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An Upper Bound for the Characteristic Variety of an Induced $\mathscr{D}$-Module

By

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Abstract

We generalise the $Car_Y^{\mathbf{Z}}(\mathfrak{M})$ upper bound of Laurent & Schapira [LS87] for the characteristic variety of the induced system of a coherent $\mathscr{D}_X$-module $\mathfrak{M}$ on a hypersurface Y of X, to the case where Y is a smooth submanifold of X of arbitrary codimension

Contents

1. Introduction

Given a complex analytic manifold X and a smooth submanifold Y of X, let $T^*X \to X$ be the cotangent bundle of X, $T^*Y \to Y$ the cotangent bundle of Y, $T_YX \xrightarrow{\lambda} Y$ the normal bundle of Y in X, $T_Y^*X \xrightarrow{\lambda^*} Y$ the conormal bundle of Y in X, and let ρ and $\bar{\omega}$ be the maps cannonically associated to the immersion $Y \xrightarrow{j} X$:

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$$T^*Y \overset{\rho}{\leftarrow} Y \times_X T^*X \overset{\varpi}{\rightarrow} T^*X.$$

Let $\mathcal{O}_X$ be the structural sheaf of X, $\mathcal{J}_Y$ the defining ideal of Y, $\mathcal{O}_Y=\mathcal{O}_X/\mathcal{J}_Y$ the structural sheaf of Y, $\mathcal{D}_X$ the sheaf of holomorphic differential operators of finite order in X, $\mathcal{D}_{X|Y}$ the restriction of $\mathcal{D}_X$ to Y, and let

$$\mathcal{D}_{Y\to X}=\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{D}_{X|Y}=\mathcal{D}_X/\mathcal{J}_Y\mathcal{D}_X$$

be the transfer bimodule from Y to X. Given a coherent $\mathcal{D}_X$-Module $\mathfrak{M}$, let $\mathfrak{M}_Y^\circ=\mathcal{D}_{Y\to X}\overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X}\mathfrak{M}$ be the induced $\mathcal{D}_Y$-Module in Y. Define

$$\mathfrak{M}_Y^k=\mathbb{H}^{-k}(\mathfrak{M}_Y^\circ)=\mathbb{T}\mathrm{or}_k^{\mathcal{D}_X}(\mathcal{D}_{Y\to X},\mathfrak{M}).$$

Kashiwara [Ka83a] proved that, if $\mathfrak{M}$ is non-characteristic for Y, then

- ∘ the cohomology of the complex $\mathfrak{M}_Y^\circ$ is concentrated in degree 0;
- ∘ $\mathfrak{M}_Y^0$ is a coherent $\mathcal{D}_Y$-module;
- ∘ $Car(\mathfrak{M}_Y^0)=\rho\varpi^{-1}Car(\mathfrak{M})$.

Consider now in $\mathcal{D}_{X|Y}$ the Kashiwara [Ka83b] *V-filtration* associated to the embedding $Y\overset{j}{\rightarrow}X$ and defined in degree k by

$$F_Y^k\mathcal{D}_X=\{P\in\mathcal{D}_{X|Y}: P\mathcal{J}_Y{}^l\subset\mathcal{J}_Y{}^{l-k}\quad\forall\, l\in\mathbb{N}\},$$

and let $F_Y^k\mathcal{D}_{Y\to X}=\frac{F_Y^k\mathcal{D}_{X|Y}}{\mathcal{J}_Y\cap\mathcal{D}_{X|Y}}$ be the degree k of the corresponding $F_Y\mathcal{D}_{Y\to X}$ quotient filtration.

Let $\mathfrak{M}$ be an arbitrary coherent $\mathcal{D}_X$-module not necessarily non characteristic for Y. In [LS87] Laurent & Schapira proved that

- ∘ $\mathfrak{M}_Y^k$ is a union of an increasing sequence of coherent $\mathcal{D}_Y$-modules.

So they could define the notion of characteristic variety of $\mathfrak{M}_Y^\circ$, $Car(\mathfrak{M}_Y^\circ)$. Moreover by [Sch85] the sheaf of graded rings $gr_Y(\mathcal{D}_X)$ is isomorphic to the subsheaf $\lambda_*\mathcal{D}_{[T_YX]}$ of rings of holomorphic differential operators of finite order on T_YX that are algebraic in the fibers, and if $F_Y\mathfrak{M}$ is a $F_Y\mathcal{D}_X$-good filtration on $\mathfrak{M}$ then the graded module of $\mathfrak{M}$ for this filtration, $gr_Y(\mathfrak{M})$, is a $gr_Y(\mathcal{D}_X)$-coherent module. Denoting by $\widehat{C}_{T_YX}(\mathfrak{M})\subset T^*T_YX$ the formal microcharacteristic variety of $\mathfrak{M}$ along Y, i.e. the characteristic variety of $\mathcal{D}_{T_YX}\otimes_{\lambda^{-1}gr_Y(\mathcal{D}_X)}gr_Y(\mathfrak{M})$, it was proved in [LS87] that

- ∘ $Car(\mathfrak{M}_Y^\circ)\subset T^*Y\cap\widehat{C}_{T_YX}(\mathfrak{M})$.

Moreover, when Y is smooth embedded hypersurface of X, in [LS87] was defined a new subset of T^*Y, denoted $Car_Y^{\mathbf{Z}}(\mathfrak{M})$, and it was proved that

- ∘ $Car(\mathfrak{M}_Y^\circ)\subset Car_Y^{\mathbf{Z}}(\mathfrak{M})\subset T^*Y\cap\widehat{C}_{T_YX}(\mathfrak{M})$,

providing a better upper bound for $Car(\mathfrak{M}_Y^\circ)$.

The aim of this work is to generalize the construction of the $Car_Y^{\mathbf{Z}}(\mathfrak{M})$ of [LS87] to the case where Y is a smooth embedded submanifold of X of arbitrary codimension.

To finish this introductory section some of the above globally defined objects are computed in a special coordinate system.

