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Analytic Discs Attached to Manifolds with Boundary

By

Luca BARACCO* and Giuseppe ZAMPIERI*

§1. Introduction

Let X be a complex manifold of dimension n, M^+ a closed half-space with boundary M, A an analytic disc of X "attached" to M^+, tangent to M at some point z_0 of $\partial A \cap M$, and intersecting $\mathring{M}^+$ in any neighbourhood of z_0. Then holomorphic functions extend from $\mathring{M}^+$ to a full neighborhood of z_0. This theorem refines the results of [1] where the boundary ∂A (instead of the whole A) was supposed to intersect $\mathring{M}^+$. The argument of the proof consists in constructing a (closed) manifold with boundary W, contained in the envelop of holomorphy of $\mathring{M}^+$ and such that $A \subset W$ but $A \not\subset \partial W$. In this situation it is easy to find a new small disc $A_1 \subset A$ with $\partial A_1 \not\subset \partial W$. We are therefore in a situation similar to [1], and get the conclusion by exhibiting a disc transversal to ∂W at z_0.

Extension of holomorphic functions by the aid of tangent discs attached to M and of "defect 0" is a particular case of a general theorem of "wedge extendibility" of CR-functions by A. Tumanov; the new part of our theorem is that no assumptions on "defect" are made.

This paper is tightly inspired to the results and the techniques by A. Tumanov [7]. We also owe to A. Tumanov a great help during private communications.

§2.

Let X be a complex manifold of dimension n, M a real submanifold of

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* Department of Math., Univ. of Padova, V. Belzoni 7, 35131 Padova, Italy

X, M^+ one of the two closed half-spaces with boundary M, $A=A(\tau)$, $\tau\in\Delta$ an analytic disc of X, $z_0=A(1)$ a point of $\partial A\cap M$, $\{B\}$ the system of spheres of center z_0. Let $C^{k,\alpha}$ be the functions whose derivatives up to the order k are α-Lipschitz continuous. We assume M to be $C^{2,\alpha}$ and A to be $C^{1,\alpha}$ up to the boundary and small. We recall the result by [1].

Theorem A. ([1, Theorem 1]) *Assume*

(i) $$\partial_\tau A(1)\in T^{\mathbf{C}}_{z_0}M,$$

(ii) $$\partial A\subset M^+,$$

(iii) $$\text{there exists } z_1\in\partial A \text{ with } z_1\in\mathring{M}^+.$$

Then for any $B\supset A$ there is $B'\subset B$ such that holomorphic functions extend from $\mathring{M}^+$ to B'.

It is essential in the previous statement that $z_1\in\partial A$ (in addition to $z_1\in\mathring{M}^+$). As for the case $z_1\in\operatorname{int}A$, we can reduce to the former case when we strengthen (ii) to "$A\subset M^+$". In this case extension from $B\cap\mathring{M}^+$ to a suitable B' holds provided that B contains z_1 ([1, Corollary 4]).

What happens when $z_1\notin\partial A$ and $A\not\subset M^+$? Holomorphic extension seems not to take place. However it holds when a sequence converging to z_0 of such points $z_1\notin\partial A$ does exist.

Theorem 1. *Let M be $C^{3,\alpha}$, and assume that ∂A is a $C^{2,\alpha}$ curve with $A(1)=z_0$. Suppose*
(i) $\partial_\tau A(1)\in T^{\mathbf{C}}_{z_0}M$

(ii) $\partial A\subset M^+$

(iii) $A\cap\mathring{M}^+\cap B'\neq\emptyset\ \forall B'$.

Then if $B\supset A$ there exists B' such that any holomorphic function extends from $B\cap\mathring{M}^+$ to B'.

Proof. When $\partial A\not\subset M$, the statement is the same as in Theorem A. For completeness, we shall treat it at the end of the proof. Assume therefore $\partial A\subset M$.

(a) Construction of a manifold with boundary W such that $W\supset A$. We assume M is described by $y_1=h(x_1,z')$, $z'\in\boldsymbol{C}^{n-1}=T^{\boldsymbol{C}}_{z_0}M$, $h(0,0)=0$, $\partial h(0,0)=0$. Set $v_2=i\partial_\tau A(1)$ $(\in T^{\boldsymbol{C}}_{z_0}M)$, choose $v_1\in T_{z_0}M$ transversal to $T^{\boldsymbol{C}}_{z_0}M$, e.g. $v_1=e_1$, and write:

$$T_{z_0}M=\boldsymbol{R}v_1\oplus\boldsymbol{C}v_2\oplus\boldsymbol{C}^{n-2}.$$

Let $w'=z'\circ A$ take $w''_0\in\boldsymbol{C}^{n-2}$, $s\in\boldsymbol{R}$ and consider the equation

$$u(\tau)=-T_1(h(u(\tau),w'(\tau)+(0,w''_0))+s \qquad \tau\in\partial\Delta \tag{1}$$

in the unknown u. Note that for $s=0$, $w''_0=0$, the "x_1-component" u of A satisfies (1). We need the following technical tool.

