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Gevrey Hypoellipticity for Grushin Operators

By

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Introduction

Toward 1966-1967, M. S. Baouendi, [1], has considered the boundary value problem and hypoellipticity on essentially the operators $P=D_y^2+y^{2k}D_x^2$, $k=1, 2, \cdots$. Shortly later, V. V. Grushin introduced in the papers [8] and [10] a wide class of degenerate elliptic operators including the above operator P with three conditions for them to be hypoelliptic. After then there have been investigated the problem of analytic and non-analytic hypoellipticity of the Grushin operators, (cf. [19], [2], [9], [27], [28], [30], [31], [11]).

The aim of this paper is to give a nearly complete answer to this problem by expanding the idea developed in [10] to utilize the operator-valued pseudo-differential operators. Such a method might be called a kind of separation of variables in x and y. The Gevrey index of each Grushin operator may be determined depending on the quasihomogeneity condition of the symbol, (cf. Condition 1.1, Condition 6.1 and Condition 7.1 and Theorem 1.2, Theorem 6.1 and Theorem 7.1 respectively). We treat the Grushin operators dividing into three groups. The operators in the first group introduced in § 1 including the above operator P are analytic hypoelliptic in the space of hyperfunctions $\mathcal{B}$ as mentioned in Theorem 1.2. The operators in the second and the third group are Gevrey hypoelliptic in the corresponding ultradistribution spaces $\mathcal{D}^{\{\theta\}'}$, $(\theta>1)$ and treated similarly as those of the first group, (cf. Theorem 6.1 and Theorem 7.1).

In § 2～§ 4, we prepare necessary steps to prove these three theorems. In § 2, it will be shown that any eigenfunction of Grushin operators of the form (2.1) satisfying ellipticity condition (2.2) belongs to the $\mathcal{S}$-space of Gel'fand-Shilov, (in fact $\mathcal{S}^{\kappa/(1+\kappa)}_{1/(1+\kappa)}$). This fact (especially $1=\frac{1}{1+\kappa}+\frac{\kappa}{1+\kappa}$) plays an

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important role. In § 3, we shall derive precise estimates of the pseudoinverse of a Grushin operator as a symbol of an operator-valued pseudodifferential operator. The notion of pseudoinverse was first introduced in [8]. Actually it is shown to be a symbol of (ρ, δ)-type. Therefore, we can apply the Gevrey calculus developed so far for the ordinary pseudodifferential operators, (cf. [12], [20], etc.). We shall give some microlocal preparation for hyperfunctions and ultradistributions in § 4, where the heat kernel method will be needed, (cf. [22], [24], [25]). In § 5, the proof of Theorem 1.2 will be completed by using these preparations. In § 6, we shall consider the operators of the second group whose typical example is given by $L=D_y^2+(x^{2l}+y^{2k})D_x^2$, $(l, k=1, 2, \cdots)$. Gevrey index for L is given by $\theta=\frac{l(1+k)}{l(1+k)-k}$, (cf. Theorem 6.1). In § 7, we shall investigate the third group which is represented by an example $M=D_y^2+y^{2k}D_x^2+D_z^2$, $(k=1, 2, \cdots)$. Gevrey index for M is given by $\theta=1+k$. Grushin operators in the above three groups are considered to be some freezing operators at the origin of more general differential operators with analytic coefficients. In § 8, we shall mention about natural extension of these operators.

Finally we note that for the example M given above, Gevrey index $\theta=1+k$ is optimal by the result of [11], where is constructed a function $u=u(x, y, z)$ satisfying the equation $Mu=0$ in a neighborhood of $(0, 0, 0)\in\mathbb{R}^3$ and strictly belonging to Gevrey class $\mathcal{G}^{(1+k)}$. Also we note that the operator of the form

$$D_y^2+y^{2k}D_x^2+y^{2l}D_z^2, \quad k\geq l\geq 0,$$

is essentially contained in the third class and it turns out to be Gevrey hypoelliptic of index $\theta=(1+k)/(1+l)$. The optimality of this index was shown in the paper [31].

Chapter 1. Analytic Hypoellipticity of a Class of Grushin Operators

§ 1. The First Group of Grushin Operators

We write $\mathbb{R}^N=\mathbb{R}^{k+n}$, $N=k+n$, $N\geq 2$, whose point is denoted by (x, y), $x=(x_1, \cdots, x_k)$, $y=(y_1, \cdots, y_n)$. We first consider the operator with the polynomial coefficients in y:

$$(1.1)\qquad P(y, D_x, D_y)=\sum_{\substack{|\gamma|\leq\kappa(m-|\beta|)\\ |\alpha+\beta|\leq m}} a_{\alpha\beta\gamma}y^\gamma D_x^\alpha D_y^\beta, \quad a_{\alpha\beta\gamma}\in\mathbb{C}, \quad \alpha\in Z_+^k, \quad \beta, \gamma\in Z_+^n,$$

where m is a positive integer and κ is a rational number with κm a positive integer fixed. The symbol of the operator P is given by

$$P(y,\xi,\eta)=\sum_{\substack{|\gamma|\le\varkappa(m-|\beta|)\\ |\alpha+\beta|\le m}} a_{\alpha\beta\gamma}y^{\gamma}\xi^{\alpha}\eta^{\beta},\quad \xi\in\boldsymbol{R}^{k},\quad \xi\in\boldsymbol{R}^{n}. \tag{1.2}$$

We suppose the following three conditions on P.

Condition 1.1. (quasihomogeneity) The symbol of the operator P possesses the following property of quasihomogeneity:

$$P(y/\lambda,\lambda^{1+\varkappa}\xi,\lambda\eta)=\lambda^{m}P(y,\xi,\eta),\quad \lambda>0. \tag{1.3}$$

Condition 1.2. (ellipticity) The operator (1.1) is elliptic for $y\neq 0$, i.e.

$$P_{m}(y,\xi,\eta)=\sum_{\substack{|\alpha+\beta|=m\\ |\gamma|=\varkappa|\alpha|}} a_{\alpha\beta\gamma}y^{\gamma}\xi^{\alpha}\eta^{\beta}\neq 0 \tag{1.4}$$

for $y\neq 0$, $\xi\in\boldsymbol{R}^{k}$, $\eta\in\boldsymbol{R}^{n}$, $|\xi|+|\eta|\neq 0$.

