

Goursat Problem for a Microdifferential Operator of Fuchsian Type and Its Application

By

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Abstract

We consider a matrix of microdifferential operators of Fuchsian type with respect to several variables and obtain the Cauchy-Kovalevskaja type theorem for the Goursat problem. Moreover, we prove the existence theorem in the framework of microfunctions under some assumptions for initial values and the operator.

§0. Introduction

The Goursat problem in the holomorphic (or the real analytic) category is treated by several authors and studied in depth. Moreover, C. Wagschal [W] extended the problem to the case of a system of integro-differential operators and obtained the Cauchy-Kovalevskaja type (namely, the unique solvability) theorem. However, it seems that the study of the Goursat problem is not so satisfactory from the microlocal point of view. Therefore in this paper, we treat a microdifferential operator of Fuchsian type with respect to several variables and consider the Goursat problem in the framework of holomorphic (or micro-) functions.

The notion of Fuchsian type (with respect to one variable) was introduced by M. S. Baouendi and C. Goulaouic [Ba-G] for a partial differential operator. This includes non characteristic type as a special case, and the Cauchy-Kovalevskaja type theorem was proved in [Ba-G]. Seeing this result, N. S. Madi [M] generalized Fuchsian type to several variable case by the name of “opérateurs de Goursat holomorphes à plusieurs variables Fuchiennes” and obtained the Cauchy-Kovalevskaja type theorem for the Goursat problem

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in the framework of holomorphic functions. We remark that Y. Laurent-T. Monterio Fernandes [La-MF] and Z. Szmydt and B. Ziemian [Sz-Zi] gave different definitions of Fuchsian type with respect to several variables respectively. On the other hand, after Baouendi-Goulaouic [Ba-G], many mathematicians have obtained almost sufficient results in Fuchsian type with respect to one variable. For example, H. Tahara [Ta] treated a Fuchsian system in the sense of Volevič and proved the Cauchy-Kovalevskaja type theorem in the complex domain. Further, as an application he obtained the existence and uniqueness theorem on an initial value problem for a Fuchsian hyperbolic system in the framework of hyperfunctions. Moreover, he proved the existence theorem on a homogeneous initial value problem for a Fuchsian microhyperbolic system of microdifferential operators in the framework of microfunctions. On the other hand, T. Oaku proved the existence theorem on an inhomogeneous initial value problem for a Fuchsian hyperbolic microdifferential operator in [O2] (cf. [O1]) and the uniqueness theorem under the F -mildness condition (but without the hyperbolicity assumption) in [O4] in the framework of microfunctions (cf. [O3]).

In this paper, we define a matrix of microdifferential operators of Fuchsian type with respect to several variables as a natural generalization of one variable case due to Tahara [Ta] or non-microlocal case due to Madi [M]. Moreover, we prove the Cauchy-Kovalevskaja type theorem for the Goursat problem in the space of holomorphic functions under the action of microdifferential operators due to J. M. Bony and P. Schapira [Bo-Sc]. As an application we solve the Goursat problem in the framework of micro-(or hyper-) functions; we prove the existence theorem for sufficiently “regular” initial data under suitable assumptions. Note that our solvability conditions are weaker than that of M. Kashiwara and P. Schapira [K-Sc1]. We hope our results will serve as a starting point for studying the Goursat problem for general systems in microlocal analysis.

We will show the plan of this paper.

In Section 1, we announce the main theorem on the Goursat problem in the complex domain after some preparations.

In Section 2, we reduce the problem to a simple case.

In Section 3, we give the proof of the main theorem.

Section 4 is devoted to applications to microlocal solvability for the Goursat problem under suitable assumptions.

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§1. Statement of Main Theorem

In this paper, we use the following notation: \mathbb{N} denotes the set of natural numbers (not containing 0) and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a subset D of some topological space, $[D]$ denotes the closure. For natural numbers $M, N \in \mathbb{N}$, and a linear space L we denote by $\text{Mat}(M, N; L)$ the space of matrices of size $M \times N$ whose components are in L . Further set

$$\left\{ \begin{array}{l} \text{Mat}(N; L) := \text{Mat}(M, N; L), \\ L^N := \text{Mat}(1, N; L), \\ L^{\oplus N} := \text{Mat}(M, 1; L). \end{array} \right.$$

In addition, if L has a norm $\| \cdot \|$, for $P = (P^{(\mu, \nu)})_{\mu, \nu=1}^N \in \text{Mat}(M, N; L)$ we set $\|P\| := \max\{\|P^{(\mu, \nu)}\|; 1 \leq \mu \leq M, 1 \leq \nu \leq N\}$. For natural numbers $d, n \in \mathbb{N}$, we use coordinates $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{C}^d$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Moreover for multi-indices $\gamma = (\gamma_1, \dots, \gamma_n)$ and $\alpha = (\alpha_1, \dots, \alpha_d)$, we set

$$\left\{ \begin{array}{ll} \partial_z^\gamma := \partial_{z_1}^{\gamma_1} \dots \partial_{z_n}^{\gamma_n}, & \partial_\tau^\alpha := \partial_{\tau_1}^{\alpha_1} \dots \partial_{\tau_d}^{\alpha_d}, \\ z^\gamma := z_1^{\gamma_1} \dots z_n^{\gamma_n}, & \tau^\alpha := \tau_1^{\alpha_1} \dots \tau_d^{\alpha_d}, \\ \gamma! := \gamma_1! \dots \gamma_n!, & \alpha! := \alpha_1! \dots \alpha_d!, \\ |\gamma| := \sum_{j=1}^n \gamma_j, & |\alpha| := \sum_{j=1}^d \alpha_j, \end{array} \right.$$

as usual. For vectors $R = (R_1, \dots, R_d)$ and $R' = (R'_1, \dots, R'_d) \in \mathbb{R}^d$, we define an order relation as follows:

$$R' \leq R \stackrel{\text{def.}}{\Leftrightarrow} R'_j \leq R_j \text{ for all } j,$$

$$R' < R \stackrel{\text{def.}}{\Leftrightarrow} R' \leq R \text{ and } R' \neq R,$$

$$R' \prec R \Leftrightarrow R'_j \prec R_j \quad \text{for all } j. \\ \text{def.}$$

For a vector $r = (r_1, \dots, r_d) \in \mathbf{R}^d$, we set $[r]_+ := ([r_1]_+, \dots, [r_d]_+)$, where $[r_j]_+ = \max\{r_j, 0\}$. We fix $m^{(v)} = (m_1^{(v)}, \dots, m_d^{(v)})$ and $k^{(v)} = (k_1^{(v)}, \dots, k_d^{(v)}) \in \mathbf{N}_0^d$ with $m^{(v)} \geq k^{(v)}$ ($1 \leq v \leq N$) and set $m = (m^{(1)}, \dots, m^{(N)})$ and $k = (k^{(1)}, \dots, k^{(N)}) \in (\mathbf{N}_0^d)^N$. For any N -tuple of (generalized) functions $f(z, \tau) = (f_1(z, \tau), \dots, f_N(z, \tau))$, we mean by $f = O(\tau^{m-k})$

$$\partial_{\tau_i}^j f|_{\tau_i=0} = 0 \quad (1 \leq v \leq N, 1 \leq i \leq d, 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1).$$

Set $\mathbf{1}_d := (1, \dots, 1) \in \mathbf{N}^d$. For a vector $R = (R_1, \dots, R_d) \in \mathbf{R}^d$ with $0 \prec R$, we set

$$B(R) := \{\tau \in \mathbf{C}^d; |\tau_j| < R_j \ (1 \leq j \leq d)\}.$$

Let $V \subset \mathbf{C}^n$ be a relatively compact open neighborhood of the origin and h_0 a positive number. We set

$$U := \{(z; \zeta) \in T^*\mathbf{C}^n; z \in V, \zeta_1 = 1, |\zeta_j| < h_0 \ (2 \leq j \leq n)\}.$$

We denote the sheaf of rings of microdifferential operators of finite order (resp. of order at most v) by \mathcal{E} (resp. $\mathcal{E}(v)$) as usual.

1.1. Definition. Let $P(z, \tau; \partial_z, \partial_\tau) = (P^{\mu, v}(z, \tau; \partial_z, \partial_\tau))_{\mu, v=1}^N$ be a matrix in $\text{Mat}(N; I([U \times B(R)]; \mathcal{E}_{\mathbf{C}^n+a}))$; that is, each $P^{\mu, v}$ is a microdifferential operator of finite order defined in some neighborhood of $[U \times B(R)]$. Then, P is said to be of *Fuchsian type with weight (k, m) (with respect to τ -variables)* if it has the following form:

$$P^{\mu, v}(z, \tau; \partial_z, \partial_\tau) = \sum_{0 \leq \alpha \leq m^{(v)}} P_\alpha^{\mu, v}(z, \tau; \partial_z) \partial_\tau^\alpha,$$

where each $P_\alpha^{\mu, v}$ is a microdifferential operator with holomorphic parameters τ and satisfies the following:

1. The order $\text{ord } P_\alpha^{\mu, v}$ of $P_\alpha^{\mu, v}$ is at most $|m^{(v)}| - |\alpha|$;
2. There exist $P_\alpha^{1, (\mu, v)}(z, \tau; \partial_z)$ and $P_\alpha^{2, (\mu, v)}(z, \tau; \partial_z)$ ($0 \leq \alpha \leq m^{(v)}$) such that $\text{ord } P_\alpha^{1, (\mu, v)} \leq 0$ and

$$P_\alpha^{\mu, v}(z, \tau; \partial_z) = \tau^{[\alpha - m^{(v)} + k^{(v)}]_+} P_\alpha^{1, (\mu, v)}(z, \tau; \partial_z) \\ + \tau^{[\alpha - m^{(v)} + k^{(v)} + \mathbf{1}_d]_+} P_\alpha^{2, (\mu, v)}(z, \tau; \partial_z).$$

1.2. Remark. (1) The Fuchsian property above is invariant under any coordinate change of z -variables, or more generally an arbitrary quantized contact transformation for $(z; \zeta)$ -variables.

(2) The Fuchsian type defined in Definition 1.1 is a natural generalization of differential operators of Fuchsian type introduced by Madi [M]; that is, if P is a differential operator of Fuchsian type in the sense of Definition 1.1, then P is of Fuchsian type in the sense of Madi. Further if $d=N=1$, a microdifferential operator of Fuchsian type is nothing but of Fuchsian type defined by Tahara (see [Ta]).

Let $T^{(v)}=(T_1^{(v)}, \dots, T_d^{(v)})$ ($1 \leq v \leq N$) be indeterminates and set

$$\tilde{T} := (T^{(1)}, \dots, T^{(N)}).$$

If P is of Fuchsian type with weight (k, m) , we define the indicial polynomial of P by

$$\mathcal{I}_P(z; \zeta; \tilde{T}) := \det \left(\sum_{m^{(v)} - k^{(v)} \leq \alpha \leq m^{(v)}} \sigma_0(P_\alpha^{1, (\mu, v)})(z, 0; \zeta) \mathcal{I}_\alpha(T^{(v)}) \right)$$

where $\mathcal{I}_\alpha(T^{(v)}) := \prod_{j=1}^d \mathcal{I}_{\alpha_j}(T_j^{(v)})$ with

$$\mathcal{I}_{\alpha_j}(T_j^{(v)}) := \begin{cases} T_j^{(v)}(T_j^{(v)} - 1) \cdots (T_j^{(v)} - \alpha_j + 1) & (\alpha_j \geq 1), \\ 1 & (\alpha_j = 0). \end{cases}$$

Let $A(z, \tau; \partial_z)$ be a microdifferential operator of finite order with holomorphic parameters τ defined in a neighborhood of $[U \times B(R)]$. Let $c \in \mathbb{C}$ and set $\Sigma := \{z \in \mathbb{C}^n; z_1 = c\}$. Let $\Omega \subset V$ be an open convex set and assume that Ω is h_0 - Σ -flat in the sense of Bony-Schapira; that is, if $z \in \Omega$, $w \in \Sigma$ and $h_0|z_j - w_j| \leq |z_1 - w_1|$ ($2 \leq j \leq n$), then it follows that $w \in \Omega \cap \Sigma$. For example, we easily see that for any $z_0 \in \Sigma$ the set $\{w \in \mathbb{C}^n; |w_1 - z_1|/h + |w_j - z_j| < r$ ($2 \leq j \leq n$)\} is h - Σ -flat. Let $f(z, \tau)$ be a holomorphic function defined on $\Omega \times B(R)$. If $p \in \mathbb{N}$, there exists a unique holomorphic function $g(z, \tau)$ on $\Omega \times B(R)$ such that

$$\begin{cases} \partial_{z_1}^p g(z, \tau) = f(z, \tau), \\ \partial_{z_1}^j g|_{z_1=c} = 0 \quad (0 \leq j \leq p-1). \end{cases}$$

Then, we define $(\partial_{z_1}^{-p})_{\Sigma} f(z, \tau) := g(z, \tau)$; that is,

$$(\partial_{z_1}^{-p})_{\Sigma} f(z, \tau) := \int_c^{z_1} \frac{(z_1 - w_1)^{p-1}}{(p-1)!} f(w_1, z') dw_1,$$

where $z' := (z_2, \dots, z_n)$. We write

$$A(z, \tau; \partial_z) = \sum_{\substack{\gamma_1 \in \mathbb{Z} \\ \gamma_2, \dots, \gamma_n \in \mathbb{N}_0}} A_{\gamma}(z, \tau) \partial_z^{\gamma}$$

formally. Then, applying the argument in the proof of Théorème 2.3.3 of Bony-Schapira [Bo-Sc] regarding τ as holomorphic parameters, we find that

$$A_{\Sigma} f(z, \tau) := \sum_{\gamma \in \mathbb{N}_0^d} A_{\gamma}(z, \tau) \partial_z^{\gamma} f(z, \tau) + \sum_{\substack{\gamma_1 < 0 \\ \gamma_2, \dots, \gamma_n \in \mathbb{N}_0}} A_{\gamma}(z, \tau) (\partial_{z_1}^{\gamma_1})_{\Sigma} \partial_z^{\gamma'} f(z, \tau)$$

is holomorphic on $\Omega \times B(R)$. Let s be a parameter with $0 < s < 1$. We fix a point $z_0 \in \Omega \cap \Sigma$ and set

$$\Omega_s := \{s(z - z_0) + z_0 \in \mathbb{C}^n; z \in \Omega\}.$$

We fix an arbitrary positive number c_0 and set

$$X_s := \{f(z) \in \mathcal{O}(\Omega_s); \|f\|_s := \sup\{|f(z)| d'_s(z)^{c_0}; z \in \Omega_s\} < \infty\},$$

where $\mathcal{O}(\Omega_s)$ denotes the space of holomorphic functions defined on Ω_s and

$$d'_s(z) := \inf\{\max_{2 \leq j \leq n} |z_j - w_j|; w \in \mathbb{C}^n \setminus \Omega_s, z_1 = w_1\}.$$

Note that X_s is a Banach space under the norm $\|f\|_s$ and that $X_{s'} \subset X_s$ and $\|f\|_s \leq \|f\|_{s'}$ for any $0 < s < s' < 1$ and $f \in X_{s'}$. Moreover, by Proposition 2.4.3 of Bony-Schapira [Bo-Sc], we see that if the operator $A(z, \tau; \partial_z)$ above is of order zero then A_{Σ} operates on X_s as a bounded linear operator; that is, there exists a constant C such that for any $f \in X_s$,

$$\sup\{\|(A_{\Sigma} f)(\cdot, \tau)\|_s; \tau \in B(R)\} \leq C \|f\|_s.$$

Consider the following condition:

[A-1]. *There exist a positive constant $C > 0$ and a neighborhood W of $[U]$ such that for any $(z; \zeta) \in [W]$ and $\beta \in \mathbb{N}_0^d$*

$$|\mathcal{I}_P(z; \zeta; \beta + m^{(1)} - k^{(1)}, \dots, \beta + m^{(N)} - k^{(N)})| \geq C \prod_{\nu=1}^N (\beta + \mathbf{1}_d)^{m^{(\nu)}}.$$

Note that if $N=1$, then [A-1] is a natural generalization of Madi's condition which is similar to the "Fuchsian ellipticity condition" due to Szmydt-Ziemian [Sz-Zi].

