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Semialgebraic Description of Teichmüller Space

By

Yohei KOMORI*

Abstract

We give a concrete semialgebraic description of Teichmüller space T_g of the closed surface group Γ_g of genus $g(\geq 2)$. Our result implies that for any $SL_2(\mathbf{R})$-representation of Γ_g, we can determine whether this representation is discrete and faithful or not by using $4g-6$ explicit trace inequalities. We also show the connectivity and contractibility of T_g from the point of view of $SL_2(\mathbf{R})$-representations of Γ_g. Previously, these properties of T_g had been proved by using hyperbolic geometry and quasi-conformal deformations of Fuchsian groups. Our method is simple and only uses topological properties of the space of $SL_2(\mathbf{R})$-representations of Γ_g.

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* Department of Mathematics
Osaka City University, Osaka 558, Japan

§1. Introduction

The Teichmüller space T_g of compact Riemann surfaces of genus $g(\geq 2)$ is the moduli space of *marked* Riemann surfaces of genus g. Thanks to the uniformization theorem due to Klein, Koebe and Poincaré, any compact Riemann surface of genus $g(\geq 2)$ can be obtained as the quotient space $G\backslash\mathbb{H}$ where $\mathbb{H}$ is the upper half plane and G is a cocompact Fuchsian group, i.e. a cocompact discrete subgroup of $PSL_2(\mathbb{R})$. As an abstract group, G is isomorphic to the surface group Γ_g, which has the following presentation

$$\Gamma_g := \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} (\alpha_i \cdot \beta_i \cdot \alpha_i^{-1} \cdot \beta_i^{-1}) = id. \rangle.$$

From this point of view, T_g can be considered as the deformation space of a Fuchsian group which is isomorphic to Γ_g and this is called *Fricke space* studied by Fricke himself and more precisely by Keen ([F], [K1], [K2], [K3]).

In this article, we consider this Fricke space from the point of view of $SL_2(\mathbb{R})$-representations of the surface group Γ_g. We treat T_g as the $PGL_2(\mathbb{R})$-adjoint quotient of the set of discrete and faithful $PSL_2(\mathbb{R})$-representations of Γ_g

$$T_g = \{\Gamma_g \to PSL_2(\mathbb{R}) : \text{discrete and faithful}\} / PGL_2(\mathbb{R})$$

where a *discrete and faithful* $PSL_2(\mathbb{R})$-representation of Γ_g means a group homomorphism from Γ_g to $PSL_2(\mathbb{R})$ which is injective and the image of Γ_g is a discrete subgroup of $PSL_2(\mathbb{R})$. Because any Fuchsian group which is isomorphic to Γ_g can be lifted to $SL_2(\mathbb{R})$ ([Pa],[S-S]), we can start from $Hom(\Gamma_g, SL_2(\mathbb{R}))$ the set of $SL_2(\mathbb{R})$-representations of Γ_g. And T_g can be considered as the set of characters of discrete and faithful $SL_2(\mathbb{R})$-representations of Γ_g.

From this point of view, we can get a real algebraic structure on T_g as follows. By using the presentation of Γ_g, $Hom(\Gamma_g, SL_2(\mathbb{R}))$ can be embeded into the product space $SL_2(\mathbb{R})^{2g}$ as the real algebraic subset $R(\Gamma)$ which is called *the space of representations* ([C-S], [Go], [M-S]). The adjoint action of $PGL_2(\mathbb{R})$ on $R(\Gamma)$ induces the action on $\mathbb{R}[R(\Gamma)]$ the affine coordinate ring of $R(\Gamma)$ and put $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$ the ring of invariants under this action. Let $X(\Gamma)$ be a real algebraic set whose affine coordinate ring is isomorphic to $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$. Then T_g can be realized as a semialgebraic subset of $X(\Gamma)$. Hence T_g is determined by finitely many polynomial equalities and inequalities on $X(\Gamma)$. This construction is essentially due to Helling [He], and later Culler-Shalen [C-S] and Morgan-Shalen [M-S] made this process more clear. And by using this procedure, Brumfiel described the real spectrum compactification of T_g [Br]. More recently, in a series of works [Sa 1], [Sa 2] and [Sa 3], Saito obtained a description of the coordinate ring for T_g as a semialgebraic set defined over $\mathbb{Z}$.

Our theme in this paper is to study the semialgebraic structure on T_g, and we mainly consider the following two things. First we describe the defining equations of T_g on $X(\Gamma)$ by using 6g-6 explicit polynomial inequalities. (Theorems 3.2 and 4.2). This problem is related to the construction of the global coordinates of T_g by means of a small number of traces of elements of Fuchsian groups which have been studied deeply by Keen ([K1], [K2], [K3]) and more recently by Okai and Okumura ([Ok], [O1], [O2]) by using hyperbolic geometry on $\mathbb{H}$ and an argument involving the fundamental polygons of Fuchsian groups. Our treatment in this paper is rather algebraic. The second is that from a real algebraic viewpoint, we also show the well known fact that T_g is a 6g-6 dimensional cell (Theorems 3.1 and 4.1) which was proved by Teichmüller himself by means of his theory of quadratic differentials and quasi-conformal mappings.

For our purposes, we only need some elementary topological properties of semialgebraic sets and some geometric properties of the space of representations $R(\Gamma)$. More precisely, the next three assertions are essential for our arguments (the following notations are defined in Sections 3 and 4):

1. Let $R_0(\Gamma)$ be the set of discrete and faithful $SL_2(\mathbb{R})$-repretations of Γ_g. Then $R_0(\Gamma)$ consists of finitely many connected components of $R(\Gamma)=Hom(\Gamma_g, SL_2(\mathbb{R}))$.
2. Let $t^{-1}(S(\Gamma))$ be the of $SL_2(\mathbb{R})$-representations ρ of Γ_g satisfying the following 2g-3 trace inequalities

$$tr(\rho([\alpha_i, \beta_i])) < -2 \ (i=1,\cdots,g)$$
$$tr(\rho([\alpha_1, \beta_1]\cdots[\alpha_j, \beta_j])) < -2 \ (j=1,\cdots,g-2)$$

where $[\alpha,\beta]:=\alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1}$. Then $t^{-1}(S(\Gamma))$ consists of $2^{4g-3}\times 2$ connected components such that each components is homemorphic to $\mathbb{R}^{6g-6}\times PSL_2(\mathbb{R})$ and this homeomorphism is $PSL_2(\mathbb{R})$-equivariant.

3. $R_0(\Gamma)\subset t^{-1}(S(\Gamma))$.

In fact, we will construct global coordinates on $t^{-1}(S(\Gamma))$, and by using these coordinates, we shall find a system of trace inequalities which characterize $R_0(\Gamma)$ in $t^{-1}(S(\Gamma))$. Finally, we obtain a semialgebraic description of T_g and a cell structure of T_g. The trace inequalities which define $t^{-1}(S(\Gamma))$ in Assertion 2 above already appeared in the paper of Seppälä-Sorvali (S-S]), and in that paper, they used these inequalities to solve the lifting problem of a Fuchsian group to $SL_2(\mathbf{R})$. At first, I proved Assertion 2 above by using a slightly complicated method involving some geometric properties of $R(\Gamma)$, and I would like to thank Professor Kyoji Saito for telling me about the result of [S-S]. These inequalities also appeared in the paper of Okumura [O2].

The remainder of this paper is organized as follows. Section 2 deals with the construction of Teichmüller space T_g following Culler-Shalen [C-S] and Morgan-Shalen [M-S]. The description of the defining inequalities and cell structure of T_g are given in Sections 3 and 4. In Section 3 we treat the case of genus $g=2$ and in Section 4, the case of genus $g\geq 3$ is discussed.

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§2. Construction of Teichmüller Space as a Semialgebraic Set

In this section we review the construction of Teichmüller space following [C-S], [M-S], [Sa 1], [Sa 2], [Sa 3].

§2.1. The Space of $SL_2(\mathbf{R})$-Representations of the Surface Group Γ

Let $g \geq 2$ be fixed. We define the (closed) surface group of genus g by the following presentation

$$\Gamma = \Gamma_g := \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] = id. \rangle$$

where $[\alpha_i, \beta_i] := \alpha_i \cdot \beta_i \cdot \alpha_i^{-1} \cdot \beta_i^{-1}$.

By using this presentation, we can embed $Hom(\Gamma, SL_2(\mathbf{R}))$ the set of $SL_2(\mathbf{R})$-representations of Γ into the product space $SL_2(\mathbf{R})^{2g}$ and let $R(\Gamma)$ denote the image of $Hom(\Gamma, SL_2(\mathbf{R}))$

$$Hom(\Gamma, SL_2(\mathbf{R})) \to R(\Gamma) \subset SL_2(\mathbf{R})^{2g}$$
$$\rho \mapsto (\rho(\alpha_1), \rho(\beta_1), \cdots, \rho(\alpha_g), \rho(\beta_g)).$$

We identify $R(\Gamma)$ and $Hom(\Gamma, SL_2(\mathbf{R}))$. In the following we also identify a representation ρ and the image $(A_1, B_1, \cdots, A_g, B_g) \in SL_2(\mathbf{R})^{2g}$ of the system of generators $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$ of Γ under ρ. $R(\Gamma)$ is a real algebraic set and we call this *the space of* $SL_2(\mathbf{R})$-*representations of* Γ. $PGL_2(\mathbf{R})$ acts on $R(\Gamma)$ from the right-hand side

$$R(\Gamma) \times PGL_2(\mathbf{R}) \to R(\Gamma)$$
$$(\rho, P) \mapsto P^{-1} \rho P.$$

We remark that although we use the system of generators $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$ of Γ to define $R(\Gamma)$, the real algebraic structure of $R(\Gamma)$ does not depend on this system of generators. In fact if we choose another system of generators of Γ consisting of N elements and embed $Hom(\Gamma, SL_2(\mathbf{R}))$ into the product space $SL_2(\mathbf{R})^N$, we get another real algebraic set but it is canonically isomorphic to $R(\Gamma)$.

Next we consider the following subset of $R(\Gamma)$

$$R'(\Gamma) := \{\rho \in R(\Gamma) \mid \rho \text{ is non abelian and irreducible}\}$$

where a representation ρ is *non abelian* if $\rho(\Gamma)$ is a non abelian subgroup of $SL_2(\mathbf{R})$ and ρ is *irreducible* if the action of $\rho(\Gamma)$ on $\mathbf{R}^2$ admits no nontrivial invariant subspaces. Hence if ρ is not irreducible (i.e., reducible) then there exists $P\in PGL_2(\mathbb{R})$ such that $P^{-1}\rho(\Gamma)P$ consists of upper triangular matrices, hence in particular $\rho(\Gamma)$ is solvable. We remark that the action of $PGL_2(\mathbb{R})$ on $R(\Gamma)$ preserves $R'(\Gamma)$. The next lemma is useful for the study of $R'(\Gamma)$.

Lemma 2.1. *For $\rho\in R'(\Gamma)$, there exist $g,h\in\Gamma$ such that $\rho(g)$ is a hyperbolic matrix, i e., $|tr(\rho(g))|>2$, and $\rho(h)$ has no fixed points in common with $\rho(g)$. In other words there exists $P\in PGL_2(\mathbb{R})$ such that*

$$P^{-1}\rho(g)P=\begin{pmatrix}\lambda & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}\quad(\lambda\neq\pm 1)$$

$$P^{-1}\rho(h)P=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\quad(b\cdot c\neq 0).$$

Proof. For $\rho\in R'(\Gamma)$, suppose that $\rho(\Gamma)$ has no hyperbolic elements. If there exists a parabolic element $\rho(g)\in\rho(\Gamma)$, there exists $P\in PGL_2(\mathbb{R})$ such that

$$P^{-1}\rho(g)P=\pm\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}\quad(q\neq 0).$$

As ρ is irreducible, there exists $h\in\Gamma$ with

$$P^{-1}\rho(h)P=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\quad(c\neq 0).$$

Then for $n\in\mathbb{N}$

$$tr(\rho(g)^n\cdot\rho(h))=(\pm 1)^n(tr(\rho(h))+ncq)$$

and $\rho(g)^n\cdot\rho(h)=\rho(g^nh)$ is hyperbolic for sufficiently large n. This is a contradiction. Next if every element of $\rho(\Gamma)-\{id\}$ is elliptic, as ρ is non abelian, there exist $g,h\in\Gamma$ with $[\rho(g),\rho(h)]\neq\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Then there exists $P\in PGL_2(\mathbb{R})$ such that

$$P^{-1}\rho(g)P=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\quad(\theta\neq 0,\pi)$$

$$P^{-1}\rho(h)P=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

with $a^2+b^2+c^2+d^2\neq 2$, and $tr(\rho([g,h]))=2+\sin^2\theta(a^2+b^2+c^2+d^2-2)>2$. This is a contradiction. Hence there exists $g\in\Gamma$ so that $\rho(g)$ is hyperbolic and

$$P^{-1}\rho(g)P=\begin{pmatrix}\lambda & 0\\ 0 & \frac{1}{\lambda}\end{pmatrix}\quad(\lambda\neq\pm 1)$$

for some $P\in PGL_2(\mathbf{R})$. Next suppose that any element of $P^{-1}\rho(\Gamma)P$ is either upper or lower triangular matrix. Because ρ is irreducible there exist $h_1, h_2\in\Gamma$ such that $P^{-1}\rho(h_1)P$ is an upper triangular matrix and $P^{-1}\rho(h_2)P$ is a lower triangular matrix. But $P^{-1}\rho(h_1h_2)P$ is not a triangular matrix. This is a contradiction. Therefore there exists $h\in\Gamma$ so that

$$P^{-1}\rho(h)P=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\quad(b\cdot c\neq 0).$$

We have another characterization of $R'(\Gamma)$. ∎

Proposition 2.1.

$$R'(\Gamma)=\{\rho\in R(\Gamma)\mid tr(\rho([a,b]))\neq 2 \textit{ for some } a,b\in\Gamma\}$$

$$=R(\Gamma)-\bigcap_{a,b\in\Gamma}\{\rho\in R(\Gamma)\mid tr(\rho([a,b]))=2\}.$$

Proof. ($\Rightarrow$) Take $g,h\in\Gamma$ which satisfy the conditions of Lemma 2.1. Then $tr([\rho(g),\rho(h)])\neq 2$.