Condition 1.3. (nonzero eigenvalue) For all $\omega\in\boldsymbol{R}^{k}$, $|\omega|=1$, the equation

$$P(y,\omega,D_{y})v(y)=0\qquad \text{in } \boldsymbol{R}^{n}_{y} \tag{1.5}$$

has no nonzero solution in $\mathcal{S}(\boldsymbol{R}^{n}_{y})$, where

$$P(y,\omega,D_{y})=\sum_{\substack{|\gamma|\le\varkappa(m-|\beta|)\\ |\alpha+\beta|\le m}} a_{\alpha\beta\gamma}y^{\gamma}\omega^{\alpha}D_{y}^{\beta}. \tag{1.6}$$

V. V. Grushin proved the following theorem in the paper [8]:

Theorem 1.1. *Under the conditions* 1.1 *and* 1.2, *the operator* P *in* (1.1) *is hypoelliptic if and only if the condition* (1.3) *holds.*

Example 1.1. (cf. [8]) The operator

$$\Delta_{y}+|y|^{2}\Delta_{x}+i\mu\frac{\partial}{\partial_{x_1}},\quad (x\in\boldsymbol{R}^{k},\quad y\in\boldsymbol{R}^{n}), \tag{1.7}$$

is hypoelliptic for $\mathcal{I}m\mu\neq 0$, while for real μ and $k>1$ it is hypoelliptic if and only if $|\mu|$ is less than the first eigen value of the operator $|y|^{2}-\Delta_{y}$ in $L_{2}(\boldsymbol{R}^{n}_{y})$. We can see that $\exp[-|y|^{2}/2]$ is an eigenfunction of this operator with eigenvalue n. Hence (1.7) is not hypoelliptic for $\mathcal{I}m\mu=0$ and $|\mu|\ge n$.

We shall prove the following theorem in § 5 after preparing some auxiliary results in § 2, § 3 and § 4.

Theorem 1.2. *Let* P *be the operator given above. Consider the equation*

$$Pu=\sum_{\substack{|\gamma|\le\kappa(m-|\beta|)\\ |\alpha+\beta|\le m}} a_{\alpha\beta\gamma}y^{\gamma}D_x^{\alpha}D_y^{\beta}u(x,y)=f(x,y) \quad in \quad \Omega, \tag{1.8}$$

where $u(x,y)\in\mathcal{B}(\Omega)$ *and* $f(x,y)\in\mathcal{A}(\Omega)$. *Then* $u(x,y)\in\mathcal{A}(\Omega)$.

Remark. We denote by $\mathcal{B}(\Omega)$ the space of hyperfunctions in Ω whose definition will be given in § 4.

§ 2. $\mathcal{S}$-spaces of Gel'fand-Shilov and the Grushin Operators

We fix the parameter ω, $|\omega|=1$, in (1.6) and we rewrite it as

$$A(y,D_y)=\sum_{|\gamma|\le\kappa(m-|\beta|)} a_{\beta\gamma}y^{\gamma}D_y^{\beta}, \quad a_{\beta\gamma}\in\mathbb{C}. \tag{2.1}$$

Then the ellipticity condition 1.2 turns into

$$A^m(y,\eta)\equiv\sum_{|\gamma|=\kappa(m-|\beta|)} a_{\beta\gamma}y^{\gamma}\eta^{\beta}\neq 0, \quad (y,\eta)\in\mathbb{R}_y^n\times\mathbb{R}_\eta^n\setminus(0,0). \tag{2.2}$$

We remark that if $A(y,D)$ satisfies the condition (2.2) then the transposed operator ${}^tA(y,D)$ also satisfies the same condition (2.2). A typical example of such an operator is given by $(d/dy)^2-y^{2k}$ for which $m=2$ and $\kappa=k$.

Let $H_{(m,\kappa)}(\mathbb{R}^n)$ denote the domain of $A(y,D)$ considered as an unbounded operator in $L_2(\mathbb{R}^n)$, equipped with the norm

$$\|u\|_{H_{(m,\kappa)}}=\sum_{|\gamma|+\kappa|\beta|\le\kappa m}\|y^{\gamma}D^{\beta}u\|_{L_2(\mathbb{R}^n)}. \tag{2.3}$$

We have the topological inclusion

$$H_{(m,\kappa)}\subset H_m(\mathbb{R}^n)\subset L_2(\mathbb{R}^n), \tag{2.4}$$

where H_m denotes the usual Sobolev space of order m.

Theorem 2.1. (cf. [8]) *There exists a positive constant* C *such that for all* $u\in H_{(m,\kappa)}$:

$$\|u\|_{(m,\kappa)}\le C\{\|Au\|+\|u\|\}. \tag{2.5}$$

where $\|\cdot\|$ *denotes the* L_2 *norm. We call* (2.5) *the Grushin inequality.*

Now we recall the definition of the space $\mathcal{S}_r^s(\mathbb{R}^n)$ following the expression of [4].

Definition 2.1. The space $\mathcal{S}_{r,A}^{s,B}(\mathbb{R}^n)$ $(r,\ s\ge 0;\ A,\ B>0)$ consists of all infinitely differentiable functions $\phi(y)$ satisfying the inequalities

$$\sup_{y\in\boldsymbol{R}^n}|y^\alpha D^\beta\phi(y)|\leq C_\phi A^{|\alpha|}B^{|\beta|}\alpha!^r\beta!^s,\quad \alpha,\beta\in Z_+^n, \tag{2.6}$$

where the constant C_ϕ depends on the function ϕ. This is a Banach space with the norm

$$|\phi|_{A,B}=\sup_{\substack{y\in\boldsymbol{R}^n\\ \alpha,\beta}}\frac{|y^\alpha D^\beta\phi|}{A^{|\alpha|}B^{|\beta|}\alpha!^r\beta!^s}. \tag{2.7}$$

The space $\mathcal{S}_r^s(\boldsymbol{R}^n)$ is defined by the inductive limit

$$\mathcal{S}_r^s(\boldsymbol{R}^n)=\varinjlim_{A,B}\mathcal{S}_{r,A}^{s,B}(\boldsymbol{R}^n).$$

Remark 2.1. We have the inclusion $\mathcal{S}_r^s(\boldsymbol{R}^n)\subset\mathcal{S}(\boldsymbol{R}^n)$ $(r,\ s\geq 0)$ by (2.6), where $\mathcal{S}(\boldsymbol{R}^n)$ is the space of rapidly decreasing functions of L. Schwartz.