The above objects in local coordinates. Let $(y, t)=(y_1, \ldots, y_{m-q}, t_1, \ldots, t_q)$ be a local coordinate system in X such that $Y=\{(y, t): t=0\}$. Then:

$$T_Y X=\{(y, \tau): y\in\mathbf{C}^{m-q}, \tau\in\mathbf{C}^q\},$$

and

$$\mathscr{D}_{Y\to X}\simeq\frac{\mathscr{D}_X}{t_1\mathscr{D}_X+\cdots+t_q\mathscr{D}_X}.$$

Let $\delta^\alpha=\delta^{(\alpha_1,\ldots,\alpha_q)}$ be the image of $\partial_{t_1}^{\alpha_1},\ldots, \partial_{t_q}^{\alpha_q}\in\mathscr{D}_X$ by the canonical projection $\mathscr{D}_X\to\mathscr{D}_{Y\to X}=\frac{\mathscr{D}_X}{t_1\mathscr{D}_X+\cdots+t_q\mathscr{D}_X}$. Then

$$F_Y^k\mathscr{D}_{Y\to X}\simeq\bigoplus_{|\alpha|\leq k}\mathscr{D}_Y\delta^\alpha,$$

and

$$\mathscr{D}_{Y\to X}\simeq\bigoplus_{k\geq 0}\bigoplus_{|\alpha|=k}\mathscr{D}_Y\delta^\alpha.$$

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2. Definition of $Car(\mathfrak{M}_Y^\bullet)$

Proposition 2.1. [LS87]. *Let $(X, \mathscr{O}_X)$ be a complex analytic manifold and let Y be a smooth submanifold of X. Let $\mathfrak{M}$ be a coherent $\mathscr{D}_X$-module. Then the $\mathscr{D}_Y$-modules $\mathfrak{M}_Y^k$ may be locally written as a union of an increasing sequence of coherent $\mathscr{D}_Y$-modules.*

Proof. Consider a local finite type free resolution of $\mathfrak{M}$:

$$0\longrightarrow{\mathscr{D}_X}^{m^p}\xrightarrow{A_{p-1}}\cdots\xrightarrow{A_0}{\mathscr{D}_X}^{m^0}\longrightarrow\mathfrak{M}\longrightarrow 0, \tag{1}$$

where $A_i\ (i=0, \ldots, p-1)$ is a $m_{i+1}\times m_i$ matrix of differential operators that acts on the right of ${\mathscr{D}_X}^{m^{i+1}}$. Tensoring (1) on the left by $\mathscr{D}_{Y\to X}\otimes_{\mathscr{D}_X}$ we get the complex

$$(\mathscr{D}_{Y\to X})^{m^p}\longrightarrow\cdots\longrightarrow(\mathscr{D}_{Y\to X})^{m^0}$$

which is quasi-isomorphic to $\mathfrak{M}_Y^\bullet$. Then

$$\begin{aligned}\mathbf{Ker}(A_{i-1}) &= \bigcup_{k\in\mathbf{N}}\mathbf{Ker}(F_Y^k{\mathscr{D}_{Y\to X}}^{m^i}\longrightarrow{\mathscr{D}_{Y\to X}}^{m^{i-1}})\\ &= \bigcup_{k\in\mathbf{N}}\mathbf{Ker}(F_Y^k{\mathscr{D}_{Y\to X}}^{m^i}\longrightarrow F_Y^{k+l}{\mathscr{D}_{Y\to X}}^{m^{i-1}})\end{aligned}$$

for a big enough $l\geq 0$. Setting

$$K_t(k) = \mathbb{K}\mathrm{er}\,(F^k_Y\mathscr{D}_{Y\to X}{}^{m^t} \longrightarrow F^{k+l}_Y\mathscr{D}_{Y\to X}{}^{m^{t-1}})$$

we have that $K_t(k) \subset K_t(k+1)$ and that $K_t(k)$ is a coherent $\mathscr{D}_Y$-module. This proves that $\mathbb{K}\mathrm{er}\,(A_{i-1})$ is a union of an increasing sequence of coherent $\mathscr{D}_Y$-modules. On the other hand we have:

$$\begin{aligned}\mathbb{I}\mathrm{m}\,(A_t) &= \bigcup_{k\in\mathbf{N}} \mathbb{I}\mathrm{m}\,(\mathscr{D}_{Y\to X}{}^{m^{t+1}} \longrightarrow \mathscr{D}_{Y\to X}{}^{m^t}) \cap F^k_Y\mathscr{D}_{Y\to X}{}^{m^t} \\ &= \bigcup_{k\in\mathbf{N}} \bigcup_{l\in\mathbf{N}} \mathbb{I}\mathrm{m}\,(F^l_Y\mathscr{D}_{Y\to X}{}^{m^{t+1}} \longrightarrow \mathscr{D}_{Y\to X}{}^{m^t}) \cap F^k_Y\mathscr{D}_{Y\to X}{}^{m^t}.\end{aligned}$$

Setting

$$I_t(k) = \bigcup_{l\in\mathbf{N}} \mathbb{I}\mathrm{m}\,(F^l_Y\mathscr{D}_{Y\to X}{}^{m^{t-1}} \longrightarrow \mathscr{D}_{Y\to X}{}^{m^t}) \cap F^k_Y\mathscr{D}_{Y\to X}{}^{m^t},$$

we see that $I_i(k)$ is a union of an increasing sequence of coherent sub-$\mathscr{D}_Y$-modules of the coherent $\mathscr{D}_Y$-module $F^k_Y\mathscr{D}_{Y\to X}{}^{m^t}$. Being $\mathscr{D}_Y$ a noetherian sheaf of rings, $I_t(k)$ is a coherent $\mathscr{D}_Y$-module. Finally we have $\mathbb{I}\mathrm{m}\,(A_t) = \bigcup_{k\in\mathbf{N}} I_t(k)$ and $I_t(k) \subset I_t(k+1)$. Hence it follows that $\mathbb{I}\mathrm{m}\,(A_t)$ is also a union of an increasing sequence of coherent $\mathscr{D}_Y$-modules. □

Now let $\mathfrak{N}$ be a left $\mathscr{D}_Y$-module, locally a union of an increasing sequence of coherent $\mathscr{D}_Y$-modules $(\mathfrak{N}_k)_{k\in\mathbf{N}}$. Then the subset

$$Car(\mathfrak{N}) := \bigcup_{k\in\mathbf{N}} Car(\mathfrak{N}_k)$$

does not depend on the sequence $(\mathfrak{N}_k)_{k\in\mathbf{N}}$ and is called the *Characteristic Variety* of $\mathfrak{N}$.

If $0 \longrightarrow \mathfrak{N}' \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{N}'' \longrightarrow 0$ is an exact sequence of $\mathscr{D}_Y$-modules of the preceding type then

$$Car(\mathfrak{N}) = Car(\mathfrak{N}') \cup Car(\mathfrak{N}'').$$

Definition 2.2. *If $\mathfrak{N}^\circ$ is a bounded complex of $\mathscr{D}_Y$-modules such that the cohomology groups are $\mathscr{D}_Y$-modules of the preceding type the* characteristic variety *of the complex $\mathfrak{N}^\circ$ is defined to be the following subset of T^*Y:*

$$Car(\mathfrak{N}^\circ) = \bigcup_{j\in\mathbf{Z}} Car(\mathbb{H}^j(\mathfrak{N}^\circ)).$$

In particular if $\mathfrak{M}$ is a coherent $\mathscr{D}_X$-module, then the characteristic variety of $\mathfrak{M}^\circ_Y$ is the following subset of T^*Y

$$Car(\mathfrak{M}^\circ_Y) := \bigcup_{j\in\mathbf{N}} Car(\mathbb{H}^{-j}(\mathfrak{M}^\circ_Y)).$$