1.3. Theorem. *Let P be a matrix of microdifferential operators defined in a neighborhood of $[U \times B(R)]$. Assume that P is of Fuchsian type with weight (k, m) and satisfies [A-1]. Then, there exist constants $r_0 > 0$ and \tilde{R} with $0 < \tilde{R} \leq R$ such that the following hold:*

Take arbitrary h and r with $0 < h < h_0$ and $0 < r < r_0$ respectively. Let Ω be any h - Σ -flat open convex subset of V with $\text{dia } \Omega \leq r$, where dia denotes the diameter. Then, there exists a constant δ such that for any \tilde{R} with $0 < \tilde{R} \leq \tilde{R}$ it follows that for any holomorphic functions $f(z, \tau) = (f_1(z, \tau), \dots, f_N(z, \tau))$ and $g(z, \tau) = (g_1(z, \tau), \dots, g_N(z, \tau))$ on $\Omega \times B(\tilde{R})$, there exists a unique holomorphic solution $u(z, \tau) = (u_1(z, \tau), \dots, u_N(z, \tau))$ of the Goursat problem

$$(G.P.) \quad \begin{cases} P_{\Sigma} u = f, \\ u - g = O(\tau^{m-k}), \end{cases}$$

and each $u_{\nu}(z, \tau)$ ($1 \leq \nu \leq N$) is holomorphic on

$$\bigcup_{0 < s < 1} \left(\Omega_s \times \left\{ \tau \in B(\tilde{R}); \prod_{j=1}^d |\tau_j| < \delta(1-s)^{|m|} \right\} \right),$$

where $|m| := \sum_{\nu=1}^N \sum_{j=1}^d m_j^{(\nu)}$.

1.4. Remark. (1) Assume that P is a differential operator. Then Theorem 1.3 is (essentially) obtained by Madi [M] (cf. [La-MF]).

(2) By definition (G.P.) is written as

$$\begin{cases} P_{\Sigma}^t(u_1, \dots, u_N) = f, \\ \partial_{\tau_i}^j u_v|_{\tau_i=0} = \partial_{\tau_i}^j g_v|_{\tau_i=0} \quad \left(\begin{array}{l} 1 \leq v \leq N, \quad 1 \leq i \leq d, \\ 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1 \end{array} \right). \end{cases}$$

Let us set

$$\Omega \times B(\tilde{R})_i := \Omega \times B(\tilde{R}) \cap \{\tau_i = 0\} \quad (1 \leq i \leq d)$$

and regard each $\Omega \times B(\tilde{R})_i$ as an open subset of \mathbb{C}^{n+d-1} . Similarly we identify $\Omega \times B(\tilde{R}) \cap \{\tau_1 = \dots = \tau_d = 0\}$ with Ω . Let $g_{v,ij}$ ($1 \leq v \leq N$, $1 \leq i \leq d$, $0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1$) be holomorphic functions on $\Omega \times B(\tilde{R})_i$ with compatibility conditions

$$(C.C.) \quad \partial_{\tau_i}^j g_{v,ij}|_{\tau_i=0} = \partial_{\tau_i}^j g_{v,i'j'}|_{\tau_i=0} \quad \left(\begin{array}{l} 1 \leq v \leq N, \quad 1 \leq i, i' \leq d, \\ 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1, \\ 0 \leq j' \leq m_{i'}^{(v)} - k_{i'}^{(v)} - 1 \end{array} \right).$$

Then we see that the Goursat problem (G.P.) is equivalent to the following problem:

$$\begin{cases} P_{\Sigma}^t(u_1, \dots, u_N) = f, \\ \partial_{\tau_i}^j u_v|_{\tau_i=0} = g_{v,ij} \quad (1 \leq v \leq N, \quad 1 \leq i \leq d, \quad 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1). \end{cases}$$

Indeed, for $g(z, \tau) = (g_1(z, \tau), \dots, g_N(z, \tau))$ it is clear that the data

$$\{g_{v,ij} = \partial_{\tau_i}^j g_v|_{\tau_i=0} \in \mathcal{O}(\Omega \times B(\tilde{R})_i); \quad 1 \leq v \leq N, \quad 1 \leq i \leq d, \quad 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1\}$$

satisfy (C.C.). Conversely, for any

$$\{g_{v,ij} \in \mathcal{O}(\Omega \times B(\tilde{R})_i); \quad 1 \leq v \leq N, \quad 1 \leq i \leq d, \quad 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1\}$$

which satisfy (C.C.), let us set

$$g_v(z, \tau) := \sum_{\alpha \in N_v} \frac{g_{v,\alpha}(z)}{\alpha!} \tau^\alpha,$$

where

$$N_v := N_0^d \setminus \{\alpha \in N_0^d; \alpha \geq m^{(v)} - k^{(v)}\}$$

and

$$g_{v,\alpha}(z) := \begin{cases} \partial_{\tau}^{\hat{\alpha}_1} g_{v,1\alpha_1}|_{\Omega} & (0 \leq \alpha_1 \leq m_1^{(v)} - k_1^{(v)} - 1), \\ \partial_{\tau}^{\hat{\alpha}_2} g_{v,2\alpha_2}|_{\Omega} & (m_1^{(v)} - k_1^{(v)} \leq \alpha_1, 0 \leq \alpha_2 \leq m_2^{(v)} - k_2^{(v)} - 1), \\ \partial_{\tau}^{\hat{\alpha}_3} g_{v,3\alpha_3}|_{\Omega} & \left(\begin{matrix} m_1^{(v)} - k_1^{(v)} \leq \alpha_1, m_2^{(v)} - k_2^{(v)} \leq \alpha_2, \\ 0 \leq \alpha_3 \leq m_3^{(v)} - k_3^{(v)} - 1 \end{matrix} \right), \\ \vdots & \vdots \\ \partial_{\tau}^{\hat{\alpha}_d} g_{v,d\alpha_d}|_{\Omega} & \left(\begin{matrix} m_i^{(v)} - k_i^{(v)} \leq \alpha_i (1 \leq i \leq d-1), \\ 0 \leq \alpha_d \leq m_d^{(v)} - k_d^{(v)} - 1 \end{matrix} \right), \end{cases}$$

with $\hat{\alpha}_i := (\alpha_1, \dots, \alpha_{i-1}, \overset{i}{0}, \alpha_{i+1}, \dots, \alpha_d)$. Then we easily see that each $g_v(z, \tau)$ is holomorphic on $\Omega \times B(\tilde{R})$ and satisfies

$$\partial_{\tau_i}^j g_v|_{\tau_i=0} = g_{v,ij} \quad (1 \leq v \leq N, 1 \leq i \leq d, 0 \leq j \leq m_i^{(v)} - k_i^{(v)} - 1).$$

Note that for example if $d=2$ or 3 , then each $g_v(z, \tau)$ is written explicitly as

$$g_v(z, \tau) = \sum_{\alpha_1=0}^{m_1^{(v)} - k_1^{(v)} - 1} \frac{g_{v,1\alpha_1}(z, \tau_2)}{\alpha_1!} \tau_1^{\alpha_1} + \sum_{\alpha_2=0}^{m_2^{(v)} - k_2^{(v)} - 1} \frac{g_{v,2\alpha_2}(z, \tau_1)}{\alpha_2!} \tau_2^{\alpha_2} - \sum_{0 \leq \alpha \leq m^{(v)} - k^{(v)} - 1_2} \frac{\partial_{\tau_1}^{\alpha_1} g_{v,2\alpha_2}(z, 0)}{\alpha!} \tau^{\alpha},$$

or

$$g_v(z, \tau) = \sum_{\alpha_1=0}^{m_1^{(v)} - k_1^{(v)} - 1} \frac{g_{v,1\alpha_1}(z, \tau_2, \tau_3)}{\alpha_1!} \tau_1^{\alpha_1} + \sum_{\alpha_2=0}^{m_2^{(v)} - k_2^{(v)} - 1} \frac{g_{v,2\alpha_2}(z, \tau_1, \tau_3)}{\alpha_2!} \tau_2^{\alpha_2} + \sum_{\alpha_3=0}^{m_3^{(v)} - k_3^{(v)} - 1} \frac{g_{v,3\alpha_3}(z, \tau_1, \tau_2)}{\alpha_3!} \tau_3^{\alpha_3} - \sum_{\substack{0 \leq \alpha_1 \leq m_1^{(v)} - k_1^{(v)} - 1 \\ 0 \leq \alpha_2 \leq m_2^{(v)} - k_2^{(v)} - 1}} \frac{\partial_{\tau_1}^{\alpha_1} g_{v,2\alpha_2}(z, 0, \tau_3)}{\alpha_1! \alpha_2!} \tau_1^{\alpha_1} \tau_2^{\alpha_2} - \sum_{\substack{0 \leq \alpha_2 \leq m_2^{(v)} - k_2^{(v)} - 1 \\ 0 \leq \alpha_3 \leq m_3^{(v)} - k_3^{(v)} - 1}} \frac{\partial_{\tau_2}^{\alpha_2} g_{v,3\alpha_3}(z, \tau_1, 0)}{\alpha_2! \alpha_3!} \tau_2^{\alpha_2} \tau_3^{\alpha_3} - \sum_{\substack{0 \leq \alpha_1 \leq m_1^{(v)} - k_1^{(v)} - 1 \\ 0 \leq \alpha_3 \leq m_3^{(v)} - k_3^{(v)} - 1}} \frac{\partial_{\tau_3}^{\alpha_3} g_{v,1\alpha_1}(z, \tau_2, 0)}{\alpha_3! \alpha_1!} \tau_3^{\alpha_3} \tau_1^{\alpha_1} + \sum_{0 \leq \alpha \leq m^{(v)} - k^{(v)} - 1_3} \frac{\partial_{\tau_1}^{\alpha_1} \partial_{\tau_2}^{\alpha_2} g_{v,3\alpha_3}(z, 0)}{\alpha!} \tau^{\alpha}$$

respectively.

1.5. Example. (1) Let us set $m^{(v)} := (2, \dots, 2)$ and $k^{(v)} := (1, \dots, 1) \in \mathcal{N}_0^d$ ($1 \leq v \leq N$) and consider

$$P(z, \tau; \partial_z, \partial_\tau) = \prod_{j=1}^d (\tau_j \partial_{\tau_j}^2 I_N + A_j(z, \tau; \partial_z) \partial_{\tau_j} + B_j(z, \tau; \partial_z)) + C(z, \tau; \partial_z).$$

Here

$$\begin{cases} A_j(z, \tau; \partial_z), B_j(z, \tau; \partial_z) \in \text{Mat}(N; \mathcal{E}(0)), \\ C(z, \tau; \partial_z) \in \text{Mat}(N; \mathcal{E}(2d)), \end{cases}$$

and $I_N \in \text{Mat}(N; \mathbf{R})$ denotes the identity matrix. Then, P is of Fuchsian type with weight $((k^{(1)}, \dots, k^{(N)}), (m^{(1)}, \dots, m^{(N)}))$ and

$$\mathcal{I}_P(z; \zeta; \beta + m^{(1)} - k^{(1)}, \dots, \beta + m^{(N)} - k^{(N)}) = \prod_{j=1}^d \det((\beta_j + 1)\beta_j + \sigma_0(A_j)(z, 0; \zeta)(\beta_j + 1)).$$

Hence if each eigenvalue of $(\sigma_0(A_j)(0, 0; 1, 0, \dots, 0))_{i,j=1}^d$ is not in $\{l \in \mathbf{Z}; l \leq 0\}$, we see that P satisfies the condition [A-1].

(2) Let us choose $N = d$ and set $m^{(v)} := (0, 0, \dots, 0, \overset{v}{2}, 0, \dots, 0)$ and $k^{(v)} := (0, 0, \dots, 0, \overset{v}{1}, 0, \dots, 0) \in \mathcal{N}_0^d$ ($1 \leq v \leq d$). Consider

$$\begin{aligned} P(z, \tau; \partial_z, \partial_\tau) &= (P^{(i,j)}(z, \tau; \partial_z, \partial_\tau))_{i,j=1}^d \\ &= (\tau_j \partial_{\tau_j}^2 \delta_{ij} + (A_{ij}(z, \tau; \partial_z) + \tau^{1^a} B_{ij}(z, \tau; \partial_z)) \partial_{\tau_j} \\ &\quad + C_{ij}(z, \tau; \partial_z) + \tau^{1^a} D_{ij}(z, \tau; \partial_z))_{i,j=1}^d, \end{aligned}$$

where $A_{ij}(z, \tau; \partial_z)$, $B_{ij}(z, \tau; \partial_z)$, $C_{ij}(z, \tau; \partial_z)$ and $D_{ij}(z, \tau; \partial_z)$ are microdifferential operators with $\text{ord } A_{ij}, \text{ord } C_{ij} \leq 0, \text{ord } B_{ij} \leq 1$ and $\text{ord } D_{ij} \leq 2$ respectively. Then, P is of Fuchsian type with weight $((k^{(1)}, \dots, k^{(N)}), (m^{(1)}, \dots, m^{(N)}))$ and

$$\mathcal{I}_P(z; \zeta; \beta + m^{(1)} - k^{(1)}, \dots, \beta + m^{(N)} - k^{(N)}) = \det((\beta_j + 1)\beta_j \delta_{ij} + \sigma_0(A_{ij})(z, 0; \zeta)(\beta_j + 1)).$$

Hence if each eigenvalue of $\sigma_0(A_{ij})(0, 0; 1, 0, \dots, 0)$ is not in $\{l \in \mathbf{Z}; l \leq 0\}$ and each $(\sigma_0(A_{ij}))_{i,j=1}^d$ is an upper triangular matrix, we see that P satisfies the condition [A-1].

§2. Preliminaries

In this section, we will make some reductions.