Remark 2.2. The norm (2.6) can be replaced by the L_2 norm.

Remark 2.3. (cf. [6], Vol. 2, Chapter Ⅳ) The space $\mathcal{S}_r^s(\boldsymbol{R}^n)$ is not trivial if one of the following conditions is satisfied:

(ⅰ) $r+s\geq 1,\ r>0,\ s>0$;
(ⅱ) $r=0,\ s>0$;
(ⅲ) $r>1,\ s=0$.

Otherwise it degenerates to the single function $\phi(y)\equiv 0$.

Theorem 2.2. (cf. [6], Vol. 2, Chapter Ⅳ) *If $0<s<1$, then every $\phi(y)\in\mathcal{S}_r^s(\boldsymbol{R}^n)$ is continued into the entire $z=y+i\eta$ space $\boldsymbol{C}^n$ and the following estimate is satisfied*:

$$|\phi(y+i\eta)|\leq C\exp[-a|y|^{1/r}+b|\eta|^{1/(1-s)}],\quad y+i\eta\in\boldsymbol{C}^n, \tag{2.8}$$

where the positive constants a and b are taken corresponding to the constants A and B in (2.6). *The converse also holds.*

As an example, the Gaussian function $\exp[-y^2]$ is a member of $\mathcal{S}_{1/2}^{1/2}$ by (2.8) with $r=s=1/2$.

The first assertion in the following theorem has been proved by Grushin in [8] and [10]. Another method of its proof using Grushin inequality will be indicated in the proof of the second assertion of the following theorem.

Theorem 2.3. *Let $A(y,\ D)$ be the operator* (2.1) *satisfying the condition*

(2.2). *Then we have the three assertions:*

(i) *If* $u\in L_2(\mathbb{R}^n)$ *and* $Au\in\mathcal{S}(\mathbb{R}^n)$, *then* $u\in\mathcal{S}(\mathbb{R}^n)$.

(ii) *If* $u\in L_2(\mathbb{R}^n)$ *and* $Au\in\mathcal{S}^{\kappa/(1+\kappa)}_{1/(1+\kappa)}$, *then* $u\in\mathcal{S}^{\kappa/(1+\kappa)}_{1/(1+\kappa)}$.

In particular, if $A(y,D)u=0$, *then we have the following inequality*

$$\|y^\alpha D^\beta u\|\leq C_0C_1^{|\alpha+\beta|}\alpha!^{1/(1+\kappa)}\beta!^{\kappa/(1+\kappa)}\|u\|,\quad(\alpha,\beta)\in Z_+^n\times Z_+^n, \tag{2.9}$$

where the constants C_0 *and* C_1 *depend only on* $M=\max|a_{\alpha\beta}|$, C *in* (2.5) *and the dimension* n. *The inequality* (2.9) *is equivalent to the following inequality* (2.10) *with different constants* C_0, C_1 *and* a:

$$|D^\beta u(y)|\leq C_0C_1^{|\beta|}\beta!^{\kappa/(1+\kappa)}\|u\|\exp[-a|y|^{1+\kappa}],\quad y\in\mathbb{R}^n. \tag{2.10}$$

(iii) *The eigenspaces of* A *are finite dimensional and included in the space* $\mathcal{S}^{\kappa/(1+\kappa)}_{1/(1+\kappa)}$.

Proof of (ii). We may suppose $u\in\mathcal{S}(\mathbb{R}^n)$ and $Au\in\mathcal{S}^{\kappa/(1+\kappa)}_{1/(1+\kappa)}(\mathbb{R}^n)$ by (i). We shall prove that there exist constants C_0 and C_1 such that

$$\|y^\alpha D^\beta u(y)\|\leq C_0C_1^{|\alpha+\beta|}\alpha!^{1/(1+\kappa)}\beta!^{\kappa/(1+\kappa)},\quad(\alpha,\beta)\in Z_+^n\times Z_+^n. \tag{2.11}$$

Suppose that there exists a number $j\geq m$ sufficiently large and a couple of the constants C_0 and C_1 such that the following estimate

$$\|y^\alpha D^\beta u\|\leq C_0C_1^{|\alpha+\beta|}[|\alpha|+\kappa|\beta|]!^{1/(1+\kappa)} \tag{2.12}$$

holds for all α and β with $|\alpha|+\kappa|\beta|\leq\kappa j$, where $[x]$ denotes the integral part of the number x. Now take α and β with $\kappa j<|\alpha|+\kappa|\beta|\leq\kappa(j+1)$. Then we can write

$$\alpha=\alpha'+\alpha'',\quad\beta=\beta'+\beta'',\quad|\alpha'|+\kappa|\beta'|=\kappa m,$$

and we have

$$|\alpha''|+\kappa|\beta''|\leq\kappa(j+1)-\kappa m\leq\kappa j.$$

We have

$$\begin{aligned}\|y^\alpha D^\beta u\|&=\|y^{\alpha'}y^{\alpha''}D^{\beta'}D^{\beta''}u\|\\&\leq\|y^{\alpha'}D^{\beta'}y^{\alpha''}D^{\beta''}u\|+\|y^{\alpha'}[y^{\alpha''},D^{\beta'}]D^{\beta''}u\|.\end{aligned}$$

First we shall deal with the last member in the right-hand side. We have by induction assumption

$$\|y^{\alpha'}[y^{\alpha''},D^{\beta'}]D^{\beta'}u\|\leq\sum_{\substack{0<\mu\leq\alpha''\\0<\mu\leq\beta'}}\binom{\beta'}{\mu}\frac{\alpha''!}{(\alpha''-\mu)!}\|y^{\alpha-\mu}D^{\beta-\mu}u\| \tag{2.13}$$

$$\leq 2^{mn} \cdot \sup_{\substack{0<\mu\leq\alpha'' \\ 0<\mu\leq\beta'}} |\alpha''|(|\alpha''|-1)\cdots(|\alpha''|-|\mu|+1)\, C_0C_1^{|\alpha+\beta|-2}[|\alpha-\mu|+\kappa|\beta-\mu|]!^{1/(1+\kappa)}$$

$$\leq 2^{mn}C_0C_1^{|\alpha+\beta|-2} \cdot \sup_{\substack{0<\mu\leq\alpha'' \\ 0<\mu\leq\beta'}} \{|\alpha''|(|\alpha''|-1)\cdots(|\alpha''|-|\mu|+1)\}^{\kappa/(\kappa+1)}[|\alpha|+\kappa|\beta-\mu|]!^{1/(1+\kappa)}$$

$$\leq 2^{mn}C_0C_1^{|\alpha+\beta|-2}[|\alpha|+\kappa|\beta|]!^{1/(1+\kappa)}.$$