3. Differential Operators on a Holomorpic Vector Bundle

Given a holomorphic vector bundle of rank q over the complex analytic manifold Y,

$$\Lambda \xrightarrow{\lambda} Y,$$

let $\theta = e_\lambda$ be the *Euler-vector field* of Λ. Given an integer k let

$$\mathcal{O}_{[\Lambda]}[k] = \{f \in \mathcal{O}_\Lambda : \theta f = kf\}$$

be the sheaf of holomorphic functions on Λ that are homogeneous of degree k in the fibers, and let

$$\mathcal{D}_{[\Lambda]}[k] = \{P \in \mathcal{D}_\Lambda : [\theta, P] = -kP\}$$

be the sheaf of holomorphic differential operators on Λ that are homogeneous of degree k in the fibers. The following proposition is clear:

Proposition 3.1. [LS87] *The map* $\lambda_* \mathcal{O}_{[\Lambda]}[0] \xrightarrow{\rho} \mathcal{O}_Y$, $f \mapsto f_{|Y}$ *is an isomorphism of* $\mathcal{O}_Y$*-modules.*

The sheaf $\lambda_* \mathcal{D}_{[\Lambda]}[0]$ acts on the left of $\lambda_* \mathcal{O}_{[\Lambda]}[0]$ and so also on $\mathcal{O}_Y$. This defines a morphism of sheaf of rings $\lambda_* \mathcal{D}_{[\Lambda]}[0] \xrightarrow{\rho} \mathcal{D}_Y$.

If (y, t) is a local trivialization of Λ such that $\lambda(y, t) = y$, then the differential operators $P \in \mathcal{D}_{[\Lambda]}[k]$ are those that may be written in that coordinate system in the form:

$$P = \sum_{|\alpha| - |\beta| = k} P_{\alpha,\beta}(y, \partial_y) t^\alpha \partial_t^\beta.$$

In particular the differential operators $P \in \mathcal{D}_\Lambda[0]$ are those that may be written in the form:

$$P = \sum_{|\alpha| = |\beta|} P_{\alpha,\beta}(y, \partial_y) t^\alpha \partial_t^\beta.$$

and we have

$$\rho(P) = P_{0,0}(y, \partial_y).$$

Thus, locally, $\lambda_* \mathcal{D}_{[\Lambda]}[0]$ is identified to

$$\mathcal{D}_Y\langle\theta\rangle := \mathcal{D}_Y[\theta_{11}, \theta_{12} \ldots, \theta_{1q}, \ldots, \theta_{q1}, \ldots, \theta_{qq}] / \{\text{commutation relations}\}$$

where, by definition, $\theta_{ij} = t_i \partial_{t_j}$, and the commutation relations between the variables θ_{ij} are the following ones:

$$[\theta_{ij}, \theta_{kl}] = [t_i \partial_{t_j}, t_k \partial_{t_l}] = \begin{cases} 0 & \text{if } j \neq k \text{ and } i \neq l \\ \theta_{il} & \text{if } j = k \text{ and } i \neq l \\ \theta_{ii} - \theta_{jj} & \text{if } j = k \text{ and } i = l \\ -\theta_{kj} & \text{if } j \neq k \text{ and } i = l \end{cases} \tag{2}$$

In particular, locally, ρ is identified to $\rho(P(y, \partial_y, \theta_{ij})) = P(y, \partial_y, 0)$

If $\mathfrak{N}$ is a coherent $\lambda_* \mathcal{D}_{[\Lambda]}[0]$-module the coherent $\mathcal{D}_Y$-module $\rho(\mathfrak{N})$ is defined by "extension" of scalars:

$$\rho(\mathfrak{N})=\mathscr{D}_Y\otimes_{\lambda_*\mathscr{D}_{[A]}[0]}\mathfrak{N},$$

thus having a characteristic variety $Car(\rho(\mathfrak{N}))$ in its own right, which is an involutive analytic subset of T^*Y.

Proposition 3.2.

(a) *If* $0\longrightarrow\mathfrak{N}'\longrightarrow\mathfrak{N}\longrightarrow\mathfrak{N}''\longrightarrow 0$ *is an exact sequence of coherent* $\lambda_*\mathscr{D}_{[A]}[0]$*-modules then*

$$Car(\rho(\mathfrak{N}))=Car(\rho(\mathfrak{N}'))\cup Car(\rho(\mathfrak{N}'')).$$

(b) *If* $\mathscr{J}$ *is coherent ideal of* $\lambda_*\mathscr{D}_{[A]}[0]$ *and if* $\mathfrak{N}=\lambda_*\mathscr{D}_{[A]}[0]/\mathscr{J}$ *then*

$$Car(\rho(\mathfrak{N}))=\{y^*\in T^*Y : \forall P\in\mathscr{J}\ \sigma(\rho(P))(y^*)=0\}$$

Proof. The problem being of local character we can set

$$\lambda_*\mathscr{D}_{[A]}[0]=\mathscr{D}_Y\langle\theta\rangle.$$

Let I be the left ideal of $\mathscr{D}_Y\langle\theta\rangle$ generated by $\theta_{11}, \theta_{12}, \ldots, \theta_{1q}, \ldots, \theta_{q1}, \ldots, \theta_{qq}$. Then:

- $\mathscr{D}_Y\simeq\dfrac{\mathscr{D}_Y\langle\theta\rangle}{I}$
- $\theta_{ij}\in I\ \forall i, j,$

and the commutation relations (2) give

- $\theta_{ij}\in I^k$ if $i\neq j$
- $\theta_{ii}-\theta_{jj}\in I^k\ \forall i, j$
- $\theta_{ii}^k-\theta_{jj}^k\in I^k\ \forall i, j\ \forall k\in\mathbb{N}.$

Let $F\mathscr{D}_Y\langle\theta\rangle$ be the non-separated filtration on $\mathscr{D}_Y\langle\theta\rangle$ defined by

$$F_k\mathscr{D}_Y\langle\theta\rangle=\begin{cases}\mathscr{D}_Y\langle\theta\rangle & \text{if } k\geq 0\\ I^{-k} & \text{if } k<0\end{cases}$$

The properties of $\mathscr{D}_Y\langle\theta\rangle$ listed above imply that the graded ring of $\mathscr{D}_Y\langle\theta\rangle$ for this filtration is isomorphic to the ring of polynomials $\mathscr{D}_Y[\bar{\theta}]$ in one variable $\bar{\theta}$ and with coefficients in $\mathscr{D}_Y$, where $\bar{\theta}$ is the image of all the $\theta_{ii}\in I^1$ $(i=1, \ldots, q)$ in the quotient I^1/I^2.

As $gr\mathscr{D}_Y\langle\theta\rangle\simeq\mathscr{D}_Y[\bar{\theta}]$ is a noetherian graded ring and

$$F_0\mathscr{D}_Y\langle\theta\rangle=\mathscr{D}_Y\langle\theta\rangle$$

is a noetherian filtered ring, proposition 1.1.8 of Chap. II of [Sch85] implies that the filtration $F\mathscr{D}_Y\langle\theta\rangle$ is a noetherian one.

