*Publ. RIMS, Kyoto Univ.*
**33** (1997), 573–598

# A Family of Rank-2 Mathematical Instanton Bundles on $P_3$

By

Valery VEDERNIKOV*

**Abstract**

A big family of rank-2 mathematical instanton bundles on the three-dimensional complex projective space $P_3$ is constructed.

## §1. Introduction

In their paper [1] G. Ellingsrud and S-A. Strömme have introduced a method to construct any stable rank-2 vector bundles on $P_3$ with even first Chern class $c_1$. However, the progress for the values of the Chern classes $c_1=0$ and $c_2=4$ was stopped by the lack of an adequate description of the variety of all pencils without fixed points in $PH^0\theta(2)$, where $\theta$ stands for any theta-characteristic on a nonsingular plane quartic curve $C$, for which the corresponding kernel splits on this quartic.

The main purpose of this paper is to show that just as in the case of extremal bundles, defined by the canonical theta-characteristic of the plane embedding of $C$, there are no pencils for which the kernel will not split. Hence, for *any* theta-characteristic on a smooth plane curve $C$ of degree $n$ there is defined an irreducible family of stable rank-2 vector bundles on $P_3$. The main step of the proof involves the definition of a singular theta-characteristic $K_S$, i.e. the one supported on the double curve $2C$. This invertible sheaf on $2C$ splits off the cokernel of a trivial subsheaf $O^3$ of $S^2F^\vee$ and provides the splitting of $adF$ on $C$ for the kernel $F$ of a pencil without fixed points.

The rest of the paper is straightforward because the sheaf $adF$ is always unstable on the defining curve $C$. Besides being a step towards a proof of

Received May 16, 1996
Communicated by Y. Miyaoka, May 16, 1997
1991 Math. Subject Classification: 14F05
* Novoslobodskaya St. 10–42, Moscow K-30, 103030, Russia; e-mail: valery@exdesign.msk.ru

the irreducibility of the moduli scheme of mathematical instanton bundles, this paper provides an instrument to calculate $h^2 EndE$ for such vector bundles on $P_3$. This is done in [2], where we use on a larger scale the properties of the singular theta-characteristic $K_S$.

This paper continues the research of the author started in [3] and originated as an attempt to formulate the results of the paper [3] in the language of the paper [1].

## §2. Some Preliminaries

2.1. Let $E$ be a stable rank-2 vector bundle over the three-dimensional complex projective space $P_3$. As usual we shall denote by $\mathbf{M}_2(0,n)$ the coarse moduli scheme of stable rank-2 vector bundles on $P_3$ with fixed Chern classes $c_1(E)=0$ and $c_2(E)=n$.
Tensoring any rank-2 vector bundle $E$ with even first Chern class with a proper line bundle one can always normalize such $E$ so that $c_1(E)=0$. The stability condition for $E$ will then be equivalent to the vanishing of $h^0E$ for the normalized $E$. Everywhere below $E^\vee = Hom(E, O_Y)$ will denote the dual sheaf of the sheaf $E$ on a nonsingular variety $Y$. The structure sheaf on $P_2$ will be denoted by $O$. If other case will occur, then the structure sheaf will be pointed out explicitly. All other notations correspond to standard ones. For instance, we write $h^1(E)$ for the dimension of the vector space $H^1(E, Y)$, where $E$ is a sheaf on a nonsingular variety $Y$. Note that $O^n$ stands for the direct sum of $n$ copies of the sheaf $O$.

2.2. For the convenience of the reader we review here some facts from [1]. Let $E$ be a stable vector bundle from $\mathbf{M}_2(0,n)$. Suppose that $P_3$ has such a point $x \in P_3$ that all the jumping lines of $E$, that pass through the point $x$, have the splitting type $O_P(-1)\oplus O_P(1)=E_P$, while $O_P^2$ is the general splitting type. Here $O_P$ is the structure sheaf of a line $P$ in $P_3$. Then one can blow up this point and write the following diagram of maps

$$\begin{array}{ccc} X & \xrightarrow{f} & P_3 \\ g\downarrow & & \\ P_2 . & & \end{array} \tag{1}$$

The condition on the blown-up point gives a locally-free sheaf on the projective space $P_2$ that parametrizes the lines through the blown-up point. We put by definition $F=R^0g_*f^*E$. We remind that in this case $X$ is a projectivized fibering over $P_2$ with an isomorphism $X=P(O\oplus O(1))$. The characteristic classes of $F$ have the following values: $c_1(F)=-n$ and $c_2(F)=n(n+1)/2$. Besides, if $\sigma=g^*O(1)$ and $\tau=f^*O_{P_3}(1)$, then $\omega_g=O_X(\sigma-2\tau)$ is the relative canonical sheaf for the morphism $g$ over $P_2$.

The main result from the paper of Ellingsrud-Strömme [1] states that the sheaf $F$ splits on the support of the sheaf $R^1g_*f^*E(-\sigma,-\tau)$. The curve $C$, which is the support of this sheaf, will be called a spectral curve of the vector bundle $E$, and when it is a nonsingular curve, then the sheaf $\theta=R^1g_*f^*E(-\sigma,-\tau)$ on $C$ is a theta-characteristic on it, i.e. there is an isomorphism $\theta^{\otimes 2}=\omega_C$. In this case $n$ is the degree of $C$ and we have an isomorphism $\omega_C=O_C(n-3)$. It is easy to see that the curve $C$ parametrizes jumping lines through the blown up point in $P_3$. The following exact sequence on $X$ is of the fundamental importance

$$0\to g^*F\to f^*E\to h^*\theta(2\sigma-\tau)\to 0. \tag{2}$$

Here $h$ is the restriction of the morphism $g$ on a ruled surface $g^{-1}(C)$. Finally, as we have already pointed out above, there is an isomorphism $F_C^\vee=\theta(1)\oplus\theta(2)$.