($\Leftarrow$) If $\rho(\Gamma)$ is abelian, $[\rho(a),\rho(b)]=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for any $a,b\in\Gamma$. If $\rho(\Gamma)$ has a non trivial invariant subspace, there exists $P\in PGL_2(\mathbf{R})$ such that any element of $P^{-1}\rho(\Gamma)P$ is an upper triangular matrix. Hence $tr([\rho(a),\rho(b)])=2$ for any $a,b\in\Gamma$. ∎

Corollary 2.1. *$R'(\Gamma)$ is open in $R(\Gamma)$.* ∎

Proposition 2.2. *$R'(\Gamma)$ has a structure of a* 6g-3 *dimensional real analytic manifold.*

Proof. We consider the mapping r

$$r: SL_2(\mathbf{R})^{2g} \to SL_2(\mathbf{R})$$

$$(A_1, B_1, \cdots, A_g, B_g) \mapsto \prod_{i=1}^{g} [A_i, B_i].$$

Then $R(\Gamma) = r^{-1}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$. We will check that the rank of the differential $(dr)_\rho$ at $\rho \in R'(\Gamma)$ is maximal. By regarding $sl_2(\mathbf{R})$, the Lie algebra of $SL_2(\mathbf{R})$, as the Lie algebra of left-invariant vector fields on $SL_2(\mathbf{R})$, $(dr)_\rho$ can be written as follows. For $\rho = (A_1, B_1, \cdots, A_g, B_g) \in R'(\Gamma)$

$$(dr)_\rho: sl_2(\mathbf{R})^{2g} \to sl_2(\mathbf{R})$$

$$(\xi_i, \eta_i)_{1 \le i \le g} \mapsto \sum_{i=1}^{g} \{\delta_{F_i A_i F_i^{-1}}(F_i \eta_i F_i^{-1}) - \delta_{F_i B_i F_i^{-1}}(F_i \xi_i F_i^{-1})\}$$

where $F_i := \Pi_{l=1}^{i-1}[A_l, B_l] A_i B_i$ and for $A \in SL_2(\mathbf{R})$ and $X \in sl_2(\mathbf{R})$, $\delta_A(X) := X - A^{-1} X A$.

Put $S_i := F_i A_i F_i^{-1}$, $S_{i+g} := F_i B_i F_i^{-1}$ $(i = 1, \cdots, g)$. Then $\{S_k\}$ is a system of generators of $\rho(\Gamma)$. As $\rho \in R'(\Gamma)$, some S_j is not the identity matrix hence S_j is hyperbolic, parabolic or elliptic. First we assume that S_j is hyperbolic. Then there exists $P \in PGL_2(\mathbf{R})$ such that

$$S_j = P \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} P^{-1} \quad (\lambda \neq \pm 1).$$

Because $\rho \in R'(\Gamma)$, there exists S_k such that

$$S_k = P \begin{pmatrix} p & q \\ r & s \end{pmatrix} P^{-1} \quad (q \neq 0 \ \ or \ \ r \neq 0).$$

If we put $X_j = P \begin{pmatrix} x_j & y_j \\ z_j & -x_j \end{pmatrix} P^{-1}$, $X_k = P \begin{pmatrix} x_k & y_k \\ z_k & -x_k \end{pmatrix} P^{-1} \in sl_2(\mathbf{R})$,

$$\delta_{S_j}(X_j) = P \begin{pmatrix} 0 & (1 - \frac{1}{\lambda^2}) y_j \\ (1 - \lambda^2) z_j & 0 \end{pmatrix} P^{-1}$$

$$\delta_{S_k}(X_k) = P \begin{pmatrix} (1 - ps - qr) x_k + pq z_k - rs y_k & * \\ * & * \end{pmatrix} P^{-1}.$$

Therefore the mapping

$$sl_2(\mathbf{R}) \times sl_2(\mathbf{R}) \to sl_2(\mathbf{R})$$

$$(X_j, X_k) \mapsto \delta_{S_j}(X_j) + \delta_{S_k}(X_k)$$

is surjective, and this shows the surjectivity of $(dr)_\rho$. By a similar argument, one can also show the surjectivity of $(dr)_\rho$ for the case that S_j is parabolic or elliptic. Hence by the implicit function theorem $R'(\Gamma)$ has the structure of a 6g-3 dimensional real analytic manifold. ■

Next we define the subset $R_0(\Gamma)$ of $R(\Gamma)$ by

$$R_0(\Gamma) := \{\rho \in R(\Gamma) \mid \rho \text{ is discrete and faithful}\} \tag{1}$$

where a representation ρ is *discrete* if $\rho(\Gamma)$ is a discrete subgroup of $SL_2(\mathbf{R})$ and ρ is *faithful* if ρ is injective. We remark that the action of $PGL_2(\mathbf{R})$ on $R(\Gamma)$ preserves $R_0(\Gamma)$. Then another characterization of $R_0(\Gamma)$ is

Proposition 2.3.

$$R_0(\Gamma) = \{\rho \in R(\Gamma) \mid \rho \textit{ is cocompact, discrete and faithful}\} \tag{2}$$

$$= \{\rho \in R(\Gamma) \mid \rho \textit{ is purely hyperbolic}\} \tag{3}$$

where a representation p is cocompact if the quotient space $\rho(\Gamma)\backslash SL_2(\mathbf{R})$ is compact with respect to the quotient topology, and ρ is called purely hyperbolic if $\rho(h)$ is hyperbolic for any $h(\neq identity) \in \Gamma$.

Proof. (1) ⇒ (2) The fundamental group of a surface $\rho(\Gamma)\backslash\mathbf{H}$ is isomorphic to the surface group Γ. Hence $\rho(\Gamma)\backslash\mathbf{H}$ is compact.
(2) ⇒ (3) Because $\rho(\Gamma)$ is discrete, any elliptic element of $\rho(\Gamma)$ is finite order. But Γ is torsion free. Thus $\rho(\Gamma)$ has no elliptic elements. Moreover if $\rho(\Gamma)$ has a parabolic element, then $\rho(\Gamma)\backslash\mathbf{H}$ has a cusp. Since $\rho(\Gamma)\backslash\mathbf{H}$ is compact, $\rho(\Gamma)$ has no parabolic elements.
(3) ⇒ (1) Faithfulness is immediate. Discreteness follows from Nielsen's theorem (see [Si, Theorem 3, p33]). ■

Proposition 2.4. $R_0(\Gamma)$ *is open and closed in* $R(\Gamma)$.

Proof. We give a sketch of a proof. We recall the inequalities of Jørgensen ([Jø], especially the argument of Proposition 1, §3):

For any $\rho \in R(\Gamma)\rho$ *is contained in* $R_0(\Gamma)$ *if and only if*

$$|tr([\rho(g), \rho(h)]) - 2| + |tr(\rho(h))^2 - 4| \geq 1$$

for any pair $g, h \in \Gamma$ *with* $gh \neq hg$.

These inequalities are closed conditions of $R_0(\Gamma)$ in $R(\Gamma)$.

The openness of $R_0(\Gamma) \subset R(\Gamma)$ follows from the next theorem due to Weil [W]:

If G is a coonnected Lie group and Γ *is a discrete group, then the set of cocompact, discrete and faithful representations from* Γ *to G is open in the set of all representations from* Γ *to G.* ∎

Next we recall the notions of a semialgebraic set. Let V be a real algebraic set with affine coordinate ring $\mathbb{R}[V]$, i.e., the ring of polynomial functions on V. A subset S of V is called *a semialgebraic subset of V* if there exist finitely many polynomial functions f_i, g_{i_1}, $\cdots$ $g_{i_{m(i)}}$ $(i = 1, \cdots, l)$ on V such that S can be written as

$$S = \bigcup_{i=1}^{l} \{x \in V \mid f_i(x) = 0,\ g_{i_1}(x) > 0,\ \cdots\ g_{i_{m(i)}}(x) > 0\}.$$

From the above definition, it follows immediately that any real algebraic set is a semialgebraic set. Moreover, it is known that any connected component of a semialgebraic set (with respect to the Euclidean topology) is also a semialgebraic set and the number of connected components of a semialgebraic set is finite (see [B-C-R] Theorem 2.4.5).

Corollary 2.2. $R_0(\Gamma)$ *consists of finitely many connected components of* $R(\Gamma)$, *hence* $R_0(\Gamma)$ *is a semialgebraic subset of* $R(\Gamma)$. ∎

The relation between $R'(\Gamma)$ and $R_0(\Gamma)$ is

Proposition 2.5. $R_0(\Gamma) \subset R'(\Gamma)$.

Proof. Let ρ be a element of $R_0(\Gamma)$. Since the surface group Γ is non abelian and ρ is injective, ρ is non abelian. Also beccause Γ is not solvable, ρ is irreducible. ∎

Corollary 2.3. *$R_0(\Gamma)$ has the structure of a 6g-3 dimensional real analytic manifold.* ∎

§2.2. The Action of $PGL_2(\mathbb{R})$ on $R'(\Gamma)$

This subsection follows the argument of Gunning (Section 9 in [Gu]). We will show that the quotient space $R'(\Gamma)/PGL_2(\mathbb{R})$ under the action defined in Subsection 2.1 has the structure of a 6g-6 dimensional real analytic manifold such that the natural projection

$$R'(\Gamma) \to R'(\Gamma)/PGL_2(\mathbb{R})$$

is a real analytic principal $PGL_2(\mathbb{R})$-bundle.

Lemma 2.2. *$PGL_2(\mathbb{R})$ acts on $R'(\Gamma)$ without fixed points.*

Proof. For $P \in PGL_2(\mathbb{R})$ and $\rho \in R'(\Gamma)$ suppose that $P^{-1}\rho P = \rho$. Then by Lemma 2.1 there exists $g \in \Gamma$ and $Q \in PGL_2(\mathbb{R})$ such that

$$Q^{-1}\rho(g)Q = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \quad (\alpha \neq \pm 1).$$

Then $P^{-1}\rho P = \rho$ implies that $Q^{-1}PQ$ is also diagonal, and if $Q^{-1}PQ$ is not the identity, then $Q^{-1}\rho(h)Q$ is also diagonal for any $h \in \Gamma$, but this contradicts the fact that $\rho \in R'(\Gamma)$. Therefore, $Q^{-1}PQ$, hence also P, is the identity. ∎

Lemma 2.3. *The action of $PGL_2(\mathbb{R})$ on $R'(\Gamma)$ is proper: i.e., every element of $R'(\Gamma)$ has an open neighborhood $U \subseteq R'(\Gamma)$ such that the closure of the set*

$$\{P \in PGL_2(\mathbb{R}) \mid (U \cdot P) \cap U \neq \phi\}$$

(where $U \cdot P := \{P^{-1}\rho P \mid \rho \in U\}$ is compact in $PGL_2(\mathbb{R})$.

Proof. By Proposition 2.2 $R'(\Gamma)$ is a real analytic manifold. Hence any ρ of $R'(\Gamma)$ has an open neighborhood $U \subset R'(\Gamma)$ such that $\bar{U}$ is compact in $R'(\Gamma)$. Because $SO(2)$ is compact, by replacing U by $U \cdot SO(2)$ if necessary, we may assume that U is invariant under the adjoint action of $SO(2)$. Suppose that there is a sequence $\{T_\nu\} \subset PGL_2(\mathbb{R})$ such that $(U \cdot T_\nu) \cap U \neq \phi$ for each ν but $\{T_\nu\}$ has no accumulation points in $PGL_2(\mathbb{R})$. T_ν can be considered as an element of $SL_2(\mathbf{R})$ or $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbf{R})$. Then by taking a subsequence we may assume that $\{T_\nu\} \subset SL_2(\mathbf{R})$ or $\{T_\nu\} \subset \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbf{R})$. First we treat the case that $\{T_\nu\} \subset SL_2(\mathbf{R})$ (a similar argument works for the other case). Then T_ν can be written as

$$T_\nu = A_\nu \cdot B_\nu$$

where $A_\nu \in SO(2)$ and $B_\nu = \begin{pmatrix} a_\nu & b_\nu \\ 0 & \frac{1}{a_\nu} \end{pmatrix}$. Because $\{T_\nu\}$ has no accumulation points in $PGL_2(\mathbf{R})$ we may assume that

$$|a_\nu| \to 0, \quad |a_\nu| \to \infty \text{ or } |b_\nu| \to \infty \text{ (as } \nu \to \infty).$$

Since U is $SO(2)$-invariant

$$(U \cdot B_\nu) \cap U \neq \phi.$$

Hence there exists $\rho_\nu \in U$ so that $B_\nu^{-1} \rho_\nu B_\nu \in U$ for each ν. Since $\bar{U}$ is compact we may assume that $\{\rho_\nu\}$ and $\{B_\nu^{-1} \rho_\nu B_\nu\}$ converge to $\rho \in \bar{U}$ and $\eta \in \bar{U}$ respectively.