Now let $gr\mathscr{D}_Y\langle\theta\rangle$ be filtered by the order of holomorphic differential operators in Y.

If $\mathfrak{N}$ is a coherent $\mathscr{D}_Y\langle\theta\rangle$-module equipped with a good $F\mathscr{D}_Y\langle\theta\rangle$-filtration the graded module of $\mathfrak{N}$ for this filtration, $gr(\mathfrak{N})$, is a graded coherent

$\mathcal{D}_Y[\bar{\theta}]$ -module whose characteristic variety $Car(gr\mathfrak{N})$ is an analytic subset of $T^*(Y)\times\mathbf{C}$.

By Proposition 1.3.1 of Chap. II of [Sch85], the characteristic variety $Car(gr\mathfrak{N})$ is independent of the choice of the good filtration on $\mathfrak{N}$ and the map that sends $\mathfrak{N}$ to $Car(gr\mathfrak{N})$ is an additive map, that is, if $0\longrightarrow\mathfrak{N}'\longrightarrow\mathfrak{N}\longrightarrow\mathfrak{N}''\longrightarrow 0$ is an exact sequence of coherent $\mathcal{D}_Y\langle\theta\rangle$ -modules then $Car(gr\mathfrak{N})=Car(gr\mathfrak{N}')\cup Car(gr\mathfrak{N}'')$.

Hence, to prove the first part of the proposition it is enough to prove that $Car(gr\mathfrak{N})=Car(\frac{\mathfrak{N}}{I\mathfrak{N}})\times\mathbf{C}$.

Suppose that $\mathfrak{N}$ is a coherent $\mathcal{D}_Y\langle\theta\rangle$ -module. Then the filtration on $\mathfrak{N}$ defined by

$$\mathfrak{N}_k=\begin{cases}\mathfrak{N} & \text{if } k\geq 0\\ I^{-k}\mathfrak{N} & \text{if } k<0\end{cases}$$

is a good filtration, and the graded module of $\mathfrak{N}$ for this filtration is

$$gr(\mathfrak{N})=\bigoplus_{k\geq 0}=\frac{I^k\mathfrak{N}}{I^{k+1}\mathfrak{N}}.$$

So, for all $k\in\mathbf{Z}$, $\frac{I^k\mathfrak{N}}{I^{k+1}\mathfrak{N}}$ is a coherent $\mathcal{D}_Y$-module and we have a surjective morphism of coherent $\mathcal{D}_Y$-modules

$$\frac{\mathfrak{N}}{I\mathfrak{N}}\xrightarrow{\bar{\theta}^k}\frac{I^k\mathfrak{N}}{I^{k+1}\mathfrak{N}}.$$

Thus

$$Car\left(\frac{I^k\mathfrak{N}}{I^{k+1}\mathfrak{N}}\right)\subset Car\left(\frac{\mathfrak{N}}{I\mathfrak{N}}\right)\subset T^*Y,$$

and

$$Car(gr\mathfrak{N})=\left(\bigoplus_{k\geq 0}Car\left(\frac{I^k\mathfrak{N}}{I^{k+1}\mathfrak{N}}\right)\right)\times\mathbf{C}=Car\left(\frac{\mathfrak{N}}{I\mathfrak{N}}\right)\times\mathbf{C}.$$

Part b) of the proposition follows from $\rho(\mathfrak{N})=\frac{\mathcal{D}_Y}{\rho(\mathcal{J})}$ where

$$\rho(\mathcal{J})=\{P(y,\partial_y,\theta_{11},\theta_{12}\ldots,\theta_{1q},\ldots,\theta_{q1},\ldots,\theta_{qq})_{|\theta_{ij}=0}:P\in\mathcal{J}\}.$$

□

Notation. For $k\in\mathbf{Z}$ the module $\mathcal{D}_{[\Lambda]}[k]$ is a coherent $\mathcal{D}_{[\Lambda]}[0]$ -bimodule (in fact it is locally free). Therefore, given a coherent $\lambda_*\mathcal{D}_{[\Lambda]}$-module $\mathfrak{N}$, we may consider the coherent $\mathcal{D}_Y$-module

$$\begin{aligned}\mathfrak{N}_{Y,k}&=\mathcal{D}_Y\otimes_{\lambda_*\mathcal{D}_{[\Lambda]}[0]}(\lambda_*\mathcal{D}_{[\Lambda]}[k]\otimes_{\lambda_*\mathcal{D}_{[\Lambda]}[0]}\mathfrak{N})\\&=\rho(\lambda_*\mathcal{D}_{[\Lambda]}[k]\otimes_{\lambda_*\mathcal{D}_{[\Lambda]}[0]}\mathfrak{N}).\end{aligned}$$

Example 3.3. *Let $P\in F_k^Y\mathscr{D}_X$ and let $\mathfrak{N}=gr^0(\frac{\mathscr{D}_X}{\mathscr{D}_XP})$, where $\frac{\mathscr{D}_X}{\mathscr{D}_XP}$ is equipped with the induced filtration $F^Y\mathscr{D}_{Y\to X}$. Then*

$$\begin{aligned}\mathfrak{N}_{Y,k}&=\mathscr{D}_Y\otimes_{\lambda_*\mathscr{D}_{[\Lambda]}[0]}\left(\lambda_*\mathscr{D}_{[\Lambda]}[k]\otimes_{\lambda_*\mathscr{D}_{[\Lambda]}[0]}gr^0\left(\frac{\mathscr{D}_X}{\mathscr{D}_XP}\right)\right)\\&=\mathscr{D}_Y\otimes_{\lambda_*\mathscr{D}_{[\Lambda]}[0]}\frac{gr^k(\mathscr{D}_X)}{gr^k(\mathscr{D}_X)\sigma_0(P)}\\&=\frac{gr^k(\mathscr{D}_{Y\to X})}{gr^k(\mathscr{D}_{Y\to X})\sigma_0(P)}.\end{aligned}$$

Example 3.4. *Given $k\geq 1$ let $P\in F_k^Y\mathscr{D}_{X|Y}\backslash F_{k-1}^Y\mathscr{D}_{X|Y}$. Then, in the special local coordinate system chosen in the introductory section,*

$$P=Q+\sum_{|\beta|=k}\partial_t^\beta Q_\beta,$$

where $Q\in F_{k-1}^Y\mathscr{D}_X$ and $Q_\beta\in F_0^Y\mathscr{D}_X$. Thus, locally,

$$gr^0\left(\frac{\mathscr{D}_X}{\mathscr{D}_XP}\right)\simeq\frac{gr^0\mathscr{D}_X}{\bigoplus_{|\alpha|=k}gr^0(\mathscr{D}_X)\tau^\alpha\sum_{|\beta|=k}\partial_\tau^\beta\hat{\sigma}(Q_\beta)}.$$