Next by the Grushin inequality (2.5)

$$\|y^{\alpha'}D^{\beta'}y^{\alpha''}D^{\beta''}u\| \leq C\{\|Ay^{\alpha''}D^{\beta''}u\|+\|y^{\alpha''}D^{\beta''}u\|\}.$$

As for the last member in the right-hand side of the above inequality we have the estimate

$$\|y^{\alpha''}D^{\beta''}u\| \leq C_0C_1^{|\alpha+\beta|-1}[|\alpha|+\kappa|\beta|]!^{1/(1+\kappa)}.$$

We have

$$\|Ay^{\alpha''}D^{\beta''}u\| \leq \|y^{\alpha''}D^{\beta''}Au\|+\Big\|\sum_{|\gamma|+\kappa|\delta|\leq\kappa m} a_{\gamma\delta}[y^{\gamma}D^{\delta}, y^{\alpha''}D^{\beta''}]u\Big\|$$

$$\leq \|y^{\alpha''}D^{\beta''}Au\|+M2^{\kappa m+2n}\Big\{\max_{\substack{|\gamma|+\kappa|\delta|\leq\kappa m \\ 0<\mu\leq\delta \\ 0<\mu\leq\alpha''}} \frac{\alpha''!}{(\alpha''-\mu)!}\|y^{\gamma+\alpha''-\mu}D^{\delta-\mu+\beta''}u\|+$$

$$\max_{\substack{|\gamma|+\kappa|\delta|\leq\kappa m \\ 0<\mu\leq\gamma \\ 0<\mu\leq\beta''}} \frac{\gamma!}{(\gamma-\mu)!}\|y^{\gamma+\alpha''-\mu}D^{\delta-\mu+\beta''}u\|\Big\},$$

where $M=\max|a_{\alpha\beta}|$. We may suppose the first term in the right-hand side is estimated by

$$C_0C_1^{|\alpha+\beta|-1}[|\alpha|+\kappa|\beta|]!^{1/(1+\kappa)}.$$

We can treat the second term in the right-hand side of the above inequality as in (2.13) and it is evaluated by

$$M2^{\kappa m+2n}C_0C_1^{|\alpha+\beta|-1} \max_{\substack{|\gamma|+\kappa|\delta|\leq\kappa m \\ 0<\mu\leq\delta \\ 0<\mu\leq\alpha'}} |\alpha''|\cdots(|\alpha''-\mu|+1)\,[|\alpha-\mu|+\kappa|\beta-\mu|]!^{1/(1+\kappa)}$$

$$\leq M2^{\kappa m+2n}C_0C_1^{|\alpha+\beta|-1}[|\alpha|+\kappa|\beta|]!^{1/(1+\kappa)}.$$

The last term in the right-hand side is estimated similarly. Thus we have obtained the estimate of the form

(2.14) $$\|y^{\alpha}D^{\beta}u\| \leq \{2^{mn}+1+C(1+2M2^{\kappa m+2n})\}C_0C_1^{|\alpha+\beta|-1}[|\alpha|+\kappa|\beta|]!^{1/(1+\kappa)}.$$

If we take the constant C_1 larger than $\{2^{mn}+1+C(1+2M2^{\varkappa m+2n})\}$, then we have (2.12) in case $j+1$. It is easy to obtain the estimate of the form (2.11) from the estimate (2.12).

The estimate (2.9) is derived by the same method as above by using the Grushin inequality (2.5) with $Au=0$.

We shall derive shortly the inequality (2.10) from the estimate (2.9). The estimate (2.9) can be modified as

$$|y|^{|\alpha|}|D^\beta u(y)| \leq C_0 C_1^{|\alpha|}|\alpha|!^{1/(1+\varkappa)}\beta!^{\varkappa/(1+\varkappa)}\|u\|.$$

Hence for $y\in\mathbb{R}^n$, $|\alpha|\leq|y|^{1+\varkappa}\leq|\alpha|+1$, $(|\alpha|=0, 1, \cdots)$, we have

$$|D^\beta u(y)| \leq C_0 C_1^{|\beta|}\beta!^{\varkappa/(1+\varkappa)} C_1^{|\alpha|}\left(\frac{|\alpha|!}{|y|^{(1+\varkappa)|\alpha|}}\right)^{1/(1+\varkappa)}\|u\|$$

$$\leq C_0 C_1^{|\beta|}\beta!^{\varkappa/(1+\varkappa)}\|u\| C_1^{|\alpha|}\left(\frac{|\alpha|!}{|\alpha|^{|\alpha|}}\right)^{1/(1+\varkappa)},$$

which is estimated by the quantity of the kind $C_0 C_1^{|\beta|}\beta!^{\varkappa/(1+\varkappa)}\|u\| \cdot \exp[-a|y|^{1+\varkappa}]$ with some positive constant a by applying the Stirling formula.

§ 3. Pseudoinverse of the Grushin Operator

Let $P(y, D_x D_y) = \sum_{\substack{|\tau|\leq\varkappa(m-|\beta|)\\ |\alpha+\beta|\leq m}} a_{\alpha\beta\tau} y^\tau D_x^\alpha D_y^\beta$ be the operator given in (1.1) satisfying the conditions 1.1, 1.2 and 1.3. For each $\xi\in\mathbb{R}^k\setminus 0$, we consider

$$P(y, \xi, D_y) = \sum_{\substack{|\tau|\leq\varkappa(m-|\beta|)\\ |\alpha+\beta|\leq m}} a_{\alpha\beta\tau} y^\tau \xi^\alpha D_y^\beta \in \mathcal{L}(H_{(m,\varkappa)}(\mathbb{R}_y^n), L_2(\mathbb{R}_y^n)).$$

Theorem 3.1. (*Grushin inequality*) *There is a positive constant C such that*

(3.1)
$$\sum_{|\tau|\leq\varkappa(m-|\beta|)} \|(|\xi|^{1/(1+\varkappa)}+|\xi||y|^\varkappa)^{m-|\beta|} D_y^\beta u\| \leq C\|P(y, \xi, D)u\|, \quad u\in H_{(m,\varkappa)}, \ |\xi|\geq 1.$$

Proof. The estimate (3.1) can be derived only by using the transformation $y=y'|\xi|^{1/(1+\varkappa)}$ in the estimate (2.5) and by the condition 1.3.