2.3. As the main result of this paper we shall construct a big family of such vector bundles from the moduli scheme $\mathbf{M}_2(0,n)$ that satisfy the following condition: $h^1E(-2)=0$. Such vector bundles are usually called rank-2 mathematical instanton bundles on $P_3$. Quite often they are defined by monads of the following specific type:

$$O^n_{P_3}(-1)\to O^{2n+2}_{P_3}\to O^n_{P_3}(1).$$

Any information, pertaining to the structure of the moduli scheme of such vector bundles, is very difficult to be obtained using the monadic or some other approach. For example, it is not yet known whether the part of $\mathbf{M}_2(0,n)$ parametrizing the isomorphism classes of such $E$ is irreducible or whether $h^2 End E$ vanishes for them. On the other hand, construction (1) gives the following resolution of the torsion sheaf $R^1g_*f^*E(-\sigma,-\tau)$ on $P_2$

$$0\to O^n(-2)\overset{m}{\to} O^n(-1)\to R^1g_*f^*E(-\sigma,-\tau)\to 0.$$

This resolution was obtained in Proposition 1.9 of [1]. The matrix $m$ can be chosen to be a symmetric matrix consisting of linear forms on $P_2$. We underline here that the process of the construction of such resolution in [1] did not use the vanishing of the sheaf $R^1g_*f^*E$. Thus, any mathematical instanton bundle on $P_3$ has such resolution for the sheaf $\theta = R^1g_*f^*E(-\sigma, -\tau)$.

2.4. The main purpose of the present paper is to prove that the kernel of a base-point free pencil from $PH^0\theta(2)$ always splits on the support of $\theta$ for *any* theta-characteristic $\theta$ on *any* plane smooth curve. We shall see that there are no restrictions on $C$ from the moduli point of view and no restrictions on $F$ from the cohomological point of view or, stated differently, the only property of $\theta$ that counts is an isomorphism $\theta^{\otimes 2} = \omega_C$. There are no cohomological properties being involved. In particular, the result applies to all vector bundles on $P_3$ that have a smooth spectral curve for some point $x \in P_3$ after applying construction (1). Therefore, there are no restrictions on the spectrum of such vector bundles and all of them may be produced using the methods from [1]. Tensoring exact sequence (2) by $O_C$, we get the following exact sequence

$$0 \to Tor_1(\theta(2), O_C) \to F_C \to O^2{}_C \to \theta(2) \to 0.$$

Since $Tor_1(\theta(2), O_C) = \theta(2) \oplus O_C(-n) = \theta^{-1}(-1)$ we obtain the following two exact sequences on $C$:

$$0 \to \theta^{-1}(-1) \to F_C \to \theta^{-1}(-2) \to 0 \tag{3}$$
$$0 \to \theta^{-1}(-2) \to O_C^2 \to \theta(2) \to 0.$$

This paper is devoted to the proof of the following fact.

**Theorem.** *Let $C$ be any nonsingular plane curve of degree $n$ and $\theta$ any theta-characteristic on it. Then any pencil without fixed points from the projective space $PH^0\theta(2)$ defines an exact sequence*

$$0 \to F \to O^2 \to \theta(2) \to 0$$

*with a rank-2 vector bundle $F$ as the kernel and an isomorphism $F_C^\vee = \theta(1) \oplus \theta(2)$ on the curve $C$.*

The extensions on $C$ of the type (3) are classified by the elements of the vector space $H^1O_C(1)$ which is empty only if $n \leq 3$. This fact was used in [1] to prove the irreducibility of $\mathbb{M}_2(0,2)$ and the two components of $\mathbb{M}_2(0,3)$.

## § Proof of the Theorem

3.1. Let us consider the kernel $F$ of the defining pencil without fixed points from $H^0\theta(2)$. We can restrict it to $C$ by tensoring the defining exact sequence with $O_C$ and, tensoring the restricted vector bundle $F_C$ with $\theta(1)$, we get the following representation for the bundle $F_C \otimes \theta(1) = F \otimes \theta(1)$ on the curve $C$

$$0 \to O_C \xrightarrow{s} F \otimes \theta(1) \to O_C(-1) \to 0. \tag{4}$$

It follows that the sheaf $F \otimes \theta(1)$ has a rank-1 subbundle that violates the stability condition. According to the general classification theory for vector bundles on algebraic curves there are precisely two types of unstable vector bundles of rank 2 on $C$:

*a*) decomposable vector bundles;

*b*) indecomposable vector bundles with a uniquely defined maximal subbundle $s$ of the type (4).

We shall show that if $C$ is a nonsingular curve, then $F_C$ cannot belong to class $b$.

Consider the exact sequence dual to the defining exact sequence for $F$. The defining pencil without fixed points from $PH^0\theta(1)$ defines a trivial rank-2 subsheaf of $F^\vee$ and provides an exact sequence

$$0 \to O^2 \to F^\vee \to \theta(1) \to 0.$$

Tensoring it with the locally free sheaf $F$, we get the following exact sequence on $P_2$

$$0 \to F^2 \to F \otimes F^\vee \to F \otimes \theta(1) \to 0. \tag{5}$$

The sheaf $F \otimes F^\vee$ is the sheaf of endomorphisms of the vector bundle $F$ and it has the following standard representation

$$0 \to O \to F \otimes F^\vee \to adF \to 0,$$

where $adF$ is the sheaf of endomorphisms with the zero trace. Since $(F^\vee \otimes F)^\vee = F^\vee \otimes F$ we have an isomorphism $F \otimes F^\vee = O \oplus adF$. In view of the stability of $F$, which is proved below in Proposition 1, the sheaf $adF$ has no sections. The isomorphism $\Lambda^2 F = O(-n)$ gives an isomorphism $F^\vee = F(n)$. Thus, the decomposition of the tensor product $F^\vee \otimes F^\vee$ into the skew-symmetric and symmetric parts provides an exact sequence

$$0 \to O(n) \to F^\vee \otimes F^\vee \to S^2F^\vee \to 0.$$

Tensoring it with $O(-n)$, one obtains a diagram on $P_2$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O & \to & F\otimes F^\vee & \to & S^2F^\vee(-n) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O & \to & EndF & \to & adF & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Therefore, we have an isomorphism $S^2F^\vee(-n)=adF$. It is easy to note that the unique maximal subbundle $O_C$ of the vector bundle $F\otimes\theta(1)$ is induced by means of the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O(-n) & \to & O & \to & O_C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F^2 & \to & F\otimes F^\vee & \to & F\otimes\theta(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_3 & \to & adF & \to & O_C(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Now to see that the induced rank-3 vector bundle $G_3$ has no sections we need to establish the stability of $F$.