For $g \in \Gamma$ put

$$\rho_\nu(g) = \begin{pmatrix} p_\nu & q_\nu \\ r_\nu & s_\nu \end{pmatrix} \tag{4}$$

$$\rho(g) = \lim_{\nu \to \infty} \rho_\nu(g) = \begin{pmatrix} p_\infty & q_\infty \\ r_\infty & s_\infty \end{pmatrix} \tag{5}$$

$$\eta(g) = \lim_{\nu \to \infty} B_\nu^{-1} \rho_\nu(g) B_\nu = \begin{pmatrix} p'_\infty & q'_\infty \\ r'_\infty & s'_\infty \end{pmatrix}. \tag{6}$$

Then

$$B_\nu^{-1} \rho_\nu B_\nu = \begin{pmatrix} p_\nu - a_\nu b_\nu r_\nu & \frac{b_\nu}{a_\nu}(p_\nu - s_\nu) + \frac{1}{a_\nu^2} q_\nu - b_\nu^2 r_\nu \\ a_\nu^2 r_\nu & a_\nu b_\nu r_\nu + s_\nu \end{pmatrix}. \tag{7}$$

Since ρ_ν, ρ, η and $B_\nu^{-1} \rho_\nu B_\nu$ are all contained in the compact set $\bar{U}$, each component of the matrices of (4), (5), (6) and (7) is bounded. If $|a_\nu| \to \infty$, the boundedness of $\{a_\nu^2 r_\nu\}$ implies that $r_\infty = \lim_{\nu \to \infty} r_\nu = 0$. If $\{a_\nu\}$ is bounded and does not converge to 0 and $|b_\nu| \to \infty$, the boundedness of $\{a_\nu b_\nu r_\nu + s_\nu\}$ also implies $r_\infty = \lim_{\nu \to \infty} r_\nu = 0$. In these cases, for any $g \in \Gamma$, $\rho(g)$ is an upper triangular matirx. If $|a_\nu| \to 0$, then $r'_\infty = \lim_{\nu\to\infty} a_\nu^2 r_\nu = 0$. In this case, for any $g \in \Gamma$, $\eta(g)$ is an upper triangular matrix. But $\rho, \eta \in \bar{U} \subset R'(\Gamma)$. This is a contradiction. ■

Proposition 2.6. *The quotient space* $R'(\Gamma)/PGL_2(\mathbf{R})$ *has the structure of a* 6g-6 *dimensional real analytic manifold such that the natural projection*

$$R'(\Gamma) \to R'(\Gamma)/PGL_2(\mathbf{R})$$

is a real analytic principal $PGL_2(\mathbf{R})$*-bundle.*

Proof. Let $\rho \in R'(\Gamma)$ be fixed. Let us define a real analytic mapping G by

$$G: PGL_2(\mathbf{R}) \to R'(\Gamma) \subset SL_2(\mathbf{R})^{2g}$$
$$P \mapsto P^{-1}\rho P.$$

We first show that for $P \in PGL_2(\mathbf{R})$ the differential $(dG_*)_P$ at P has maximal rank. We may assume that $P \in SL_2(\mathbf{R})$ (For the case $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} P \in SL_2(\mathbf{R})$ is essentially the same procedure). Then

$$(dG)_P: sl_2(\mathbf{R}) \to sl_2(\mathbf{R})^{2g}$$
$$X \mapsto (P^{-1}\rho(\gamma_j)PX - XP^{-1}\rho(\gamma_j)P)_{1\le j\le 2g}$$

where $\{\gamma_j\}$ is the system of generators of Γ. If $(dG)_P(X) = 0$ for $X \in sl_2(\mathbf{R})$, then $P^{-1}\rho(g)PX = XP^{-1}\rho(g)P$ for all $g \in \Gamma$. $\rho \in R'(\Gamma)$ implies that $X = 0$. Hence $(dG)_P$ is injective. Therefore the mapping G is regular and there exists an open neighborhood Δ of the identity $I \in PGL_2(\mathbf{R})$ such that $G: \Delta \to G(\Delta)$ is a real analytic homeomorphism. Thus, there exist an open neighborhood U of ρ and a real analytic submanifold V of U so that V and $G(\Delta)$ are transversal at ρ. Then the real analytic mapping defined by

$$\Delta \times V \to R'(\Gamma)$$
$$(P, \eta) \mapsto P^{-1}\eta P$$

is regular at (I, ρ). We may assume that this mapping is a real analytic homeomorphism $\Delta \times V \simeq U$. To complete the proof, it is enough to show that this mapping extends to a real analytic homeomorphism from $PGL_2(\mathbf{R}) \times V$

into $R'(\Gamma)$. For this purpose it is only necessary to check that after restricting V if necessary, no two points of V are in the same $PGL_2(\mathbb{R})$-orbit. Suppose that there are sequences $(\rho_\nu), (\eta_\nu) \subset V$ and $P_\nu \in PGL_2(\mathbb{R})$ such that $P_\nu^{-1}\rho_\nu P_\nu = \eta_\nu$ and $\lim_{\nu\to\infty}\rho_\nu = \lim_{\nu\to\infty}\eta_\nu = \rho$. Then Lemma 2.3 shows that by taking subsequence, we may assume that P_ν converges to some $P \in PGL_2(\mathbb{R})$. Since

$$P^{-1}\rho P = \lim_{\nu\to\infty} P_\nu^{-1}\rho_\nu P_\nu = \rho$$

P is identity by Lemma 2.2. Hence $P_\nu \in \Delta$ for sufficiently large ν. But

$$P_\nu^{-1}\rho_\nu P_\nu = \eta_\nu = P^{-1}\eta_\nu P$$

and $\Delta \times V \simeq U$. This is a contradiction. ∎

§2.3. The Space of Characters of Γ

As we have seen in Subsection 2.1, $R(\Gamma)$ has the structure of a real algebraic set. Let $\mathbb{R}[R(\Gamma)]$ be its affine coordinate ring i.e., the ring of polynomial functions on $R(\Gamma)$. Then the action of $PGL_2(\mathbb{R})$ on $R(\Gamma)$ induces the action of $PGL_2(\mathbb{R})$ on $\mathbb{R}[R(\Gamma)]$

$$PGL_2(\mathbb{R}) \times \mathbb{R}[R(\Gamma)] \to \mathbb{R}[R(\Gamma)]$$
$$(P, f(\rho)) \mapsto f(P^{-1}\rho P)$$

and let $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$ be the ring of invariants of this action. For example the function $\tau_h \in \mathbb{R}[R(\Gamma)]$ $(h \in \Gamma)$ on $R(\Gamma)$ defined by

$$\tau_h(\rho) := tr(\rho(h))$$

the trace of $\rho(h)$ for $\rho \in R(\Gamma)$ is an element of $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$. In fact $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$ is generated by the τ_h $(h \in \Gamma)$ and is a finitely generated $\mathbb{R}$-subalgebra of $\mathbb{R}[R(\Gamma)]$ (see [He], [Ho], [Pr]).

Let $X(\Gamma)$ be a real algebraic set whose affine coordinate ring $\mathbb{R}[X(\Gamma)]$ is isomorphic to $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$. And let $I_h \in \mathbb{R}[X(\Gamma)]$ correspond to $\tau_h \in \mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$. Then $\mathbb{R}[X(\Gamma)]$ is generated by the I_h $(h \in \Gamma)$ as an $\mathbb{R}$-algebra. The injection

$$\mathbb{R}[X(\Gamma)] \cong \mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})} \subsetneq \mathbb{R}[R(\Gamma)]$$

induces a polynomial mapping

$$t: R(\Gamma) \to X(\Gamma).$$

Because $\mathbb{R}[R(\Gamma)]^{PGL_2(\mathbb{R})}$ is generated by the τ_h ($h \in \Gamma$), for a represectation $\rho \in R(\Gamma)$, $t(\rho)$ can be considered as the *character* χ_ρ of ρ

$$\chi_\rho : \Gamma \to \mathbb{R}$$
$$h \mapsto tr(\rho(h)) = \tau_h(\rho).$$

Therefore the image $t(R(\Gamma)) \subset X(\Gamma)$ of $R(\Gamma)$ under the mapping t can be considered as the set of characters of $SL_2(\mathbb{R})$-representations of Γ. We call $X(\Gamma)$ *the space of characters of* Γ.

Moreover any element of $X(\Gamma) - t(R(\Gamma))$ can be considered as a character of an $SU(2)$-representation. To explain this we need to review briefly the theory of $SL_2(\mathbb{C})$-representations of Γ following [C-S] and [M-S]. Let $R_{\mathbb{C}}(\Gamma)$ be the set of $SL_2(\mathbb{C})$-representations of Γ, then $R_{\mathbb{C}}(\Gamma)$ has the structure of a complex algebraic set and let $\mathbb{C}[R_{\mathbb{C}}(\Gamma)]$ be its affine coordinate ring. $PGL_2(\mathbb{C})$ acts on $R_{\mathbb{C}}(\Gamma)$ and also on $\mathbb{C}[R_{\mathbb{C}}(\Gamma)]$. Let $\mathbb{C}[R_{\mathbb{C}}(\Gamma)]^{PGL_2(\mathbb{C})}$ denote the ring of invariants of this action and let $X_{\mathbb{C}}(\Gamma)$ be a complex algebraic set whose affine coordinate ring $\mathbb{C}[X_{\mathbb{C}}(\Gamma)]$ is isomorphic to $\mathbb{C}[R_{\mathbb{C}}(\Gamma)]^{PGL_2(\mathbb{C})}$. Then the injection

$$\mathbb{C}[X_{\mathbb{C}}(\Gamma)] \cong \mathbb{C}[R_{\mathbb{C}}(\Gamma)]^{PGL_2(\mathbb{C})} \subsetneq \mathbb{C}[R_{\mathbb{C}}(\Gamma)]$$

induces the polynomial map

$$t_{\mathbb{C}}: R_{\mathbb{C}}(\Gamma) \to X_{\mathbb{C}}(\Gamma)$$

which is surjective. Since $R_{\mathbb{C}}(\Gamma)$, $t_{\mathbb{C}}$ and $X_{\mathbb{C}}(\Gamma)$ are all defined over $\mathbb{Q}$, we can consider $X_{\mathbb{R}}(\Gamma)$, the set of real-valued points of $X_{\mathbb{C}}(\Gamma)$. Then we can consider $X_{\mathbb{R}}(\Gamma)$ as the set of real-valued characters of $SL_2(\mathbb{C})$-representations of Γ. It is known that any element of $X_{\mathbb{R}}(\Gamma)$ is either a character of $SL_2(\mathbb{R})$ or $SU(2)$-representation of Γ ([M-S] Proposition 3.1.1).

If we consider the polynomial function $tr_h \in \mathbb{C}[R_{\mathbb{C}}(\Gamma)]$ ($h \in \Gamma$) on $R_{\mathbb{C}}(\Gamma)$ defined by

$$tr_h(\rho) := tr(\rho(h))$$

for $\rho \in R_{\mathbb{C}}(\Gamma)$, then tr_h is an element of $\mathbb{C}[R_{\mathbb{C}}(\Gamma)]^{PGL_2(\mathbb{C})}$, and we shall denote the corresponding element of $\mathbb{C}[X_{\mathbb{C}}(\Gamma)]$ also by tr_h for the sake of simplicity. Then by regarding $R(\Gamma)$ as the set of real-valued points of $R_{\mathbb{C}}(\Gamma)$, we obtain a natural surjective homomorphism from $\mathbb{R}[X_{\mathbb{R}}(\Gamma)]$, the affine coordinate ring of $X_{\mathbf{R}}(\Gamma)$, to $\mathbf{R}[X(\Gamma)]$

$$\mathbf{R}[X_{\mathbf{R}}(\Gamma)] \to \mathbf{R}[X(\Gamma)]$$
$$tr_h \mapsto I_h .$$

Therefore there is a canonical injection from $X(\Gamma)$ to $X_{\mathbf{R}}(\Gamma)$. Hence any element of $X(\Gamma)$ is either contained in $t(R(\Gamma))$ or can be considered as a character of a $SU(2)$-representation of Γ.

Next, we define the following subsets of $X(\Gamma)$

$$X'(\Gamma) := t(R'(\Gamma))$$
$$U(\Gamma) := \{\chi \in X(\Gamma) \mid I_{[a,b]}(\chi) \neq 2 \text{ for some } a, b \in \Gamma\}$$
$$= X(\Gamma) - \bigcap_{a,b \in \Gamma} \{\chi \in X(\Gamma) \mid I_{[a,b]}(\chi) = 2\}.$$

Then $U(\Gamma)$ is open in $X(\Gamma)$. By Proposition 2.1 $t^{-1}(X'(\Gamma)) = R'(\Gamma)$ and $X'(\Gamma) \subset U(\Gamma)$.

Proposition 2.7. *$X'(\Gamma)$ is open in $U(\Gamma)$. Hence $X'(\Gamma)$ is open in $X(\Gamma)$.*

Proof. Let $V(\Gamma)$ be the set of characters of $SU(2)$-representations of Γ. Since $SU(2)$ is compact, $V(\Gamma)$ is compact in $X_{\mathbf{R}}(\Gamma)$. Hence $U(\Gamma) = X'(\Gamma) \cup (U(\Gamma) \cap V(\Gamma))$ and $(U(\Gamma) \cap V(\Gamma))$ is compact in $U(\Gamma)$. Therefore it is enough to show that $X'(\Gamma) \cap (U(\Gamma) \cap V(\Gamma)) = \phi$. For $\rho \in R'(\Gamma)$, by Lemma 2.1 there exists $g \in \Gamma$ with $|tr(\rho(g))| = |I_g(\rho)| > 2$. On the other hand for any $SU(2)$-representation η of Γ

$$|tr(\eta(h))| = |I_h(\eta)| \leq 2 \quad \text{for any } h \in \Gamma.$$

Therefore $X'(\Gamma) \cap (U(\Gamma) \cap V(\Gamma)) = \phi$. ■

Next we will show that the restriction of the mapping t to $R'(\Gamma)$

$$t : R'(\Gamma) \to X'(\Gamma)$$

is a principal $PGL_2(\mathbf{R})$-bundle. By the result of Subsection 2.2, it is enough to show that $X'(\Gamma)$ is the $PGL_2(\mathbf{R})$ adjoint quotient of $R'(\Gamma)$. For this purpose, we need to prepare two lemmas which are $SL_2(\mathbf{R})$ versions of the results in

[C-S] and [M-S].

Lemma 2.4. ([C-S, Proposition 1.5.2])
For $\rho_1, \rho_2 \in R'(\Gamma)$, we assume that $t(\rho_1)=t(\rho_2)$, in other words, they have the same character $\chi_{\rho_1}=\chi_{\rho_2}$. Then there is a $P \in PGL_2(\mathbf{R})$ such that $\rho_2=P^{-1}\rho_1 P$.