Since

$$\mathscr{D}_Y=\frac{gr^0\mathscr{D}_X}{gr^0\mathscr{D}_X(\tau_1\partial_{\tau 1},\ \ldots,\ \tau_1\partial_{\tau q},\ \ldots,\ \tau_q\partial_{\tau 1},\ \ldots,\ \tau_q\partial_{\tau q})},$$

it follows that

$$\begin{aligned}\mathfrak{N}_{Y,0}&=\mathscr{D}_Y\otimes_{gr^0\mathscr{D}_\lambda}gr^0\left(\frac{\mathscr{D}_X}{\mathscr{D}_XP}\right)\\&=\frac{gr^0\mathscr{D}_X}{gr^0\mathscr{D}_X(\tau_1\partial_{\tau 1},\ \ldots,\ \tau_q\partial_{\tau q})}\otimes_{gr^0\mathscr{D}_\lambda}gr^0\left(\frac{\mathscr{D}_X}{\mathscr{D}_XP}\right)\\&=0.\end{aligned}$$

Proposition 3.5. [LS87]
(i) Let $\mathfrak{M}$ be a coherent $\lambda_*\mathscr{D}_{[\Lambda]}$-module and let $\mathfrak{N}$ be a coherent sub-$\lambda_*\mathscr{D}_{[\Lambda]}[0]$-module of $\mathfrak{M}$ that generates $\mathfrak{M}$ over $\lambda_*\mathscr{D}_{[\Lambda]}$. Then

$$\mathfrak{S}(\mathfrak{M}):=\bigcup_{k\in\mathbf{Z}}Car(\mathfrak{N}_{Y,k})$$

is a subset of T^*Y which does not depend on the choice of $\mathfrak{N}$.
(ii) If $0\longrightarrow\mathfrak{M}'\longrightarrow\mathfrak{M}\longrightarrow\mathfrak{M}''\longrightarrow 0$ is an exact sequence of coherent $\lambda_*\mathscr{D}_{[\Lambda]}$-modules then

$$\mathfrak{S}(\mathfrak{M})=\mathfrak{S}(\mathfrak{M}')\cup\mathfrak{S}(\mathfrak{M}'').$$

Proof. (i) Let $\mathfrak{N}$ and $\mathfrak{N}'$ two coherent $\lambda_*\mathscr{D}_{[\Lambda]}[0]$-modules that generate $\mathfrak{M}$. As $\mathfrak{N}$ is a generator of $\mathfrak{M}$ we have

$$\mathfrak{N}'=\sum_{k\in\mathbf{Z}}(\lambda_*\mathcal{D}_{[\Lambda]}[k]\mathfrak{N})\cap\mathfrak{N}'$$

and so $\mathfrak{N}'=\bigcup_{k\in\mathbf{N}}\mathfrak{N}'^{(k)}$ where

$$\mathfrak{N}'^{(k)}=\sum_{-k\leq j\leq k}(\lambda_*\mathcal{D}_{[\Lambda]}[j]\mathfrak{N})\cap\mathfrak{N}'.$$

The sequence $(\mathfrak{N}'^{(k)})_{k\in\mathbf{Z}}$ is a sequence of coherent $\lambda_*\mathcal{D}_{[\Lambda]}[0]$-modules of $\mathfrak{N}'$, and being $\mathfrak{N}'$ of finite type this sequence must stabilize. Let k_0 be an integer such that $\mathfrak{N}'=\mathfrak{N}'^{(k_0)}$ and let $\mathfrak{N}''=\sum_{-k_0\leq j\leq k_0}(\lambda_*\mathcal{D}_{[\Lambda]}[j]\mathfrak{N})$.

Then

$$\bigcup_{k\in\mathbf{Z}}Car(\mathfrak{N}'_{Y,k})\subset \underset{k\in\mathbf{Z}}{Car}(\mathfrak{N}''_{Y,k})=\bigcup_{k\in\mathbf{Z}}Car(\mathfrak{N}_{Y,k}).$$

Reversing the roles of $\mathfrak{N}$ and $\mathfrak{N}'$ we get the first part of the proposition.

(ii) It is enough to prove that if $0\longrightarrow\mathfrak{N}'\longrightarrow\mathfrak{N}\longrightarrow\mathfrak{N}''\longrightarrow 0$ is an exact sequence of coherent $\lambda_*\mathcal{D}_{[\Lambda]}[0]$-modules then

$$Car(\mathfrak{N}_{Y,k})=Car(\mathfrak{N}'_{Y,k})\cup Car(\mathfrak{N}''_{Y,k}).$$

But this is an immediate consequence of Proposition 3.2 and of the flatness of $\lambda_*\mathcal{D}_{[\Lambda]}[k]$ over $\lambda_*\mathcal{D}_{[\Lambda]}[0]$. □

4. Definition of $Car_Y^{\mathbf{Z}}(\mathfrak{M})$

Now let $\Lambda=T_YX\overset{\lambda}{\longrightarrow}Y$ be the normal bundle of Y in X. Let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$-module and let $F_Y\mathfrak{M}$ be a good filtration on $\mathfrak{M}$. Then the graded module for this filtration, $gr_Y\mathfrak{M}$, is a coherent $\mathcal{D}_{[\Lambda]}$-module and $\mathfrak{N}=gr_Y^0(\mathfrak{M})$ generates $\mathfrak{M}$ over $\mathcal{D}_{[\Lambda]}$. Thus we can associate to $gr_Y(\mathfrak{M})$ the subset $\mathfrak{S}(gr_Y\mathfrak{M})$ of T^*Y. By Proposition (3.5), the functor $\mathfrak{N}\mapsto\mathfrak{S}(\mathfrak{N})$ is an additive one. By Proposition 1.3.1. of Chap. II of [Sch85], $\mathfrak{S}(gr_Y\mathfrak{M})$ is independent of the choice of the good $F_Y\mathcal{D}_X$-filtration and the functor $\mathfrak{M}\mapsto\mathfrak{S}(gr_Y\mathfrak{M})$ is an additive one. Therefore we have the following proposition

Proposition 4.1. *Let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$-module and let $F_Y\mathfrak{M}$ be a good $F_Y\mathcal{D}_X$-filtration on $\mathfrak{M}$. Then*

(i) $\mathfrak{S}(gr_Y\mathfrak{M})$ is a subset of T^*Y and does not depend on the choice of the good $F_Y\mathcal{D}_X$-filtration on $\mathfrak{M}$.