Considering

$$\begin{cases} P_{\Sigma} u = f - P_{\Sigma} g, \\ u = O(\tau^{m-k}), \end{cases}$$

we may assume that $g \equiv 0$. Further, if P is of Fuchsian type with weight (k, m) and $k \neq m$, then define P_1 by

$$P_1 := P \cdot \begin{pmatrix} \tau^{m_1 - k_1} & & 0 \\ & \ddots & \\ 0 & & \tau^{m_N - k_N} \end{pmatrix}.$$

Since a direct calculation shows that P_1 is of Fuchsian type with weight (m, m) and $\mathcal{I}_{P_1}(z; \zeta; \beta, \dots, \beta) = \mathcal{I}_P(z; \zeta; \beta + m^{(1)} - k^{(1)}, \dots, \beta + m^{(N)} - k^{(N)})$, we may assume that $k = m$. In this case, we can write P as

$$\begin{aligned} P^{(\mu, \nu)}(z, \tau; \partial_z, \partial_{\tau}) &= \sum_{0 \leq \alpha \leq m^{(\nu)}} P_{\alpha}^{1, (\mu, \nu)}(z, \tau; \partial_z) \tau^{\alpha} \partial_{\tau}^{\alpha} \\ &+ \sum_{0 \leq \alpha \leq m^{(\nu)}} P_{\alpha}^{2, (\mu, \nu)}(z, \tau; \partial_z) \tau^{\alpha + 1} \partial_{\tau}^{\alpha}, \end{aligned}$$

and for simplicity set

$$\mathcal{I}_P(z; \zeta; \beta) := \mathcal{I}_P(z; \zeta; \beta, \dots, \beta).$$

Moreover, P can be written as $P(z, \tau, \partial_z, \partial_{\tau}) = Q + Q' + A$, where

$$\begin{aligned} Q^{(\mu, \nu)} &= \sum_{0 \leq \alpha \leq m^{(\nu)}} Q_{\alpha}^{(\mu, \nu)}(z; \partial_z) \tau^{\alpha} \partial_{\tau}^{\alpha}, \\ Q'^{(\mu, \nu)} &= \sum_{0 \leq \alpha \leq m^{(\nu)}} \sum_{i=1}^d \tau_i Q'_{i, \alpha}^{(\mu, \nu)}(z, \tau; \partial_z) (\partial_{\tau} \tau)^{\alpha}, \\ A^{(\mu, \nu)} &= \sum_{0 \leq \alpha \leq m^{(\nu)}} \tau^{1_{\alpha}} A_{\alpha}^{(\mu, \nu)}(z, \tau; \partial_z) (\partial_{\tau} \tau)^{\alpha}. \end{aligned}$$

Here $Q_\alpha^{(\mu, \nu)}$, $Q'_{i,\alpha}^{(\mu, \nu)}$ and $A_\alpha^{(\mu, \nu)}$ are microdifferential operators with $\text{ord } Q_\alpha^{(\mu, \nu)}$, $\text{ord } Q'_{i,\alpha}^{(\mu, \nu)} \leq 0$ and $\text{ord } A_\alpha^{(\mu, \nu)} \leq |m^{(\nu)}| - |\alpha|$, and $(\partial_\tau \tau)^\alpha$ denotes $\prod_{j=1}^d (\partial_{\tau_j} \tau_j)^{\alpha_j}$. Further, Q satisfies [A-1]. Indeed, we have only to remark the following:

(1) We can write

$$P_\alpha^{1,(\mu, \nu)}(z, \tau; \partial_z) = P_\alpha^{1,(\mu, \nu)}(z, 0; \partial_z) + \sum_{i=1}^d \tau_i Q'_{i,\alpha}{}''^{(\mu, \nu)}(z, \tau; \partial_z),$$

with $\text{ord } P_\alpha^{1,(\mu, \nu)}$, $\text{ord } Q'_{i,\alpha}{}''^{(\mu, \nu)} \leq 0$.

(2) Since $\text{ord } P_m^{2,(\mu, \nu)} \leq 0$, we can regard $\tau^{1d} P_m^{2,(\mu, \nu)}(z, \tau; \partial_z)$ as included in $\sum_{i=1}^d \tau_i Q'_{i,\alpha}{}''^{(\mu, \nu)}(z, \tau; \partial_z)$.

(3) We have

$$\tau_i \tau_j \partial_{\tau_i} u = \begin{cases} \tau_i \partial_{\tau_i}(\tau_j u) & (i \neq j), \\ (\tau_i \partial_{\tau_i} - 1)(\tau_j u) & (i = j), \end{cases}$$

and

$$\tau_i^j \partial_{\tau_i}^j = (\partial_{\tau_i} \tau_i - 1) \cdots (\partial_{\tau_i} \tau_i - j).$$

Let $\mathcal{O}(B(R); X_s)$ be the set of X_s -valued holomorphic functions defined on $B(R)$. Then, we easily see that

$$\begin{aligned} \mathcal{O}(B(R); X_s) &= \{f(z, \tau) \in \mathcal{O}(\Omega_s \times B(R)); 0 < \forall R' < R, \\ &\quad \sup\{|f(z, \tau)| d'_s(z)^{r_0}; z \in \Omega_s, \tau \in B(R')\} < \infty\}. \end{aligned}$$

Therefore it follows that for any $0 < s < 1$

$$\mathcal{O}(\Omega \times B(R)) \subset \mathcal{O}(B(R); X_s).$$

Hence Theorem 1.3 is reduced to the following theorem:

2.1. Theorem. *Let P be a matrix of microdifferential operators of Fuchsian type with weight (m, m) . Assume that P satisfies [A-1]. Then, there exist constants $r_0 > 0$ and \tilde{R} with $0 < \tilde{R} \leq R$ such that the following hold:*

Take arbitrary h and r such that $0 < h < h_0$ and $0 < r < r_0$ respectively. Let Ω be any h - Σ -flat open convex subset of V with $\text{dia } \Omega \leq r$. Then, there exists a constant δ such that for any s_0 and \tilde{R} with $0 < s_0 < 1$ and $0 < \tilde{R} \leq \tilde{R}$ it follows that for any $f(z, \tau) = {}^t(f_1(z, \tau), \dots, f_N(z, \tau)) \in \mathcal{O}(B(\tilde{R}); X_{s_0})^{\oplus N}$, there exists a unique

holomorphic solution $u(z, \tau) = (u_1(z, \tau), \dots, u_N(z, \tau))$ of the Goursat problem

(G.P.) $P_\Sigma u = f.$

Moreover, for any s with $0 < s < s_0$, each $u_\nu(z, \tau)$ ($1 \leq \nu \leq N$) is holomorphic on $\Omega_s \times B(R')$, where R' is an arbitrary vector which satisfies $0 < R' < \tilde{R}$ and $\prod_{j=1}^d R'_j < \delta(s_0 - s)^{|m|}$.

In fact, if we prove Theorem 2.1, we see that $u(z, \tau)$ is holomorphic on

$$\begin{aligned} & \bigcup_{\substack{0 < s_0 < 1 \\ 0 < R' < \tilde{R}}} \bigcup_{0 < s < s_0} \left(\Omega_s \times \left\{ \tau \in B(R'); \prod_{j=1}^d |\tau_j| < \delta(s_0 - s)^{|m|} \right\} \right) \\ &= \bigcup_{0 < s < 1} \left(\Omega_s \times \left\{ \tau \in B(\tilde{R}); \prod_{j=1}^d |\tau_j| < \delta(1 - s)^{|m|} \right\} \right) \end{aligned}$$

by the uniqueness. Hence we obtain Theorem 1.3.

Let $u(T) = \sum_{\alpha \in \mathbb{N}_0^d} u_\alpha T^\alpha$ and $U(T) = \sum_{\alpha \in \mathbb{N}_0^d} U_\alpha T^\alpha$ be formal power series with indeterminaties $T = (T_1, \dots, T_d)$, where $u_\alpha \in X_s$ and $U_\alpha \geq 0$. We say that $U(T)$ is a majorant of $u(T)$ if $\|u_\alpha\| \leq U_\alpha$ for any α . In this case, we write $u(T) \ll U(T)$.

Define

$$E(B(R); X_s) := \left\{ u = \sum_{\alpha \in \mathbb{N}_0^d} u_\alpha T^\alpha \in X_s[[T]]; \exists C > 0, u(T) \ll C\Phi_R(T) \right\},$$

where $\Phi_R(T)$ denotes $\prod_{j=1}^d \frac{R_j}{R_j - T_j}$. Then, we see the following (see Wagschal [W] for details):

1. $E(B(R); X_s)$ becomes a Banach space under the norm

$$\|u\|_{R,s} := \inf \{ C; u(T) \ll C\Phi_R(T) \};$$

2. if $0 < R \leq R'$ and $0 < s \leq s' < 1$, then it follows that $E(B(R'); X_{s'}) \subset E(B(R); X_s)$ and

$$\|u\|_{R,s} \leq \|u\|_{R',s'};$$

3. as a set

$$\bigcap_{0 < R' < R} E(B(R'); X_s) = \mathcal{O}(B(R); X_s).$$

Hereafter, we also use the notation

$$u(z, \tau) = \sum_{\alpha \in \mathbb{N}_0^d} u_\alpha(z) \tau^\alpha$$

for

$$u(z, T) = \sum_{\alpha \in \mathbb{N}_0^d} u_\alpha(z) T^\alpha \in E(B(R); X_s).$$

We prove some results about Bony-Schapira's action on $E(B(R); X_s)$:

2.2. Proposition. *Let $A(z, \tau; \partial_z)$ be a microdifferential operator of order at most ν with holomorphic parameters τ defined in a neighborhood of $[U \times B(R)]$. Then, there exists a constant $r_0 > 0$ such that the following hold:*

Take arbitrary h and r such that $0 < h < h_0$ and $0 < r < r_0$ respectively. Let Ω be any h - Σ -flat open convex subset of V with $\text{dia } \Omega \leq r$. Then, there exists a constant C_A such that for any s' and R' with $0 < s' < 1$ and $0 < R' \leq R$ respectively,

(1) *if $\nu > 0$, then for any $0 < s < s'$ and $u \in E(B(R'); X_{s'})$ it follows that*

$$\|A_\Sigma u\|_{R', s} \leq \frac{C_A}{(s' - s)^\nu} \|u\|_{R', s'};$$

(2) *if $\nu \leq 0$, then for any $0 \leq s < s'$ and $u \in E(B(R'); X_{s'})$ it follows that*

$$\|A_\Sigma u\|_{R', s} \leq C_A \|u\|_{R', s'}.$$

Proof. (1) Since A is defined in a neighborhood of $[U \times B(R)]$, we can find a relatively compact open set U' and a vector R_0 such that $[U] \Subset U'$, $R < R_0$ and A is defined in a neighborhood of $[U' \times B(R_0)]$. Thus, there exist positive constants K and M such that for any $j \leq \nu$

$$\sup\{|A_j(z, \tau; \zeta)|; (z, \zeta) \in [U'], \tau \in [B(R_0)]\} \leq KM^{\nu-j}(\nu-j),$$

where $\sum_{j \leq \nu} A_j(z, \tau; \zeta)$ denotes the total symbol of $A(z, \tau; \partial_z)$ and each $A_j(z, \tau; \zeta)$ is homogeneous of degree j with respect to ζ -variables. Thus by the Taylor expansion we can write

$$A_j(z, \tau; \zeta) = \sum_{\beta \in \mathbb{N}_0^d} \tau^\beta A_j^{(\beta)}(z; \zeta).$$

Therefore, by Cauchy's inequality for any $j \leq v$ we have

$$\sup\{|A_j^{(\beta)}(z; \zeta)|; (z; \zeta) \in [U']\} \leq \frac{KM^{v-j}}{R_0^\beta} (v-j)!.$$

Set

$$A^{(\beta)}(z; \partial_z) := \sum_{j \leq v} A_j^{(\beta)}(z; \partial_z).$$

Then, we easily see that

$$A(z, \tau; \partial_z) = \sum_{\beta \in \mathbb{N}_0^d} \tau^\beta A^{(\beta)}(z; \partial_z),$$

$$A^{(\beta)}(z; \partial_z) \in I([U']; \mathcal{E}_{C^n}(v)).$$

Note that taking $0 < r_0 < 1/M$, we can assume that Bony-Schapira's actions of all the microdifferential operators under consideration are well-defined under the assumption of Proposition 2.2 since we can take a constant M in common (see the argument of the proof of Lemma 2.3 below). We remark the following lemma:

2.3. Lemma. (cf. Oaku [O2]). *There exists a constant C such that for any $0 < s < s' < 1$, $\beta \in \mathbb{N}_0^d$ and $v \in X_{s'}$, it follows that*

$$\|A^{(\beta)}_{\Sigma} v\|_s \leq \frac{C}{(s' - s)^v R_0^\beta} \|v\|_{s'}.$$

Proof of Lemma 2.3. Since $\text{ord } \partial_{z_1}^{-v} A^{(\beta)} \leq 0$, we have

$$A^{(\beta)}_{\Sigma} = \partial_{z_1}^v (\partial_{z_1}^{-v} A^{(\beta)})_{\Sigma}$$

by Proposition 2.1.2 of Bony-Schapira [Bo-Sc]. Denote the total symbol of $\partial_{z_1}^{-v} A^{(\beta)}$ by $\sum_{j \geq 0} B_{-j}^{(\beta)}(z; \zeta)$ where each $B_{-j}^{(\beta)}(z; \zeta)$ is homogeneous of degree $-j$ with respect to ζ -variables. Then we may assume that for any $j \geq 0$

$$\sup\{|B_{-j}^{(\beta)}(z; \zeta)|; (z, \zeta) \in [U]\} \leq \frac{KM^j}{R_0^\beta} j!.$$

Thus by the proof of Proposition 2.4.3 of [Bo-Sc] we can find a constant $C' > 0$ (not depending on s and s') such that for any $z \in \Omega_s$ and $\beta \in N_0^d$

$$|(\partial_{z_1}^{-\nu} A^{(\beta)})_{\Sigma} v(z)| \leq \frac{KC'}{d'_s(z)^{c_0} R_0^\beta} \|v\|_{s'}.$$

Hence, setting $d_s^1(z) := \inf\{|z_1 - w_1|; w \in C^n \setminus \Omega_{s'}, z_j = w_j (2 \leq j \leq n)\}$ and $z' := (z_2, \dots, z_n)$ we have

$$\begin{aligned} |\partial_{z_1}^{\nu} (\partial_{z_1}^{-\nu} A^{(\beta)})_{\Sigma} v(z)| &= \left| \frac{\nu!}{2\pi\sqrt{-1}} \oint_{|z_1 - w_1| = d_s^1(z)} \frac{(\partial_{z_1}^{-\nu} A^{(\beta)})_{\Sigma} v(w_1, z')}{(w_1 - z_1)^{\nu+1}} dw_1 \right| \\ &\ll \left(\frac{2}{d_s^1(z)}\right)^{\nu} \nu! \sup\{|\partial_{z_1}^{-\nu} A^{(\beta)} v(w_1, z')|; |z_1 - w_1| = d_s^1(z)\} \\ &\ll \left(\frac{2}{d_s^1(z)}\right)^{\nu} \frac{KC' \nu!}{R_0^\beta} \|v\|_{s'} \sup\{d'_s(w_1, z')^{-c_0}; |z_1 - w_1| = d_s^1(z)\}. \end{aligned}$$

Setting $\tilde{w}_1 := z_1 + 2(w_1 - z_1)$ for any w with $|z_1 - w_1| = d_s^1(z)$ we have $(\tilde{w}_1, z') \in [\Omega_{s'}]$. If $2|z_j - \tilde{w}_j| \leq d'_s(z)$ ($2 \leq j \leq n$), then we have $(z_1, z' + 2(\tilde{w}' - z')) \in [\Omega_{s'}]$. Therefore by the convexity of $[\Omega_{s'}]$, we have $(w_1, \tilde{w}') \in [\Omega_{s'}]$. Thus it follows that $2d'_s(w_1, z') \geq d'_s(z)$. Hence

$$\begin{aligned} |A^{(\beta)}_{\Sigma} v(z)| &= |\partial_{z_1}^{\nu} (\partial_{z_1}^{-\nu} A^{(\beta)})_{\Sigma} v(z)| \\ &\leq \frac{2^{\nu+c_0} KC' \nu!}{d'_s(z)^{c_0} d_s^1(z)^{\nu} R_0^\beta} \|v\|_{s'}. \end{aligned}$$

Setting $c' := \text{dis}(z_0, C^n \setminus \Omega)$, we easily see that $\text{dis}(\Omega_s, C^n \setminus \Omega_{s'}) \geq c'(s' - s)$. Therefore it follows that for any $z \in \Omega_s$

$$\begin{aligned} |A^{(\beta)}_{\Sigma} v(z)| &\leq \frac{2^{\nu+c_0} KC' \nu!}{d'_s(z)^{c_0} (c'(s' - s))^{\nu} R_0^\beta} \|v\|_{s'} \\ &\leq \frac{2^{\nu+c_0} KC' \nu!}{d'_s(z)^{c_0} (c'(s' - s))^{\nu} R_0^\beta} \|v\|_{s'}. \end{aligned}$$

Hence we may take

$$C := \frac{2^{v+c_0} K C' v!}{(c')^v}.$$

□

End of Proof of Proposition 2.2. By definition, for any $u(z, \tau) = \sum_{\gamma \in \mathbb{N}_0^d} u_\gamma(z) \tau^\gamma \in E(B(R); X_{s'})$ we have

$$A_\Sigma u(z, \tau) = \sum_{\beta, \gamma \in \mathbb{N}_0^d} (A^{(\beta)}_\Sigma u_\gamma)(z) \tau^{\beta+\gamma}$$

formally. Hence remarking

$$\Phi_{R_0}(T) \Phi_R(T) \ll \Phi_{R_0}(R) \Phi_R(T) \ll \Phi_{R_0}(R) \Phi_R(T),$$

by Lemma 2.3 we have

$$\begin{aligned} A_\Sigma u(z, T) &= \sum_{\beta, \gamma \in \mathbb{N}_0^d} (A^{(\beta)}_\Sigma u_\gamma)(z) T^{\beta+\gamma} \\ &\ll \sum_{\beta \in \mathbb{N}_0^d} \frac{C T^\beta}{(s'-s)^v R_0^\beta} \sum_{\gamma \in \mathbb{N}_0^d} \|u_\gamma\|_{s'} T^\gamma \\ &\ll \frac{C \Phi_{R_0}(T)}{(s'-s)^v} \|u\|_{R', s'} \Phi_R(T) \\ &\ll \frac{C \Phi_{R_0}(R)}{(s'-s)^v} \|u\|_{R', s'} \Phi_R(T). \end{aligned}$$

Thus we may set

$$C_A := C \Phi_{R_0}(R).$$

Proof of (2) is similar. □

2.4. Remark. For the argument to the proof of Proposition 2.2, we may assume that if we take $r_0 > 0$ small enough, then Bony-Schapira's actions of finite numbers of microdifferential operators under consideration are well-defined on Ω under the following assumption:

Ω is an h - Σ -flat open convex subset of V with $\text{dia } \Omega \leq r$ for some h and r such that $0 < h < h_0$ and $0 < r < r_0$.