Proof. By Lemma 2.1 and the assumption $\chi_{\rho_1}=\chi_{\rho_2}$, there exist $g \in \Gamma$ and $Q_1, Q_2 \in PGL_2(\mathbf{R})$ such that

$$Q_1^{-1}\rho_1(g)Q_1 = Q_2^{-1}\rho_2(g)Q_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \quad (\alpha \neq \pm 1).$$

Because ρ_1 is irreducible, there exist $h \in \Gamma$ and a diagonal matrix $R_1 \in PGL_2(\mathbf{R})$ such that

$$(Q_1 R_1)^{-1}\rho_1(h)(Q_1 R_1) = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} \quad (c_1 \neq 0).$$

Moreover because $\chi_{\rho_1}([g,h])=\chi_{\rho_2}([g,h])\neq 2$, there is a diagonal matrix $R_2 \in PGL_2(\mathbf{R})$ such that

$$(Q_2 R_2)^{-1}\rho_2(h)(Q_2 R_2) = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix} \quad (c_2 \neq 0).$$

For any $\gamma \in \Gamma$ put

$$(Q_1 R_1)^{-1}\rho_1(\gamma)(Q_1 R_1) = \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}$$

$$(Q_2 R_2)^{-1}\rho_2(\gamma)(Q_2 R_2) = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}.$$

Then $\chi_{\rho_1}(\gamma)=\chi_{\rho_2}(\gamma)$ and $\chi_{\rho_1}(g\gamma)=\chi_{\rho_2}(g\gamma)$ implies that $p_1=p_2$ and $s_1=s_2$. If we put $\gamma=h$, then we obtain that $a_1=a_2$ and $d_1=d_2$. Then $det(\rho_1(h))=det(\rho_2(h))=1$ shows $c_1=c_2$. If we take $h\gamma$ for our γ, then we obtain $r_1=r_2$ and $q_1=q_2$. Therefore if we take $P=(Q_1R_1)(Q_2R_2)^{-1}$, then $\rho_2=P^{-1}\rho_1 P$. ■

Lemma 2.5. ([M-S, Lemma 3.1.7])
For a subset U of $X'(\Gamma)$ we assume that $t^{-1}(U)$ is open in $R'(\Gamma)$, hence open in $[R(\Gamma)]$. Then U is open in $X'(\Gamma)$ hence in $X(\Gamma)$.

Proof. Suppose that there exists a sequence $\{\chi_i\} \subset X'(\Gamma) - U$ so that χ_i converges to $\chi \in U$. Let $\rho \in R'(\Gamma)$ satisfy $\chi_\rho = \chi$. By Lemma 2.1 we may assume that there exist $g, h \in \Gamma$ such that

$$\rho(g) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \quad (\lambda \neq \pm 1)$$

$$\rho(h) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \quad (c \neq 0).$$

Since $I_{[g.h]}(\chi) \neq 2$, by taking a subsequence of $\{\chi_i\}$ we may suppose that

$$|I_g(\chi_i)| > 2 \quad \text{and} \quad I_{[g.h]}(\chi_i) \neq 2.$$

Let $\rho_i \in R'(\Gamma) - t^{-1}(U)$ satisfy $\chi_{\rho_i} = \chi_i$. Then since $|I_g(\chi_i)| > 2$ we may suppose that

$$\rho_i(g) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{pmatrix} \quad (\lambda_i \neq \pm 1).$$

Moreover as $I_g(\chi_i) \to I_g(\chi)$ we may assume that $\lambda_i \to \lambda$.

For $\gamma \in \Gamma$ put

$$\rho_i(\gamma) = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}, \quad \rho(\gamma) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Then $I_{\gamma g}(\chi_i) \to I_{\gamma g}(\chi)$ and $I_{\gamma g^2}(\chi_i) \to I_{\gamma g^2}(\chi)$ shows that $x_i \to x$ and $w_i \to w$. On the other hand, since $I_{[g.h]}(\chi_i) \neq 2$, we may suppose that

$$\rho_i(h) = \begin{pmatrix} a_i & 1 \\ c_i & d_i \end{pmatrix} \quad (c_i \neq 0).$$

Then replacing γ by h, we have $a_i \to a$, $d_i \to d$ and $c_i \to c$. Moreover replacing γ by $h\gamma$, we have $y_i \to y$ and $z_i \to z$. Hence ρ_i converges to ρ. But $\rho_i \in R'(\Gamma) - t^{-1}(U)$ and $\rho \in t^{-1}(U)$. This is a contradiction. ∎

By Lemmas 2.4 and 2.5, we conclude that

Proposition 2.8. *$t: R'(\Gamma) \to X'(\Gamma)$ can be considered as the quotient map of $R'(\Gamma)$ under the action of $PGL_2(\mathbb{R})$, i.e.,*

$$X'(\Gamma) = R'(\Gamma)/PGL_2(\mathbf{R}).$$

Therefore by the result of Subsection 2.2, $t: R'(\Gamma) \to X'(\Gamma)$ is a principal $PGL_2(\mathbf{R})$-bundle. ■

Define the closed subset $X_0(\Gamma)$ of $X(\Gamma)$ by

$$X_0(\Gamma) := \{\chi \in X(\Gamma) \mid I_{[g,h]}(\chi) - 2| + |I_h(\chi)^2 - 4| \geq 1 \quad \text{for } g, h \in \Gamma \text{ with } gh \neq hg\}.$$

Then the proof of Proposition 2.4 implies that $t(R_0(\Gamma)) \subset X_0(\Gamma)$.

Proposition 2.9. 1. $X_0(\Gamma) = t(R_0(\Gamma))$.
2. *$X_0(\Gamma)$ is open in $X'(\Gamma)$ hence open in $X(\Gamma)$.*
3. $t^{-1}(X_0(\Gamma)) = R_0(\Gamma)$.

Proof. 1. Any representation of Γ to $SL_2(\mathbf{C})$ is discrete and faithful if and only if it satisfies the inequalities of Jørgensen which we saw in the proof of Proposition 2.4. But there are no discrete and faithful $SU(2)$-representations of Γ because $SU(2)$ is compact and Γ is an infinite group. Hence $X_0(\Gamma) \subset t(R(\Gamma))$ and it follows that $X_0(\Gamma) = t(R_0(\Gamma))$.

2. $R_0(\Gamma) \subset R'(\Gamma)$ implies $X_0(\Gamma) \subset X'(\Gamma)$. Because $R_0(\Gamma)$ is open in $R(\Gamma)$ and $t: R'(\Gamma) \to X'(\Gamma)$ is an open map by Proposition 2.6, $X_0(\Gamma)$ is open in $X'(\Gamma)$.

3. This is immediate from the proof of Proposition 2.4. ■

Corollary 2.4. *$X_0(\Gamma)$ is open and closed in $X(\Gamma)$. Therefore $X_0(\Gamma)$ consists of finitely many connected components of $X(\Gamma)$. Hence it is a semialgebraic subset of $X(\Gamma)$.* ■

Corollary 2.5. *$t: R_0(\Gamma) \to X_0(\Gamma)$ is also a principal $PGL_2(\mathbf{R})$-bundle. Hence $X_0(\Gamma)$ can be considered as the $PGL_2(\mathbf{R})$ adjoint quotient of $R_0(\Gamma)$, i.e., $X_0(\Gamma) = R_0(\Gamma)/PGL_2(\mathbf{R})$.* ■

We summarize the results of this subsection as the following diagram

$$\begin{array}{ccccccc} R(\Gamma) & \supset & R'(\Gamma) & \supset & R_0(\Gamma) & & \\ t\downarrow & & \downarrow & & \downarrow & PGL_2(\mathbf{R})\text{-bundle} & \\ X(\Gamma) & \supset & X'(\Gamma) & \supset & X_0(\Gamma) & = R_0(\Gamma)/PGL_2(\mathbf{R}). & \end{array}$$

§2.4. The Relation Between $SL_2(\mathbf{R})$- and $PSL_2(\mathbf{R})$-Representations of Γ

Next we consider the relation between $SL_2(\mathbf{R})$- and $PSL_2(\mathbf{R})$-representations of the surface group Γ.

The group $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ ($\cong(\mathbf{Z}/2\mathbf{Z})^{2g}$) acts on $R(\Gamma)$ as follows. For any $\mu \in Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ and $\rho \in R(\Gamma)$, we define the representation $\mu \cdot \rho R(\Gamma)$ by

$$\mu \cdot \rho(h) := \mu(h) \cdot \rho(h) \quad (\text{for all } h \in \Gamma)$$

where $\mathbf{Z}/2\mathbf{Z}$ acts on $SL_2(\mathbf{R})$ by multiplication by ± 1.

Proposition 2.10. ([Pa], [S-S], [O2])

Let $\xi : \Gamma \to PSL_2(\mathbf{R})$ be a discrete and faithful $PSL_2(\mathbf{R})$ representation. Let $A_i, B_i \in SL_2(\mathbf{R})$ denote any representatives of $\xi(\alpha_i)$, $\xi(\beta_i) \in PSL_2(\mathbf{R})$ $(i=1, \cdots, g)$. Then

$$\prod_{i=1}^{g} [A_i, B_i] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, ξ can always be lifted to a representation $\rho \in R_0(\Gamma)$ and the set of all liftings of ξ is equal to the $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ orbit of ρ in $R_0(\Gamma)$.

$$\begin{array}{ccc} & & SL_2(\mathbf{R}) \\ & {}^{\rho}\nearrow & \downarrow proj. \\ \Gamma & \xrightarrow{\xi} & PSL_2(\mathbf{R}) \end{array}$$

Proof. We briefly review what Seppälä and Sorvali showed in their paper [S-S].

Let ξ be a discrete and faithful $PSL_2(\mathbf{R})$ representation. Suppose $A_i, B_i \in SL_2(\mathbf{R})$ $(i=1, \cdots, g)$ denote any representatives of $\xi(\alpha_i), \xi(\beta_i) \in PSL_2(\mathbf{R})$. Then they showed that

$$tr([A_i, B_i]) < -2 \ (i=1, \cdots, g)$$

$$tr([A_1, B_1] \cdots [A_j, B_j]) < -2 \ (j=2, \cdots, g-1).$$

In particular

$$tr([A_g, B_g]) < -2$$

$$tr([A_1, B_1] \cdots [A_{g-1}, B_{g-1}]) < -2.$$

We may suppose that $[A_1, B_1] \cdots [A_{g-1}, B_{g-1}]$ is a diagonal matrix. Then $[A_g, B_g]$ must be also diagonal, hence the above inequalities imply the conclusion. ■

Corollary 2.6. 1. *$Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ acts on $R_0(\Gamma)$ and the quotient space $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})\backslash R_0(\Gamma)$ can be considered as the set of discrete and faithful $PSL_2(\mathbf{R})$-representations of Γ.*

2. *Through the mapping t, $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})$ also acts on $X_0(\Gamma)$ and the quotient space $Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})\backslash X_0(\Gamma)$ can be considered as the $PGL_2(\mathbf{R})$-adjoint quotient of the set of discrete and faithful $PSL_2(\mathbf{R})$-representations of Γ.* ■

We call this set *the Teichmüller space* T_g

$$\begin{aligned} T_g &:= Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})\backslash X_0(\Gamma) \\ &= Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})\backslash R_0(\Gamma)/PGL_2(\mathbf{R}). \end{aligned}$$

Remark. ([Sa 1]) For any $\rho \in R_0(\Gamma)$, let π be the projection

$$\pi : \mathbf{H} \to M := \rho(\Gamma)\backslash \mathbf{H}$$

and put $p := \pi(\sqrt{-1}) \in M$. Then the isomorphism from $\rho(\Gamma)$ to the fundamental group $\pi_1(M,p)$ of M with a base point p

$$\imath : \rho(\Gamma) \simeq \pi_1(M,p)$$

is uniquely determined. Put $a_i := \imath(\rho(\alpha_i)), \ b_i := \imath(\rho(\beta_i)) \in \pi_1(M, \rho) \ \ (i=1, \cdots, g)$. We say a representation ρ is *orientation preserving* (resp. *reversing*), if, with respect to the orientation on M coming from the complex structure on $\mathbf{H}$,

$$\langle [a_i], [b_i] > = \delta_{ij} \ (resp. \ -\delta_{ij}), \ \langle [a_i], [a_j] \rangle = \langle [b_i], [b_j] \rangle = 0$$

$(i,j=1,\cdots,g)$ where $[a_i]$ is the homology class of a_i and $<,>$ is the intersection pairing on $H_1(M,\mathbb{Z})$. If $R_0^+(\Gamma)$ (resp. $R_0^-(\Gamma)$) denotes the set of orientation preserving (resp. reversing), discrete and faithful $SL_2(\mathbb{R})$-representations of Γ, then $R_0(\Gamma)$ is the disjoint union of $R_0^+(\Gamma)$ and $R_0^-(\Gamma)$ and

$$X_0(\Gamma)=R_0(\Gamma)/PGL_2(\mathbb{R})=R_0^+(\Gamma)/PSL_2(\mathbb{R}).$$

$$\begin{aligned}T_g &= Hom(\Gamma,\mathbb{Z}/2\mathbb{Z})\backslash X_0(\Gamma)\\ &= Hom(\Gamma,\mathbb{Z}/2\mathbb{Z})\backslash R_0^+(\Gamma)/PSL_2(\mathbb{R}).\end{aligned}$$ ∎

Proposition 2.3 implies $|I_h|>2$ (*for all* $h(\neq identity)\in\Gamma$) on $X_0(\Gamma)$. Hence the sign of I_h is constant on each connected component of $X_0(\Gamma)$. This means that $Hom(\Gamma,\mathbb{Z}/2\mathbb{Z})$ permutes the set of connected components of $X_0(\Gamma)$ freely. Thus

Corollary 2.7. *The quotient map $X_0(\Gamma)\to T_g$ is an unramified $(\mathbb{Z}/2\mathbb{Z})^{2g}$-covering. Hence by taking any lifting of this mapping, we can consider T_g as a finite union of connected components of $X_0(\Gamma)$. Therefore T_g can be considered as a semialgebraic subset of $X_0(\Gamma)$.* ∎

Corollary 2.8. *If $\pi_0(X_0(\Gamma))$ denotes the number of connected components of $X_0(\Gamma)$, the order of $Hom(\Gamma,\mathbb{Z}/2\mathbb{Z})$ divides $\pi_0(X_0(\Gamma))$. In particular*

$$2^{2g}\leq\pi_0(X_0(\Gamma)).$$

∎

We summarize the result of this subsection as the following diagram.

$$\begin{array}{ccccccc}
Hom(\Gamma,SL_2(\mathbb{R})) & = & R(\Gamma) & \supset & R_0(\Gamma) & & \\
 & & {}^t\downarrow & & \downarrow & & \\
 & & X(\Gamma) & \supset & X_0(\Gamma) & = & R_0(\Gamma)/PGL_2(\mathbb{R}) \\
 & & & & \downarrow & & \\
 & & & & T_g & = & Hom(\Gamma,\mathbb{Z}/2\mathbb{Z})\backslash X_0(\Gamma)
\end{array}$$

§3. Semialgebraic Description of Teichmüller Space T_g $(g=2)$

In this section, by construction global coordinates on $X_0(\Gamma)$, we will give a semialgebraic description of the Teichmüller space T_2, which we will use to show its connectivity and contractibility. For this purpose, we need to find some semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$ containing $X_0(\Gamma)$ whose presentation as a semialgebraic set and topological structure are both simple.