Lemma 2. *Let h be $C^{k,\alpha}$. Then for any w' in $C^{k-1,\alpha}$ small, there is an unique solution to* (1) $u(\tau)=u_{s,w''_0}(\tau)$, $\tau\in\partial\Delta$, *which belongs to* $C^{k-1,\alpha}|_{\tau,s,w''_0}$.

Proof. For $j\le k-1$, let

$$F\colon C^{j,\alpha}(\partial\Delta,\boldsymbol{R})\times\boldsymbol{C}^{n-2}\times\boldsymbol{R}\to C^{j,\alpha}(\partial\Delta,\boldsymbol{R})$$

$$(u,w''_0,s)\mapsto u+T_1(h(u,w'+(0,w''_0)))-s.$$

Let $v=h(u,w'+(0,w''_0))$ in $\partial\Delta$, and define

$$D_{w''_0,s}=(u+iv,w'+(0,w''_0)) \quad \text{in } \partial\Delta.$$

We have:

$$D_{w''_0,s} \text{ extends holomorphically to } \Delta \text{ with } D_{w''_0,s}(1)=s$$

if and only if

$$F=0.$$

If h is $C^{k,\alpha}$, then F is C^1 (as application between functional spaces) and we have

$$F'(\dot u,w''_0,s)=\dot u+T_1(\partial_x h\dot u+\partial_{w''}hw''_0+\partial_{\bar w''}h\bar w'')-s.$$

We have

$$\begin{cases}F(0,0,0)=0\\ F'(\dot u)=\dot u+T_1(\partial_x h\dot u).\end{cases}$$

Thus the equation $F=0$ has solution u in $C^{j,\alpha}(\partial\Delta,\boldsymbol{R})$ (and this depends $C^{k-1-j,\alpha}$

on data w, x, ζ and on w_0'', s. (Cf. [5])). Q.E.D.

In particular, since h is $C^{3,\alpha}$ and A is $C^{2,\alpha}$ up to the boundary, then setting $w'=z'\circ A$, (1) has an unique solution

$$u=u_{w_0'',s}(\tau)\in C^{2,\alpha}|_{\tau,w_0'',s}.$$

With $v=T_1(u)+h(s,w_0)$, define $D=D_{w_0'',s}$ as in the proof of Lemma 2. Clearly for $s=0$ and $w_0''=0$, D equals the initial disc A. Note also that for any s and w_0'', D is attached to M.

We have that

$$D: \boldsymbol{C}^{n-2}_{w_0''}\times\Delta_\tau\times\boldsymbol{R}_s\to\boldsymbol{C}^n$$
$$(w_0'',\tau,s)\mapsto D_{w_0'',s}(\tau)$$

verifies

$$\mathrm{rank}^{\boldsymbol{R}}\partial_{w_0''\tau,s}D=2n-1$$

and therefore the range of D is a manifold S of class $C^{1,\alpha}$ with boundary $\partial S=D(\boldsymbol{C}^{n-2}\times\partial\Delta_\tau\times\boldsymbol{R}_s)$ in a neighborhood of $z_0=D_{0,0}(1)$. Note that ∂S is generic and $\partial S\subset M$. Note also that $A\subset S$. Define

$$W=\bigcup_{\zeta_1}-ie_1\zeta_1+S.$$

It is clear that any holomorphic function extends from $\mathring{M}^+$ to $W\cup\mathring{M}^+$. In a neighborhood B_1 of $z_0=0$, possibly smaller than B, we can describe W by:

(2) $$y_1=g(x_1,x_2+i\zeta_2,z'')+\zeta_1 \qquad \zeta_1\le 0,\ \zeta_2\le 0,$$

and

(3) $$S: W\cap\{\zeta_1=0\}.$$

(b) Perturbation of W and of A such that $\partial A\not\subset W$. We perturb g for $\zeta_2>\varepsilon$ to a new function $\tilde{g}\ge g$ such that $\{y_1<\tilde{g}\}\subset W\cup\mathring{M}^+$ and such that the initial disc $A=(u+iv,w)$ verifies in a point $\tau_1\in\Delta$:

(4) $$v(\tau_1)<\tilde{g}(u(\tau_1,w(\tau_1)).$$

We also suppose $z_1=A(\tau_1)$ close to z_0. Let us still write g instead of $\tilde{g}$ and denote by S and W the manifolds with boundary defined by (2) and (3) for this new g. Thus (4) is equivalent to

$$z_1 \in A \cap \mathring{W}.$$

Take $\tilde{\Delta} \subset \Delta$ such that $\tau_1 \in \partial\tilde{\Delta}$ and $\partial\tilde{\Delta} = \partial\Delta$ at 1. Set $\tilde{A} = A \circ \Phi$ where Φ is an analytic diffeomorphism $\Delta \underset{\Phi}{\rightarrow} \tilde{\Delta}$. Let us write (2) as

$$(5) \qquad \begin{cases} y_1 = g(x_1, x_2, z'') + \zeta_1 \\ y_2 = \zeta_2 \end{cases}$$

with ζ_1, $\zeta_2 \leq 0$. In these coordinates S is defined by $\zeta_1 = 0$ and ∂S by $\zeta_1 = \zeta_2 = 0$. Note that $\partial\tilde{A} = \partial A$ at z_0 and that $\partial A \subset \partial S$; it follows that $\zeta_1 = 0$, $\zeta_2 = 0$ at $\tau = 1$. We shall now use only the new disc $\tilde{A}$, and call again A.

(c) Construction of a transversal disc. Let $\zeta_1(\tau), \zeta_2(\tau)$ be defined by (5) over A_1. Consider the system:

$$(6) \qquad \begin{cases} u_1 = -T_1(g(u_1(\tau), u_2(\tau) + i\zeta_2(\tau), w''(\tau)) + (1-\eta)\zeta_1(\tau)), & \tau = e^{i\theta} \\ u_2(\tau) = -T_1(\zeta_2(\tau)), & \tau = e^{i\theta}. \end{cases}$$

There exists an unique solution $(u_1, u_2) = (u_{1\eta}, u_{2\eta})$ in $C^{1,\alpha}|_\theta$. Moreover if we set $v_1 = T_1(u_1)$, $v_2 = T_1(u_2)$, and

$$A_\eta = (u_{1\eta} + iv_{1\eta}, u_{2\eta} + iv_{2\eta}, w'')$$

we have that

$$(7) \qquad \partial_\tau A_\eta \quad \text{is} \quad C^1|_\eta.$$

In fact $\partial_\eta u_{2\eta} \equiv 0$, whereas $\partial_\eta u_{1\eta}$ is a solution of:

$$\partial_\eta u_{1\eta} + T_1(\partial_{x_1} g \partial_\eta u_{1\eta} - \zeta_1) = 0.$$

Since ∂g and ζ_1 are $C^{2,\alpha}$, then $\partial_\eta u_{1\eta}$ is $C^{1,\alpha}(\partial\Delta, \boldsymbol{R})$; in particular, with $\tau = e^{i\theta}$, $\partial_\theta \partial_\eta u_{1\eta}$ exists and is continuous.

As for $\partial_\theta \partial_\eta v_{1\eta}$, we begin by remarking that

$$v_{1\eta} = g(u_{1\eta}, u_2 + i\zeta_2, w'') + (1-\eta)\zeta_1 \text{ on } \partial\Delta.$$

It follows that

$$\partial_\eta v_{1\eta} = \partial_{x_1} g(u_{1\eta}, w)\partial_\eta u_\eta - \zeta \text{ belongs to } C^{1,\alpha}(\partial\Delta, \boldsymbol{R}).$$

We derive with respect to θ and obtain:

$$\begin{aligned}\partial_\theta\partial_\eta v_{1\eta} &= \partial_\theta(\partial_{x_1} g(u_{1\eta}, w)\partial_\eta u_\eta - \zeta)\\ &= \partial^2_{x_1} g \partial_\theta u_{1\eta}\partial_\eta u_{1\eta} + \partial_{x_1} g \partial_\theta \partial_\eta u_{1\eta} - \partial_\theta \zeta_1 .\end{aligned}$$

It is easy to check that all the terms on the right have at least class C^0; thus also $\partial_\theta\partial_\eta v_{1\eta}$ is C^0. Finally Cauchy-Riemann equations yield:

$$\partial_\theta(u_{1\eta} + iv_{1\eta}) = ie^{i\theta}\partial_\tau(u_{1\eta} + iv_{1\eta})$$

and we are done. It follows from (7):

$$\partial_\tau A_\eta = \partial_\tau A + \eta\partial_\tau\partial_\eta A_\eta|_{\eta=0} + o(\eta). \tag{8}$$