Thus hereafter, we often omit the argument that Bony-Schapira's action

of a microdifferential operator in question is well-defined.

§3. Proof of Main Theorem

In this section, we prove Theorem 2.1. Note that our method of proof is similar to that of [M] and [O2]. We inherit the notation and assumption of preceding section.

First, we prove the following:

3.1. Proposition. *There exists a constant $r_0 > 0$ such that for any constants h and r with $0 < r < r_0$ and $0 < h < h_0$, the following hold:*

Let Ω be any h - Σ -flat open convex subset of V such that $\text{dia } \Omega \leq r$. Then there exists a linear operator \mathcal{H}_Q such that

- (1) *for any R', s and α such that $0 < R' \leq R$, $0 < s < 1$ and $0 \leq \alpha^{(v)} \leq m^{(v)}$ ($1 \leq v \leq N$)*

$$\begin{pmatrix} (\partial_\tau \tau)^{\alpha^{(1)}} & & 0 \\ & \ddots & \\ 0 & & (\partial_\tau \tau)^{\alpha^{(N)}} \end{pmatrix} \cdot \mathcal{H}_Q : E(B(R'); X_s)^{\oplus N} \rightarrow E(B(R'); X_s)^{\oplus N}.$$

Further there exists constant C_Q (not depending on R', s and $\alpha^{(v)}$) such that

$$(3.1) \quad \left\| \begin{pmatrix} (\partial_\tau \tau)^{\alpha^{(1)}} & & 0 \\ & \ddots & \\ 0 & & (\partial_\tau \tau)^{\alpha^{(N)}} \end{pmatrix} \cdot \mathcal{H}_Q f \right\|_{R',s} \leq C_Q \|f\|_{R',s};$$

- (2) $Q_\Sigma \cdot \mathcal{H}_Q = \mathcal{H}_Q \cdot Q_\Sigma = \text{id} : E(B(R); X_s)^{\oplus N} \rightarrow E(B(R); X_s)^{\oplus N}.$

Proof. Take a relatively open set W' such that $[W] \in W'$ and Q is defined on $[W']$. Shrinking W and W' if necessary, we may assume that for any $(z; \zeta) \in [W']$ and $\beta \in N_0^d$

$$(3.2) \quad |\mathcal{I}_Q(z; \zeta; \beta)| \geq C \prod_{v=1}^N (\beta + \mathbf{1}_d)^{m^{(v)}}.$$

There exist positive constants K and M such that for all $j \geq 0$, $1 \leq \mu, v \leq N$ and $0 \leq \alpha \leq m^{(v)}$

$$(3.3) \quad \sup\{ |Q_{\alpha, -}^{(\mu, v)}(z; \zeta)|; (z; \zeta) \in [W'] \} \leq KM^j j!,$$

where $\sum_{j \geq 0} Q_{\alpha, -j}^{(\mu, \nu)}(z; \zeta)$ denotes the total symbol of $Q_{\alpha}^{(\mu, \nu)}$ and each $Q_{\alpha, -j}^{(\mu, \nu)}(z; \zeta)$ is homogeneous of degree $-j$ with respect to ζ -variables. For any $f(z, \tau)$, $u(z, \tau) \in E(B(R); X_s)^{\oplus N}$, we write $f(z, \tau) = \sum_{\beta} f_{\beta}(z) \tau^{\beta}$ and $u(z, \tau) = \sum_{\beta} u_{\beta}(z) \tau^{\beta}$ respectively. For any $\beta \in N_0^d$ we define $\tilde{Q}_{\beta}(z; \partial_z) \in \text{Mat}(N; \Gamma([W']; \mathcal{E}_{C^n}(0)))$ by

$$(3.4) \quad \tilde{Q}_{\beta}^{(\mu, \nu)}(z; \partial_z) := \sum_{0 \leq \alpha \leq m^{(\nu)}} Q_{\alpha}^{(\mu, \nu)}(z; \partial_z) \mathcal{I}_{\alpha}(\beta) \in \Gamma([W']; \mathcal{E}_{C^n}(0)).$$

Then, the formal calculation implies that

$$(3.5) \quad (Q(z, \tau; \partial_z, \partial_{\tau})_{\Sigma} u(z, \tau))_{\mu} = \sum_{\beta \in N_0^d} \sum_{\nu=1}^N \tilde{Q}_{\beta}^{(\mu, \nu)}(z; \partial_z)_{\Sigma} u_{\nu, \beta}(z) \tau^{\beta}.$$

Thus the equation $Q_{\Sigma} u = f$ is equivalent to

$$\sum_{\nu=1}^N \tilde{Q}_{\beta}^{(\mu, \nu)}_{\Sigma} u_{\nu, \beta} = f_{\mu, \beta}$$

for any μ and β . Since $\det(\sigma_0(\tilde{Q}_{\beta}^{(\mu, \nu)})(z; \zeta)) = \mathcal{I}_Q(z; \zeta; \beta)$, the assumption (3.2) implies that each $\tilde{Q}_{\beta}(z; \partial_z)$ is invertible on $[W']$. Set $S_{\beta}(z; \partial_z) := \tilde{Q}_{\beta}(z; \partial_z)^{-1} \in \text{Mat}(N; \Gamma([W']; \mathcal{E}_{C^n}(0)))$.

3.2. Lemma. *There exist constants K_1 and M_1 such that for all $j \geq 0$, $1 \leq \mu, \nu \leq N$ and $\beta \in N_0^d$*

$$(3.6) \quad \sup\{ |S_{\beta, -j}^{(\mu, \nu)}(z; \zeta)|; (z; \zeta) \in [W'] \} \leq \frac{K_1}{(\beta + \mathbf{1}_d)^{m^{(\mu)}}} M_1^j j!.$$

Here $\sum_{j \geq 0} S_{\beta, -j}^{(\mu, \nu)}(z; \zeta)$ denotes the total symbol of $S_{\beta}^{(\mu, \nu)}(z; \partial_z)$ and each $S_{\beta, -j}^{(\mu, \nu)}(z; \zeta)$ is homogeneous of degree $-j$ with respect to ζ -variables.

Proof of Lemma 3.2. Set $\tilde{R}_{\beta}(z; \zeta) := \sigma_0(\tilde{Q}_{\beta})(z; \zeta)^{-1}$. There exists a constant $\tilde{C} > 0$ such that for any $0 \leq \alpha \leq m^{(\nu)}$ and $\beta \in N_0^d$

$$(3.7) \quad |\mathcal{I}_{\alpha}(\beta)| \leq \tilde{C}(\beta + \mathbf{1}_d)^{m^{(\nu)}}.$$

Let $\Delta^{(\mu, \nu)}(z; \zeta)$ be the $(\mu; \nu)$ -th cofactor of $\sigma_0(\tilde{Q}_{\beta})(z; \zeta)$. Then by (3.4) and (3.7) there exists a constant $C' > 0$ such that

$$\sup\{|\Delta^{(\mu, \nu)}(z; \zeta)|; (z; \zeta) \in [W']\} \leq \prod_{l \neq \nu} C'(\beta + \mathbf{1}_d)^{m^{(l)}}.$$

Hence by the assumption (3.2),

$$(3.8) \quad \sup\{|\tilde{R}_\beta^{(\mu, \nu)}(z; \zeta)|; (z; \zeta) \in [W']\} \leq \frac{C'}{C(\beta + \mathbf{1}_d)^{m^{(\mu)}}}.$$

Now, for a microdifferential operator $A(z; \partial_z)$ of order zero defined in a neighborhood of $[W]$ with the total symbol $\sum_{j \geq 0} A_{-j}(z; \partial_z)$ where each $A_{-j}(z; \zeta)$ is homogeneous of degree $-j$ with respect to ζ -variables, $N_0^{[W]}(A; \rho)$ denotes the formal norm in the sense of L. Boutet de Monvel and P. K ree [Bou-K]; that is,

$$N_0^{[W]}(A; \rho) := \sum_{\substack{0 \leq j \\ \tilde{\alpha}, \tilde{\beta} \in N_0^d}} \frac{2(2n)^{-j} j!}{(|\tilde{\alpha}| + j)! (|\tilde{\beta}| + j)!} \sup\{|\partial_z^{\tilde{\alpha}} \partial_\zeta^{\tilde{\beta}} A_{-j}(z; \zeta)|; (z; \zeta) \in [W]\} \rho^{2j + |\tilde{\alpha}| + |\tilde{\beta}|},$$

where ρ is a parameter.

3.3. Sublemma. *There exist positive constants L_1 and c_1 such that*

$$N_0^{[W]}(\tilde{R}_\beta^{(\mu, \nu)}; \rho) \ll \sum_{j=0}^{\infty} \frac{L_1}{(\beta + \mathbf{1}_d)^{m^{(\mu)}}} \left(\frac{\rho}{c_1}\right)^j.$$

Proof of Sublemma 3.3. By Cauchy's inequality and (3.8), we have

$$\sup\{|\partial_z^{\tilde{\alpha}} \partial_\zeta^{\tilde{\beta}} \tilde{R}_\beta^{(\mu, \nu)}(z; \zeta)|; (z; \zeta) \in [W]\} \leq \frac{C' \tilde{\alpha}! \tilde{\beta}!}{C(\beta + \mathbf{1}_d)^{m^{(\mu)}}} \left(\frac{1}{c'}\right)^{|\tilde{\alpha}| + |\tilde{\beta}|},$$

where c' is a some constant depending only on W and W' . Thus it follows that

$$\begin{aligned} N_0^{[W]}(\tilde{R}_\beta^{(\mu, \nu)}; \rho) &\ll \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{2 \tilde{\alpha}! \tilde{\beta}!}{|\tilde{\alpha}!| |\tilde{\beta}!|} \left(\frac{\rho}{c'}\right)^{|\tilde{\alpha}| + |\tilde{\beta}|} \frac{C'}{C(\beta + \mathbf{1}_d)^{m^{(\mu)}}} \\ &\ll \sum_{j=0}^{\infty} \binom{j+2n-1}{j} \left(\frac{\rho}{c'}\right)^j \frac{2C'}{C(\beta + \mathbf{1}_d)^{m^{(\mu)}}} \\ &\ll \sum_{j=0}^{\infty} \left(\frac{2\rho}{c'}\right)^j \frac{2^{2n} C'}{C(\beta + \mathbf{1}_d)^{m^{(\mu)}}}. \end{aligned}$$

Thus we may set $L_1 := 2^{2n}C'/C$ and $c_1 := c'/2$. □

Proof of Lemma 3.2 continued. Next, set

$$R_\beta(z; \partial_z) := I_N - \tilde{Q}_\beta(z; \partial_z) \tilde{R}_\beta(z; \partial_z) \in \text{Mat}(N; I([\mathcal{W}']; \mathcal{E}_{\mathbf{c}^n}(-1))),$$

where I_N denotes the identity matrix of size $N \times N$. Therefore denoting the total symbol of $R_\beta^{(\mu, \nu)}(z; \partial_z)$ by $\sum_{j>0} R_{\beta, -j}^{(\mu, \nu)}(z; \zeta)$ where each $R_{\beta, -j}^{(\mu, \nu)}(z; \zeta)$ is homogeneous of degree $-j$ with respect to ζ -variables, for any $j > 0$ we have

$$R_{\beta, -j}^{(\mu, \nu)}(z; \zeta) = - \sum_{\substack{j=p+|\gamma| \\ p \geq 0, |\gamma| > 0}} \sum_{l=1}^N \frac{1}{\gamma!} \partial_\zeta^\gamma \left(\sum_{0 \leq \alpha \leq m^{(l)}} Q_{\alpha, -p}^{(\mu, l)}(z; \zeta) \mathcal{I}_\alpha(\beta) \right) \cdot \partial_z^\gamma \tilde{R}_\beta^{(l, \nu)}(z; \zeta).$$

There exist constants $K' > 0$ and $L' > 0$ such that for all $j \geq 0$

$$(3.9) \quad \sup\{|\sum_{0 \leq \alpha \leq m^{(v)}} Q_{\alpha, -j}^{(\mu, v)}(z; \zeta)|; (z; \zeta) \in [\mathcal{W}']\} \leq K' M^j j!.$$

3.4. Sublemma. *There exist positive constants L_2 and c_2 such that*

$$N_0^{|\mathcal{W}|}(R_\beta^{(\mu, \nu)}; \rho) \ll L_2 \sum_{p=2}^\infty \left(\frac{\rho}{c_2}\right)^p.$$

Proof of Sublemma 3.4. Take $W \in W'' \in W'$. By (3.8), (3.9) and Cauchy's inequality we have

$$\begin{aligned} \sup\{|\partial_z^\gamma \tilde{R}_\beta^{(\mu, \nu)}(z; \zeta)|; (z; \zeta) \in [\mathcal{W}'']\} &\leq \frac{C' \gamma!}{C(\beta + \mathbf{1}_d)^{m^{(u)}}} \left(\frac{1}{c''}\right)^{|\gamma|}, \\ \sup\{|\partial_\zeta^\gamma \left(\sum_{0 \leq \alpha \leq m^{(v)}} Q_{\alpha, -j}^{(\mu, v)}\right)(z; \zeta)|; (z; \zeta) \in [\mathcal{W}'']\} &\leq \gamma! \left(\frac{1}{c''}\right)^{|\gamma|} K' M^j j!. \end{aligned}$$