3.2. **Proposition 1.** *If $\theta$ is a theta-characteristic on a plane smooth curve $C$ and $F$ is the kernel of two sections from $\theta(2)$ without fixed points, then $F$ is a rank-2 stable vector bundle on $P_2$.*

*Proof.* The sheaf $F$ is not normalized. Let $n$ be the degree of $C$. Then, if $n$ is even, we have an isomorphism $F_n=F(n/2)$ for the normalization of $F$. Therefore, there is an exact sequence

$$0 \to O^2(-n/2) \to F_n \to \theta(-n/2) \to 0.$$

Because $h^0\theta(1-n/2)=h^1\theta(n/2-1)(n-3)=h^1\theta(n/2-1)$ it is enough to see that $\deg\theta(n/2-1)>2g-2$, where $g=(n-1)(n-2)/2$ is the genus of the smooth curve $C$. This gives the vanishing of $h^1\theta(n/2-1)$. But $\theta$ is a theta-characteristic on $C$ and so $\deg\theta=g-1$. The final count gives for the degree of $\theta(n/2-1)$ the value $(n^2-5n)/2$ and since $2g-2=n^2-3n$ the proof is complete. The case when $n$ is odd is treated similarly because $F_n=F((n-1)/2)$. The proof of Proposition 1 is finished. This result gives $h^0F=0$ and $h^0 adF=0$.

3.3. We restrict now exact sequence (5) to $C$ tensoring it with $O_C$. There is the following exact sequence

$$0 \to K \to F_C \otimes F^\vee \to F \otimes \theta(1) \to 0. \tag{6}$$

One can easily determine the kernel $K$ in (6). Tensoring exact sequence (4) with $F_C \otimes \theta^{-1}(-1)$, we have an exact sequence

$$0 \to \theta^{-1} \otimes F(-1) \to F_C \otimes F \to \theta^{-1}(-2) \otimes F_C \to 0.$$

Using isomorphisms $\theta^{-1}(-1)\otimes F_C=\theta^{-1}(-1)\otimes F(n)=F\otimes\theta(2)$ and $\theta^{-1}(-2)\otimes F=F\otimes\theta(1)$ as well as isomorphisms $\theta^{\otimes 2}=O_C(n-3)$ and $F^\vee=F(n)$ we obtain the next diagram on $C$

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & O_C(1) & & & & O_C & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & F\otimes\theta(2) & \to & F_C\otimes F^\vee & \to & F\otimes\theta(1) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & O_C & & & & O_C(-1) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0. & &
\end{array} \tag{7}$$

Indeed, the kernel $K$ from (6) is isomorphic to $F\otimes\theta(2)$. So the restriction of $F\otimes F^\vee$ to $C$ has at least 3 linearly independent sections in accordance with (7). Our aim will be to prove that there are such sections of this sheaf that give the following exact sequence

$$0 \to O_C^2 \to F_C \otimes F^\vee \to \tilde{R}_2 \to 0,$$

whose factor-sheaf $\tilde{R}_2$ is torsion free on $C$. This means that $\tilde{R}_2$ must be locally free on $C$. The proof of this fact will be divided into several steps.

3.4. **Proposition 2.** *If a locally free sheaf $F$ on $P_2$ is the kernel of a pencil without fixed points from $PH^0\theta(2)$, where $\theta$ is a theta-characteristic on a plane smooth curve $C$ of degree $n$ $(n>1)$, then there exists a pencil without fixed points from $PH^0\theta(1)$ and its kernel $E_1$ is connected with $F$ by means of the following exact sequence*

$$0 \to E_1 \to O^4 \to F^\vee \to 0.$$

*Proof.* It is easy to construct a pencil without fixed points from $PH^0\theta(1)$. We have $h^0\theta(1) \geq n$ for $n>1$. Consider a section of $\theta^2(1)$ with an invertible factor-sheaf. It is enough to take two sections of $\theta(1)$ without common zeroes. We have an exact sequence

$$0 \to O_C \to \theta^2(1) \to O_C(n-1) \to 0.$$

Tensoring it with $\theta^{-1}(-1)$ we obtain an exact sequence

$$0 \to \theta^{-1}(-1) \to O_C^2 \to \theta(1) \to 0.$$

The composition of projections $O^2 \to O_C^2 \to \theta(1) \to 0$ defines an exact sequence on $P_2$

$$0 \to E_1 \to O^2 \to \theta(1) \to 0,$$

where $E_1$ is the kernel of the projection. It is obvious that $E_1$ is a stable (if $n \geq 3$) rank-2 vector bundle on $P_2$ with Chern classes $c_1(E_1) = -n$, $c_2(E_1) = n(n-1)/2$. Indeed, the stability of it is checked using the method of Proposition 1. Restricting $E_1$ on $C$ we get an exact sequence

$$0 \to \theta^{-1}(-2) \to E_1 \otimes O_C \to \theta^{-1}(-1) \to 0.$$

One can now easily connect vector bundles $E_1$ and $F$. Due to $h^1O=0$ any section of $\theta(1)$ can be lifted to a section of $F^\vee$. This gives the following commutative diagram connecting two resolutions of a torsion sheaf $\theta(1)$ on $P_2$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & E_1 & \to & E_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O^2 & \to & O^4 & \to & O^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O^2 & \to & F^\vee & \to & \theta(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
\tag{8}
$$

The middle column of (8) proves the statement of the Proposition.