(ii) if $0\longrightarrow\mathfrak{M}'\longrightarrow\mathfrak{M}\longrightarrow\mathfrak{M}''\longrightarrow 0$ is an exact sequence coherent $\mathcal{D}_X$-modules then

$$\mathfrak{S}(gr_Y\mathfrak{M})=\mathfrak{S}(gr_Y\mathfrak{M}')\cup\mathfrak{S}(gr_Y\mathfrak{M}'').$$

This proposition enables us to make the following definition, as in [LS87]:

Definition 4.2. *Let $\mathfrak{M}$ be a coherent $\mathcal{D}_X$-module and let $F_Y\mathfrak{M}$ be a good $F_Y\mathcal{D}_X$-filtration on $\mathfrak{M}$. We define*

$$Car_Y^{\mathbf{Z}}(\mathfrak{M}) := \mathfrak{S}(gr_Y\mathfrak{M}) = \bigcup_{k\in\mathbf{Z}} Car(\mathfrak{N}_{Y,k})$$

where $\mathfrak{N}_{Y,k} = \mathcal{D}_Y \otimes_{\lambda_*\mathcal{D}_{[A]}[0]} \lambda_*\mathcal{D}_{[A]}[k] \otimes_{\lambda_*\mathcal{D}_{[A]}[0]} gr_Y^0(\mathfrak{M})$.

The goal of the remaining sections is to prove that $Car_Y^{\mathbf{Z}}(\mathfrak{M})$ is an upper bound for $Car(\mathfrak{M}_Y^{\circ})$.

5. The Case Where $\mathfrak{M}$ is the Module Defined by One Operator

Let $P \in \mathcal{D}_{X|Y}$ and $\mathfrak{M} = \mathcal{D}_X/\mathcal{D}_X P$. Then

$$0 \longrightarrow \mathcal{D}_X \xrightarrow{.P} \mathcal{D}_X \longrightarrow \mathfrak{M} \longrightarrow 0$$

is a free resolution of $\mathfrak{M}$. So, in the derived category,

$$\mathfrak{M}_Y^{\circ} \simeq \mathcal{D}_{Y\to X} \xrightarrow{.P} \mathcal{D}_{Y\to X} \simeq \frac{\mathcal{D}_X}{\mathcal{J}} \xrightarrow{.P} \frac{\mathcal{D}_X}{\mathcal{J}}$$

and

$$\mathfrak{M}_Y^0 = \mathbf{Ker}\ P, \quad \mathfrak{M}_Y^1 = \mathbf{Coker}\ P.$$

Let $P \in F_Y^0\mathcal{D}_X$. Then $(F_Y^k\mathcal{D}_{Y\to X}).P \subset F_Y^k\mathcal{D}_{Y\to X}$. Hence we can define

$$F_Y^k\mathcal{D}_{Y\to X} \xrightarrow{\widehat{\rho}_k(P)} F_Y^k\mathcal{D}_{Y\to X}$$

where $\widehat{\rho}_k(P)$ is the restriction of P to $F_Y^k\mathcal{D}_{Y\to X}$.

Consider now the action of P on the vectors δ^α of the base $(\delta^\gamma)_{|\gamma|\le k}$ of $F_Y^k\mathcal{D}_{Y\to X} = \bigoplus_{|\gamma|\le k} \mathcal{D}_Y\delta^\gamma$. If P is locally formally written as

$$P = \sum_{|\alpha|\ge|\beta|} P_{\alpha,\beta}(y,\partial_y)\, t^\alpha \partial_t^\beta$$

then

$$\delta^\gamma.P = \sum_{|\alpha|\ge|\beta|} P_{\alpha,\beta}(y,\partial_y)\frac{\gamma!}{(\gamma-\alpha)!}\delta^{\gamma-\alpha+\beta}.$$

Let $A(\gamma,\theta)$ the coefficient of δ^θ in the expression of $\delta^\gamma.P$. Then

$$A(\gamma,\theta) = \sum_{0\le\alpha\le\gamma} P_{\alpha,\theta-\gamma+\alpha}(y,\partial_y)\frac{\gamma!}{(\gamma-\alpha)!}.$$

Ordering the base $(\delta^\gamma)_{|\gamma|\le k}$ in such a way that all the δ^γ with $|\gamma|=i$ have orders lower than the δ^γ with $|\gamma|=i+1$, it follows that the matrix $A(k)$ of $\widehat{\rho}(k)$ in such a base is block-lower-triangular:

$$A(k)=\begin{pmatrix} A_{00} & 0 & \cdots & 0 \\ A_{10} & A_{11} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{k0} & A_{k1} & \cdots & A_{kk} \end{pmatrix},$$

where A_{ij} is the matrix $A(k)(\gamma, \theta)_{|\gamma|=i,|\theta|=j}$. Clearly that $A(k-1)$ is the matrix $A(k)$ with the last row and last column of blocks omitted. Thus

$$Car\left(\frac{F_Y^k\mathscr{D}_{Y\to X}}{F_Y^k\mathscr{D}_{Y\to X}\hat{\rho}_k(P)}\right)=\bigcup_{j=1}^{k}\{y^*\in T^*Y:\ \ det(A_{jj})(y^*)=0\}$$

where det is the determinant of Sato-Kashiwara [SK75].

But $\{y^*\in T^*Y:\ det(A_{kk})(y^*)=0\}$ is precisely $Car\left(\frac{gr_Y^k(\mathscr{D}_{Y\to X})}{gr_Y^k(\mathscr{D}_{Y\to X})\,\rho_k(P)}\right)$ where $gr_Y^k(\mathscr{D}_{Y\to X})\xrightarrow{\rho_k(P)}gr_Y^k(\mathscr{D}_{Y\to X})$ is the linear morphism whose matrix in base $(\delta^\gamma)_{|\gamma|=k}$ is A_{kk}. Observe now that $\rho_k(P)$ is the morphism $gr_Y^k\mathscr{D}_{Y\to X}\xrightarrow{\sigma_0(P)}gr_Y^k\mathscr{D}_{Y\to X}$. Then, from example (3.4) it follows that

$$\circ\ P\in F_Y^0\mathscr{D}_{X|Y}\Rightarrow Car\left(\left(\frac{\mathscr{D}_X}{\mathscr{D}_X.P}\right)_Y^{\circ}\right)=\bigcup_{k\in\mathbf{N}}\rho\left(\lambda_*\mathscr{D}_{[A]}[k]\otimes_{\lambda_*\mathscr{D}_{[A]}[0]}gr_Y^0\frac{\mathscr{D}_X}{\mathscr{D}_X.P}\right).$$

and thus the following property is true:

$$\circ\ P\in F_Y^0\mathscr{D}_{X|Y}\Rightarrow\begin{cases}Car(\mathfrak{M}_Y^{\circ})\subset Car_Y^{\mathbf{Z}}(\mathfrak{M})\\ Car_Y^{\mathbf{Z}}(\mathfrak{M})=\bigcup_{k\in\mathbf{Z}}\{y^*\in T^*Y:\ \ det(\rho_k(P))(y^*)=0\}\end{cases}$$

From example (3.3) and from definition of $Car_Y^{\mathbf{Z}}(\mathfrak{M})$ it follows that

$$\circ\ k>0 \text{ and } P\in F_Y^k\backslash F_Y^{k-1}\mathscr{D}_X\Rightarrow Car_Y^{\mathbf{Z}}(\mathfrak{M})=T^*Y.$$

Hence we have proved the following proposition

Proposition 5.1. *Let $P\in\mathscr{D}_{X|Y}$ and $\mathfrak{M}=\frac{\mathscr{D}_X}{\mathscr{D}_X P}$. Then*

(*a*) $Car(\mathfrak{M}_Y^{\circ})\subset Car_Y^{\mathbf{Z}}(\mathfrak{M})$.