Thus it follows that for any $j > 0$

$$\begin{aligned} \sup\{|\sum_{\substack{j=p+|\gamma| \\ p \geq 0}} R_{\beta, -j}^{(\mu, \nu)}(z; \zeta)|; (z; \zeta) \in [\mathcal{W}'']\} &\leq \sum_{\substack{j=p+|\gamma| \\ p \geq 0}} \gamma! \left(\frac{1}{(c'')^2}\right)^{|\gamma|} NK' L_1 M^p p! \\ &\leq NK' L_1 \sum_{\substack{j=p+|\gamma| \\ p \geq 0}} \left(\frac{M}{(c'')^2}\right)^j j! \end{aligned}$$

$$\leq 2^{n-1}NK'L_1 \left(\frac{2M}{(c'')^2}\right)^j j!.$$

Therefore, setting $L'' := 2^{n-1}NK'L_1$ we have

$$\sup\{|\partial_z^{\tilde{\alpha}} \partial_{\zeta}^{\tilde{\beta}} R_{\beta}^{(\mu, \nu)}(z; \zeta); (z; \zeta) \in [W]\} \leq \tilde{\alpha}! \tilde{\beta}! L'' \left(\frac{1}{c'''}\right)^{|\tilde{\alpha} + \tilde{\beta}|} \left(\frac{2M}{(c'')^2}\right)^j j!.$$

Since we may assume that $c''' \leq 1$ and $(c'')^2 \leq 2M$, we obtain

$$\begin{aligned} N_0^{[W]}(R_{\beta}^{(\mu, \nu)}; \rho) &\ll \sum_{\substack{\tilde{\alpha}, \tilde{\beta} \geq 0 \\ j \geq 1}} \frac{L'' \tilde{\alpha}! \tilde{\beta}! (j!)^2}{n(|\tilde{\alpha}| + j)! (|\tilde{\beta}| + j)!} \left(\frac{1}{c'''}\right)^{|\tilde{\alpha} + \tilde{\beta}|} \left(\frac{2M}{(c'')^2}\right)^j \rho^{2j + |\tilde{\alpha} + \tilde{\beta}|} \\ &\ll \frac{L''}{n} \sum_{p = |\tilde{\alpha} + \tilde{\beta}| + 2j \geq 2} \left(\frac{1}{c'''}\right)^{|\tilde{\alpha} + \tilde{\beta}|} \left(\frac{2M}{(c'')^2}\right)^j \rho^p \\ &\ll \frac{2^{2n} L''}{n} \sum_{p=2}^{\infty} \left(\frac{2M}{(c'')^2 c'''}\right)^p \rho^p. \end{aligned}$$

Thus we may set $L_2 := 2^{2n} L'' / n$ and $c_2 := (c'')^2 c''' / (2M)$. □

End of Proof of Lemma 3.2. By Sublemmas 3.3 and 3.4, we see that if ρ_0 is a sufficiently small positive constant, then

$$(3.10) \quad \begin{cases} N_0^{[W]}(\tilde{R}_{\beta}^{(\mu, \nu)}; \rho_0) \leq \frac{2L}{(\beta + \mathbb{1}_d)^{m(\mu)}}, \\ N_0^{[W]}(R_{\beta}^{(\mu, \nu)}; \rho_0) \leq \frac{1}{2N}. \end{cases}$$

Since we see that $S_{\beta} = \sum_{j=0}^{\infty} \tilde{R}_{\beta} R_{\beta}^j$, setting $K_1 := 4L$, by (3.10) we have

$$N_0^{[W]}(S_{\beta}; \rho_0) \leq \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \frac{2L}{(\beta + \mathbb{1}_d)^{m(\mu)}} = \frac{K_1}{(\beta + \mathbb{1}_d)^{m(\mu)}}.$$

Thus setting $M_1 := 2n / (\rho_0^2)$ we have

$$\sup\{|S_{\beta}^{(\mu, \nu)}(z; \zeta); (z; \zeta) \in [W]\} \leq \frac{K_1}{(\beta + \mathbb{1}_d)^{m(\mu)}} \left(\frac{2n}{\rho_0^2}\right)^j j!$$

$$= \frac{K_1}{(\beta + \mathbf{1}_d)^{m(\mu)}} M_1^j j^l.$$

Hence the proof of Lemma 3.2 is completed. □

End of Proof of Proposition 3.1. Let us take $r_0 \leq \min\{M^{-1}, M_1^{-1}\}$. Then, by Lemma 3.2 we see that Q_{β_Σ} and S_{β_Σ} are well-defined on Ω with $\text{dia } \Omega < r_0$. Moreover, the equation $\sum_{v=1}^N \tilde{Q}_\beta^{(\mu, v)}_\Sigma u_{v, \beta} = f_{\mu, \beta}$ is equivalent to $u_{\mu, \beta} = \sum_{v=1}^N S_\beta^{(\mu, v)}_\Sigma f_{v, \beta}$. Thus if we set \mathcal{H}_Q by

$$(\mathcal{H}_Q f)_\mu(z, \tau) := \sum_{v=1}^N \sum_{\beta \in \mathbb{N}_0^d} S_\beta^{(\mu, v)}_\Sigma f_{v, \beta}(z) \tau^\beta,$$

then we have

$$(\partial_\tau \tau)^{\alpha(\mu)} (\mathcal{H}_Q f)_\mu(z, \tau) = \sum_{\beta \in \mathbb{N}_0^d} (\beta + \mathbf{1}_d)^{\alpha(\mu)} S_\beta^{(\mu, v)}_\Sigma f_{v, \beta}(z) \tau^\beta.$$

Thus we easily see that all the properties are satisfied. □

Next, we prove the following:

3.5. Proposition. *There exist constants $r_0 > 0$ and \mathring{R} with $0 < \mathring{R} \leq R$ such that for any constants h and r with $0 < r < r_0$ and $0 < h < h_0$, the following hold:*

Let Ω be any h - Σ -flat open convex subset of V such that $\text{dia } \Omega \leq r$. Then, for any R' and s such that $0 < R' \leq \mathring{R}$ and $0 < s < 1$, the following equation is uniquely solvable on $E(B(R'); X_s)$:

$$(3.11) \quad (Q + Q')_\Sigma u = f.$$

Precisely, the operator $(Q + Q')_\Sigma$ is invertible on $E(B(R'); X_s)$.

Proof. We use the notation of the proof of Proposition 3.1. For any $f, u \in E(B(R); X_s)^{\oplus N}$, we write $f(z, \tau) = \sum_{\beta} f_{\beta}(z) \tau^\beta$ and $u(z, \tau) = \sum_{\beta} u_{\beta}(z) \tau^\beta$ respectively.

Thus, by Proposition 3.1, the equation (3.11) is equivalent to

$$(3.12) \quad (I + Q'_\Sigma \mathcal{H}_Q) Q'_\Sigma u = f.$$

Set $v := Q'_\Sigma u \in E(B(R); X_s)^{\oplus N}$. Since

$$(Q'_\Sigma \mathcal{H}_Q v)_\mu = \sum_{l,v=1}^N \sum_{0 \leq \alpha \leq m^{(l)}} \sum_{i=1}^d \tau_i Q'^{(\mu,l)}_{i,\alpha} (\partial_i \tau)^\alpha \mathcal{H}_Q^{(l,v)} v_v,$$

by Propositions 2.2 and 3.1, there exist constants C_Q and $C_{i,\alpha}^{(l,v)}$ such that

$$\|(Q'_\Sigma \mathcal{H}_Q v)_\mu\|_{R',s} \leq C_Q \sum_{l,v=1}^N \sum_{0 \leq \alpha \leq m^{(l)}} \sum_{i=1}^d R'_i C_{i,\alpha}^{(l,v)} \|v_v\|_{R',s}.$$

Thus, if we choose \mathring{R} as small as

$$C_Q \sum_{\mu,v=1}^N \sum_{0 \leq \alpha \leq m^{(\mu)}} \sum_{i=1}^d R'_i C_{i,\alpha}^{(\mu,v)} \leq C_Q \sum_{\mu,v=1}^N \sum_{0 \leq \alpha \leq m^{(\mu)}} \sum_{i=1}^d \mathring{R}_i C_{i,\alpha}^{(\mu,v)} < 1,$$

we can see that $I + Q'_\Sigma \mathcal{H}_Q$ is invertible on $E(B(R'); X_s)$. Hence we see that $\mathcal{H}_{Q+Q'} := \mathcal{H}_Q (I + Q'_\Sigma \mathcal{H}_Q)^{-1}$ is the inverse of $(Q + Q')_\Sigma$. \square

Now, we are ready to prove Theorem 2.1. Choose \mathring{R} as in Proposition 3.5 and take $0 < \tilde{R} \leq \mathring{R}$. Recall that the equation under consideration is the following:

$$(Q + Q' + A)_\Sigma u = f \in \mathcal{O}(B(\tilde{R}); X_{s_0})^{\oplus N}.$$

First we will prove the existence of a solution $u(z, \tau)$. By Proposition 3.5, the problem is equivalent to

$$u + \mathcal{H}_{Q+Q'} A_\Sigma u = \mathcal{H}_{Q+Q'} f,$$

(recall that $\mathcal{H}_{Q+Q'} = ((Q + Q')_\Sigma)^{-1}$). Set

$$(3.13) \quad u_p := \begin{cases} \mathcal{H}_{Q+Q'} f & (p=0), \\ \mathcal{H}_{Q+Q'} (f - A_\Sigma u_{p-1}) & (p \in \mathbb{N}). \end{cases}$$

Moreover, set

$$(3.14) \quad v_p := \begin{cases} \begin{pmatrix} (\partial_\tau \tau)^{m^{(1)}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\partial_\tau \tau)^{m^{(N)}} \end{pmatrix} u_0 & (p=0), \\ \begin{pmatrix} (\partial_\tau \tau)^{m^{(1)}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\partial_\tau \tau)^{m^{(N)}} \end{pmatrix} (u_p - u_{p-1}) & (p \in N). \end{cases}$$

Then we easily see that $v_0 \in E(B(R'); X_{s_0})^{\oplus N}$ and $v_p \in E(B(R'); X_s)^{\oplus N}$ ($p \in N$) for any $0 < s < s_0$ and $0 < R' < \tilde{R}$. Since each $(\partial_\tau \tau)^{m^{(v)}-\alpha}$ ($0 \leq \alpha < m^{(v)}$) satisfies the assumption of Proposition 3.1 in the case where $N=1$, we can write

$$(3.15) \quad (\partial_\tau \tau)^\alpha = \mathcal{H}^{m^{(v)}-\alpha} (\partial_\tau \tau)^{m^{(v)}}, \quad \text{where } \mathcal{H}^{m^{(v)}-\alpha} := \mathcal{H}_{(\partial_\tau \tau)^{m^{(v)}-\alpha}}.$$

Hence for $0 \leq \alpha < m^{(v)}$ we have $(\partial_\tau \tau)^\alpha v_{p,v} = \mathcal{H}^{m^{(v)}-\alpha} v_{p,v}$. Thus it follows that

$$(3.16) \quad \begin{aligned} v_{p,\mu} &= -(\partial_\tau \tau)^{m^{(\mu)}} \sum_{l,v=1}^N \sum_{0 \leq \alpha \leq m^{(v)}} \mathcal{H}_{Q+Q'}^{(\mu,l)} \tau^{1_a} A_\alpha^{(l,v)}(z, \tau; \partial_z)_\Sigma (\partial_\tau \tau)^\alpha v_{p-1,v} \\ &= -(\partial_\tau \tau)^{m^{(\mu)}} \sum_{l,v=1}^N \sum_{0 \leq \alpha \leq m^{(v)}} \mathcal{H}_{Q+Q'}^{(\mu,l)} \tau^{1_a} A_\alpha^{(l,v)}(z, \tau; \partial_z)_\Sigma \mathcal{H}^{m^{(v)}-\alpha} v_{p-1,v}. \end{aligned}$$

By Propositions 2.2 and 3.5 there exists a constant M_1 such that for any $0 < s \leq s_0$ and $0 < R'' \leq R'$ it follows that for any $v \in E(B(R''); X_s)^{\oplus N}$

$$(3.17) \quad \begin{aligned} & \left\| \left\| \begin{pmatrix} (\partial_\tau \tau)^{m^{(1)}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\partial_\tau \tau)^{m^{(N)}} \end{pmatrix} \mathcal{H}_{Q+Q'} v \right\|_{R',s} \right\| \\ &= \left\| \left\| \begin{pmatrix} (\partial_\tau \tau)^{m^{(1)}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\partial_\tau \tau)^{m^{(N)}} \end{pmatrix} \mathcal{H}_Q (I + Q'_\Sigma \mathcal{H}_Q)^{-1} v \right\|_{R'',s} \right\| \\ &\leq M_1 \|v\|_{R'',s}. \end{aligned}$$

We will prove that there exists a constant M_2 such that for any $0 < s < s_0$ and $0 < R'' \leq R'$ we have

$$(3.18)\text{-}p \quad \|v_p\|_{R'',s} \leq M_1 \|f\|_{R',s_0} \left(\frac{M_1 M_2 \prod_{j=1}^d R'_j}{(s_0 - s)^{|m|}} \right)^p.$$

Since $v_0 \in E(B(R'); X_{s_0})^{\oplus N}$, by (3.17) we have

$$\|v_0\|_{R'',s} \leq \|v_0\|_{R',s_0} \leq M_1 \|f\|_{R',s_0}.$$

Thus we have (3.18)–0. By Proposition 2.2, for any $s < s' < s_0$ and $0 \leq \alpha < m^{(\nu)}$ we have

$$(3.19) \quad \|A_\alpha^{(l,\nu)} \Sigma v_{p,\nu}\|_{R'',s} \leq \frac{C_\alpha^{(l,\nu)}}{(s' - s)^{|m^{(\nu)}| - |\alpha|}} \|v_{p,\nu}\|_{R'',s'},$$

where each $C_\alpha^{(l,\nu)}$ is a constant not depending on s, s' and R'' . Hence we obtain

$$\|\tau^{1^d} A_\alpha^{(l,\nu)} \Sigma v_{p,\nu}\|_{R'',s} \leq \frac{C_\alpha^{(l,\nu)} \prod_{j=1}^d R''_j}{(s' - s)^{|m^{(\nu)}| - |\alpha|}} \|v_{p,\nu}\|_{R'',s'}.$$

Note the following lemma:

3.6. Lemma. *The following equality holds:*

$$\mathcal{H}^{m^{(\nu)} - \alpha} v_{p,\nu}(z, \tau) = \int_{[0,1]^{|m^{(\nu)}| - |\alpha|}} v_{p,\nu}(z, \sigma\tau) d\sigma,$$

where $\sigma := (\sigma_{1,1}, \dots, \sigma_{1,m_1^{(\nu)} - \alpha_1}, \dots, \sigma_{d,1}, \dots, \sigma_{d,m_d^{(\nu)} - \alpha_d}) \in [0, 1]^{|m^{(\nu)}| - |\alpha|}$,

$$\sigma\tau := \left(\prod_{j=1}^{m_1^{(\nu)} - \alpha_1} \sigma_{1,j} \tau_{1,j}, \dots, \prod_{j=1}^{m_d^{(\nu)} - \alpha_d} \sigma_{d,j} \tau_{d,j} \right) \text{ and } d\sigma := \prod_{i=1}^d \prod_{j=1}^{m_i^{(\nu)} - \alpha_i} d\sigma_{i,j}.$$

Proof of Lemma 3.6. We have only to remark that for any $\gamma \in \mathbb{N}_0^d$

$$\int_{[0,1]^{|m^{(\nu)}| - |\alpha|} } (\sigma\tau)^\gamma d\sigma = \frac{\tau^\gamma}{(\gamma + \mathbb{1}_d)^{m^{(\nu)} - \alpha}}.$$

□

End of Proof of Theorem 2.1. Choose M_2 as

$$M_2 \geq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} C_\alpha^{(l,v)} e^{|\alpha|}.$$