The following diagram on $C$ follows from the definition of $E_1$

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \theta^{-1}(-2) & & & & \theta(2) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & E_{1C} & \to & O_C^4 & \to & F_C^\vee & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \theta^{-1}(-1) & & & & \theta(1) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0. & &
\end{array}
$$

3.5. **Proposition 3.** *There are imbeddings of vector bundles $E_1 \to F^\vee$ and their cokernels can be only rank-1 sheaves on a curve of degree $2n$ in $P_2$.*

*Proof.* According to the definition of $F^\vee$ there is the following exact sequence on $P_2$

$$0 \to (E_1{}^\vee)^2 \to E_1{}^\vee \otimes F^\vee \to E_1{}^\vee \otimes \theta(1) \to 0.$$

Indeed, it is enough to tensor by $E_1{}^\vee$ the dual to the defining exact sequence for $F$. Since the dual to the defining exact sequence of $E_1$ is the exact sequence

$$0 \to O^2 \to E_1{}^\vee \to \theta(2) \to 0$$

we have $h^0 E_1{}^\vee \geq 2$. So $h^0 E_1{}^\vee \otimes F^\vee \neq 0$ and there are non-trivial homomorphisms $\alpha \in Hom(E_1, F^\vee)$. Any such homomorphism $\alpha$ can be viewed either as a map $E_1 \to F^\vee$ or as a map $F \to E_1{}^\vee$. We claim that there are injective maps among such homomorphisms. Indeed, if $h^0E_1{}^\vee \otimes F^\vee > 2h^0E_1{}^\vee$, then there are sections in $H^0E_1{}^\vee \otimes F^\vee$ which do not factorize through an ideal sheaf corresponding to the factor sheaf of $E_1$ by some invertible sheaf $O(m)$. For if we assume that such a factorization is necessary, then we get a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O(m) & \to & F^\vee & \to & V_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_1 & \to & (F^\vee)^2 & \to & Q_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J(-n+m) & \to & F^\vee & \to & V_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

with the induced cokernels $V_i$, $i = 1, 2$, $Q_3$ and $J$ being an ideal sheaf. Using the values of $c_2(E_1)$ and $c_2(F^\vee)$ and calculating $c_2(Q_3)$ in two ways by means of the above diagram, we deduce that such a factorization is only possible for $m = -n$. Hence, only sections from the canonically imbedded subspace $H^0(E_1^\vee)^2$ in $H^0E_1^\vee \otimes F^\vee$ could be used to carry out such factorization. Thus, if $2h^0E_1^\vee = h^0E_1^\vee \otimes F^\vee$, then $h^0O(n-1) \leq 2h^0E_1^\vee$ as shows the long cohomology sequence

$$0 \to H^0(E_1^\vee)^2 \to H^0(E_1^\vee \otimes F^\vee) \to H^0(E_1^\vee \otimes \theta(1)) \to H^1(E_1^\vee)^2 \to H^1(E_1^\vee \otimes F^\vee) \to$$

and the exact sequence

$$0 \to O_C(n-1) \to E_1^\vee \otimes \theta(1) \to O_C(n) \to 0.$$

However, the last inequality is impossible due to the isomorphism $H^1E_1^\vee = H^0\theta(-2)$ which follows from the definition of $E_1$. We have by Clifford's theorem for a special linear system defined by $\theta(-2)$ an inequality $h^0\theta(-2) \leq (\deg\theta(-2) - 1)/2$. Since $h^0\theta(-2) \leq n(n-7)/4 + 1$ we conclude that $2h^1E_1^\vee$ never exceeds $n(n+1)/2$.

So there are imbeddings $E_1 \to F^\vee$. Note that in case of mathematical instanton bundles we have $h^1E_1^\vee = 0$ by the definition of the resolution for $\theta$. Hence our statement is obvious in case $\theta$ has a resolution of the form as in 2.3. It remains to remark that the cokernel of any such imbedding cannot be a rank-2 vector bundle on a degree $n$ curve $S$ in $P_2$. Indeed, denote some cokernel by $K_S$. If $K_S \otimes O_S$ has rank 2 on $S$, then $F_S^\vee$ also has rank 2 on $S$ and we conclude that the kernel $E_1$ must have Chern classes of $F$ in accordance with the restriction sequence of $F$ to $S$ for $F^\vee(-n)=F$. But $c_2(F) \neq c_2(E_1)$ by the definition of $E_1$ and, therefore, $K_S$ must have rank 1 on its support and can never be the restriction of $F^\vee$ to some degree $n$ curve in $P_2$. The Proposition is proved.

3.6. **Proposition 4.** *The support of the cokernel of any imbedding $E_1 \to F^\vee$ is the curve $2C$ in $P_2$.*

*Proof.* The proof will be a consequence of the following important property of the vector bundles $E_1$ and $F$: any trivial rank-2 subsheaf $O^2$ of $F^\vee$ has as its cokernel the sheaf $\theta(1)$ on $C$. Stated differently it means that any variation of a trivial subsheaf $O^2$ in $F^\vee$ cannot change the support of the corresponding cokernel and, therefore, the map $Gr(1, PH^0F^\vee) \to PH^0O(n)$, defined by assigning the support of the cokernel to any rank-2 trivial subsheaf in $F^\vee$ is a constant map. Indeed, suppose there is a subsheaf $O^2$ of the vector bundle $F^\vee$ whose cokernel $L$ is supported along a curve $S$ of degree $n$ in $P_2$ which is different from $C$. Without loss of generality one can assume $S$ to be an irreducible curve. Hence, our assumption gives $Tor_1(O_S, O_C)=0$. Because $F^\vee = F(n)$ we have the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & O^2 & \to & O^2 & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F^\vee & \to & O^2(n) & \to & L_S(n) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \theta(1) & \to & M(n) & \to & L_S(n) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array} \tag{9}$$

Here $M$ is the induced factor sheaf. The support of $M$ is an algebraic scheme, which is not integral and consists of the two curves $S$ and $C$. We have for the value of Euler characteristic the equality $\gamma(O_S \otimes O_C)=n^2$ due to the assumption $Tor_1(O_S, O_C)=0$. The middle column of (9) defines a resolution of $M$

$$0 \to O^2(-n) \to O^2 \to M \to 0.$$

So we have $h^0 M=2$. But $H^0 M \subseteq H^0 L_S$ and hence there should be sections of $M$, i.e. maps $O_M \to M$ that have a nontrivial kernel isomorphic to $O_C(-n)$ and that induce an imbedding $O_S \to M$. Since $h^1 O^2(-n)=0$ such an imbedding can be lifted to a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O(-n) & \to & O & \to & O_S & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O^2(-n) & \to & O^2 & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O(-n) & \to & O & \to & O_C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array} \tag{10}$$

We used the standard exact sequence

$$0 \to O_C(-n) \to O_{SC} \to O_S \to 0$$

and the fact that $\operatorname{supp} M$ is not integral. The right column of (10) gives that $M$ has sections that factorize through $O_S(-n)$ because $h^0 M=2$. So we obtain finally that $M=O_S \oplus O_C$. The representation of $M$ as the lower row of (9) gives now a contradiction to the assumption $Tor_1(O_S, O_C)=0$. We conclude that $C=S$ and $M=O_C^2$. Therefore, $L=\theta(2)$ and the statement is proved. So the map $Gr(1, PH^0 F^\vee) \to PH^0 O(n)$ defined by choosing the support of any trivial rank-2 subsheaf in $F^\vee$ after factorization is a constant map. It is worthwhile to point out that it is only the support that remains constant because varying $O^2$ in $F^\vee$ we vary the epimorphisms $F^\vee \to \theta(2)$.