§3.1. Definition of the Semialgebraic Subset $S(\Gamma)$ of $X(\Gamma)$

We define an open semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$ by

$$S(\Gamma):=\{\chi\in X(\Gamma)\mid I_{c_1}(\chi)<-2\}$$

where $c_1:=[\alpha_1,\beta_1]=[\alpha_2,\beta_2]^{-1}\in\Gamma$.

Proposition 3.1. $S(\Gamma)\subset X'(\Gamma)$. *Hence* $t^{-1}(S(\Gamma))\xrightarrow{t}S(\Gamma)$ *is a* $PGL_2(\mathbf{R})$-*bundle, and* $S(\Gamma)$ *can be considered as the* $PGL_2(\mathbf{R})$-*adjoint quotient of* $t^{-1}(S(\Gamma))$, *i.e.*,

$$S(\Gamma)=t^{-1}(S(\Gamma))/PGL_2(\mathbf{R}).$$

Proof. First we show

$$S(\Gamma)\cap(X(\Gamma)-t(R(\Gamma)))=\phi.$$

As we have seen in Subsection 2.3, any element of $X(\Gamma)-t(R(\Gamma))$ can be considered as a character of an SU(2)-representation of Γ. Thus for $\chi\in X(\Gamma)-t(R(\Gamma))$

$$|I_h(\chi)|\leq 2 \qquad \text{for } h\in\Gamma.$$

This means that $S(\Gamma)\subset t(R(\Gamma))$. On the other hand, Proposition 2.1 shows that $S(\Gamma)\subset X'(\Gamma)$. ∎

The next result is due to Keen, Okumura and Seppälä-Sorvali ([K1], [K2], [K3], [O1], [O2], [S-S]).

Proposition 3.2. $X_0(\Gamma)\subset S(\Gamma)$.

Proof. Any element $\rho=(A_1,B_1,A_2,B_2)$ of $R_0(\Gamma)$ induces a discrete and faithful $PSL_2(\mathbf{R})$-representation of Γ. Hence we have seen in the proof of Proposition 2.10 that

$$tr([A_1,B_1])<-2.$$

This completes the proof. ∎

The above arguments give rise to the following diagram.

Corollary 3.1.

$$\begin{array}{ccccccc} R(\Gamma) & \supset & R'(\Gamma) & \supset & t^{-1}(S(\Gamma)) & \supset & R_0(\Gamma) \\ {\scriptstyle t}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(\Gamma) & \supset & X'(\Gamma) & \supset & S(\Gamma) & \supset & X_0(\Gamma) \end{array}$$

∎

§3.2. Topological Structure of $S(\Gamma)$

In this subsection, by constructing global coordinates on $S(\Gamma)$, we will show that $S(\Gamma)$ consists of $2^4 \times 2$ connected components such that each component is a 6 dimensional cell. For this purpose we need some preliminaries.

Let us define a polynomial mapping f from $X(\Gamma)$ to $\mathbf{R}^6$: For any $\chi \in X(\Gamma)$, let

$$f(\chi) := (I_{\alpha_1}(\chi), I_{\beta_1}(\chi), I_{\alpha_1\beta_1}(\chi), I_{\alpha_2}(\chi), I_{\beta_2}(\chi), I_{\alpha_2\beta_2}(\chi)).$$

By the definition of I_h ($h \in \Gamma$), for any $\rho \in R(\Gamma)$

$$f \circ t(\rho) = (tr(\rho(\alpha_1)), tr(\rho(\beta_1)), \cdots, tr(\rho(\alpha_2\beta_2))).$$

$$\begin{array}{ccc} R(\Gamma) & & \\ {\scriptstyle t}\downarrow & \overset{f\circ t}{\searrow} & \\ X(\Gamma) & \overset{f}{\rightarrow} & \mathbf{R}^6 \end{array}$$

We denote the coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ on $\mathbf{R}^6$ by $(\vec{x}, \vec{y})$ for the sake of simplicity. Next we define a polynomial function $\kappa(x, y, z)$ on $\mathbf{R}^3$ by

$$\kappa(x, y, z) := x^2 + y^2 + z^2 - xyz - 2.$$

Easy calculation shows the following lemma ([F], [Go]).

Lemma 3.1. 1. *For any $A, B \in SL_2(\mathbf{R})$*

$$\kappa(tr(A), tr(B), tr(AB)) = tr([A, B]).$$

2. *If* $(x, y, z) \in \mathbf{R}^3$ *satisfies* $\kappa(x, y, z) < -2$, *then*

$$|x| > 2, \quad |y| > 2, \quad |z| > 2 \text{ and } x \cdot y \cdot z > 0.$$

∎

In particular if we put

$$V_- = \{(\vec{x}, \vec{y}) \in \mathbf{R}^6 \mid \kappa(\vec{x}) = \kappa(\vec{y}) < -2\}$$

then from the definition of $S(\Gamma)$, we have $f(S(\Gamma)) \subset V_-$. In fact we will see in Proposition 3.3 that $f(S(\Gamma)) = V_-$.

Lemma 3.2. $V_- \subset \mathbf{R}^6$ *consists of* 2^4 *connected components such that each components is a* 5-*dimensional cell. More precisely, put* $U := V_- \cap \{(\vec{x}, \vec{y}) \in \mathbf{R}^6 \mid x_i > 0, y_i > 0\ (i = 1, 2)\}$ *and define the action of* $(\mathbf{Z}/2\mathbf{Z})^4$ *on* $\mathbf{R}^6$ *by multiplication by* ± 1 *on the* x_i *and* y_i $(i = 1, 2)$. *Then* U *is a* 5 *dimensional cell and* V_- *can be written as*

$$V_- = \coprod_{\gamma \in (\mathbf{Z}/2\mathbf{Z})^4} \gamma(U) \quad (\textit{disjoint union}).$$

Proof. For $r < -2$ put

$$W_r := \{(x, y, z) \in \mathbf{R}^3 \mid \kappa(x, y, z) = r,\ x > 0,\ y > 0,\ z > 0\}$$

and $u := x - y$, $v := x + y$ for $(x, y, z) \in W_r$. Then by Lemma 3.1.2

$$v = \sqrt{\frac{z+2}{z-2} u^2 - \frac{4}{z-2}(2 + r - z^2)} > 0.$$

Hence the next mapping is a homeomorphism, and consequently W_r is a 2 demensional cell.

$$W_r \simeq \mathbf{R} \times \{z \in \mathbf{R} \mid z > 2\}.$$
$$(x, y, z) \mapsto (u, z)$$

Since U is a fiber bundle over the base space $\{r \in \mathbf{R} \mid r < -2\}$ with fibers $W_r \times W_r$, U is a 5 dimensional cell and by Lemma 3.1.2

$$V_- = \coprod_{\gamma \in (\mathbf{Z}/2\mathbf{Z})^4} \gamma(U).$$

■

The next lemma can be shown directly by calculation, but it is a key lemma in the whole story of this section.

Lemma 3.3. *Let* $(A, B) \in SL_2(\mathbb{R})^2$ *be a pair of hyperbolic matrices which satisfies the following condition*

$$[A, B] = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \quad (\lambda < -1). \cdots 1)$$

Put $(x, y, z) := (tr(A), tr(B), tr(AB))$. *Then* $\kappa(x, y, z) < -2$, *and there exists a constant* $k \in \mathbb{R}^* := \mathbb{R} - \{0\}$ *such that* A *and* B *can be written as*

$$A = \begin{pmatrix} \frac{1}{\lambda+1}x & \frac{1}{k}\left\{\frac{\lambda}{(\lambda+1)^2}x^2 - 1\right\} \\ k & \frac{1}{\lambda+1}x \end{pmatrix}$$

$$\cdots 2)$$

$$B = \begin{pmatrix} \frac{1}{\lambda+1}y & \frac{1}{k}\left\{\frac{1}{\lambda+1}z - \frac{\lambda}{(\lambda+1)^2}xy\right\} \\ k\,\dfrac{\frac{\lambda}{(\lambda+1)^2}y^2 - 1}{\frac{1}{\lambda+1}z - \frac{\lambda}{(\lambda+1)^2}xy} & \frac{\lambda}{\lambda+1}y \end{pmatrix}.$$

Conversely for any $k \in \mathbb{R}^*$ *and* $(x, y, z) \in \mathbb{R}^3$ *with* $\kappa(x, y, z) < -2$, *define* $\lambda < -1$ *by* $\lambda + \frac{1}{\lambda} = \kappa(x, y, z)$. *Then the pair of matrices* $(A, B) \in SL_2(\mathbb{R})^2$ *defined by condition* 2) *satisfies* 1) *and* $(x, y, z) = (tr(A), tr(B), tr(AB))$. ■

Since the pair $(A, B) \in SL_2(\mathbb{R})^2$ defined by the above condition 2) is uniquely determined by $k \in \mathbb{R}^*$ and $(x, y, z) \in \mathbb{R}^3$ with $\kappa(x, y, z) < -2$, we write it as

$$(A, B)=(A(x, y, z, k), B(x, y, z, k)).$$

Now we can show the main result of this subsection.

Proposition 3.3. *$S(\Gamma)$ consists of $2^4 \times 2$ connected components such that each component is a 6-dimensional cell.*

Proof. First, we define the mapping Ψ

$$\Psi : t^{-1}(S(\Gamma)) \to \mathbf{R}^* \times V_- \times PGL_2(\mathbf{R}).$$

as follows: For any $\rho=(A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$, we first diagonalize $[A_1, B_1]$. More precisely, by using Lemma 3.3, we can choose $P \in PGL_2(\mathbf{R})$ uniquely such that by use of the notations of Lemma 3.3, (PA_iP^{-1}, PB_iP^{-1}) $(i=1,2)$ can be written as

$$PA_1P^{-1}=A(tr(A_1), tr(B_1), tr(A_1B_1), 1)$$

$$PB_1P^{-1}=B(tr(A_1), tr(B_1), tr(A_1B_1), 1)$$

$$PA_2P^{-1}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A(tr(A_2), tr(B_2), tr(A_2B_2), k)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$PB_2P^{-1}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B(tr(A_2), tr(B_2), tr(A_2B_2), k)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $k \in \mathbf{R}^*$ is some constant. We define the mapping Ψ by

$$\Psi : t^{-1}(S(\Gamma)) \to \mathbf{R}^* \times V_- \times PGL_2(\mathbf{R})$$

$$\rho \mapsto (k, f \circ t(\rho), P).$$

Lemma 3.3 implies that Ψ is bijective and, in fact, a homeomorphism. From the difinition, Ψ is $PGL_2(\mathbf{R})$-equivariant. Hence it induces a homeomorphism Φ from $S(\Gamma)$ to $\mathbf{R}^* \times V_-$ as follows.

$$\begin{array}{ccc} t^{-1}(S(\Gamma)) & \overset{\Psi}{\simeq} & \mathbf{R}^* \times V_- \times PGL_2(\mathbf{R}) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle proj.} \\ S(\Gamma) & \overset{\Phi}{\simeq} & \mathbf{R}^* \times V_- \end{array}$$

Moreover by Lemma 3.2, $\mathbf{R}^* \times V_-$ consists of $2^4 \times 2$ connected components such that each component is a 6-dimensional cell. ■

§3.3. Cell Structure of Teichmüller Space T_2

Next we consider conditions which characterize the connected components of $X_0(\Gamma)$ in $S(\Gamma)$. By the definition of Φ in the proof of Proposition 3.3, the first component k of Φ can be considered as a function on $S(\Gamma)$.

Proposition 3.4. *Suppose that $U \subseteq S(\Gamma)$ is a connected component on which the function $I_{\alpha_1} \cdot I_{\alpha_2} \cdot k$ is negative. Then there exists $\chi \in U$ such that χ is not contained in $X_0(\Gamma)$.*

Proof. First we remark that on a connected component U of $S(\Gamma)$, the signs of the functions $I_{\alpha_1} \cdot I_{\alpha_2}$, and k are constant. We consider $(\vec{x}, \vec{y}) \in V_-$ satisfying $|x_i| = |y_i| = 4$ $(i = 1, 2, 3)$. Then there are 2^4 points of V_- satisfying this condition. Since $f(U)$ is a connected component of V_-, take $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$ with $t(\rho) \in U$ and $f \circ t(\rho) = (\vec{x}, \vec{y})$. If $I_{\alpha_1}(t(\rho)) \cdot I_{\alpha_2}(t(\rho)) = tr(A_1) \cdot tr(A_2) = 16 > 0$, then by using the presentation of $\rho = (A_1, B_1, A_2, B_2)$ in the proof of Proposition 3.3, $tr(A_1 A_2) = -2 - k - \frac{4}{k}$ where we denote $k(\rho)$ by k for the sake of simplicity. Hence if $k(\rho) = k = -2$ (*i.e.*, $I_{\alpha_1} \cdot I_{\alpha_2} \cdot k < 0$ on U), then $tr(A_1 A_2) = 2$ and this means that $A_1 A_2 \in SL_2(\mathbf{R})$ is a parabolic matrix, thus $t(\rho)$ is not contained in $X_0(\Gamma)$. Similar argument holds for the case $I_{\alpha_1}(\rho) \cdot I_{\alpha_2}(\rho) = tr(A_1) \cdot tr(A_2) = -16 < 0$. ■

Since $X_0(\Gamma)$ consists of finitely many connected components of $X(\Gamma)$ by Corollary 2.4, Proposition 3.4 implies that there are 16 connected components of $S(\Gamma)$ on which the function $I_{\alpha_1} \cdot I_{\alpha_2} \cdot k$ is negative. Hence the number of connected components of $X_0(\Gamma)$, $\pi_0(X_0(\Gamma))$ is less than or equal to 16. On the other hand, as the argument in Subsection 2.4 implies $\pi_0(X_0(\Gamma)) \geq 16$, we get following result.