Then, by (3.19), Proposition 2.2, Lemma 3.6 and (3.18)–0 we have

$$\begin{aligned} \|v_{1,\mu}\|_{R'',s} &= \left\| (\partial_\tau \tau)^{m^{(\mu)}} \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \mathcal{H}_{Q+Q}^{(\mu,l)} \tau^{1d} A_\alpha^{(l,v)}(z, \tau; \partial_z) \mathcal{H}^{m^{(v)}-\alpha} v_{0,v} \right\|_{R'',s} \\ &\leq M_1 \left\| \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \tau^{1d} A_\alpha^{(l,v)}(z, \tau; \partial_z) \mathcal{H}^{m^{(v)}-\alpha} v_{0,v} \right\|_{R'',s} \\ &\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1 C_\alpha^{(l,v)} \prod_{j=1}^d R_j''}{(s_0 - s)^{|m^{(v)}| - |\alpha|}} \|\mathcal{H}^{m^{(v)}-\alpha} v_{0,v}\|_{R'',s_0} \\ &= \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1 C_\alpha^{(l,v)} \prod_{j=1}^d R_j''}{(s_0 - s)^{|m^{(v)}| - |\alpha|}} \left\| \int_{[0,1]^{|m^{(v)}| - |\alpha|}} v_{0,v}(z, \sigma \tau) d\sigma \right\|_{R'',s_0} \\ &\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1 C_\alpha^{(l,v)} \prod_{j=1}^d R_j''}{(s_0 - s)^{|m^{(v)}| - |\alpha|}} \int_{[0,1]^{|m^{(v)}| - |\alpha|}} \|v_{0,v}\|_{\sigma R'',s_0} d\sigma \\ &\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1^2 C_\alpha^{(l,v)} \prod_{j=1}^d R_j'' \|f\|_{R',s_0}}{(s_0 - s)^{|\alpha|}} \\ &\leq \frac{M_1^2 M_2 \prod_{j=1}^d R_j'' \|f\|_{R',s_0}}{(s_0 - s)^{|\alpha|}}. \end{aligned}$$

Thus we have (3.18)–1. Now assume (3.18)– p for a $p \in \mathbb{N}$ and take arbitrary s and s' with $0 < s < s' < s_0$. Then, by (3.19), Proposition 2.2 and Lemma 3.6 we have

$$\begin{aligned} \|v_{p+1,\mu}\|_{R'',s} &= \left\| (\partial_\tau \tau)^{m^{(\mu)}} \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \mathcal{H}_{Q+Q}^{(\mu,l)} \tau^{1d} A_\alpha^{(l,v)}(z, \tau; \partial_z) \mathcal{H}^{m^{(v)}-\alpha} v_{p,v} \right\|_{R'',s} \\ &\leq M_1 \left\| \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \tau^{1d} A_\alpha^{(l,v)}(z, \tau; \partial_z) \mathcal{H}^{m^{(v)}-\alpha} v_{0,v} \right\|_{R'',s} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1 C_\alpha^{(l,v)} \prod_{j=1}^d R_j''}{(s' - s)^{|m^{(v)}| - |\alpha|}} \|\mathcal{H}^{m^{(v)} - \alpha} v_{p,v}\|_{R',s'} \\
&\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1 C_\alpha^{(l,v)} \prod_{j=1}^d R_j''}{(s' - s)^{|m^{(v)}| - |\alpha|}} \int_{[0,1]^{|m^{(v)}| - |\alpha|}} \|v_{p,v}\|_{\sigma R',s} d\sigma \\
&\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{M_1^{p+2} M_2^p C_\alpha^{(l,v)} \prod_{j=1}^d R_j'' \|f\|_{R',s_0}}{(s' - s)^{|m^{(v)}| - |\alpha|}} \\
&\quad \times \int_{[0,1]^{|m^{(v)}| - |\alpha|}} \frac{(\prod_{j=1}^d (\prod_{i=1}^{m^{(v)} - \alpha_j} \sigma_{j,i} R_j''))^p}{(s_0 - s')^{|m|p}} d\sigma \\
&= \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{C_\alpha^{(l,v)} M_1^{p+2} M_2^p (\prod_{j=1}^d R_j'')^{p+1} \|f\|_{R',s_0}}{(s_0 - s')^{|m|p} (s' - s)^{|m^{(v)}| - |\alpha|} (p+1)^{|m^{(v)}| - |\alpha|}}.
\end{aligned}$$

Hence, choosing $s' - s = (s_0 - s)/(p+1)$ we have

$$\begin{aligned}
\|v_{p+1,\mu}\|_{R'',s} &\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{C_\alpha^{(l,v)} M_1^{p+2} M_2^p (\prod_{j=1}^d R_j'')^{p+1} \|f\|_{R',s_0}}{(s_0 - s)^{|m|(p+1) - |\alpha|}} \left(\frac{p+1}{p}\right)^{|m|p} \\
&\leq \sum_{l,v=1}^N \sum_{0 \leq \alpha < m^{(v)}} \frac{C_\alpha^{(l,v)} M_1^{p+2} M_2^p e^{|m|} (\prod_{j=1}^d R_j'')^{p+1} \|f\|_{R',s_0}}{(s_0 - s)^{|m|(p+1)}} \\
&\leq \frac{M_1^{p+2} M_2^{p+1} (\prod_{j=1}^d R_j'')^{p+1} \|f\|_{R',s_0}}{(s_0 - s)^{|m|(p+1)}}.
\end{aligned}$$

Hence we can prove (3.18)-(p+1). Set $\delta := 1/(M_1 M_2)$ and take an arbitrary vector R'' which satisfies $0 < R'' \leq R'$ and $\prod_{j=1}^d R_j'' < \delta (s_0 - s)^{|m|}$. Then it follows that

$$\sum_{p=0}^{\infty} \|v_p\|_{R'',s} \leq M_1 \|f\|_{R',s_0} \sum_{p=0}^{\infty} \left(\frac{M_1 M_2 \prod_{j=1}^d R_j''}{(s_0 - s)^{|m|}} \right)^p < \infty.$$

Thus there exists $v(z, \tau) \in E(B(R''); X_s)^{\oplus N}$ such that $\sum_{j=0}^p v_j$ converges to v as p tends to infinity in $E(B(R''); X_s)^{\oplus N}$. Set

$$u := \begin{pmatrix} \mathcal{H}^{m(1)} & & 0 \\ & \ddots & \\ 0 & & \mathcal{H}^{m(N)} \end{pmatrix} v.$$

Then it is obvious that $P_{\Sigma}u = f$ holds and u is holomorphic on

$$\bigcup_{0 < s < s_0} \left(\Omega_s \times \left\{ \tau \in B(R'); \prod_{j=1}^d |\tau_j| < \delta(s_0 - s)^{|m|} \right\} \right).$$

Hence the existence is proved. Next we will show the uniqueness. Assume that $u(z, \tau) \in E(B(R''); X_s)^{\oplus N}$ satisfies $P_{\Sigma}u = 0$. Then, by the preceding argument for some positive constant C we have

$$\|u\|_{R'',s} \leq C \left(\frac{M_1 M_2 \prod_{j=1}^d R_j''}{(s_0 - s)^{|m|}} \right)^p$$

for any $0 < s < s_0 < 1$ and $p \in \mathbb{N}$. Therefore, choosing $M_1 M_2 \prod_{j=1}^d R_j'' < (s_0 - s)^{|m|}$ and letting p tend to infinity, we see that $u \equiv 0$ in some neighborhood of $(z, \tau) = (z_0, 0)$. Thus the uniqueness follows. Therefore the proof is complete. □

§4. Applications

Let M be $R_x^n \times R_t^d$ with its complexification $X := C_x^n \times C_t^d = Y \times C^d$ and π_M the canonical projection $T_M^*X \rightarrow M$. Set $N := R^n \cong M \cap \{t=0\} \subset M$, $L := X \cap \{\text{Im } z = 0\} = R^n \times C^d$, $\tilde{A} := T_L^*X \cong T_N^*Y \times C^d$ and $A := T_M^*X \cap \tilde{A}$. We denote the sheaf of microfunctions on T_M^*X (resp. T_N^*Y) by \mathcal{C}_M (resp. \mathcal{C}_N) as usual. Further, let \mathcal{C}_L be the sheaf of microfunctions with holomorphic parameters on \tilde{A} ; that is,

$$\mathcal{C}\mathcal{O}_L := \mu_L(\mathcal{O}_X) \otimes or_{N/Y}[n],$$

where μ_L denotes Sato's microlocalization functor along L and $or_{N/Y}$ denotes the relative orientation sheaf (see [K-Sc2] and [S-K-K]). The sheaf \mathcal{B}_M of hyperfunctions on M and the sheaf $\mathcal{B}\mathcal{O}_L$ of hyperfunctions with holomorphic parameters on L are defined by $\mathcal{B}_M := \mathcal{C}_M|_M$ and $\mathcal{B}\mathcal{O}_L := \mathcal{C}\mathcal{O}_L|_L$ respectively. Let

ρ be a natural mapping $N \times_M T_M^*X \ni (x, 0; \sqrt{-1}(\langle \zeta, dx \rangle + \langle \eta, dt \rangle)) \mapsto (x; \sqrt{-1} \langle \zeta, dx \rangle) \in T_N^*Y$. Then, we have the following canonical morphisms:

$$\mathcal{C}\mathcal{O}_L|_L \twoheadrightarrow \mathcal{C}_M|_L, \quad \rho_!(\mathcal{C}_M|_{N \times_M T_M^*X}) \rightarrow \mathcal{C}_N.$$

Set $p_0 := (0; \sqrt{-1}dx_1) \in T_N^*Y$ and assume that $P(x, t; \partial_x, \partial_t)$ is a matrix of microdifferential operators of Fuchsian type with weight (k, m) defined in some neighborhood of $\rho^{-1}(p_0)$, then the following morphism is induced:

$$P: \rho_!(\mathcal{C}_M|_{N \times_M T_M^*X})_{p_0} \rightarrow \rho_!(\mathcal{C}_M|_{N \times_M T_M^*X})_{p_0},$$

where $\rho_!(\mathcal{C}_M|_{N \times_M T_M^*X})_{p_0}$ denotes the stalk at p_0 .

Consider the following condition:

[A-2]. $\det(\sigma_{|m^{(v)}|}(P^{(\mu, v)})(z, \tau; \zeta, \eta)) = \tau^{\tilde{k}} \tilde{P}(z, \tau; \zeta, \eta)$ for a function $\tilde{P} (\tilde{k} := \sum_{v=1}^N k^{(v)} \in N_0^d)$ which satisfies the following condition:

There exist positive constants h_0, M and v_i with $v_i \geq 1$ ($1 \leq i \leq d$) such that $\tilde{P}(z, t; \zeta, \eta)$ never vanishes on the set

$$(4.1) \quad \left\{ (z, t; \zeta, \eta) \in \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{C}^n \times \mathbb{C}^d; |z|, |t| < h_0, |\zeta_j| < h_0 |\zeta_1| \ (2 \leq j \leq n), \right. \\ \left. \exists \lambda > M \left(\sum_{j=1}^n |\operatorname{Im} z_j| + \sum_{j=2}^n |\operatorname{Im}(\zeta_j / \zeta_1)| \right), |\operatorname{Im}(\eta_i / \zeta_1)| = v_i \lambda \ (1 \leq i \leq d) \right\}.$$

4.1. Example. (1) Let P be as in Example 1.5 (2) with $d=2$. Set

$$\begin{cases} b_{ij}(z, \tau; \zeta) := \sigma_1(B_{ij})(z, \tau; \zeta), \\ d_{ij}(z, \tau; \zeta) := \sigma_2(D_{ij})(z, \tau; \zeta), \end{cases}$$

and suppose that $b_{21}(z, \tau; \zeta) = d_{21}(z, \tau; \zeta) = 0$. Then

$$\det(\sigma_2(P^{(i, j)})(z, \tau; \zeta, \eta)) \\ = \tau_1 \tau_2 \det \begin{pmatrix} \eta_1^2 + \tau_2(b_{11}\eta_1 + d_{11}) & \tau_1 b_{12}\eta_2 + d_{12} \\ 0 & \eta_2^2 + \tau_1(b_{22}\eta_2 + d_{22}) \end{pmatrix}$$

$$= \tau_1 \tau_2 (\eta_1^2 + \tau_2 (b_{11} \eta_1 + d_{11})) (\eta_2^2 + \tau_1 (b_{22} \eta_2 + d_{22})).$$

Hence if $\eta_1^2 + \tau_2 (b_{11}(z, \tau; \zeta) \eta_1 + d_{11}(z, \tau; \zeta))$ (resp. $\eta_2^2 + \tau_1 (b_{22}(z, \tau; \zeta) \eta_2 + d_{22}(z, \tau; \zeta))$) is hyperbolic with respect to η_1 (resp. η_2), then we see that [A-2] is satisfied (cf. Kashiwara-Kawai [K-K]).

(2) Condition [A-2] is satisfied if

$$(4.2) \quad \tilde{P}(x, t; \zeta, \eta) = \prod_{i=1}^d P_i(x, t; \zeta, \eta_1, \dots, \eta_i),$$

where each P_i satisfies the following condition: there exist positive constants h_0 and M_i ($1 \leq i \leq d$)

such that $P_i(z, t; \zeta, \eta_1, \dots, \eta_i)$ never vanishes on the set

$$\left\{ (z, t; \zeta, \eta_1, \dots, \eta_i) \in \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{C}^{n+i}; |z|, |t| < h_0, |\zeta_j| < h_0 |\zeta_1| \ (2 \leq j \leq n), \right. \\ \left. |\operatorname{Im}(\eta_i / \zeta_1)| > M_i \left(\sum_{j=1}^n |\operatorname{Im} z_j| + \sum_{j=2}^n |\operatorname{Im}(\zeta_j / \zeta_1)| + \sum_{j=1}^{i-1} |\operatorname{Im}(\eta_j / \zeta_1)| \right) \right\}.$$

4.2. Theorem. Assume that P satisfies [A-1] and [A-2]. Then, for any microfunctions with holomorphic parameters

$$(4.3) \quad f(x, t), g(x, t) \in \rho_* (\mathcal{C}_L |_{N \times T_M^* X})_{p_0}^{\oplus N},$$

there exists an N -tuple of microfunctions

$$u(x, t) \in \rho_! (\mathcal{C}_M |_{N \times T_M^* X})_{p_0}^{\oplus N}$$

such that u is a solution of the Goursat problem

$$(G.P.) \quad \begin{cases} P(x, t; \partial_x, \partial_t) u(x, t) = f(x, t), \\ u(x, t) - g(x, t) = O(t^{m-k}). \end{cases}$$

4.3. Remark. The uniqueness of $u(x, t)$ in Theorem 4.2 will be discussed in a forthcoming paper (cf. [O3] and [O4]).

Proof of Theorem 4.2. We may assume that $g \equiv 0$. We remark that the

method of proof is similar to Oaku [O2].