We see now that a vector bundle $F$ defined as the kernel of a pencil of sections without fixed points of a theta-characteristic twisted by $O(2)$ on a plane nonsingular curve of degree $n$ is isomorphic to $O^2$ outside the unique curve $C$ by virtue of which it was constructed. One obtains immediately from this fact that outside $C$ the sheaves $E_1$ and $F$ are isomorphic. This fact already proves the statement of the Proposition because the cokernel of any imbedding $E_1 \to F^\vee$ must now be concentrated along $C$ (as a set). A formal procedure of the proof is analogous to the algorithm we used above replacing this time the left column of (9) with an exact sequence

$$0 \to E_1 \to F^\vee \to K_S \to 0.$$

We leave the details to the reader. The Proposition is proved.

3.7. We deduce from Propositions 3 and 4 that $K_S$ is an invertible sheaf on $2C$, i.e. $K_S \otimes O_{2C} = K_S$ and the rank of $K_S$ as an $O_{2C}$-sheaf is 1. The following exact sequence gives the reduction of the structure sheaf $O_{2C}$

$$0 \to O_C(-n) \to O_{2C} \to O_C \to 0.$$

**Definition 1.** *The cokernel of any imbedding $E_1 \to F^\vee$ is called the singular theta-characteristic associated with $\theta$ and denoted by $K_S$.*

3.8. In 3.1. we have defined a vector bundle $G_3$ which can be represented as

$$0 \to O \to (F^\vee)^2 \to G_3(n) \to 0$$

because $F^\vee = F(n)$. We have the following obvious exact sequence

$$0 \to E_1 \to (F^\vee)^2 \to F^\vee \oplus K_S \to 0.$$

The composite map of the section of $(F^\vee)^2$ from the previous exact sequence into the sheaf $F^\vee \oplus K_S$ must be an imbedding. Indeed, if it were not so, we would have a section of $K_S$. Because $K_S$ is an $O_{2C}$-sheaf the sheaf $G_3(n)$ would contain as a subsheaf a sheaf of ideals $J_W(n)$, i.e. the factor sheaf of $E_1$ by $O(-2n)$. It is easy to see that this is equivalent to the following diagram, where $\Sigma_W$ is a skyscaper sheaf,

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & \rightarrow & \Sigma_W & \rightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & J_W(n) & \rightarrow & G_3(n) & \rightarrow & F\oplus\Sigma_W & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & O(n) & \rightarrow & G_3{}^{\vee\vee}(n) & \rightarrow & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & \rightarrow & \Sigma_W & \rightarrow & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

from which we obtain $h^0G_3 \neq 0$. But that contradicts to the definition of $G_3$ since $h^0F=0$ in view of the stability of $F$. Hence, the definition of $G_3$ and the result of Proposition 4 give the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & E_1 & \rightarrow & E_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & O & \rightarrow & (F^{\vee})^2 & \rightarrow & G_3(n) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & O & \rightarrow & F^{\vee}\oplus K_S & \rightarrow & K_S\oplus J_Y(n) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $J_Y(n)$ is an ideal sheaf defined by the following exact sequence

$$0 \rightarrow O \rightarrow J_Y(n) \rightarrow \theta(1) \rightarrow 0.$$

Using the above diagram we can project the sheaf $(F^{\vee})^2$ onto the sheaf $K_S$ and write the following diagram of the projection, where $M_4$ is its kernel

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & O & \to & O & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_4 & \to & (F^\vee)^2 & \to & K_S & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & V_3 & \to & G_3(n) & \to & K_S & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array} \tag{11}$$

We conclude immediately that $c_i(M_4)=0$ for $i=1, 2$. Moreover, $M_4$ must be a reflexive sheaf on $P_2$ since $Ext^2(K_S, O)=0$. This is the reason that makes the induced sheaf $V_3$ to be reflexive too. Our aim now will be to prove that $V_3=O^3$. Consider the diagram of the composite map of $M_4$ into $F^\vee$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & F^\vee & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & M_4 & \to & (F^\vee)^2 & \to & K_S & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & F^\vee & & & & \\ & & & & \downarrow & & & & \\ & & & & 0. & & & & \end{array}$$

We see that the composite map $F^\vee \to (F^\vee)^2 \to K_S$ must be surjective onto $K_S$. Indeed, if it is a zero map, then we have an isomorphism $M_4=F^\vee \oplus E_1$ and this provides an ideal sheaf $J_W(n)$ which is imbedded into $G_3(n)$. It is enough to recall that $h^0 E_1=0$. But such an imbedding was prohibited above. The map $F^\vee \to K_S$ cannot have as its image some $O_{2C}$-subsheaf of the sheaf $K_S$. In this case the assumption that the second Chern class of the resulting factor sheaf is distinct from zero leads to a contradiction after checking Chern classes of $M_4$. Hence the only case that is left is when the image of this map is some $O_C$-subsheaf of the sheaf $K_S$. Because the rank of $K_S \otimes O_C$ is one, we obtain after factorization the exact sequence of the reduction of the structure sheaf of $K_S$ which is represented by the right column of the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_1 & \to & F^\vee & \to & K_{red} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_4 & \to & (F^\vee)^2 & \to & K_S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_2 & \to & F^\vee & \to & K_S \otimes O_C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Here $G_i, i=1, 2$, are the induced locally free sheaves while $K_{red}$ is the kernel of the reduction map. Counting the ranks of the kernels of the map corresponding to the restriction of the middle column to $C$ we obtain a contradiction because $K_{red} \otimes O_C$ has rank-1 as an $O_C$-sheaf. Therefore, there is only one possibility left and it is represented by the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E_1 & \to & F^\vee & \to & K_S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_4 & \to & (F^\vee)^2 & \to & K_S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F^\vee & \to & F^\vee & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

We have here $M_4 = F^\vee \oplus E_1$ as it was shown above.

