uniformly in t. Therefore, the solutions to (1.2) in Theorems 3.1 and 3.2 are extended bounded-continuously for $t \geqq T$. □

In addition, the existence of a unique solution with $\|D^5 x\|$ bounded and continuous was obtained in [12].

§4. Spatially Periodic Case

In this section, the domain of s is restricted to $I \equiv (0,1)$, and the norms in $L^2(I)$ and in the Sobolev space $W_2^n(I)$ $(n=1,2,\cdots)$ are denoted by $\|\cdot\|_I$ and $\|\cdot\|_I^{(n)}$, respectively. Let P^n be the completion with respect to $\|\cdot\|_I^{(n)}$ of the space where every element g belongs to $C^\infty[0,1]$ and $D^j g(0) = D^j g(1)$ for any $j=0,1,2,\cdots$. Then, imposing

(4.1) $$v(s,t)=v(s+1,t)$$

on (2.4), we have

Theorem 4.1. *Let* $T>0$, $v_0\in P^3$ *and* $a\in \boldsymbol{R}$. *Then, for each* $\varepsilon\in(0,\varepsilon_0]$ *with* $0<\varepsilon_0<(4c_0T\|v_{0s}\|_I^8)^{-1}$, there exists a unique solution of (2.4) with (1.5) and (4.1) such that $v\in C(0,T;P^3)\cap L^2(0,T;P^5)$, *and* $v_t\in L^2(0,T;P^1)$. *Moreover*, (2.5) *and* $\langle\|v\|_I^{(3)}\rangle_T\leqq c_a$ *are valid, where* c_a *depends only on* $|a|$, v_0, T *and* ε_0.

Proof. The proof is divided into two parts. One is to establish the existence of a temporally local solution as in Proposition 2.1. It is possible because (2.1) with (2.2) and the spatially periodic condition is solved by the Fourier expansion analogously to Lemma 2.1. The other is to derive (2.5) and the a priori estimate in the theorem. The former is clear from the proof of Lemma 2.2. In order to get the latter by the method in the proof of Lemma 2.3, we should verify the validity of the multiplicative inequality for an element in P^n.

Let $g=(g^1,g^2,g^3)\in P^n$, $n\geqq 2$. Then, for $k=1,2,\cdots,n-1$, we have

$$\|D^kg\|_I^2=-\int_I D^{k-1}g\cdot D^{k+1}g\,\mathrm{d}s\leqq\|D^{k-1}g\|_I\|D^{k+1}g\|_I,$$

$$|D^kg(s)|^2=\sum_{i=1}^{3}2\int_{s_i}^{s}D^kg^iD^{k+1}g^i\,\mathrm{d}s\leqq 2\|D^kg\|_I\|D^{k+1}g\|_I.$$

Here s_i $(i=1,2,3)$ is a point on $[0,1]$ satisfying $D^kg^i(s_i)=0$, whose existence is obtained from $\int_I D^kg\mathrm{d}s=0$. From this, we see that the multiplicative inequality is valid for an element in P^n. □

In the same way as in the preceding section, we can establish

Theorem 4.2. *Let* $x_{0s}\in P^3$ *and* $a\neq 0$. *Then, for any* $T>0$, *there exists a unique solution* x *of* (1.2) *with* (1.6) *and* (1.8) *such that* $|x_s|=1$, $x_s\in C(0,T;P^3)$ *and* $x_t\in C(0,T;P^1)$. *Moreover, if* $x_0\in P^4$, *then* $x\in C(0,T;P^4)$.

Theorem 4.3. *The solution of* (1.2), (1.6) *and* (1.8) *in Theorem* 4.2 *converges strongly in* $W_2^3(I)$ *and weakly in* $W_2^4(I)$ *uniformly on* $[0,T]$ *as* $a\to 0$. *The limit function is a unique solution of* (1.1), (1.6) *and* (1.8). *It has the same properties* as in Theorem 4.2 but $x_t \in C(0, T; P^2)$.

Furthermore, we can repeat the above arguments infinitely many times.

Theorem 4.4. *Let* x *be a solution to* (1.2), (1.6) *and* (1.8) *for* $a \in \boldsymbol{R}$ *which is guaranteed by Theorem* 4.2 *or* 4.3. *Then it is extended so that* $\|x(\cdot, t)\|_I$ *is continuous in* $t \in [0, \infty)$, *and* $\|x_s(\cdot, t)\|_I^{(3)}$ *and* $\|x_t(\cdot, t)\|_I^{(1)}$ (*or* $\|x_t(\cdot, t)\|_I^{(2)}$ *if* $a=0$) *are bounded and continuous in* $t \in [0, \infty)$.

References

[1] Ablowitz, M. J. and Segur, H., *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

[2] Da Rios, L. S., On the motion of an unbounded fluid with a vortex filament of any shape, *Rend. Circ. Mat. Palermo*, **22** (1906), 117–135, (in Italian).

[3] Fukumoto, Y. and Miyazaki, T., Three-dimensional distortions of a vortex filament with axial velocity, *J. Fluid Mech.*, **222** (1991), 369–416.

[4] Hasimoto, H., A soliton on a vortex filament, *J. Fluid Mech.*, **51** (1972), 477–485.

[5] Hirota, R., Exact envelope-soliton solutions of a nonlinear wave equation, *J. Math. Phys.*, **14** (1973), 805–809.

[6] Kato, T., Nonstationary flows of viscous and ideal fluids in R^3, *J. Funct. Anal.*, **9** (1972), 296–305.

[7] ———, Quasi-linear equations of evolution, with applications to partial differential equations, *Lect. Note Math.* **448**, *Spectral Theory and Differential Equations*, ed. Everitt, W. N., Springer-Verlag, Berlin, Heidelberg, New York, 1975, pp. 25–70.

[8] Koiso, N., Vortex filament equation and semilinear Schrödinger equation, to appear in *Proceedings of the 4th International Research Institute on Nonlinear Waves*, Gakkotosho, Tokyo.

[9] Lakshmanan, M., Ruijgrok, T. W. and Thompson, C. J., On the dynamics of a continuum spin system, *Physica A*, **84** (1976), 577–590.

[10] Laurey, C., Le problème de Cauchy pour une équation de Schrödinger non linéaire d'ordre 3, *C. R. Acad. Sci. Paris*, **315** (1992), 165–168.

[11] ———, The Cauchy problem for a third order nonlinear Schrödinger equations, to appear in *Nonlinearity*.

[12] Nishiyama, T., On the initial value problems for the vortex filament equations, to appear in *Adv. Math. Sci. Appl.*

[13] Nishiyama, T. and Tani, A., Solvability of the localized induction equation for vortex motion, *Comm. Math. Phys.*, **162** (1994), 433–445.

[14] ———, Initial and initial-boundary value problems for a vortex filament with or without axial flow, *SIAM J. Math. Anal.*, **27** (1996), 1015–1023.

[15] Ricca, R.L., Rediscovery of Da Rios equations, *Nature*, **352** (1991), 561–562.

[16] Solonnikov. V. A., On boundary value problems for linear parabolic systems of differential equations of general form, *Trudy Mat. Inst. Steklov*, **83** (1965), 3–163, (in Russian), (Its English translation is in *Proc. Steklov Inst. Math.*, **83** (1967), 1–184).

[17] Sulem, P. L., Sulem, C. and Bardos, C., On the continuous limit for a system of classical spins, *Comm. Math. Phys.*, **107** (1986), 431–454.

[18] Zakharov, V. E. and Shabat, A. B., Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Soviet Phys. JETP*, **34** (1972), 62–69.