We denote by $P^{(\mu)}(y, \xi, D) = \partial_\xi^\mu P(y, \xi, D)$ as usual. Then we can derive the following estimates from (3.1):

Theorem 3.2. *There is a constant C such that*

$$\|P^{(\alpha)}(y,\xi,D)u\|\leq C|\xi|^{-|\alpha|}\|P(y,\xi,D)u\|,\quad u\in H_{(m,\varkappa)},\ |\xi|\geq 1. \tag{3.2}$$

Grushin, [7], has shown that the index of the operator $P(y,\xi,D_y)$ is equal to zero when $n>1$ under the conditions 1.1, 1.2 and 1.3. When $n=1$, the index of the operator $P(y,\xi,D)$ is not necessarily equal to zero, depending on the topological characteristics of the symbol $P(y,\xi,\eta)$. We remark that transpose ${}^tP(y,\xi,D)$ also satisfies the conditions 1.1 and 1.2.

Theorem 3.3. (i) *In the case $n>1$, there is an inverse $G(\xi)\in\mathcal{L}(L_2(\mathbb{R}^n), H_{(m,\varkappa)})$ of $P(y,\xi,D)$ such that*

$$G(\xi)P(y,\xi,D)=I \quad in \quad H_{(m,\varkappa)}(\mathbb{R}^n_y), \tag{3.3}$$

$$P(y,\xi,D)G(\xi)=I \quad in \quad L_2(\mathbb{R}^n_y). \tag{3.4}$$

(ii) *In the case $n=1$, let $\prod(\xi)$ be the orthogonal projection on the kernel of ${}^tP(y,\xi,D)$ in $L_2(\mathbb{R}^n_y)$. Then there is a pseudoinverse $G(\xi)\in\mathcal{L}(L_2(\mathbb{R}_y), H_{(m,\varkappa)}(\mathbb{R}_y))$ of $P(y,\xi,D)$ such that*

$$G(\xi)P(y,\xi,D)=I \quad in \quad H_{(m,\varkappa)}(\mathbb{R}_y), \tag{3.5}$$

$$P(y,\xi,D)G(\xi)=I-\prod(\xi) \quad in \quad L_2(\mathbb{R}_y). \tag{3.6}$$

Theorem 3.4. *Let $G(\xi)$ be the inverse of $P(y,\xi,D)$ when $n>1$ or let $G(\xi)$ be the pseudoinverse when $n=1$. Then there are constants C_0 and C_1 such that*

$$\|G^{(\alpha)}(\xi)\|_{(m)}\leq C_0C_1^{|\alpha|}\alpha!|\xi|^{-|\alpha|},\quad \alpha\in Z^k_+,\quad |\xi|\geq 1, \tag{3.7}$$

where $\|\cdot\|_{(m)}$ denotes the operator norm on $\mathcal{L}(L_2(\mathbb{R}^n), H_{(m,\varkappa)}(\mathbb{R}^n))$.

Proof. (i) The case $n>1$. By using the relations (3.3) and (3.4) we have

$$G^{(\alpha)}(\xi)=-\sum_{0<\mu\leq\alpha}\binom{\alpha}{\mu}G^{(\alpha-\mu)}(\xi)P^{(\mu)}(y,\xi,D)G(\xi),\quad \alpha\in Z^n_+, \tag{3.8}$$

from where we obtain immediately the estimates of the kind (3.7) by induction procedure.

(ii) The case $n=1$. Let Π be the orthogonal projection on the null space of ${}^tP(y,\omega,D)$, $(\xi\in\mathbb{R}^k,\ \omega=\xi/|\xi|)$ in $L_2(\mathbb{R}_y)$. Then the distribution kernel of Π is given in the form:

$$\Pi(y,t)=\sum_{j=1}^{r}u_j(y)\overline{u}_j(t),$$

where the u_j satisfy (2.9) and (2.10). By the quasihomogeneity, we can see the distribution kernel of $\Pi(\xi)$ of null space of ${}^tP(y,\xi,D)$ is given by

$$|\xi|^{1/(1+\varkappa)}\sum_{j=1}^{r}u_j(|\xi|^{1/(1+\varkappa)}y)\bar{u}_j(|\xi|^{1/(1+\varkappa)}t). \tag{3.9}$$

We can show that there are another couple of constants C_0 and C_1 such that

$$\|\Pi^{(\alpha)}(\xi)\|_{(0)}\leq C_0C_1^{\alpha}\alpha!|\xi|^{-\alpha},\quad \alpha=0,1,2,\cdots,\quad |\xi|\geq 1. \tag{3.10}$$

This is equivalent to the estimation of the following type for $u=u_j$.

$$\|\partial_\xi^{\alpha}u(|\xi|^{1/(1+\varkappa)}y)\|\leq C_0C_1^{|\alpha|}\alpha!|\xi|^{-|\alpha|},\quad \alpha\in Z_+^k,\quad |\xi|\geq 1. \tag{3.11}$$

It can be obtained by a careful application of the following formula for the derivatives in ξ of a composition of two functions $u(t)$ and $t=|\xi|^{1/(1+\varkappa)}y$ under the assumption (2.9).

Lemma 3.1. (*The formula of Faà di Bruno,* cf. [17]) *Let I be an open interval in $\mathbb{R}$ and suppose that $f\in C^\infty(I)$. Assume that f takes real values in an open interval J and $g\in C^\infty(J)$. Then the derivatives of $h=g(f(t))$ are given by*

$$h^{(n)}(t)=\sum\frac{n!}{k_1!k_2!\cdots k_n!}g^{(k)}(f(t))\left(\frac{f^{(1)}(t)}{1!}\right)^{k_1}\cdots\left(\frac{f^{(n)}(t)}{n!}\right)^{k_n},$$

where $k=k_1+k_2+\cdots+k_n$ and the sum is taken over all $k_1, k_2, \cdots k_n$ for which $k_1+2k_2+\cdots+nk_n=n$.