3.9. **Proposition 5.** *There is an equality* $h^1 F \otimes E_1 = 1$.

*Proof.* This equality means that any non-trivial extension of the form

$$0 \to E_1 \to M_4 \to F^\vee \to 0$$

must be isomorphic to the bundle $O^4$ from Proposition 2. Indeed, such extensions are classified by the elements of the vector space $H^1 F \otimes E_1$. We have to show that besides $M_4 = E_1 \oplus F^\vee$ and $M_4 = O^4$ there are no other extensions in $H^1F\otimes E_1$. In view of the equality $c_i(M_4)=0$ it will be enough to see that any non-trivial extension from $H^1E_1\otimes F$ is semistable. This will provide an isomorphism $M_4=O^4$. The dual exact sequence

$$0\to F\to M_4^\vee\to E_1^\vee\to 0$$

gives an extension from $H^1F\otimes E_1$ defined by the same cocycle. We conclude from this that $M_4$ must be selfdual for a nonzero cocycle. There is the following obvious isomorphism $M_4=O\oplus V_3$, where the locally free sheaf $V_3$ is defined by diagram (11). The selfduality of $M_4$ entails the selfduality of $V_3$. The two definitions of $M_4$ provide the following commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & O & \to & O & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_1 & \to & M_4 & \to & F^\vee & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_1 & \to & V_3 & \to & J_Z(n) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

where the section of the sheaf $F^\vee$ is induced by the section of $M_4$ and $J_Z(n)$ is the induced factor sheaf. Now, if $V_3$ is not semistable, then one must have $h^0V_3(-1)\neq 0$. But then $h^0V_3\neq 0$ too and because $h^0E_1=0$ we obtain an induced section of the sheaf $J_Z(n)$, which gives the following commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & O & \to & O & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_1 & \to & V_3 & \to & J_Z(n) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E_1 & \to & V_2 & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

However, we have $h^1 O = 0$ and this section of the factor sheaf $J_Z(n)$ lifts to a section of the sheaf $F^\vee$ and that gives the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & O & \to & O & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O^2 & \to & F^\vee & \to & \theta(1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O & \to & J_Z(n) & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

We obtain an isomorphism $L = \theta(1)$ using the uniqueness of the unstable curve $C$ proved in Proposition 4. The definition of the vector bundle $E_1$ gives an isomorphism $V_2 = O^2$ and, therefore, $M_4 = O^4$ even if $V_3$ is not semistable. The Proposition is proved. Using now the isomorphism $M_4 = O^4$ we see that the locally free sheaf $G_3(n)$ has a trivial rank-3 subsheaf $O^3$ and the corresponding cokernel is a rank-1 $O_{2C}$-sheaf $K_S$ on $P_2$.

3.10. **Proposition 6.** *There is the following exact sequence for the symmetric square $S^2F^\vee$ of the vector bundle $F^\vee$*

$$0 \to O^3 \to S^2F^\vee \to K_S \oplus O_C(n-1) \to 0.$$

*Proof.* Using the definition of $G_3$ from 3.1 we have an exact sequence

$$0 \to G_3(n) \to S^2F^\vee \to O_C(n-1) \to 0.$$

This exact sequence provides a trivial subsheaf of $S^2F^\vee$ by means of the subsheaf induced from $G_3(n)$ and that gives rise to the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & O^3 & \to & O^3 & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G_3(n) & \to & S^2F^\vee & \to & O_C(n-1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K_S & \to & M_2 & \to & O_C(n-1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

The lower row of this diagram is supported on $C$. It remains to determine the rank of the sheaf $M_2 \otimes O_C$ on the curve $C$. Since the rank of $K_S \otimes O_C$ equals 1 the rank of $M_2 \otimes O_C$ can be only 1 or 2. If its rank is 1, then $M_2$ must be an invertible $O_{3C}$-sheaf, i.e. $M_2 \otimes O_{3C} = M_2$, where $O_{3C}$ is a multiplicity 3 structure on $C$ defined by the exact sequence

$$0 \to O_{2C}(-n) \to O_{3C} \to O_C \to 0.$$

Let us assume that $M_2 \otimes O_C = O_C(n-1)$. This isomorphism is equivalent to the fact that $M_2$ carries the structure of an invertible $O_{3C}$-sheaf. Tensoring the diagram defining $M_2$ by $O_C$, we get the following diagram on the curve $C$ in $P_2$

$$\begin{array}{ccccccccccc} & & & & 0 & & 0 & & & & \\ & & & & \downarrow & & \downarrow & & & & \\ & & 0 & \to & L_2 & \to & L_2 & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O_C(-1) & \to & G_3(n)_C & \to & S^2F_C^\vee & \to & O_C(n-1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O_C(-1) & \to & K_S \otimes O_C & \to & O_C(n-1) & \to & O_C(n-1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0. & & \end{array}$$

Here $O_C(-1) = Tor_1(O_C(n-1), O_C)$ and $L_2$ is the induced kernel. The lower row of the diagram gives an isomorphism $K_S \otimes O_C = O_C(-1)$ and, hence, $G_3(n)_C$ decomposes into the direct sum $G_3(n)_C = O_C(-1) \oplus L_2$. Tensoring the defining exact sequence for $G_3(n)$ from 2.1 by $O_C$ and dualizing, we have an exact sequence on $C$

$$0 \to O_C(1) \oplus L_2 \to (F_C)^2 \to O_C \to 0.$$

Thus $h^0 F_C^2(-1) \neq 0$. But exact sequence (3) shows that $h^0 F_C^2 = 0$. This contradiction shows that $M_2 \otimes O_C \neq O_C(n-1)$. This means that $O_{3C}$ cannot be the structure sheaf of $M_2$. Because the rank of $M_2 \otimes O_C$ is 2, the only possibility left is the splitting $M_2 = K_S \oplus O_C(n-1)$. The Proposition is proved.