Step 1. Set

$$W := \{(z; \zeta) \in \mathbb{C}^{2n}; |z| < h_0, |\zeta_j| < h_0 |\zeta_1| \ (2 \leq j \leq n)\}$$

and choose constants h and c such that

$$(4.4) \quad 0 < h < \min\{h_0/(4n^2), 1/2\}, \ 0 < c < \min\{r_0, h_0\},$$

where $r_0 > 0$ is a constant determined by P and W by Theorem 1.3. Then, setting

$$(4.5) \quad \begin{cases} \Sigma := \{z \in \mathbb{C}^n; z_1 = \sqrt{-1}c\}, \\ \Omega' := \{z \in \mathbb{C}^n; |z_j| + |z_1 - \sqrt{-1}c|/h_0 < c/h \ (2 \leq j \leq n)\}, \end{cases}$$

we see that Ω' is h - Σ -flat, $\text{dia } \Omega' < r_0$ and $\Omega' \subset \{z \in \mathbb{C}^n; |z| < h\}$. Choose constants a and a_1 such that

$$(4.6) \quad 0 < a < a_1 < (2n^2 - 1)c.$$

Then we have $\{z \in \mathbb{C}^n; |z| < a_1\} \subset \Omega'$ by (4.5) and (4.6). On the other hand, by Theorem 2.2 of Kataoka-Tose [Kt-To] and the argument to the proof of Theorem 3.8.1 of Kashiwara-Kawai-Kimura [K³], we can find holomorphic functions $F(z, \tau) = (F_1(z, \tau), \dots, F_N(z, \tau))$ defined in a complex neighborhood of

$$(4.7) \quad \begin{aligned} & \left\{ (z, \tau) \in \mathbb{C}^{n+d}; |\text{Re } z|, |\text{Re } \tau| < h_1, |\text{Im } \tau| < h_2, h \sum_{j=2}^n |\text{Im } z_j| < 2(n-1)\text{Im } z_1 < h' \right\} \\ & \cup \{(x, t) \in \mathbb{R}^{n+d}; a \leq |x| < h_1, |t| < h_1\} \end{aligned}$$

with some constants $h', h_2 > 0$ and $0 < h_0 < h_1$ such that on $\rho^{-1}(p_0)$

$$\begin{pmatrix} \text{sp}_M(\mathfrak{b}(F_1)) \\ \vdots \\ \text{sp}_M(\mathfrak{b}(F_N)) \end{pmatrix} = f.$$

Here $\text{sp}_M: \pi_M^{-1}\mathcal{B}_M \rightarrow \mathcal{C}_M$ denotes the spectral morphism from the sheaf of hyperfunctions to that of microfunctions and $\mathfrak{b}(F_\nu)$ denotes the boundary value as a hyperfunction of F_ν . Set

$$\tilde{D} := \{(x, t) \in \mathbf{R}^{n+d}; |x| < h_1, |t| < h_1\},$$

$$\tilde{G} := \{(y, s) \in \mathbf{R}^{n+d}; |s| < h_2, h \sum_{j=2}^n |y_j| < (n-1)y_1\},$$

$$\tilde{G}' := \{(y, s) \in \mathbf{R}^{n+d}; |s| < h_2, h \sum_{j=2}^n |y_j| < 2(n-1)y_1\},$$

$$\tilde{G}_{h'} := \{(y, s) \in \mathbf{R}^{n+d}; |s| < h_2, h \sum_{j=2}^n |y_j| < (n-1)(y_1 + h')\}.$$

Then, we have

$$\tilde{G}_{h'} = (\tilde{G}_{h'} \cap \{(y, s) \in \mathbf{R}^{n+d}; y_1 < h'\}) \cup (\tilde{G}_{h'} \cap \tilde{G}'),$$

$$F \in \mathcal{O}(\tilde{D} + \sqrt{-1}(\tilde{G}' \cap \{(y, s) \in \mathbf{R}^{n+d}; y_1 < h'\}))^{\oplus N}.$$

Therefore, since $\tilde{D} + \sqrt{-1}\tilde{G}_{h'}$ is a Stein open set we can decompose F into

$$(4.8) \quad F = \tilde{f} + f',$$

with

$$(4.9) \quad \begin{cases} \tilde{f} \in \mathcal{O}(\tilde{D} + \sqrt{-1}(\tilde{G}_{h'} \cap \tilde{G}'))^{\oplus N}, \\ f' \in \mathcal{O}(\tilde{D} + \sqrt{-1}(\tilde{G}_{h'} \cap \{(y, s) \in \mathbf{R}^{n+d}; y_1 < h'\}))^{\oplus N}. \end{cases}$$

Since an N -tuple f' is holomorphic in a complex neighborhood of \tilde{D} by (4.7), (4.8) and (4.9), we see that an N -tuple \tilde{f} is holomorphic in some complex neighborhood of

$$(4.10) \quad (\tilde{D} + \sqrt{-1}\tilde{G}) \cup \{(x, t) \in \mathbf{R}^{n+d}; a \leq |x| < h_1, |t| < h_1\},$$

and on $\rho^{-1}(p_0)$

$$\begin{pmatrix} \text{sp}_M(\mathfrak{b}(\tilde{f}_1)) \\ \vdots \\ \text{sp}_M(\mathfrak{b}(\tilde{f}_N)) \end{pmatrix} = f.$$

Step 2. Set

$$(4.11) \quad \begin{cases} D := \{x \in \mathbf{R}^n; |x| < h_1\} \\ G := \{y \in \mathbf{R}^n; h \sum_{j=2}^n |y_j| < (n-1)y_1\}. \\ \Omega := \Omega' \cap (\mathbf{R}^n + \sqrt{-1}G) = \Omega' \cap (D + \sqrt{-1}G). \end{cases}$$

Then, by (4.10) and (4.11) we see that Ω is h_0 - Σ -flat, $\text{dia } \Omega < r_0$ and an N -tuple \tilde{f} is holomorphic on

$$\Omega \times B(h_2 \mathbf{1}_d),$$

(recall that $\mathbf{1}_d := (1, \dots, 1) \in \mathbf{N}^d$). Consider the following:

$$\begin{cases} P_\Sigma \tilde{u} = \tilde{f}, \\ \tilde{u} = O(\tau^{m-k}). \end{cases}$$

Thus, by Theorem 1.3, there exists a constant δ with $0 < \delta < 1$ such that there exist a unique holomorphic solution \tilde{u} on

$$\bigcup_{0 < s < 1} \left(\Omega_s \times \left\{ \tau \in B(h_2 \mathbf{1}_d); \prod_{j=1}^d |\tau_j| < \delta(1-s)^{|m|} \right\} \right),$$

where

$$\Omega_s := \{sz + (1-s)\sqrt{-1}(c, 0, \dots, 0) \in \mathbf{C}^n; z \in \Omega\}.$$

Step 3. Take any x_0 with $a \leq |x_0| \leq a_1$. Note that $\tilde{f}(z, \tau)$ is holomorphic at $(z, \tau) = (x_0, 0)$. Hence we can choose sufficiently small constants c' and T_1 with $0 < c' < c$ such that

$$\Omega'' := \left\{ z \in \mathbf{C}^n; 2|z_j - x_{0,j}| + \frac{2}{h_0}|z_1 - x_{0,1} - \sqrt{-1}c'| < \frac{c'}{h} \right\} \in \Omega'$$

and

$$\tilde{f} \text{ is holomorphic on } \Omega'' \times \{\tau \in \mathbf{C}^d; |\tau_j| < T_1 \ (1 \leq j \leq d)\}.$$

Set

$$\Sigma' := \{z \in \mathbf{C}^n; z_1 = x_{0,1} + \sqrt{-1}c'\},$$

then we see that Ω'' is h_0 - Σ -flat and $\Sigma' \cap \Omega'' \subset \Omega_{s_1}$ for some $0 < s_1 < 1$. Therefore, by the argument to Théorème 2.5.1 of Bony-Schapira [Bo-Sc] $P_\Sigma \tilde{u} - P_{\Sigma'} \tilde{u}$ is holomorphic on $\Omega'' \times B(\delta(1-s_1)^{|m|} \mathbf{1}_d)$. Since $P_\Sigma \tilde{u} = \tilde{f}$ is holomorphic on

$\Omega'' \times B(T_1 \mathbf{1}_d)$, $P_{\Sigma} \cdot \tilde{u}$ is holomorphic on $\Omega'' \times B(\min\{T_1, \delta(1-s_0)^{|m|} \mathbf{1}_d\})$. Thus by Theorem 1.3, there exists an N -tuple \tilde{u}' of holomorphic functions on a neighborhood of $\Omega' \times \{0\}$ such that

$$\begin{cases} P_{\Sigma} \cdot \tilde{u}' = P_{\Sigma} \cdot \tilde{u} \\ \tilde{u}' = O(\tau^{m-k}). \end{cases}$$

By the uniqueness of \tilde{u}' , we have $\tilde{u}' = \tilde{u}$ on a neighborhood of $(\Omega \cap \Omega'') \times \{0\}$ because $x_0 \in \Omega''$. Since $\{x \in \mathbb{R}^n; a \leq |x| \leq a_1\}$ is compact, it follows that \tilde{u} is analytically continued to some neighborhood of

$$(4.12) \quad U_1 := \bigcup_{0 < s < 1} \left(\Omega_s \times \left\{ \tau \in B(h_2 \mathbf{1}_d); \prod_{j=1}^d |\tau_j| < \delta(1-s)^{|m|} \right\} \right) \cup \{(z, \tau) \in \mathbb{C}^{n+d}; a \leq |\operatorname{Re} z| \leq a_1, |\operatorname{Im} z| < a', |\tau| < T_2\}$$

with some T_2 and $a' > 0$.

Step 4. Fix arbitrary $\sigma = (\sigma_1, \dots, \sigma_d) \in \{-1, 1\}^d$. For simplicity, set

$$D' := \{(z, t) \in \mathbb{C}^n \times \mathbb{R}^d; |\operatorname{Re} z| < a_1, |\operatorname{Im} z| < a'\} \subset \mathbb{C}^n \times \mathbb{R}^d.$$

Then, by the result of Step 3 we see that there exists a positive constant δ_1 such that \tilde{u} is holomorphic in some complex neighborhood of

$$(4.13) \quad \begin{aligned} & \left\{ (z, t) \in D'; \sum_{j=1}^d |\sigma_j v_j t_j| < T_2, \operatorname{Im} z_1 > \delta_1 \left(\sum_{j=1}^d |\sigma_j v_j t_j| \right)^{1/|m|} + \frac{h}{n-1} \sum_{j=2}^n |\operatorname{Im} z_j| \right\} \\ & \cup \left\{ (z, t) \in D'; \sum_{j=1}^d |\sigma_j v_j t_j| < T_2, \operatorname{Im} z_1 > \delta_1 \left(\prod_{j=1}^d |\sigma_j v_j t_j| \right)^{1/|m|} + \frac{h}{n-1} \sum_{j=2}^n |\operatorname{Im} z_j| \right\} \\ & \cup \left\{ (z, \tau) \in \mathbb{C}^{n+d}; a \leq |\operatorname{Re} z| \leq a_1, |\operatorname{Im} z| < a', \sum_{j=1}^d |\sigma_j v_j t_j| < T_2 \right\}. \end{aligned}$$

Choose positive constants M_1 and T_0 such that

$$(4.14) \quad \left\{ \begin{aligned} & \max\{|m|nM, \delta_1/a'\} < M_1, \\ & 0 < T_0 < \min\{T_2, 1\}, \\ & \exp(M_1 T_0) < 2, \\ & 2n \exp(M_1 T_0) M_1 T_0^{1/|m|} < h_0, \\ & \delta_1 T_0^{1/|m|} < \frac{a'}{2}, \end{aligned} \right.$$

and set

$$\varphi_{2,\varepsilon}^\sigma(y', t) := \left(M_1 \left(\sum_{j=1}^d \sigma_j v_j t_j \right)^{1/|m|} + nh \right) \sqrt{|y'|^2 + \varepsilon^2},$$

where $y' := (y_2, \dots, y_n)$ and $\varepsilon > 0$. Let $\varphi_1(T)$ be a real valued real analytic function in a neighborhood of $\{T \in \mathbf{R}; 0 < T \leq T_0\}$ satisfying for some $h'' > 0$

$$(4.15) \quad \begin{cases} \varphi_1(T_0) > \delta_1 T_0^{1/|m|}, \\ \varphi_1(T) \geq h'' & \text{for any } 0 < T \leq T_0, \\ \left(\frac{d\varphi_1}{dT} \right)(T) > 0 & \text{for any } 0 < T \leq T_0. \end{cases}$$

Hence there exists a small constant c_1 such that if $0 < T < c_1$ or $T_0 - c_1 < T < T_0$, then

$$(4.16) \quad \varphi_1(T) > \delta_1 T^{1/|m|}.$$

Set

$$\begin{aligned} \varphi_{1,\varepsilon}^\sigma(t) &:= \varphi_1 \left(\sum_{j=1}^d \sigma_j v_j t_j \right), \\ \varphi_\varepsilon^\sigma(y', t) &:= (\varphi_{1,\varepsilon}^\sigma(t) + \varphi_{2,\varepsilon}^\sigma(y', t)) \exp \left(M_1 \sum_{j=1}^d \sigma_j v_j t_j \right). \end{aligned}$$

Then we easily see that

$$\frac{\partial \varphi_\varepsilon^\sigma}{\partial t_j}(y', t) = \sigma_j v_j \lambda \quad (1 \leq j \leq d),$$

where

$$\begin{aligned} \lambda &:= M_1 \varphi_\varepsilon^\sigma(y', t) + \exp \left(M_1 \sum_{j=1}^d \sigma_j v_j t_j \right) \\ &\quad \times \left(\frac{d\varphi_1}{dT} \left(\sum_{j=1}^d \sigma_j v_j t_j \right) + \frac{M_1}{|m|} \left(\sum_{j=1}^d \sigma_j v_j t_j \right)^{1/|m|-1} \sqrt{|y'|^2 + \varepsilon^2} \right). \end{aligned}$$

Moreover by (4.4) and (4.14) we have

$$(4.17) \quad \begin{aligned} \lambda &> M_1 \left(\varphi_\varepsilon^\sigma(y', t) + \frac{M_1}{|m|} \left(\sum_{j=1}^d \sigma_j v_j t_j \right)^{1/|m|-1} |y'| \right) \\ &> M \left(\varphi_\varepsilon^\sigma(y', t) + \sum_{j=2}^n |y_j| \right), \end{aligned}$$

and

$$(4.18) \quad \sum_{j=2}^n \left| \frac{\partial \varphi_\varepsilon^\sigma}{\partial z_j} \right| = \sum_{j=2}^n \frac{1}{2} \left| \frac{\partial \varphi_\varepsilon^\sigma}{\partial y_j} \right| < \frac{n-1}{2} (M_1 T_0^{1/|m|} + nh) \exp(M_1 T_0) < \frac{h_0}{2}.$$

By (4.10), \tilde{f} is holomorphic in complex neighborhood of

$$(4.19) \quad \left\{ (z, t) \in D'; 0 \leq \sigma_j v_j t_j \ (1 \leq j \leq d), \sum_{j=1}^d \sigma_j v_j t_j < T_0, \operatorname{Im} z_1 > \frac{h}{n-1} \sum_{j=2}^n |\operatorname{Im} z_j| \right\}.$$

4.4. Lemma. *Assume that \tilde{u} is holomorphic in a complex neighborhood of*

$$\left\{ (z, t) \in D'; 0 < \sigma_j v_j t_j \ (1 \leq j \leq d), 0 < \sum_{j=1}^d \sigma_j v_j t_j < T_0, \operatorname{Im} z_1 > \varphi_\varepsilon^\sigma(\operatorname{Im} z', t) \right\}.$$

Then \tilde{u} is analytically continued to some complexification of

$$\left\{ (z, t) \in D'; 0 < \sigma_j v_j t_j \ (1 \leq j \leq d), 0 < \sum_{j=1}^d \sigma_j v_j t_j < T_0, \operatorname{Im} z_1 = \varphi_\varepsilon^\sigma(\operatorname{Im} z', t) \right\}.$$

Proof of Lemma 4.4. We can apply the argument to Lemma 4.3 of Kashiwara-Kawai [K-K]. Set

$$V := \left\{ (z, t) \in D'; 0 < \sigma_j v_j t_j \ (1 \leq j \leq d), 0 < \sum_{j=1}^d \sigma_j v_j t_j < T_0, \operatorname{Im} z_1 = \varphi_\varepsilon^\sigma(\operatorname{Im} z', t) \right\}.$$