To apply the formula of Faà di Bruno we need the following combinatorial lemma which follows from a particular application of the above formula.

Lemma 3.2. (cf. [17]) *For each posilive integer n and positive real number R,*

$$\sum\frac{k!}{k_1!k_2!\cdots k_n!}R^k=R(1+R)^{n-1}$$

holds, where $k=k_1+k_2+\cdots+k_n$ and the sum is taken over all $k_1, k_2, \cdots, k_n$ for which $k_1+2k_2+\cdots+nk_n=n$.

Now by using the relations (3.5) and (3.6), we can legitimate the following formula

$$G^{(\alpha)}(\xi)=-\sum_{0<\nu\leq\alpha}\binom{\alpha}{\nu}G(\xi)P^{(\nu)}(y,\xi,D)G^{(\alpha-\nu)}(\xi)-G(\xi)\Pi^{(\alpha)}(\xi). \tag{3.12}$$

This yields the estimates of the kind (3.7) by applying the estimates (3.2) and (3.10). Q.E.D.

We remark that the estimate (3.7) means $G(\xi)$ is a symbol of an operator-

valued parametrix of $P(y, D_x, D_y)$ as an operator-valued analytic pseudo-differential operator, which we shall recall in § 5.

§ 4. The Heat Kernel Method and the Wave Front Sets

We begin this section with remembering the definition of ultra-distributions and hyperfunctions.

Definition 4.1. Let Ω be an open subset of $\boldsymbol{R}^n$ and $\varphi \in C^\infty(\Omega)$. Then we say that φ is in $\mathcal{E}^{\{s\}}(\Omega)$ with $s>1$ if for any compact subset K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{x\in K} |D^\alpha \varphi(x)| \leq C_0 C_1^{|\alpha|} \alpha!^s, \quad \alpha \in Z_+^n. \tag{4.1}$$

We denote by $\mathcal{D}^{\{s\}}(\Omega)$ the subspace of $\mathcal{E}^{\{s\}}(\Omega)$ which consists of functions with compact support in Ω. The topology of such spaces is defined as follows:

(1) We say a sequence $\{f_j(x)\} \subset \mathcal{E}^{\{s\}}(\Omega)$ converges to zero in $\mathcal{E}^{\{s\}}(\Omega)$, $s>1$, if for any compact subset K of Ω there is a constant C such that

$$\sup_{\substack{x\in K \\ \alpha}} \frac{|D^\alpha f_j(x)|}{C^{|\alpha|}\alpha!^s} \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty. \tag{4.2}$$

(1) We say a sequence $\{f_j(x)\} \subset \mathcal{D}^{\{s\}}(\Omega)$ converges to zero if there is a compact set K of Ω such that $\operatorname{supp} f_j \subset K$, $j=1, 2, \cdots$, and $f_j \longrightarrow 0$ in $\mathcal{E}^{\{s\}}(\Omega)$ as $j \longrightarrow \infty$.

We denote by $\mathcal{D}^{\{s\}'}(\Omega)$ and $\mathcal{E}^{\{s\}'}(\Omega)$ the strong dual spaces of $\mathcal{D}^{\{s\}}(\Omega)$ and $\mathcal{E}^{\{s\}}(\Omega)$ respectively. Let $u \in \mathcal{E}^{\{s\}'}(\boldsymbol{R}^n)$ with the support contained in a compact set K, $(s>1)$. Then for any $h>0$ there is a constant $C>0$ satisfying

$$|u(\varphi)| \leq C \sup_{\substack{x\in K \\ \alpha}} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|}\alpha!^s}, \quad \varphi \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n). \tag{4.3}$$

Definition 4.2. A linear form u on the space $A = A(\boldsymbol{C}^n)$ of entire analytic functions in $\boldsymbol{C}^n$ is called an analytic functional supported by a compact set K of $\boldsymbol{R}^n$ if for every complex neighborhood ω of K there exists a positive constant C_ω such that

$$|u(\varphi)| \leq C_\omega \sup_{\omega} |\varphi|, \quad \varphi \in A. \tag{4.4}$$

The space of such analytic functionals is denoted by $A'[K]$.

We set $A'(\mathbb{R}^n) \equiv \bigcup_K A'[K]$ and the support of $u \in A'(\mathbb{R}^n)$ is the smallest set $K \subset \mathbb{R}^n$ such that $u \in A'[K]$, (cf. Remark 4.2).

Now we define the space of hyperfunctions:

Definition 4.3. (cf. [22], [24]) Let Ω_j, $j = 1, 2, \cdots$, be bounded open subsets of $\mathbb{R}^n$ with $\bigcup_{j=1}^{\infty} \Omega_j = \mathbb{R}^n$. We say $u \in \mathcal{B}(\mathbb{R}^n)$ if $u = \{u_j\}_{j=1}^{\infty}$, $u_j \in A'[\bar{\Omega}_j]$, $u_j = u_k$ in $\Omega_j \cap \Omega_k$ for all j and k.

Next we recall the n-dimensional heat kernel, $(n \geq 1)$:

$$E(x, t) = (4\pi t)^{-n/2} \exp[-|x|^2/4t], \quad x \in \mathbb{R}^n, \quad t > 0. \tag{4.5}$$

We note that $E(\cdot, t)$ may be considered as an entire function of the order 2 in $\mathbb{C}^n$ for every $t > 0$.

Theorem 4.1. (cf. [22], [24]) (i) *Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then there exists $U(x, t) \in C^{\infty}(\mathbb{R}^{n+1}_+)$, $\mathbb{R}^{n+1}_+ = \{(x, t); x \in \mathbb{R}^n, 0 < t < \infty\}$. satisfying the following conditions:*

$$(\partial/\partial t - \Delta) U(x, t) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+; \tag{4.6}$$

For any compact set $K \subset \mathbb{R}^n$ there exist positive integer $N = N(K)$ and a positive constant C such that

$$\sup_{x \in K} |U(x, t)| \leq C(t^{-N} + 1), \quad t > 0, \tag{4.7}$$

and $U(x, t) \to u$ as $t \to 0_+$ in the sense that for every $\varphi \in C_0^{\infty}(\mathbb{R}^n)$

$$u(\varphi) = \lim_{t \to 0+} \int U(x, t) \varphi(x) dx. \tag{4.8}$$

Conversely, let $U(x, t) \in C^{\infty}(\mathbb{R}^{n+1}_+)$ satisfy the conditions (4.6) and (4.7). Then there exists a unique $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfying (4.8).