3.11. Proposition 6 gives $h^0 S^2 F^\vee(-n+1) = 1$. Because $F$ is stable we have $h^0 S^2 F^\vee(-n) = h^0 adF = 0$. It means that the unique section of $S^2 F^\vee(-n+1)$ provides a torsion free factor sheaf $R_2(1-n)$. There is the following exact sequence on $P_2$

$$0 \to O(n-1) \to S^2 F^\vee \to R_2 \to 0.$$

We know that $h^0 K_S(-n+1) = 0$. So one obtains the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O(-1) & \to & O(n-1) & \to & O_C(n-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & O^3 & \to & S^2 F^\vee & \to & K_S \oplus O_C(n-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_2 & \to & R_2 & \to & K_S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Here $T_2$ is the induced factor sheaf which is torsion free. The diagram arises as the result of the composite map $O(n-1) \to S^2 F^\vee \to K_S \oplus O_C(n-1)$. Restricting this diagram to $C$ we obtain a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & O_C(n-1) & \rightarrow & O_C(n-1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L_1 & \rightarrow & S^2F_C^{\vee} & \rightarrow & K_S\otimes O_C\oplus O_C(n-1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L_1 & \rightarrow & R_2\otimes O_C & \rightarrow & K_S\otimes O_C & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Here $L$ is a rank-1 sheaf on $C$ which should be invertible because it is a rank-1 subsheaf of the vector bundle $S^2F_C^{\vee}$ on the curve $C$. From here we deduce immediately that the sheaf $adF_C=S^2F_C^{\vee}(-n)$ can be represented as the following extension on $C$

$$0\rightarrow O_C(-1)\rightarrow adF_C\rightarrow \tilde{R}_2\rightarrow 0,$$

where the factor sheaf $\tilde{R}_2=R_2\otimes O_C$ is torsion free on $C$. In other words, $Ext^1(\tilde{R}_2, O_C)=0$. We have the dual exact sequence

$$0\rightarrow \tilde{R}_2^{\vee}\rightarrow adF_C\rightarrow O_C(1)\rightarrow 0.$$

This is equivalent to the fact that the following exact sequence is valid on $C$

$$0\rightarrow \tilde{R}_2^{\vee}\rightarrow adF_C\oplus O_C\rightarrow O_C(1)\oplus O_C\rightarrow 0.$$

In view of the isomorphism $adF_C\oplus O_C=F^{\vee}\otimes F_C$ we have obtained a surjective map on $C$ of the sheaf $EndF_C$ onto the sheaf $O_C(1)\oplus O_C$. On the other hand, using diagram (7) we have an inclusion $O_C(1)\rightarrow adF_C$. This inclusion determines an inclusion $O_C\oplus O_C(1)\rightarrow F\otimes F_C$ and, therefore, the sheaf $O_C\oplus O_C(1)$ is a direct summand of $F^{\vee}\otimes F_C$. Upon consideration of the selfduality of $F^{\vee}\otimes F$ we obtain a final isomorphism $F^{\vee}\otimes F_C=O_C^2\oplus O_C(1)\oplus O_C(-1)$. Diagram (7) becomes now the following diagram of maps on $C$

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & O_C(1) & & & & O_C & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & F_C \otimes \theta(2) & \to & O_C^2 \oplus O_C(1) \oplus O_C(-1) & \to & F_C \otimes \theta(1) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & O_C & & & & O_C(-1) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0. & &
\end{array}
$$

The remaining part of the proof is now straightforward. Indeed, we can choose a subbundle $O_C^2$ in $F^\vee \otimes F_C$. It defines an exact sequence

$$0 \to O_C^2 \to F^\vee \otimes F_C \to O_C(1) \oplus O_C(-1) \to 0.$$

It gives the following diagram of maps on $C$

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & O_C^2 & & & \\
 & & & r \downarrow & & & \\
0 \to & F \otimes \theta(2) & \to & F^\vee \otimes F_C & \xrightarrow{s} & F \otimes \theta(1) & \to 0 \\
 & & & \downarrow & & & \\
 & & & O_C(-1) \oplus O_C(1) & & & \\
 & & & \downarrow & & & \\
 & & & 0. & & &
\end{array}
\tag{12}
$$

Consider the composition of maps $rs: O_C^2 \to F \otimes \theta(1)$. It is enough to remark that the image of the map $rs$ must be isomorphic to $O_C$. Indeed, $rs$ cannot map onto a rank-2 subbundle of the vector bundle $F \otimes \theta(1)$ since $h^0 F \otimes \theta(1) = 1$ by the definition. For the further proof, note that the map $rs$ cannot be a zero map because $F \otimes \theta(2) \neq O_C^2$ in view of the isomorphism $\Lambda^2 F \otimes \theta(1) = O_C(-1)$. So we have at out disposal the following commutative diagram where $L$ is an invertible sheaf on $C$, i.e. the image of $rs(O_C^2)$,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L^\vee & \to & O_C^2 & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F\otimes\theta(2) & \to & F^\vee\otimes F_C & \to & F\otimes\theta(1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L(1) & \to & O_C(1)\oplus O_C(-1) & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array} \tag{13}$$

Here $L$ is the induced factor sheaf on $C$. The upper row of (13) gives $h^0L^\vee(-1)=0$. However, looking at the following diagram of maps

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & L & & & & \\ & & & & \gamma\downarrow & & & & \\ 0 & \to & L^\vee(-1) & \to & F\otimes\theta(1) & \xrightarrow{\delta} & L & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & L & & & & \\ & & & & \downarrow & & & & \\ & & & & 0, & & & & \end{array}$$

which corresponds to the left column of (13), we note that the composition of maps $\gamma\delta : L \to L$ is either zero or an isomorphism. If it is a zero map, then we have an exact sequence

$$0 \to L \to L^\vee(-1) \to \Delta_L \to 0,$$

with $\Delta_L$ being some skyscraper sheaf on $C$. This exact sequence gives an equality $h^0L=0$ because $h^0L^\vee(-1)=0$. But then we would have an equality $h^0F_C\otimes\theta(1)=0$ in accordance with the middle row of diagram (13). This is impossible since we have an equality $h^0F\otimes\theta(1)\neq 0$. Therefore, the previous diagram shows that $L=L^\vee(-1)$ and that is why the factor sheaf $L$ should be an invertible sheaf on $C$. Since $h^0L(-1)=0$ we obtain an equality $h^0L=1$. This gives an inclusion $O_C\subset L$. On the other hand, we have $h^0L^\vee\neq 0$, as shows the upper row of diagram (13). That is why there is an inclusion

$L \subset O_C$. We get an isomorphism $L = O_C$. It means that we have obtained an exact sequence

$$0 \to O_C(-1) \to F \otimes \theta(1) \to O_C \to 0.$$

The Theorem is proved.