Set $r = y_1 - g$. We prove now that

$$\Re\mathrm{e}\langle \partial r \circ A,\ \partial_\tau \partial_\eta A_\eta \rangle|_{\tau=1} < 0. \tag{9}$$

In fact one finds a real function λ on $\partial\Delta$ such that $\lambda\partial r \circ A$ extends holomorphically to Δ and notice that then:

$$\langle \lambda\partial r \circ A,\ \partial_\eta A_\eta \rangle = \langle \lambda\partial_1 r \circ A,\ \partial_\eta A_\eta \rangle$$

is a holomorphic function. The real part φ of this holomorphic function verifies $\varphi|_{\partial\Delta} = -\zeta\lambda$ whence: $\varphi|_{\partial\Delta} \geq 0$, $\varphi(1) = 0$, $\varphi(\tau_1) > 0$. (Here τ_1 is such that $A(\tau_1) = z_1$.) Then Hopf Lemma implies (9). Plugging together (8) and (9) we get

$$\Re\mathrm{e}\langle \partial r, \partial_\tau A_\eta \rangle(1) < 0. \tag{10}$$

(d) Construction of a dihedron. We denote again by ζ_1, ζ_2, w'' the components of this new disc A_η and solve the equations:

$$\begin{cases} u_1 = -T_1(g(u_1(\tau), u_2(\tau) + i\zeta_2(\tau), w''(\tau) + w''_0) + \zeta_1(\tau)) + s \\ u_2 = -T_1(\zeta_2(\tau)) \\ v_1 = -T_1 u_1 + g(s, 0, w''_0) \\ v_2 = T_1(u_2). \end{cases} \tag{11}$$

Let $A = A_{s,w''_0}$ be defined by (11). Note that

$$\begin{cases} \partial A \subset W \text{ and } \partial A \subset S \text{ at } z_0 \\ A \text{ is transversal to } M \text{ (for } s, w'' \text{ small).} \end{cases} \tag{12}$$

Define

$$S_1 = \{A_{s.w_0''}(\tau); \tau \in \Delta, s \text{ and } w_0'' \in \boldsymbol{R} \times \boldsymbol{C}^{n-2}\}. \tag{13}$$

We have

$$\begin{cases} \partial S_1 \text{ generic} \\ \partial S_1 = \partial S \text{ at } z_0 \text{ (hence } \partial S_1 \subset M) \end{cases} \tag{14}$$

where the latter of (14) follows from the former of (12). By using S_1 and M one gets a dihedron V with non-proper tangent cone, such that any f holomorphic on $\mathring{M}^+ \cap B$, for $B \supset A$, is extended to V at z_0.

(e) Conclusion of the proof in the case $\partial A \subset M$. The dihedron V has generic edge ∂S_1. We approximate V by an increasing sequence V_ν of domains with C^2 boundary such that

$$V_\nu \subset V, \ \partial V_\nu \supset \partial S_1 \quad \forall \nu.$$

It is obvious that for large ν, V_ν has at least one Levi-pseudoconcavity. Then germs of holomorphic functions extends from V_ν to a full neighborhood of z_0 (according to a famous theorem by Hans Lewy).

(f) The case $\partial A \not\subset M$ (cf. [1]). In this case the proof is simpler. Let M be defined by $y_1 = h(x, z')$ $(z' = (z_2, z_3, \cdots))$ with $h(0) = 0$, $\partial h(0) = 0$, write $A = (u + iv, w')$, and put $\zeta_1 = v - (h \circ A)$. Solve

$$u_\eta = -T_1(h(u_\eta(\tau), w'(\tau) + (1-\eta)(\zeta_1(\tau)). \tag{15}$$

This produces a family of discs $A_\eta = (u_\eta + iv_\eta, w')$ $(v_\eta = T_1(u_\eta))$ which verify by the same argument as above:

$$\Re e < \partial r \circ A, \ \partial_\tau A_\eta > |_{\tau=1} < 0$$

for any sufficiently small η. Let ζ_1, w' still denote the components of A_η and solve

$$u(\tau) = -T_1(h(u(\tau), z'(\tau) + w_0') + \zeta_1(\tau)) + s. \tag{16}$$

By taking the union of the discs $A_{s,w_0'}(\tau)$ one gets a manifold S_1 such that $\partial S_1 \subset M^+$ and $S_1 \cap M$ is a generic manifold. We thus get a dihedron V with edge $S_1 \cap M$ and such that functions extend from $\mathring{M}^+ \cap B$ to V at z_0 $(B \supset A)$. The conclusion is the same as above. Q.E.D.

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