Theorem 3.1. $\pi_0(X_0(\Gamma)) = 16$. *Thus*

$$T_2 = Hom(\Gamma, \mathbf{Z}/2\mathbf{Z}) \backslash X_0(\Gamma)$$

is a connected 6-dimensional cell, and, in particular, contractible. ■

§3.4. Semialgebraic Structure of Teichmüller Space T_2

The preceding argument shows that $X_0(\Gamma)$ can be presented as the following subset of $X(\Gamma)$:

$$\begin{aligned} X_0(\Gamma) &= \{\chi \in S(\Gamma) \mid I_{\alpha_1}(\chi) \cdot I_{\alpha_2}(\chi) \cdot k(\chi) > 0\} \\ &= \{\chi \in X(\Gamma) \mid I_{c_1} < -2 \text{ and } I_{\alpha_1}(\chi) \cdot I_{\alpha_2}(\chi) \cdot k(\chi) > 0\} \end{aligned}$$

where $c_1 = [\alpha_1, \beta_1] \in \Gamma$. This presentation induces the following semialgebraic description of $X_0(\Gamma)$ in $X(\Gamma)$.

Theorem 3.2. *$X_0(\Gamma)$ can be described as a semialgebraic subset of $X(\Gamma)$ as follows*

$$X_0(\Gamma) = \{\chi \in X(\Gamma) \mid I_{c_1}(\chi) < -2, \frac{(I_{c_1}(\chi) + 2) \cdot I_{\alpha_1\alpha_2}(\chi)}{I_{\alpha_1}(\chi) \cdot I_{\alpha_2}(\chi)} > 2\}.$$

Hence for any representation $\rho = (A_1, B_1, A_2, B_2) \in R(\Gamma)$, ρ is a discrete and faithful $SL_2(\mathbf{R})$-representation of Γ if and only if

$$tr([A_1, B_1]) < -2 \quad \textit{and} \quad \frac{(tr([A_1, B_1]) + 2) \cdot tr(A_1 A_2)}{tr(A_1) \cdot tr(A_2)} > 2.$$

Proof. For any $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$, calculating $tr(A_1 A_2)$ gives

$$\begin{aligned} &k(\rho)^2 + \left(tr(A_1 A_2) - \frac{2tr(A_1) \cdot tr(A_2)}{tr([A_1, B_1]) + 2} \right) k(\rho) \\ &+ \left(\frac{tr(A_1)^2}{tr([A_1, B_1]) + 2} - 1 \right) \left(\frac{tr(A_2)^2}{tr([A_1, B_1]) + 2} - 1 \right) = 0. \end{aligned}$$

If we consider this as a quadratic equation in $k(\rho)$, the constant term is positive. Hence the sign of $k(\rho)$ and the sign of the coefficients of the linear term of this equation are opposite each other. Thus, for $\rho = (A_1, B_1, A_2, B_2) \in t^{-1}(S(\Gamma))$,

$$tr(A_1) \cdot tr(A_2) \cdot k(\rho) > 0 \Leftrightarrow \frac{(tr([A_1, B_1]) + 2) \cdot tr(A_1 A_2)}{tr(A_1) \cdot tr(A_2)} > 2.$$

■

Remark. The various connected components of $X_0(\Gamma)$ are separated by the action of $Hom(\Gamma, \mathbb{Z}/2\mathbb{Z})$, i.e., the sign conditions on the functions I_{α_1}, I_{β_2}, I_{α_2} and I_{β_2}. Therefore, by adding these 4 conditions, we obtain a semialgebraic description of T_2 by means of 6 polynomial inequalities (see Corollary 2.7). ∎

Corollary 3.2. *The function $k=k(\chi)$ on $S(\Gamma)$ can be written as follows*

$$k=\frac{1}{\lambda^4-1}I_{\alpha_1\alpha_2}-\frac{\lambda^2}{\lambda^4-1}I_{\alpha_1c_1\alpha_2c_1^{-1}}+\frac{2}{\lambda^2+1}\frac{I_{\alpha_1}\cdot I_{\alpha_2}}{I_{c_1}+2}$$

where $\lambda=\lambda(\chi)$ is the function on $S(\Gamma)$ defined by

$$\lambda(\chi)+\frac{1}{\lambda(\chi)}=I_{c_1}(\chi)<-2 \quad and \quad \lambda(\chi)<-1.$$

Especially, the point of T_2 is completely determined by the 8 functions $I_{\alpha_i}, I_{\beta_i}, I_{\alpha_i\beta_i}$ $(i=1,2)$, $I_{\alpha_1\alpha_2}$ and $I_{\alpha_1c_1\alpha_2c_1^{-1}}$.

Proof. For $\rho = (A_1,B_1,A_2,B_2) \in t^{-1}(S(\Gamma))$, by comparing $tr(A_1A_2)=I_{\alpha_1\alpha_2}(t(\rho))$ and $tr(A_1[A_1,B_1]A_2[A_1,B_1]^{-1})=I_{\alpha_1c_1\alpha_2c_1^{-1}}(t(\rho))$, we obtain

$$I_{\alpha_1\alpha_2}(t(\rho))=-\frac{1}{k}\left(\frac{\lambda}{(\lambda+1)^2}x^2-1\right)\left(\frac{\lambda}{(\lambda+1)^2}y^2-1\right)-k+\frac{2\lambda}{(\lambda+1)^2}xy$$

$$I_{\alpha_1c_1\alpha_2c_1^{-1}}(t(\rho))=-\frac{1}{\lambda^2k}\left(\frac{\lambda}{(\lambda+1)^2}x^2-1\right)\left(\frac{\lambda}{(\lambda+1)^2}y^2-1\right)-\lambda^2k+\frac{2\lambda}{(\lambda+1)^2}xy$$

where $x=tr(A_1)$, $y=tr(A_2)$, $\lambda<-1$ with $\lambda+\frac{1}{\lambda}=tr([A_1,B_1])$ and $k=k(\rho)$. Subtracting λ^2 times the second equation from the first equation then gives the desired formula. ∎

§4. Semialgebraic Description of Teichmüller Space T_g $(g\geq 3)$

In this section, we assume $g\geq 3$. We show the connectivity and contractibility of the Teichmüller space T_g, as well as give a semialgebraic description of it, by means of arguments similar to those Section 3.

§4.1. Definition of the Semialgebraic Subset $S(\Gamma)$ of $X(\Gamma)$

We define an open semialgebraic subset $S(\Gamma)$ of $X(\Gamma)$ by

$$S(\Gamma):=\{\chi\in X(\Gamma)\mid I_{c_i}(\chi)<-2\ (i=1,\cdots,g)\\ I_{d_j}(\chi)<-2\ (j=2,\cdots,g-2)\}$$

where $c_i:=[\alpha_i,\beta_i]\in\Gamma$ and $d_j:=c_1c_2\cdots c_j$.

Similar arguments to those used to prove Propositions 3.1 and 3.2 show

Proposition 4.1. $S(\Gamma)\subset X'(\Gamma)$. *Hence* $t^{-1}(S(\Gamma))\xrightarrow{t}S(\Gamma)$ *is a* $PGL_2(\mathbb{R})$-*bundle and can be considered as the* $PGL_2(\mathbb{R})$-*adjoint quotient of* $t^{-1}(S(\Gamma))$ *i.e.*,

$$S(\Gamma)=t^{-1}(S(\Gamma))/PGL_2(\mathbb{R}).$$

■

Proposition 4.2. $X_0(\Gamma)\subset S(\Gamma)$. ■

Moreover if a representation $\rho=(A_1,B_1,\cdots,A_g,B_g)$ is contained in $R_0(\Gamma)$, the representation $\rho_j:=(A_j,B_j,\cdots,A_g,B_g,A_1,B_1,\cdots,A_{j-1},B_{j-1})$ $(j=2,\cdots,g)$ is well-defined and also an element of $R_0(\Gamma)$, hence we have

Corollary 4.1. *For* $\chi\in X_0(\Gamma)$, $I_{c_ic_{i+1}}(\chi)<-2$ $(i=2,\cdots,g)$ *where we assume that* $c_{g+1}=c_1$. ■

The above arguments give rise to the following diagram.

Corollary 4.2.

$$\begin{array}{ccccccc} R(\Gamma) & \supset & R'(\Gamma) & \supset & t^{-1}(S(\Gamma)) & \supset & R_0(\Gamma) \\ {\scriptstyle t}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(\Gamma) & \supset & X'(\Gamma) & \supset & S(\Gamma) & \supset & X_0(\Gamma) \end{array}$$

■

§4.2. Topological Structure of $S(\Gamma)$

In this subsection, by constructing global coordinates on $S(\Gamma)$, we will show that $S(\Gamma)$ consists of $2^{2g}\times 2^{2g-3}$ connected components such that each component is a 6g-6 dimensional cell. For this purpose we need some preliminaries.

First we define a polynomial mapping f from $X(\Gamma)$ to $\mathbf{R}^{3g}$ by

$$f(\chi):=(I_{\alpha_1}(\chi), I_{\beta_1}(\chi), I_{\alpha_1\beta_1}(\chi), \cdots, I_{\alpha_g}(\chi), I_{\beta_g}(\chi), I_{\alpha_g\beta_g}(\chi))$$

for $\chi\in X(\Gamma)$.

$$\begin{array}{ccc} R(\Gamma) & & \\ {\scriptstyle t}\downarrow & \overset{f\circ t}{\searrow} & \\ X(\Gamma) & \overset{f}{\rightarrow} & \mathbf{R}^{3g} \end{array}$$

Let $(\vec{x}_1, \cdots, \vec{x}_g)$ denote the coordinates $(x_{11}, x_{12}, x_{13}, \cdots, x_{g1}, x_{g2}, x_{g3})$ on $\mathbf{R}^{3g}$. We define a semialgebraic subset V_- by

$$V_-:=\{(\vec{x}_1, \cdots, \vec{x}_g)\in\mathbf{R}^{3g}\mid \kappa(\vec{x}_i)<-2\ (i=1, \cdots, g)\}$$

where $\kappa(x, y, z)$ is a polynomial function on $\mathbf{R}^3$ defined in Subsection 3.2. Then from the definition of $S(\Gamma)$, we obtain that $f(S(\Gamma))\subset V_-$. In fact we will see in the proof of Proposition 4.3 that $f(S(\Gamma))=V_-$.

The next lemma can be proved by the same argument as that used to prove Lemma 3.2.

Lemma 4.1. *$V_-\subset\mathbf{R}^{3g}$ consists of 2^{2g} connected components such that each component is a 3g-dimensional cell. More precisely, put*

$$U:=V_-\cap\{(\vec{x}_1, \cdots, \vec{x}_g)\in\mathbf{R}^{3g}\mid x_{ij}>0\ (i=1, \cdots, g;\ \ j=1, 2)\}$$

and define the action of $\mathbf{Z}/2\mathbf{Z})^{2g}$ on $\mathbf{R}^{3g}$ by multiplication by ±1 on the x_{ij} $(i=1, \cdots, g;\ j=1, 2)$. Then U is a 3g-dimensional cell and V_- can be represented as

$$V_-=\coprod_{\gamma\in(\mathbf{Z}/2\mathbf{Z})^{2g}}\gamma(U)\quad (\textit{disjoint union}).$$

■

The next lemma, which is shown by elementary calculation, is a key lemma in this section.

Lemma 4.2. 1. *For a pair of hyperbolic matrices $(C_1, C_2) \in SL_2(\mathbf{R})^2$, assume that C_1 is diagonal*

$$C_1 = \begin{pmatrix} \eta & 0 \\ 0 & \frac{1}{\eta} \end{pmatrix} \quad (\eta < -1).$$

If the traces of C_1, C_2 and C_1C_2 satisfy

$$x := tr(C_1) < -2, \; y := tr(C_2) < -2 \text{ and } z := tr(C_1C_2) < -2 \cdots 1)$$

then there exists a unique $m \in \mathbf{R}^*$ such that C_2 can be written as follows.

$$C_2 = \begin{pmatrix} \dfrac{\eta z - y}{\eta^2 - 1} & m \\ \dfrac{1}{m}\left\{\dfrac{\eta(\eta y - z)(\eta z - y)}{(\eta^2-1)^2} - 1\right\} & \dfrac{\eta(\eta y - z)}{\eta^2 - 1} \end{pmatrix}. \qquad \cdots 2)$$

Conversely, for any constant $m \in \mathbf{R}^*$ and $(x, y, z) \in \mathbf{R}^3$ with $x < -2$, $y < -2$ and $z < -2$, if we let η be the unique real number < -1 such that $\eta + \frac{1}{\eta} = x$, and define $C_1 = \begin{pmatrix} \eta & 0 \\ 0 & \frac{1}{\eta} \end{pmatrix}$ and C_2 by the condition 2), then $(x, y, z) = (tr(C_1), tr(C_2), tr(C_1C_2))$ as the condition 1).

2. *Moreover for such a pair $(C_1, C_2) \in SL_2(\mathbf{R})^2$, we can diagonalize C_1C_2 and C_2 by using the following matrices $P, Q \in SL_2(\mathbf{R})$.*

$$P := \begin{pmatrix} 1 & -\dfrac{m\tau\eta}{\tau^2 - 1} \\ \dfrac{\tau(\eta^2 - 1) - \eta(\eta z - y)}{m\eta(\eta^2 - 1)} & \dfrac{\tau\eta(\eta z - y) - (\eta^2 - 1)}{(\eta^2 - 1)(\tau^2 - 1)} \end{pmatrix}$$

where $\tau < -1$ with $\tau + \frac{1}{\tau} = z = tr(C_1C_2)$ and $C_1C_2 = P\begin{pmatrix} \tau & 0 \\ 0 & \frac{1}{\tau} \end{pmatrix}P^{-1}$.

$$Q:=\begin{pmatrix} 1 & -\dfrac{m\xi}{\xi^2-1} \\ \dfrac{\xi(\eta^2-1)-(\eta z-y)}{m(\eta^2-1)} & \dfrac{\xi(\eta z-y)-(\eta^2-1)}{(\eta^2-1)(\xi^2-1)} \end{pmatrix}$$

where $\xi<-1$ *with* $\xi+\frac{1}{\xi}=y=tr(C_2)$ *and* $C_2=Q\begin{pmatrix}\xi & 0\\ 0 & \frac{1}{\xi}\end{pmatrix}Q^{-1}$. ∎

In the following we write these C_2, P and Q by $C(x,y,z,m)$, $P(x,y,z,m)$ and $Q(x,y,z,m)$.