3.12. *Remark* 1. The invertible sheaf $K_S$ on the double curve $2C$ is a theta-characteristic on it.

We shall restrict ourselves to the case of mathematical instanton bundles. The other cases are dealt with similarly. So there is a resolution for $\theta(1)$

$$0 \to O^n(-1) \to O^n \to \theta(1) \to 0.$$

The following diagram corresponds to the projection of the sheaf $(F^\vee)^2$ onto the sheaf $K_S$

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O & \to & O^4 & \to & O^3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & O & \to & (F^\vee)^2 & \to & G_3(n) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \to & K_S & \to & K_S & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

Using exact sequence

$$0 \to O^2 \to F^\vee \to \theta(1) \to 0$$

and the equality $h^1 O = 0$, we easily construct a resolution of $F^\vee$

$$0 \to O^n(-1) \to O^{n+2} \to F^\vee \to 0.$$

Thus, the middle column of the above commutative diagram provides a resolution of the sheaf $K_S$

$$0 \to O^{2n}(-1) \to O^{2n} \to K_S \to 0.$$

We leave to the reader to check that the matrix defining $K_S$ can be chosen symmetric, thus $K_S(1) = Ext^1(K_S, O)$. The statement of the remark is now obvious.

## §4. A Family of Rank-2 Mathematical Instanton Bundles

4.1. It is well known that any plane smooth curve can be represented as the determinant of a symmetric matrix, the entries of which are linear forms of three variables on $P_2$. This corresponds to the following exact sequence on $P_2$

$$0 \to O^n(-2) \overset{m}{\to} O^n(-1) \to \theta \to 0,$$

in which $\operatorname{Supp}\theta$ can stand for any plane smooth curve from $PH^0O(n)$. It is a classical fact that the number of such representations for a fixed curve equals the number of nonzero theta-constants on that curve.

It is easy to see that the set of all such cokernels $\theta$ is parametrized by the set of all quadratic forms on $O^n(-2)$ with values in $O(-3)$. Hence, each $m$ is defined by an element from $\Gamma(S^2(O^n(2) \otimes O^n(2)))(-3)$. We denote by $M$ the dense open subset of this set that consists of such $m$, the support of $\theta$ for which is nonsingular. The group $\mathbf{Aut}(O^n(-2))$ acts on $M$. We denote the factor set by $\mathbf{M}_0$. Each point of $\mathbf{M}_0$ is a point which is the class of a quadratic form on $O^n(-2)$ with values in $O(-3)$ whose support is a nonsingular plane curve in $P_2$. Hence, $\mathbf{M}_0$ is irreducible and open. It is clear that the dimension of $\mathbf{M}_0$ equals $n(n+3)/2$. Consider now a fibering over $\mathbf{M}_0$ defined as follows. The fibre over a point $(\theta, C)$ will be isomorphic to $Gr^0(1, PH^0\theta(2))$. Here $Gr^0$ stands for a dense open subset of the Grassmann manifold $Gr(1, PH^0\theta(2))$, which consists of linear systems $P_1 \subset PH^0\theta(2)$ without fixed points. It is obvious that such a fibering is irreducible over $\mathbf{M}_0$. We denote it by $\mathbf{G}_1$. Finally, we consider the projective space $P_4 = PH^0(O_C^2 \oplus O_C(1))$. Each point of this projective space defines with the help of the exact sequence

$$0 \to F \to O^2 \to \theta(2) \to 0$$

a surjective map $\lambda: g^*F \to g^*\theta(\sigma, \tau)$, where $g$ is the projectivized fibering $P(O \oplus O(1))$ from 2.2. As it is known, a mathematical instanton bundle of rank 2 up to an automorphism is defined by the given data as the kernel of the map $\lambda$, i.e. $E = \ker\lambda$. For the details of this construction we refer to [1]. But the projective space $PH^0(O_C^2 \oplus O_C(1))$, generally speaking, depends on the curve $C$ and one should add to the already constructed fibering $\mathbb{G}_1$ one more fibering with the fibre isomorphic to $P_4$ over each point of $\mathbb{G}_1$. We denote this fibering by $\mathbb{G}_0$. Note that the fibre over each point is nothing but the projective space $PH^0(EndF_C)=P_4$. Therefore, any point from $\mathbb{G}_0$ defines an exact sequence

$$0 \to E \to g^*F^\vee \to h^*\theta(\sigma, \tau) \to 0,$$

whose kernel, being trivial on the exceptional divisor in $X$, pulls down to $P_3$ via $R^0 f_* E$. The equality $h^0\theta = h^1 E(-2) = 0$ ensures that we obtain a mathematical instanton bundle. Hence, we have obtained a map

$$i: \mathbb{G}_0 \to \mathbb{M}_2(0, n).$$

The map $i$ defines an irreducible family of mathematical instanton bundles of rank 2 on $P_3$ which is the image of an open irreducible variety of dimension $n(n+11)/2$. It is clear that $i$ cannot be an imbedding if $n \geq 4$. Thus, $i$ has fibres of positive dimension if $n \geq 4$.


## Acknowledgements

The author thanks very much the referee for a very professional report and for pointing some misprints and Professor Y. Miyaoka for communicating the paper to the editorial board. Another thanks go to Professor A.G. Alexandrov for his moral support when I worked on the problem. The editor-in-chief Professor Y. Ihara has carefully followed the passing through of the manuscript and I wish to express my most heartfelt gratitude to him.


## References


[1] Ellingsrud, G., Strömme, S-A., Stable Rank-2 Vector Bundles on $P_3$ with $c_1=0$ and $c_2=3$, *Math. Ann.*, **255** (1981), 123–135.

[2] Vedernikov, V., On Deformations of Stable Rank-2 Vector Bundles on $P_3$, *Manuscript submitted to Publ. RIMS.*

[3] ———, The Moduli of Super-Null-Correlation Bundles on $P_3$, *Math. Ann.*, **276** (1987), 365–383.