(ii) *Let $u \in \mathcal{D}^{\{s\}'}(\mathbb{R}^n)$, $s > 1$. Then there exists $U(x, t) \in C^{\infty}(\mathbb{R}^{n+1}_+)$ satisfying the heat equation (4.6), and for any compact set $K \subset \mathbb{R}^n$ and for any $\varepsilon > 0$ there exists a constant $C_{\varepsilon, K} > 0$ such that*

$$\sup_{x \in K} |U(x, t)| \leq C_{\varepsilon, K} \exp[(\varepsilon/t)^{1/(2s-1)}], \quad t > 0, \tag{4.9}$$

and $U(x, t) \to u$ in the sense that for every $\varphi \in \mathcal{D}^{\{s\}}(\mathbb{R}^n)$ the relation (4.8) holds.

Conversely, let $U(x, t) \in C^{\infty}(\mathbb{R}^{n+1}_+)$ satisfy the conditions (4.6) and (4.9). Then there exists a unique $u \in \mathcal{D}^{\{s\}'}(\mathbb{R}^n)$ satisfying (4.8) for every $\varphi \in \mathcal{D}^{\{s\}}(\mathbb{R}^n)$.

(iii) *Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then there exists $U(x, t) \in C^{\infty}(\mathbb{R}^{n+1}_+)$ satisfying the*

heat equation (4.6), *and for any compact* $K \subset \boldsymbol{R}^n$ *and for any* $\varepsilon > 0$ *there exists a constant* $C_{\varepsilon,K} > 0$ *such that*

$$\sup_{x \in K} |U(x, t)| \leq C_{\varepsilon,K} \exp[\varepsilon/t], \quad t > 0, \tag{4.10}$$

and $U(x, t) \to u$ *as* $t \to 0_+$ *in the sense that* $U(x, t) - U_j(x, t) \rightrightarrows 0$ *as* $t \to 0_+$ *in* Ω_j, $j = 1, 2, \cdots$, *where* $u = \{u_j\}$, $\boldsymbol{R}^n = \cup_{j=1}^{\infty} \Omega_j$ *and*

$$U_j(x, t) = \int E(x-y, t) u_j(y) dy, \quad j = 1, 2, \cdots. \tag{4.11}$$

Conversely, let $U(x, t)$ *satisfy the heat equation* (4.6) *and the estimate* (4.10). *Then there exists unique* $u \in \mathcal{B}(\boldsymbol{R}^n)$ *satisfying* (4.11).

(iv) *Let* $u \in A'[K]$ *with a compact set* $K \in \boldsymbol{R}^n$ *and set*

$$U(x, t) \equiv u_y(E(x-y, t)), \quad (x, t) \in \boldsymbol{R}^{n+1}_+. \tag{4.12}$$

Then $U(x, t)$ *satisfies the heat equation* (4.6) *in* $\boldsymbol{R}^{n+1}_+$ *and* $U(\cdot, t) \in A$ *for each* $t > 0$. *Furthermore for every* $\varepsilon > 0$ *we have*

$$|U(x, t)| \leq C_\varepsilon \exp\left[\frac{\varepsilon - \mathrm{dis}(x, K)^2}{4t}\right] \quad \text{in} \quad \boldsymbol{R}^{n+1}_+. \tag{4.13}$$

$U(x, t) \to u$ *as* $t \to 0_+$ *in the following sense*:

$$u(\varphi) = \lim_{t \to 0} \int_{\boldsymbol{R}^n} U(x, t) \chi(x) \varphi(x) dx, \quad \varphi \in A(\boldsymbol{C}^n), \tag{4.14}$$

where $\chi \in C_0^\infty(\boldsymbol{R}^n)$ *and* $\chi = 1$ *in a neighborhood of* K.

Conversely, every C^∞*-function* $U(x, t)$ *satisfying the heat equation* (4.6) *and the condition* (4.13) *can be expressed in the form* (4.12) *with unique* $u \in A'\{K\}$.

Remark 4.1. We have the inclusion

$$\mathcal{D}'(\boldsymbol{R}^n) \subset \mathcal{D}^{\{s\}'}(\boldsymbol{R}^n) \in \mathcal{B}(\boldsymbol{R}^n), \quad (s > 1); \ \mathcal{E}'(\boldsymbol{R}^n) \subset A'(\boldsymbol{R}^n).$$

Remark 4.2. Let $u \in \mathcal{B}(\boldsymbol{R}^n)$. Then $\mathrm{supp}\, u$ is the complement of the largest open set $0_u \subset \boldsymbol{R}^n$ such that $U(x, t) \rightrightarrows 0$ as $t \to 0_+$ in 0_u. (cf. (4.13)).

Now we remember the definition of the wave front sets.

Definition 4.4. (cf. [22], [24]). Let $u \in A'(\boldsymbol{R}^n)$ and $(x_0, \xi_0) \in T^*(\boldsymbol{R}^n) \setminus 0$. Then we say (i) $(x_0, \xi_0) \notin WF_A(u)$ if and only if there is a conic neighborhood Γ of (x_0, ξ_0) and there are positive constants C and c satisfying the inequality

$$|u_y(\exp[-i\langle y, \xi\rangle - |\xi|(\beta - y)^2/2])| \leq Ce^{-c|\xi|}, \quad (\beta, \xi) \in \Gamma; \tag{4.15}$$

(ii) $(x_0, \xi_0) \notin WF_{\{s\}}(u)$ $(1<s<\infty)$ if and only if there is a conic neighborhood Γ of (x_0, ξ_0) and there are positive constants C and c such that

$$|u_y(\exp[-i\langle y, \xi\rangle - |\xi|(\beta-y)^2/2])| \leq Ce^{-|\xi|^{1/s}}, \quad (\beta, \xi) \in \Gamma; \tag{4.16}$$