Proposition 4.3. *$S(\Gamma)$ consists of $2^{2g}\times 2^{2g-3}$ connected components such that each component is a 6g-6 dimensional cell.*

Proof. We construct a mapping Ψ

$$\Psi: t^{-1}(S(\Gamma))\to V_-\times\{w\in\mathbb{R}\mid w<-2\}^{g-3}\times(\mathbb{R}^*)^{g-3}\times(\mathbb{R}^*)^g\times PGL_2(\mathbb{R})$$

as follows.

For $\rho=(A_1,B_1,\cdots,A_g,B_g)\in t^{-1}(S(\Gamma))$, put

$$(\vec{x}_1,\cdots,\vec{x}_g):=f\circ t(\rho)\in V_-\ \text{(where } \vec{x}_i:=(x_{i1},x_{i2},x_{i3}))$$

$$C_i:=[A_i,B_i]\ (i=1,\cdots,g)$$

$$u_i:=tr(C_i)=\kappa(\vec{x}_i)\ (i=1,\cdots,g)$$

$$D_k:=C_1\cdots C_k\ (k=1,\cdots,g-1)$$

$$w_k:=tr(D_k)\ (k=1,\cdots,g-1).$$

We remark that

$$D_1=C_1$$

$$w_1=u_1$$

$$w_{g-1}=u_g.$$

The definition of $S(\Gamma)$ implies that

$$w_1<-2,\quad u_2<-2,\quad \text{and } w_2<-2.$$

Lemma 4.2.1 shows that there exists a unique $R \in PGL_2(\mathbf{R})$ such that

$$RC_1R^{-1} = \begin{pmatrix} \eta_1 & 0 \\ 0 & \frac{1}{\eta_1} \end{pmatrix} \quad (\eta_1 < -1 \text{ with } \eta_1 + \frac{1}{\eta_1} = w_1)$$

$$RC_2R^{-1} = C(w_1, u_2, w_2, 1).$$

Then by Lemma 4.2.2 there exists $P_1 = P(w_1, u_2, w_2, 1)$ such that

$$RD_2R^{-1} = P_1 \begin{pmatrix} \eta_2 & 0 \\ 0 & \frac{1}{\eta_2} \end{pmatrix} P_1^{-1} \ (\eta_2 < -1 \text{ with } \eta_2 + \frac{1}{\eta_2} = w_2).$$

Similarly because

$$w_2 < -2, \ u_3 < -2 \text{ and } w_3 < -2$$

Lemma 4.2.1 shows that there exists a constant $m_1 \in \mathbf{R}^*$ such that

$$RC_3R^{-1} = P_1 C(w_2, u_3, w_3, m_1) P_1^{-1}$$

and by Lemma 4.2.2 there exists $P_2 = P(w_2, u_3, w_3, m_1)$ such that

$$RD_3R^{-1} = P_1P_2 \begin{pmatrix} \eta_3 & 0 \\ 0 & \frac{1}{\eta_3} \end{pmatrix} P_2^{-1} P_1^{-1} \quad (\eta_3 < -1 \text{ with } \eta_3 + \frac{1}{\eta_3} = w_3).$$

Inductively, for $j = 2, \cdots, g-1$, because

$$w_{j-1} < -2, \ u_j < -2, \quad \text{and} \quad w_j < -2$$

Lemma 4.2 shows

$$RC_jR^{-1} = P_1 \cdots P_{j-2} C(w_{j-1}, u_j, w_j, m_{j-2}) P_{j-2}^{-1} \cdots P_1^{-1}$$

$$RD_jR^{-1} = P_1 \cdots P_{j-1} \begin{pmatrix} \eta_j & 0 \\ 0 & \frac{1}{\eta_j} \end{pmatrix} P_{j-1}^{-1} \cdots P_1^{-1}$$

where $m_{j-2} \in \mathbf{R}^*$ with $m_0 = 1$, $P_{j-1} = P(w_{j-1}, u_j, w_j, m_{j-2})$ with $P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\eta_j < -1$ with $\eta_j + \frac{1}{\eta_j} = w_j$.

Moreover, RC_gR^{-1} can be written as

$$RC_gR^{-1}=P_1\cdots P_{g-2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\eta_{g-1} & 0\\ 0 & \frac{1}{\eta_{g-1}}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}P_{g-2}^{-1}\cdots P_1^{-1}.$$

On the other hand, by Lemma 3.3,

$$RA_1R^{-1}=A(\vec{x}_1,k_1)$$
$$RB_1R^{-1}=B(\vec{x}_1,k_1)$$

for some $k_1\in\mathbf{R}^*$ where we denote $A(x_{11},x_{12},x_{13},k_1)$ by $A(\vec{x}_1,k_1)$. By Lemma 4.2.2 there exist $Q_2=Q(w_1,u_2,w_2,1)$ and $k_2\in\mathbf{R}^*$ such that

$$RA_2R^{-1}=Q_2A(\vec{x}_2,k_2)Q_2^{-1}$$
$$RB_2R^{-1}=Q_2B(\vec{x}_2,k_2)Q_2^{-1}.$$

Inductively, for $j=2,\cdots,g-1$

$$RA_jR^{-1}=P_1\cdots P_{j-2}Q_jA(\vec{x}_j,k_j)Q_j^{-1}P_{j-2}^{-1}\cdots P_1^{-1}$$
$$RB_jR^{-1}=P_1\cdots P_{j-2}Q_jB(\vec{x}_j,k_j)Q_j^{-1}P_{j-2}^{-1}\cdots P_1^{-1}$$

where $Q_j=Q(w_{j-1},u_j,w_j,m_{j-2})$ and $k_j\in\mathbf{R}^*$. Moreover

$$RA_gR^{-1}=P_1\cdots P_{g-2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}A(\vec{x}_g,k_g)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}P_{g-2}^{-1}\cdots P_1^{-1}$$
$$RB_gR^{-1}=P_1\cdots P_{g-2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}B(\vec{x}_g,k_g)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}P_{g-2}^{-1}\cdots P_1^{-1}$$

for some $k_g\in\mathbf{R}^*$. Now we can define the mapping Ψ

$$t^{-1}(S(\Gamma))\xrightarrow{\Psi} V_-\times\{w\in\mathbf{R}\mid w<-2\}^{g-3}\times(\mathbf{R}^*)^{g-3}\times(\mathbf{R}^*)^g\times PGL_2(\mathbf{R})$$
$$\rho\mapsto(f\circ t(\rho),w_2,\cdots,w_{g-2},m_1,\cdots,m_{g-3},k_1,\cdots,k_g,R).$$

Lemma 4.2 shows that this mapping is bijective and homeomorphic. Ψ induces a homeomorphism Φ as follows

$$\begin{array}{ccc} t^{-1}(S(\Gamma)) & \overset{\Psi}{\simeq} & V_- \times \{w \in \mathbf{R} \mid w < -2\}^{g-3} \times (\mathbf{R}^*)^{g-3} \times (\mathbf{R}^*)^g \times PGL_2(\mathbf{R}) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \text{proj.}} \\ S(\Gamma) & \overset{\Phi}{\to} & V_- \times \{w \in \mathbf{R} \mid w < -2\}^{g-3} \times (\mathbf{R}^*)^{g-3} \times (\mathbf{R}^*)^g. \end{array}$$

Thus by Lemma 4.1, $S(\Gamma)$ consists of $2^{2g} \times 2^{2g-3}$ connected components such that each component is a 6g-6 dimensional cell. ■

§4.3. Cell Structure of Teichmüller Space T_g

In the following, by using the global coordinate functions on $S(\Gamma)$ constructed in the previous subsection, we give conditions which characterize the connected components of $X_0(\Gamma)$ in $S(\Gamma)$.

Proposition 4.4. *On $X_0(\Gamma)$, the component m_j $(j=1,\cdots,g-3)$ of the mapping Φ is positive.* ■

This is equivalent to the next proposition for the space of representations.

Proposition 4.5. *For $\rho=(A_1,B_1,\cdots,A_g,B_g)\in R_0(\Gamma)$, the value $m_j(\rho)$ of the component m_j $(j=1,\cdots,g-3)$ of the mapping Ψ at ρ is positive.*

To prove this, we need the following elementary lemma.

Lemma 4.3. *If the matrices P, $C(m)\in SL_2(\mathbf{R})$ $(m<0)$ satisfy*

$$P=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ (a>0,\ b<0,\ c<0,\ d>0) \ \cdots (\star)$$

$$C(m)=\begin{pmatrix} x & m \\ \frac{1}{m}(xw-1) & w \end{pmatrix} \quad (x>0,\ w<0,\ m<0,\ x+w<-2)$$

then there exists $m_0<0$ such that $PC(m_0)P^{-1}$ is a lower triangular matrix. Moreover if $P_1,P_2\in SL_2(\mathbf{R})$ satisfy the above condition $(\star)$, then P_1P_2 also satisfies the above condition $(\star)$.

Proof of Proposition 4.5. We use the notations of the proof of Proposition 4.3. We apply the above Lemma 4.3 to P_1 and $C(w_2,u_3,w_3,m_1)$ to conclude that $m_1>0$ on $R_0(\Gamma)$ because $\rho(\Gamma)$ is a discrete purely hyperbolic subgrourp of $SL_2(\mathbb{R})$, RC_1R^{-1} and RC_3R^{-1} cannot have a common fixed point. Hence P_2 and P_1P_2 also satisfy the condition (*) in Lemma 4.3 and by applying it to P_1P_2 and $C(w_3,u_4,w_4,m_2)$, we get $m_2>0$ on $R_0(\Gamma)$. Successive repetition of this procedure proves the Proposition. ∎

Proposition 4.6. *On $X_0(\Gamma)$, the product of components $x_{i1}\cdot k_i$ of the mapping Φ is positive $(i=1,\cdots,g)$.* ∎

This is equivalent to the next proposition for the space of representations.

Proposition 4.7. *For $\rho=(A_1,B_1,\cdots,A_g,B_g)\in R_0(\Gamma)$, the value $x_{i1}(\rho)\cdot k_i(\rho)$ of the product of components x_{i1} and k_i of the mapping Ψ at ρ is positive $(i=1,\cdots,g)$.*

Proof. We use the notations in the proof of Proposition 4.3. For $\rho=(A_1,B_1,\cdots,A_g,B_g)\in R_0(\Gamma)$, a computation shows that

$$\begin{aligned} tr(A_1C_2)&=tr(RA_1R^{-1}RC_2R^{-1})\\ &=tr(A(\vec{x}_1,k_1)C(w_1,u_2,w_2,1))\\ &=\frac{1}{k_1}\left(\frac{x_{11}^2}{w_1+2}-1\right)\left\{\frac{\eta_1(\eta_1u_2-w_2)(\eta_1w_2-u_2)}{(\eta_1^2-1)^2}-1\right\}\\ &\quad+k_1+\frac{x_{11}(w_2+u_2)}{w_1+2}. \end{aligned}$$

Suppose that there exists a connected component U of $R_0(\Gamma)$ such that U contains a representation ρ with $x_{11}\cdot k_1<0$. Then the function $x_{11}\cdot k_1$ is negative on U. Moreover there exist $t<-2$ and $\rho\in U$ such that

$$-|x_{11}|=w_1=w_2=u_2=t \quad\text{and}\quad |k_1|=-\frac{t^2}{t+2}+1.$$

Then since $x_{11}\cdot k_1<0$

$$\begin{aligned} tr(A_1C_2)&=\pm\frac{2t^2}{t+2}\mp 2\left(\frac{t^2}{t+2}-1\right)\\ &=\pm 2. \end{aligned}$$

This contradicts the pure hyperbolicity of $\rho\in R_0(\Gamma)$ (see Proposition 2.3). Hence

$x_{11} \cdot k_1$ is positive on $R_0(\Gamma)$.

Next we will show that the function $x_{j1} \cdot k_j$ $(j=2,\cdots,g)$ is also positive on $R_0(\Gamma)$. For $\rho=(A_1,B_1,\cdots,A_g,B_g)\in R_0(\Gamma)$, Proposition 4.2 and Corollary 4.1 show that

$$trC_j<-2, \quad trC_{j+1}<-2 \quad \text{and} \quad tr(C_jC_{j+1})<-2 \ (j=1,\cdots,g).$$

Hence by Lemma 4.2 there exists a unique $R_j \in PGL_2(\mathbf{R})$ such that

$$R_jC_jR_j^{-1}=\begin{pmatrix}\lambda_j & 0\\ 0 & \frac{1}{\lambda_j}\end{pmatrix}$$

$$R_jC_{j+1}R_j^{-1}=C(trC_j, trC_{j+1}, tr(C_jC_{j+1}), 1)$$

where $\lambda_j<-1$ with $\lambda_j+\frac{1}{\lambda_j}=trC_j$ $(j=1,\cdots,g)$. We remark that $R_1=R$ and $\lambda_1=\eta_1$. Then by Lemma 3.3 there exists $\hat{k}_j\in\mathbf{R}^*$ $(j=1,\cdots,g)$ such that

$$R_jA_jR_j^{-1}=A(\vec{x}_j,\hat{k}_j).$$

We remark that $\hat{k}_1=k_1$. Moreover, the argument just applied to show that $x_{11}\cdot k_1>0$ on $R_0(\Gamma)$ also implies (by permuting the A's and B's as in the paragraph preceding Corollary 4.1) that $x_{j1}\cdot\hat{k}_j>0$ on $R_0(\Gamma)$ $(j=1,\cdots,g)$. In the following, we will show that as functions on $R_0(\Gamma)$, k_j and $\hat{k}_j$ have the same signs on each connected component of $R_0(\Gamma)$. More precisely, for $\rho\in R_0(\Gamma)$, there exists $s_j\in\mathbf{R}^*$ $(j=1,\cdots,g)$ such that $s_j^2\cdot\hat{k}_j=k_j$. By proving this claim, we will thus obtain that $x_{j1}\cdot\hat{k}_j>0$ on $R_0(\Gamma)$ is equivalent to $x_{j1}\cdot k_j>0$ on $R_0(\Gamma)$. This will complete our proof.