We regard V as a real analytic manifold of dimension $2n + d - 1$ with coordinates $x := \operatorname{Re} z$, $y := \operatorname{Im} z$ and t . Let \tilde{V} be a complex neighborhood of V and set $\tilde{\mathcal{A}}_V := H^0(v_V(\mathcal{O}_{\tilde{V}}))$, where $v_V(*)$ denotes the specialization functor along V (see [K-Sc2] and [S-K-K]). Since \tilde{u} is analytic on

$$\left\{ (z, t) \in D'; 0 < \sigma_j v_j t_j \ (1 \leq j \leq d), 0 < \sum_{j=1}^d \sigma_j v_j t_j < T_0, \operatorname{Im} z_1 > \varphi_\varepsilon^\sigma(\operatorname{Im} z', t) \right\},$$

it follows that \tilde{u} defines a section of $\mathcal{A}_V^{\oplus N}$ on the set

$$\begin{aligned} & \left\{ (x, y, t) + \sqrt{-1} \left(c + 2\operatorname{Re} \left\langle \zeta', \frac{\partial \varphi_\varepsilon^\sigma}{\partial z'} \right\rangle \right) \frac{\partial}{\partial x_1} + 2\operatorname{Im} \left\langle \zeta', \frac{\partial \varphi_\varepsilon^\sigma}{\partial z'} \right\rangle \frac{\partial}{\partial y_1} \right. \\ & \quad \left. + \left\langle \operatorname{Im} \zeta', \frac{\partial}{\partial x'} \right\rangle - \left\langle \operatorname{Re} \zeta', \frac{\partial}{\partial y'} \right\rangle \right\} \in T_V \tilde{V}; \zeta' \in \mathbb{C}^{n-1}, c > 0 \Big\} \\ & = \left\{ (x, y, t) + \left(\sqrt{-1} c \frac{\partial}{\partial z_1} + \sqrt{-1} c \frac{\partial}{\partial \bar{z}_1} + \left\langle \zeta', \frac{\partial}{\partial z'} \right\rangle + 2\sqrt{-1} \frac{\partial \varphi_\varepsilon^\sigma}{\partial z'} \frac{\partial}{\partial z_1} \right. \right. \\ & \quad \left. \left. - \left\langle \bar{\zeta}', \frac{\partial}{\partial \bar{z}'} \right\rangle - 2\sqrt{-1} \frac{\partial \varphi_\varepsilon^\sigma}{\partial \bar{z}'} \frac{\partial}{\partial \bar{z}_1} \right) \right\} \in T_V \tilde{V}; \zeta' \in \mathbb{C}^{n-1}, c > 0 \Big\} \end{aligned}$$

Indeed, for any $(x, y, t) \in V$, $\zeta' \in \mathbb{C}^{n-1}$ and $c > 0$, choosing a positive constant ε' small enough we have

$$\begin{aligned} & \operatorname{Im} \left(z_1 - \varepsilon' (\sqrt{-1} (c + 2 \left\langle \zeta', \frac{\partial \varphi_\varepsilon^\sigma}{\partial z'} \right\rangle) - \varphi_\varepsilon^\sigma(y' + \varepsilon' \operatorname{Im} \zeta', t)) \right) \\ & = y_1 + \varepsilon' (c + \left\langle \operatorname{Im} \zeta', \frac{\partial \varphi_\varepsilon^\sigma}{\partial y'} \right\rangle) - \varphi_\varepsilon^\sigma(y', t) - \varepsilon' \left\langle \operatorname{Im} \zeta', \frac{\partial \varphi_\varepsilon^\sigma}{\partial y} \right\rangle + O((\varepsilon')^2) \\ & = \varepsilon' c + O((\varepsilon')^2) > 0. \end{aligned}$$

Thus we see that each boundary value $b(\tilde{u}_v)$ ($1 \leq v \leq N$) as a hyperfunction is well-defined. Hence, denoting by SS the singularity spectrum of a hyperfunction, we obtain that $\operatorname{SS}(b(\tilde{u}_v)) \subset Z$, where

$$Z := \left\{ (x, y, t; \sqrt{-1} dz_1 + 2 \left\langle \frac{\partial \varphi_\varepsilon^\sigma}{\partial z'}, dz' \right\rangle + \left\langle \frac{\partial \varphi_\varepsilon^\sigma}{\partial t} + \sqrt{-1} a, dt \right\rangle) \in T_V^* \tilde{V}; a \in \mathbb{R}^d \right\}.$$

We see that P is defined on Z , and by (4.17) we have

$$\begin{cases} \left| \operatorname{Im} \left(\frac{\eta_i}{\zeta_1} \right) \right| = \nu_i \lambda & (1 \leq i \leq d), \\ \lambda > M \left(\varphi_\varepsilon^\sigma(y', t) + \sum_{j=2}^n |y_j| \right) = M \left(y_1 + \sum_{j=2}^n |y_j| \right). \end{cases}$$

Moreover, by (4.18) we have

$$\left| \frac{\zeta_j}{\zeta_1} \right| = \left| 2 \frac{\partial \varphi_\varepsilon^\sigma}{\partial z_j} \right| = \left| \frac{\partial \varphi_\varepsilon^\sigma}{\partial y_j} \right| < h_0 \quad (2 \leq j \leq n).$$

and

$$\left| \operatorname{Im} \left(\frac{\zeta_j}{\zeta_1} \right) \right| = \left| \operatorname{Im} \left(-2\sqrt{-1} \frac{\partial \varphi_\varepsilon^\sigma}{\partial z_j} \right) \right| = \left| \frac{\partial \varphi_\varepsilon^\sigma}{\partial x_j} \right| = 0 \quad (2 \leq j \leq n).$$

Hence by assumption [A-2], P is invertible on Z . Further by (4.19), we have

$$P \begin{pmatrix} \operatorname{sp}_V(\mathbf{b}(\tilde{u}_1)) \\ \vdots \\ \operatorname{sp}_V(\mathbf{b}(\tilde{u}_N)) \end{pmatrix} = \begin{pmatrix} \operatorname{sp}_V(\mathbf{b}((P_\Sigma \tilde{u})_1)) \\ \vdots \\ \operatorname{sp}_V(\mathbf{b}((P_\Sigma \tilde{u})_N)) \end{pmatrix} = \begin{pmatrix} \operatorname{sp}_V(\mathbf{b}(\tilde{f}_1)) \\ \vdots \\ \operatorname{sp}_V(\mathbf{b}(\tilde{f}_N)) \end{pmatrix} = 0.$$

Thus it follows that

$$\begin{pmatrix} \operatorname{sp}_V(\mathbf{b}(\tilde{u}_1)) \\ \vdots \\ \operatorname{sp}_V(\mathbf{b}(\tilde{u}_N)) \end{pmatrix} = 0.$$

On the other hand, it is easy to see that for any $1 \leq j \leq n$

$$\frac{\partial}{\partial \bar{z}_j} \begin{pmatrix} \mathbf{b}(\tilde{u}_1) \\ \vdots \\ \mathbf{b}(\tilde{u}_N) \end{pmatrix} = 0.$$

Hence the proof is complete. □

Step 5. Set

$$(4.20) \quad W^\sigma := \left\{ (z, t) \in D'; 0 \leq \sigma_j \nu_j t_j \ (1 \leq j \leq d), 0 < \sum_{j=1}^d \sigma_j \nu_j t_j < T_0, \right. \\ \left. \operatorname{Im} z_1 > \delta_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + \frac{h}{n-1} \sum_{j=2}^n |\operatorname{Im} z_j| \right\} \\ \cup \left\{ (z, \tau) \in \mathbb{C}^{n+d}; a < |\operatorname{Re} z| \leq a_1, |\operatorname{Im} z| < a', \right. \\ \left. 0 \leq \sigma_j \nu_j t_j \ (1 \leq j \leq d), \sum_{j=1}^d \sigma_j \nu_j t_j < T_0 \right\},$$

$$(4.21) \quad Z_\varepsilon^\sigma := \widetilde{Z}_\varepsilon^\sigma \setminus W^\sigma,$$

where

$$\widetilde{Z}_\varepsilon^\sigma := \left\{ (z, t) \in D'; 0 \leq \sigma_j \nu_j t_j \ (1 \leq j \leq d), \ 0 < \sum_{j=1}^d \sigma_j \nu_j t_j < T_0, \ \operatorname{Im} z_1 = \varphi_\varepsilon^\sigma(\operatorname{Im} z', t) \right\}.$$

Note that by (4.13) and (4.15) \tilde{u} is analytic in some neighborhood of $W^\sigma \cup \{(z, t) \in D'; \prod_{j=1}^d t_j = 0, \operatorname{Im} z_1 = \varphi_\varepsilon^\sigma(\operatorname{Im} z', t)\}$. Choosing ε sufficiently large, we see that $Z_\varepsilon^\sigma = \emptyset$. We will show that Z_ε^σ is compact.

Case 1. If $0 < \sum_{j=1}^d \sigma_j \nu_j t_j < c_1$ or $T_0 - c_1 < \sum_{j=1}^d \sigma_j \nu_j t_j < T_0$, then by (4.16)

$$\varphi_\varepsilon^\sigma(\operatorname{Im} z', t) > \varphi_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right) + nh |\operatorname{Im} z'| > \delta_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + h \sum_{j=2}^n |\operatorname{Im} z_j|.$$

Hence $(z, \tau) \notin Z_\varepsilon^\sigma$.

Case 2. If $a < |\operatorname{Re} z| < a_1$, then $(z, \tau) \notin Z_\varepsilon^\sigma$ by definition.

Case 3. Since $M_1 a' > \delta_1$ by (4.14), we can take a'' as $M_1 a' > M_1 a'' > \delta_1$. Therefore, if $a'' < |\operatorname{Im} z'| < a'$, then we have

$$\begin{aligned} \varphi_\varepsilon^\sigma(\operatorname{Im} z', t) &> M_1 \left(\left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + nh \right) |\operatorname{Im} z'| \\ &> \delta_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + h \sum_{j=2}^n |\operatorname{Im} z_j|. \end{aligned}$$

Hence $(z, t) \notin Z_\varepsilon^\sigma$.

Case 4. Since $2h < 1$ and $2\delta_1 T_0^{1/|m|} < a'$ by (4.4) and (4.14), we can choose a''' such that $ha' + \delta_1 T_0^{1/|m|} < a''' < a'$. Therefore, if $\operatorname{Im} z_1 = \varphi_\varepsilon^\sigma(\operatorname{Im} z', t)$, $|\operatorname{Re} z| < a_1$, $|\operatorname{Im} z'| < a'$ and $a''' \operatorname{Im} z_1 < a'$, then

$$\operatorname{Im} z_1 > a''' > \delta_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + \frac{h}{n-1} \sum_{j=2}^n |\operatorname{Im} z_j|.$$

Hence $(z, t) \in W^\sigma$; that is, $(z, t) \notin Z_\varepsilon^\sigma$.

Thus we see that Z_ε^σ is compact. Therefore, by the “sweeping-out” method we see that \tilde{u} is analytically continued to

$$\left\{ (z, t) \in D'; 0 \leq \sigma_j \nu_j t_j \ (1 \leq j \leq d), \ 0 < \sum_{j=1}^d \sigma_j \nu_j t_j < T_0, \right.$$

$$\text{Im } z_1 > \exp(M_1 \sum_{j=1}^d \sigma_j \nu_j t_j)$$

$$\times \left(\varphi_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right) + \left(M_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + nh \right) |\text{Im } z'| \right) \left. \right\}.$$

For any $\varepsilon > 0$ and T' with $0 < T' < T_0$, we can find φ_1 as $\varphi_1(T') < \varepsilon$. Thus, we can conclude that \tilde{u} is analytic on some complex neighborhood of

$$U^\sigma := \left\{ (z, t) \in D'; 0 \leq \sigma_j \nu_j t_j \ (1 \leq j \leq d), \ 0 < \sum_{j=1}^d \sigma_j \nu_j t_j < T_0, \right.$$

$$\text{Im } z_1 > \exp \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right) \left(M_1 \left(\sum_{j=1}^d \sigma_j \nu_j t_j \right)^{1/|m|} + nh \right) |\text{Im } z'| \left. \right\}.$$

On the other hand, there exists a sufficiently small positive constant ε such that

$$\{(x + \sqrt{-1}(s, 0, \dots, 0), t) \in \mathbb{C}^n \times \mathbb{R}^d; |x| + |t| < \varepsilon, 0 < s < \varepsilon\} \subset U_1 \cup \bigcup_{\sigma \in (-1, 1)^d} U^\sigma.$$

Thus, by Theorem 3.1.1 of [S-K-K] Chapter I each boundary value $b(\tilde{u}_\nu)$ ($1 \leq \nu \leq N$) as a hyperfunction is well-defined at $(x, t) = (0, 0)$. Further, it is easy to see that $u(x, t) := (\text{sp}_M(b(\tilde{u}_1)), \dots, \text{sp}_M(b(\tilde{u}_N))) \in \rho_!(\mathcal{C}_M|_{N \times_M T_M^* X})_{\rho_0}^{\oplus N}$. This completes the proof of Theorem 4.2. □

4.5. Corollary. *Let P be a matrix of an analytic differential operators of Fuchsian type defined on a neighborhood of $(x, t) = (0, 0)$. Assume [A-1] and the following:*

[A-3]. $\det(\sigma_{|m^{(v)}|}(P^{(u, v)})(z, \tau; \zeta, \eta)) = \tau^{\tilde{k}} \tilde{P}(z, \tau; \zeta, \eta)$ for a function \tilde{P} ($\tilde{k} := \sum_{v=1}^N k^{(v)} \in \mathbb{N}_0^d$) satisfying the following:

For any $\zeta^* \in \mathbb{R} \setminus \{0\}$, there exists $\Theta \in GL(n; \mathbb{R})$ such that $\Theta(1, 0, \dots, 0) = \zeta^*$ and that $\tilde{P}(z, t; \Theta \zeta, \eta)$ satisfies the condition in [A-2].

Then, for any hyperfunctions with holomorphic parameters

$$(4.22) \quad f(x, t), g(x, t) \in (\mathcal{B}\mathcal{O}_{L|M})_0^{\oplus N},$$

there exists an N -tuple of hyperfunctions

$$u(x, t) \in (\mathcal{B}_M)_0^{\oplus N}$$

such that u has t as real analytic parameters and is a solution of the Goursat problem

$$(G.P.) \quad \begin{cases} P(x, t; \partial_x, \partial_t) u(x, t) = f(x, t), \\ u(x, t) - g(x, t) = O(t^{m-k}). \end{cases}$$

Proof. We may assume that $g \equiv 0$. By the flabbiness of the sheaves of microfunctions and hyperfunctions and by (4.22), we can find a representation

$$f(x, t) = \sum_{j=1}^J F_j f_j(x, t) + \sqrt{-1}\Gamma_j 0,$$

where each N -tuple $F_j = \tilde{f}_j + f_j'$ is of type (4.8). Moreover by [A-3], we may assume that we can apply the proof of Theorem 4.2; that is, for any $1 \leq j \leq J$ we can find an N -tuple \tilde{u}_j of holomorphic functions on some infinitesimal wedge-type domain $D + \sqrt{-1}\Gamma_j' 0$ such that

$$\begin{cases} P\tilde{u}_j = \tilde{f}_j, \\ \tilde{u}_j = O(t^{m-k}). \end{cases}$$

On the other hand, since an N -tuple $f' := \sum_{j=1}^J f_j'$ is analytic in a neighborhood of $(x, t) = (0, 0)$ there exists a unique N -tuple u' of analytic functions such that

$$\begin{cases} Pu' = f', \\ u' = O(t^{m-k}), \end{cases}$$

by the argument to the proof of Theorem 1.3 and Madi's result [M]. Thus an N -tuple of hyperfunctions

$$u(x, t) = (u_1(x, t), \dots, u_N(x, t)) := \sum_{j=1}^J u_j(x, t) + \sqrt{-1}\Gamma_j' 0 + u'(x, t)$$

is a solution and has t as real analytic parameters. □

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