(*b*) *Moreover if $P\in F_Y^0\mathscr{D}_X$ then*

$$Car_Y^{\mathbf{Z}}(\mathfrak{M})=\bigcup_{k\in\mathbf{Z}}\{y^*\in T^*Y:\ \ det(\rho_k(P))(y^*)=0\}.$$

6. The Case Where $\mathfrak{M}$ is the Module Defined by a Coherent Ideal

Let $\mathscr{J}$ be a coherent ideal of $\mathscr{D}_X$; then $gr_Y(\frac{\mathscr{D}_X}{\mathscr{J}})$ is generated by $gr_Y^0(\frac{F_Y^0\mathscr{D}_X}{\mathscr{J}^0})$ where $\mathscr{J}_0=\mathscr{J}\cap F_Y^0\mathscr{D}_X$. Therefore

$$(3)\qquad Car_Y^{\mathbf{Z}}\left(\frac{\mathscr{D}_X}{\mathscr{J}}\right)=\bigcup_{k\in\mathbf{Z}}Car\left((\mathscr{D}_Y\otimes_{\lambda_*\mathscr{D}_{[A]}[0]}\lambda_*\mathscr{D}_{[A]}[k])\otimes_{\lambda_*\mathscr{D}_{[A]}[0]}gr_Y^0\left(\frac{F_Y^0\mathscr{D}_X}{\mathscr{J}_0}\right)\right)$$

$$(4) \qquad = \bigcup_{k\in\mathbf{Z}} Car\left(gr^k\mathcal{D}_{Y\to X}\otimes_{\lambda_*\mathcal{D}_{[\Lambda]}[0]}\frac{\lambda_*\mathcal{D}_{[\Lambda]}[0]}{gr^0_Y(\mathcal{J}_0)}\right)$$

$$(5) \qquad = \bigcup_{k\in\mathbf{Z}} Car\left(\frac{gr^k\mathcal{D}_{Y\to X}}{gr^0_Y(\mathcal{J}_0)}\right)$$

Let P be an element of $\mathcal{J}$ and let $\sigma_0(P)$ be the image of P in $gr^0_Y(\mathcal{J}_0)$. Then

$$\frac{gr^k\mathcal{D}_{Y\to X}}{gr^k\mathcal{D}_{Y\to X}\sigma_0(P)} \longrightarrow \frac{gr^k\mathcal{D}_{Y\to X}}{gr^k\mathcal{D}_{Y\to X}gr^0_Y(\mathcal{J}_0)} \longrightarrow 0,$$

is an exact sequence of $gr^0\mathcal{D}_{Y\to X}$-modules. Thus, taking into account equation (3),

$$Car^{\mathbf{Z}}_Y\left(\frac{\mathcal{D}_X}{\mathcal{J}}\right)\subset \bigcap_{P\in\mathcal{J}} Car^{\mathbf{Z}}_Y\left(\frac{\mathcal{D}_X}{\mathcal{D}_X.P}\right),$$

i.e.

$$Car^{\mathbf{Z}}_Y\left(\frac{\mathcal{D}_X}{\mathcal{J}}\right)\subset \bigcup_{k\in\mathbf{Z}} \{y^*\in T^*Y\colon det(\rho_k(P))(y^*)=0 \quad \forall P\in\mathcal{J}\}.$$

The following proposition shows that the above inclusion is in fact an equality.

Proposition 6.1. *Let $\mathcal{J}$ be a coherent ideal of $\mathcal{D}_X$. Then*

$$Car^{\mathbf{Z}}_Y\left(\frac{\mathcal{D}_X}{\mathcal{J}}\right)= \bigcup_{k\in\mathbf{Z}} \{y^*\in T^*Y\colon det(\rho_k(P))(y^*)=0 \quad \forall P\in\mathcal{J}\}.$$

Proof. For each $k\in\mathbb{Z}$ let us denote by $\mathcal{L}_k$ the following $\mathcal{D}_Y$-module:

$$\mathcal{L}_k=\mathcal{D}_Y\otimes_{\lambda_*\mathcal{D}_{[\Lambda]}[0]}\lambda_*\mathcal{D}_{[\Lambda]}[k]=\begin{cases}\bigoplus_{|\alpha|=k}\mathcal{D}_Y\partial_t^\alpha & \text{if } k\geq 0\\ \bigoplus_{|\alpha|=-k}\mathcal{D}_Y t^\alpha & \text{if } k\leq 0\end{cases}.$$

Then we have a commutative diagram of ring homomorphisms

$$\begin{array}{ccc}\lambda_*\mathcal{D}_{[\Lambda]}[0] & \longrightarrow & End_{\mathcal{D}_Y}(\mathcal{L}_k).\\ \uparrow & \nearrow_{\rho_k} & \\ \mathcal{J}_0=\mathcal{J}\cap F^0_Y\mathcal{D}_X & & \end{array}$$

We denote by $\mathcal{N}_k$ the $\mathcal{D}_Y$-module $\mathcal{L}_k\rho_k(\mathcal{J}_0)$. With this notation, we see that, to prove the proposition, it is enough to prove that, for all $k\in\mathbb{Z}$,

$$Car\left(\frac{\mathcal{L}_k}{\mathcal{N}_k}\right)\supset \bigcap_{Q\in\rho_k(\mathcal{J}_0)} Car\left(\frac{\mathcal{L}_k}{\mathcal{L}_kQ}\right).$$