For $j=2,\cdots,g-2$ put

$$\hat{R}_j:=Q_j^{-1}P_{j-2}^{-1}\cdots P_1^{-1}R.$$

Then we have

$$\hat{R}_jC_j\hat{R}_j^{-1}=\begin{pmatrix}\lambda_j & 0\\ 0 & \frac{1}{\lambda_j}\end{pmatrix}$$

$$\hat{R}_jC_{j+1}\hat{R}_j^{-1}=Q_j^{-1}P_{j-1}C(w_j,u_{j+1},w_{j+1},m_{j-1})P_{j-1}^{-1}Q_j$$

$$\hat{R}_jA_j\hat{R}_j^{-1}=A(\vec{x}_j,k_j).$$

Proposition 4.5 implies that $m_j>0$ on $R_0(\Gamma)$ $(j=1,\cdots,g-3)$. Hence the signs of entries of the matrices $C(w_j,u_{j+1},w_{j+1},m_{j-1})$, P_{j-1} and Q_j are $\begin{pmatrix}+ & +\\ - & -\end{pmatrix}$,

$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ and $\begin{pmatrix} + & + \\ - & - \end{pmatrix}$. Therefore there exists $s_j \in \mathbf{R}^*$ such that

$$\begin{pmatrix} s_j & 0 \\ 0 & \frac{1}{s_j} \end{pmatrix} \hat{R}_j C_j \hat{R}_j^{-1} \begin{pmatrix} \frac{1}{s_j} & 0 \\ 0 & s_j \end{pmatrix} = \begin{pmatrix} \lambda_j & 0 \\ 0 & \frac{1}{\lambda_j} \end{pmatrix}$$

$$\begin{pmatrix} s_j & 0 \\ 0 & \frac{1}{s_j} \end{pmatrix} \hat{R}_j C_{j+1} \hat{R}_j^{-1} \begin{pmatrix} \frac{1}{s_j} & 0 \\ 0 & s_j \end{pmatrix} = C(trC_j, trC_{j+1}, tr(C_jC_{j+1}), 1)$$

$$\begin{pmatrix} s_j & 0 \\ 0 & \frac{1}{s_j} \end{pmatrix} \hat{R}_j A_j \hat{R}_j^{-1} \begin{pmatrix} \frac{1}{s_j} & 0 \\ 0 & s_j \end{pmatrix} = A(\vec{x}_j, s_j^{-2} k_j).$$

Hence

$$R_j = \begin{pmatrix} s_j & 0 \\ 0 & \frac{1}{s_j} \end{pmatrix} \hat{R}_j \text{ and } k_j = s_j^2 \hat{k}_j.$$

Similarly put

$$\hat{R}_{g-1} = Q_{g-1}^{-1} P_{g-3}^{-1} \cdots P_1^{-1} R$$

$$\hat{R}_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_{g-2}^{-1} \cdots P_1^{-1} R.$$

Then a similar argument shows that there exist $s_{g-1}, s_g \in \mathbf{R}^*$ such that

$$R_{g-1} = \begin{pmatrix} s_{g-1} & 0 \\ 0 & \frac{1}{s_{g-1}} \end{pmatrix} \hat{R}_{g-1} \text{ and } k_{g-1} = s_{g-1}^2 \hat{k}_{g-1}$$

$$R_g = \begin{pmatrix} s_g & 0 \\ 0 & \frac{1}{s_g} \end{pmatrix} \hat{R}_g \text{ and } k_g = s_g^2 \hat{k}_g.$$

■

The above arguments show that

$$X_0(\Gamma) \subset \{\chi \in S(\Gamma) \mid m_j > 0 \ (j = 1, \cdots, g-3),\ x_{i1} k_i > 0 \ (i = 1, \cdots, g)\}.$$

Then by considering the number of connected components of $X_0(\Gamma)$, we see that $\pi_0(X_0(\Gamma))$ is less than or equal to 2^{2g}. On the other hand we have seen in Subsection 2.4 that $\pi_0(X_0(\Gamma)) \geq 2^{2g}$. Hence we get the following result.

Theorem 4.1. $\pi_0(X_0(\Gamma)) = 2^{2g}$. *Thus*

$$T_g = Hom(\Gamma, \mathbf{Z}/2\mathbf{Z})\backslash X_0(\Gamma)$$

is a connected $(6g-6)$*-dimensional cell, and, in particular, contractible.* ∎

§4.4. Semialgebraic Structure of Teichmüller Space T_g

Now $X_0(\Gamma)$ can be written as

$$X_0(\Gamma) = \{\chi \in S(\Gamma) \mid m_j > 0 \ (j=1, \cdots, g-3),\ x_{i1}k_i > 0 \ (i=1, \cdots, g)\}.$$

In the following we will rewrite the above presentation of $X_0(\Gamma)$ by using polynomial inequalities in the I_h $(h \in \Gamma)$.

Proposition 4.8. *For a representation* $\rho = (A_1, B_1, \cdots, A_g, B_g) \in t^{-1}(S(\Gamma))$ *we denote by* $m_j(\rho)$ $(j=1, \cdots, g-3)$ *by* m_j *for the sake of simplicity. Then*

$$m_j > 0 \ (j=1, \cdots, g-3)$$

if and only if

$$trD_{j+1}(trD_j trD_{j+2} + trC_{j+1} trC_{j+2}) - 2(trD_j trC_{j+2} + trC_{j+1} trD_{j+2})$$
$$> \{(trD_{j+1})^2 - 4\} tr(D_j C_{j+2}) \ (j=1, \cdots, g-3)$$

where $C_i := [A_i, B_i]$ $(i=1, \cdots, g)$ *and* $D_j := C_1 \cdots C_j$ $(j=1, \cdots, g-1)$.

Proof. We use the notations of the proof of Proposition 4.3. We calculate $tr(D_j C_{j+2})$ $(j=1, \cdots, g-3)$ for $\rho = (A_1, B_1, \cdots, A_g, B_g) \in t^{-1}(S(\Gamma))$.

$$\begin{aligned} tr(D_j C_{j+2}) &= tr(RD_jR^{-1}RC_{j+2}R^{-1}) \\ &= tr(P_1 \cdots P_{j-1} \begin{pmatrix} \eta_j & 0 \\ 0 & \frac{1}{\eta_j} \end{pmatrix} P_{j-1}^{-1} \cdots P_1^{-1} \\ &\quad \times P_1 \cdots P_j C(w_{j+1}, u_{j+2}, w_{j+2}, m_j) P_j^{-1} \cdots P_1^{-1}) \\ &= tr(P_j^{-1} \begin{pmatrix} \eta_j & 0 \\ 0 & \frac{1}{\eta_j} \end{pmatrix} P_j C(w_{j+1}, u_{j+2}, w_{j+2}, m_j)) \\ &= \frac{w_{j+1}(w_j w_{j+2} + u_{j+1} u_{j+2}) - 2(w_j u_{j+2} + u_{j+1} w_{j+2})}{w_{j+1}^2 - 4} \\ &\quad + \frac{m_{j-1}}{m_j} \cdot \frac{-\eta_{j+1} - \eta_{j+1}^3 \eta_j^2 + \eta_{j+1}^2 \eta_j u_{j+1}}{(\eta_{j+1}^2 - 1)^2} \end{aligned}$$

$$\times\left\{\frac{\eta_{j+1}(\eta_{j+1}u_{j+2}-w_{j+2})(\eta_{j+1}w_{j+2}-u_{j+2})}{(\eta_{j+1}^2-1)^2}-1\right\}$$

$$+\frac{m_j}{m_{j-1}}\cdot\frac{\eta_j^2\eta_{j+1}^{-1}+\eta_{j+1}-\eta_j u_{j+1}}{\eta_j^2}.$$

Hence we get the following quadratic equation in $\frac{m_j}{m_{j-1}}$

$$\left(\frac{\eta_j^2\eta_{j+1}^{-1}+\eta_{j+1}-\eta_j u_{j+1}}{\eta_j^2}\right)\cdot\left(\frac{m_j}{m_{j-1}}\right)^2$$

$$+\left\{\frac{w_{j+1}(w_j w_{j+2}+u_{j+1}u_{j+2})-2(w_j u_{j+2}+u_{j+1}w_{j+2})}{w_{j+1}^2-4}-tr(D_jC_{j+2})\right\}$$

$$\times\left(\frac{m_j}{m_{j-1}}\right)+\left(\frac{-\eta_{j+1}-\eta_{j+1}^3\eta_j^2+\eta_{j+1}^2\eta_j u_{j+1}}{(\eta_{j+1}^2-1)^2}\right)$$

$$\times\left\{\frac{\eta_{j+1}(\eta_{j+1}u_{j+2}-w_{j+2})(\eta_{j+1}w_{j+2}-u_{j+2})}{(\eta_{j+1}^2-1)^2}-1\right\}$$

$$=0 \qquad (j=1,\cdots,g-3).$$

One can easily check that the coefficients of $(\frac{m_j}{m_{j-1}})^2$ and constant terms are negative. Hence if we put $m_0=1$, then

$$m_j>0\ (j=1,\cdots,g-3)$$

$$\Leftrightarrow \frac{m_j}{m_{j-1}}>0 \quad (j=1,\cdots,g-3)$$

$$\Leftrightarrow \frac{w_{j+1}(w_j w_{j+2}+u_{j+1}u_{j+2})-2(w_j u_{j+2}+u_{j+1}w_{j+2})}{w_{j+1}^2-4}-tr(D_jC_{j+2})>0$$

$$(j=1,\cdots,g-3)$$

Since $w_{j+1}^2-4>0$, we have prove our assertion. ■

We put

$$S'(\Gamma):=\{\chi\in S(\Gamma)\,|\,m_j(\chi)>0\ (j=1,\cdots,g-3)\}.$$

Proposition 4.9. *For $\rho=(A_1,B_1,\cdots,A_g,B_g)\in t^{-1}(S'(\Gamma))$ we write $x_{i1}(\rho)\cdot k_i(\rho)$ $(i=1,\cdots,g)$ by $x_{i1}\cdot k_i$ for the sake of simplicity. Then*

$$x_{i1}\cdot k_i>0\ (i=1,\cdots,g)$$

if and only if

$$\frac{tr([A_i,B_i][A_{i+1},B_{i+1}])+tr[A_{i+1},B_{i+1}]}{tr[A_i,B_i]+2}<\frac{tr(A_i[A_{i+1},B_{i+1}])}{trA_i}.$$

Proof. We use the notations of the proof of Proposition 4.7. We compute $tr(A_iC_{i+1})$ $(i=1,\cdots,g)$ for $\rho=(A_1,B_1,\cdots,A_g,B_g)\in t^{-1}(S'(\Gamma))$.

$$\begin{aligned}tr(A_iC_{i+1})&=tr(R_iA_iR_i^{-1}R_iC_{i+1}R_i^{-1})\\&=tr(A(\vec{x}_i,\hat{k}_i)C(trC_i,trC_{i+1},tr(C_iC_{i+1}),1))\\&=\hat{k}_i+\frac{trA_i\cdot(tr(C_iC_{i+1})+trC_{i+1})}{trC_i+2}+\frac{1}{\hat{k}_i}\left(\frac{(trA_i)^2}{trC_i+2}-1\right)\\&\quad\times\left\{\frac{(\lambda_itrC_{i+1}-tr(C_iC_{i+1}))(\lambda_itr(C_iC_{i+1})-trC_{i+1})}{(trC_i+2)(\lambda_i-1)^2}-1\right\}.\end{aligned}$$

Hence we get the following quadratic equation in $\hat{k}_i$.

$$\begin{aligned}&\hat{k}_i^2+\left\{\frac{trA_i\cdot(tr(C_iC_{i+1})+trC_{i+1})}{trC_i+2}-tr(A_iC_{i+1})\right\}\hat{k}_i\\&\quad+\left(\frac{(trA_i)^2}{trC_i+2}-1\right)\left\{\frac{(\lambda_itrC_{i+1}-tr(C_iC_{i+1}))(\lambda_itr(C_iC_{i+1})-tr(C_{i+1})}{(trC_i+2)(\lambda_i-1)^2}-1\right\}\\&=0\end{aligned}$$

As the constant term is positive

$$\begin{aligned}&x_{i1}\cdot k_i>0\ (i=1,\cdots,g)\\\Leftrightarrow\ &x_{i1}\cdot\hat{k}_i>0\ (i=1,\cdots,g)\\\Leftrightarrow\ &\frac{tr(C_iC_{i+1})+trC_{i+1}}{trC_i+2}<\frac{tr(A_iC_{i+1})}{trA_i}\ (i=1,\cdots,g).\end{aligned}$$

■

The above considerations give rise to the following semialgebraic presentation of $X_0(\Gamma)$.

Theorem 4.2. *For* $\alpha_i,\beta_i\in\Gamma$, *put* $c_i:=[\alpha_i,\beta_i]$ $(i=1,\cdots,g)$, *and* $d_j:=c_1\cdots c_j$

$(j=1,\cdots,g-1)$. *Then* $\chi\in X(\Gamma)$ *is contained in* $X_0(\Gamma)$ *if and only if* χ *satisfies the following* $4g-6$ *inequalities on* I_h $(\in\Gamma)$.

$$I_{c_i}(\chi)<-2\ (i=1,\cdots,g),$$

$$I_{d_j}(\chi)<-2\ (j=2,\cdots,g-2),$$

$$\frac{I_{c_kc_{k+1}}(\chi)+I_{c_{k+1}}(\chi)}{I_{c_k}(\chi)+2}<\frac{I_{\alpha_kc_{k+1}}(\chi)}{I_{\alpha_k}(\chi)}\ (k=1,\cdots,g),$$

$$\begin{aligned}&I_{d_{l+1}}(\chi)(I_{d_l}(\chi)I_{d_{l+2}}(\chi)+I_{c_{l+1}}(\chi)I_{c_{l+2}}(\chi))\\&\quad>2(I_{d_l}(\chi)I_{c_{l+2}}(\chi)+I_{c_{l+1}}(\chi)I_{d_{l+2}}(\chi))+(I_{d_{l+1}}(\chi)^2-4)I_{d_lc_{l+2}}(\chi)\\&\quad(l=1,\cdots,g-3)\end{aligned}$$

where $c_{g+1}=c_1$.

Hence by adding $2g$ *inequalities which consist of the sign conditions of* I_{α_i}, I_{β_i} $(i=1,\cdots,g)$, *we can also describe* T_g *by* $6g-6$ *polynomial inequalities in* $X(\Gamma)$. ∎

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