Let $e_1, \ldots, e_s$ be a basis of the free $\mathcal{D}_Y$-module $\mathcal{L}_k$ and we write $\mathcal{L}_k=\bigoplus_{t=1}^{s}\mathcal{D}_Y e_t$. We denote by $\mathcal{A}_k$ the ring of $\mathcal{D}_Y$-endomorphisms $End_{D_Y}(\mathcal{L}_k)$. Then $\mathcal{L}_k$ is a left-right $(\mathcal{D}_Y, \mathcal{A}_k)$-bimodule and the functor

$$\begin{aligned} \mathfrak{Mod}(\mathcal{D}_Y) &\xrightarrow{\mathbf{Hom}(\mathcal{L}_k,)} \mathfrak{Mod}(\mathcal{A}_k) \\ \mathfrak{M} &\longmapsto \mathbf{Hom}_{\mathcal{D}_Y}(\mathcal{L}_k, \mathfrak{M}) \end{aligned}$$

is an equivalence of categories. The correspondence

$$\begin{aligned} \mathfrak{Mod}(\mathcal{A}_k) &\xrightarrow{\mathcal{L}_k\otimes_{\mathcal{A}_k}} \mathfrak{Mod}(\mathcal{D}_Y) \\ \mathcal{Q} &\longmapsto \mathcal{L}_k\otimes_{\mathcal{A}_k}\mathcal{Q} \end{aligned}$$

is a left adjoint functor. This gives a correspondence

$$\begin{array}{ccc} \mathcal{L}_k & \longleftrightarrow & \mathcal{A}_k \\ \cup & & \cup \\ \mathcal{N}\,(submodule) & \longleftrightarrow & \mathcal{J}\,(ideal) \end{array},$$

between submodules of $\mathcal{L}_k$ and ideals of $\mathcal{A}_k$. Thus one has a bijective correspondence

$$\frac{\mathcal{L}_k}{\mathcal{L}_k\mathcal{J}} = \mathcal{L}_k\otimes_{\mathcal{A}_k}\left(\frac{\mathcal{A}_k}{\mathcal{J}}\right) \longleftrightarrow \frac{\mathcal{A}_k}{\mathcal{J}}.$$

For each $(u_1, \dots, u_s) \in \mathcal{L}_k^{\oplus s}$ let $\phi(u_1, \dots, u_s) : \mathcal{L}_k \longrightarrow \mathcal{L}_k$ be the homomorphism of free $\mathcal{D}_Y$-modules defined by $\phi(u_1, \dots, u_s)(e_i) = u_i$. Then $\mathcal{L}_k^{\oplus s} \longrightarrow \mathcal{A}_k$, $(u_1, \dots, u_s) \mapsto \phi(u_1, \dots, u_s)$ is an isomorphism. Now $\mathcal{J} \simeq \mathcal{N}_k^{\oplus s}$. Hence we have

$$\bigcap_{P\in\rho_k(\mathcal{J}_0)} Car\left(\frac{\mathcal{L}_k}{\mathcal{L}_k.P}\right) = \bigcap_{(u_1,\dots,u_s)\in\mathcal{N}_k^{\oplus s}} Car\left(\frac{\mathcal{L}_k}{\sum_{i=1}^{s}\mathcal{D}_Y u_i}\right)$$

Suppose now that $p \in T^*Y\backslash Car(\frac{\mathcal{L}_k}{\mathcal{N}_k})$. Then there is some $Q \in \mathcal{D}_Y$ such that $Qe_i \in \mathcal{N}_k\,(1 \leq i \leq s)$ and $\sigma(Q)(p) \neq 0$. Setting $u_i = Qe_i\,(i=1, \dots, s)$ it follows that $\frac{\mathcal{L}_k}{\sum_{i=1}^{s}\mathcal{D}_Y u_i}$ is isomorphic as a $\mathcal{D}_Y$-module to $\bigoplus_{i=1}^{s}\frac{\mathcal{D}_Y}{\mathcal{D}_Y Q}$, implying $p \notin Car(\frac{\mathcal{L}_k}{\sum_{i=1}^{s}\mathcal{D}_Y u_i})$. □

7. Main Theorem

Now everything is prepared for the statement and proof of the main theorem.

Theorem 7.1. *Let X be a complex analytic manifold, Y a smooth submanifold of X and $\mathfrak{M}$ a coherent $\mathcal{D}_X$-module. Then*

$$Car(\mathfrak{M}_Y^{\bullet}) \subset Car_Y^{\mathbb{Z}}(\mathfrak{M}).$$

Proof. Let θ be a point in $T^*Y\backslash Car_Y^{\mathbb{Z}}(\mathfrak{M})$. Given a section u of $\mathfrak{M}$ let $\mathcal{J}\subset\mathcal{D}_X$ be the annihilator of u. By proposition (6.1) there exists a $P\in\mathcal{J}$ such that $det(\rho_k(P))(\theta)\neq 0$ for all $k\in\mathbb{Z}$. In fact, if this was not the case, then we would have $\theta\in Car_Y^{\mathbb{Z}}(\frac{\mathcal{D}_X}{\mathcal{J}})$ and, since $\mathcal{D}_X/\mathcal{J}\longrightarrow\mathfrak{M}$, $P\mapsto P.u$ is an injective morphism, one would conclude by proposition (4.1) that $\theta\in Car_Y^{\mathbb{Z}}(\mathfrak{M})$.

As the module $\mathfrak{M}$ is locally of finite type there is a local system of generators $(u_1, \dots, u_s)$ of $\mathfrak{M}$ and for each u_i one operator P_i, such that $P_iu_i=0$ and $det(\rho_k(P_i))(\theta)\neq 0$ for all $k\in\mathbb{Z}$ and $1\leq i\leq s$.

Let us denote $\mathcal{L}=\bigoplus_{i=1}^{s}\frac{\mathcal{D}_X}{\mathcal{D}_X P_i}$ and let $\mathcal{L}\overset{\phi}{\longrightarrow}\mathfrak{M}$ be the morphism that sends u_i to the class of 1 modulo $\mathcal{D}_X.P_i$. Let $\mathcal{N}$ be $\mathrm{Ker}(\phi)$. Then there is an exact sequence of left $\mathcal{D}_X$-modules:

$$0\longrightarrow\mathcal{N}\longrightarrow\mathcal{L}\longrightarrow\mathfrak{M}\longrightarrow 0.$$

Applying the functor $\mathcal{D}_{Y\to X}\otimes_{\mathcal{D}_X}$ to the above exact sequence we get a long exact sequence of cohomology

$$\cdots\longrightarrow\mathcal{N}_Y^k\longrightarrow\mathcal{L}_Y^k\longrightarrow\mathfrak{M}_Y^k\longrightarrow\cdots\longrightarrow\mathfrak{M}_Y^0\longrightarrow 0. \tag{6}$$

As the theorem was already proved for modules of type $\frac{\mathcal{D}_X}{\mathcal{D}_X P}$ and there exists a $k_0\in\mathbb{Z}$ such that $\mathcal{N}^{k_0}=0$, one may assume, as an induction hypothesis, that $Car(\mathfrak{M}_Y^k)\subset Car_Y^{\mathbb{Z}}(\mathfrak{M})$ for all coherent $\mathcal{D}_X$-module $\mathfrak{M}$ and all $k\leq k_0$. Now, from the long exact sequence (6), it follows that

$$Car(\mathfrak{M}_Y^{k_0+1})\subset Car(\mathcal{L}_Y^{k_0+1})\cup Car(\mathcal{N}_Y^{k_0}),$$

implying that $\theta\notin Car(\mathfrak{M}_Y^{k_0+1})$. Hence, by induction, we finally conclude that

$$\theta\notin\bigcup_{k\in\mathbb{N}} Car(\mathfrak{M}_Y^{-k})=:Car(\mathfrak{M}_Y^{\circ}),$$

finishing the proof of the theorem. □

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