(iii) $(x_0, \xi_0) \notin WF(u)$ if and only if there is a conic neighborhood Γ of (x_0, ξ_0) and there are positive constants C_N, $N=0, 1, 2, \cdots$, such that

$$|u_y(\exp[-i\langle y, \xi\rangle - |\xi|(\beta-y)^2/2])| \leq C_N|\xi|^{-N}, \quad (\beta, \xi) \in \Gamma. \tag{4.17}$$

The following local expression formula of hyperfunctions plays an essential role in the microlocal calculus, (cf. [22], [33]):

Let $u \in A'[K]$. Then for any $x_0 \in \mathbb{R}^n$ and $\delta>0$ we have

(4.18)

$$u(x) = (2\pi)^{-n} \iint_{\substack{|\beta-x_0|\leq 2\delta \\ |\xi|\geq 1}} u_y(\exp[i\langle x-y, \xi\rangle - (\beta-y)^2|\xi|/2]) \left(\frac{|\xi|}{2\pi}\right)^{n/2} d\beta d\xi + w_\delta(x),$$

where $w_\delta(x) \in \mathcal{B}(\mathbb{R}^n)$ and $w_\delta(x)$ is analytic in $\{x; |x-x_0|<\delta\}$. The first term in the right hand side of (4.18) is also in $\mathcal{B}(\mathbb{R}^n)$, which is the initial value of a solution $R(x, t)$ of the heat equation satisfying the estimate of the kind (4.10) and given by

$$R(x, t) = (2\Pi)^{-n} \iint_{\substack{|\beta-x_0|\leq 2\delta \\ |\xi|\geq 1}} u_y(\exp[i\langle x-y, \xi\rangle - (\beta-y)^2|\xi|/2]) \left(\frac{|\xi|}{2\Pi}\right)^{n/2} e^{-t\xi^2} d\beta d\xi$$

§ 5. Analytic Hypoellipticity of $P(y, D_x, D_y)$ in the Space of Hyper-functions. Proof of Theorem 1.2.

Assume that $P(y, D_x, D_y)$ be the operator given in § 1. Let Ω be a bounded open set in $\mathbb{R}^{k+n}_{x,y}$ containing the origin and let us consider a partial differential equation

$$P(y, D_x, D_y)u(x, y) = f(x, y) \quad \text{in} \quad \Omega, \tag{5.1}$$

where we assume $u \in \mathcal{B}(\Omega)$ and $f \in \mathcal{A}(\Omega)$ which denotes the set of all real analytic functions on Ω.

According to the ellipticity of P for $y \neq 0$, $u(x, y)$ must be real analytic in $\Omega \setminus \{(x, 0); x \in \mathbb{R}^k\}$, and the operator P is partially elliptic with respect to y everywhere, so that there is no analytic wave front set of u in the direction $(0, 0; \xi, \eta)$, $|\eta|>c|\xi|$ for any fixed $c>0$, (cf. [22], [24]). Therefore, in order to prove the analyticity of u in a neighborhood of the origin, it will be sufficient to prove that there is no analytic wave front set of u in the direction $(0, 0; \xi, 0)$, $\xi \neq 0$.

We shall use the local expression formula (4.18) with respect to any solution $u\in\mathcal{B}(\Omega)$ to the equation (5.1) for a sufficiently small $\delta>0$ with $N=k+n$:

$$(5.2)\quad u(x,y)=(2\pi)^{-N}\iint_{\substack{|\beta_1|+|\beta_2|\le 2\delta\\ |\xi|+|\eta|\ge 1}} u_{x',y'}\Big(\exp\Big[i\langle x-x',\xi\rangle+i\langle y-y',\eta\rangle-\{(\beta_1-x')^2+(\beta_2-y')^2\}\frac{\sqrt{\xi^2+\eta^2}}{2}\Big]\Big)\Big(\frac{\sqrt{\xi^2+\eta^2}}{2\pi}\Big)^{N/2}d\beta_1 d\beta_2 d\xi d\eta+w_\delta(x,y)$$
$$\equiv\tilde{u}(x,y)+w_\delta(x,y).$$

We write the integrand of $\tilde{u}(x,y)$ in (5.2) as $F(x,y,\beta_1,\beta_2,\xi,\eta)$. Since there is no analytic wave front set of u in the η-direction in a neighborhood of the origin, we have by (4.15)

$$(5.3)\quad |F(x,y,\beta_1,\beta_2,\xi,\eta)|\le Ce^{-c|\eta|},\quad |\beta_1|+|\beta_2|\le 2\delta,\quad |\eta|\ge c'|\xi|,$$

where $(x,y)\in\boldsymbol{R}^{k+n}$ and $c'>0$ can be taken arbitrarily.

On the other hand, we can see that (4.4) yields the estimate

$$(5.4)\quad |F(x,y,\beta_1,\beta_2,\xi,\eta)|\le C_\varepsilon e^{\varepsilon(|\xi|+|\eta|)},\quad (\xi,\eta)\in\boldsymbol{R}^N,\quad \varepsilon>0.$$

Now let us consider the function $\tilde{u}(x,t;y)$ for $t>0$:

$$(5.5)\quad \tilde{u}(x,t;y)=(2\pi)^{-N}\iint_{\substack{|\beta_1|+|\beta_2|\le 2\delta\\ |\xi|+|\eta|\ge 1}} F(x,y,\beta_1,\beta_2,\xi,\eta)e^{-t\xi^2}d\beta_1 d\beta_2 d\xi d\eta.$$

Then we have $\tilde{u}(x,t;y)\in C^\infty(\boldsymbol{R}^k\times(0<t<\infty)\times\boldsymbol{R}^n)$ satisfying

$$(5.6)\quad \Big(\frac{\partial}{\partial t}-\Delta_x\Big)\tilde{u}(x,t;y)=0\quad\text{in}\quad \boldsymbol{R}^k_x\times(0<t<\infty)\times\boldsymbol{R}^n_y.$$

Furthermore, by (5.3) and (5.4), we can easily show that for arbitrary intervals $I_\delta=\{x\in\boldsymbol{R}^k;\ |x|<\delta\}$, $(\delta>0)$, and $I_\mu=\{y\in\boldsymbol{R}^n;\ |y|<\mu\}$, $(\mu>0)$, we have the estimate of the form

$$(5.7)\quad \|\tilde{u}(x,t;y)\|_{H_m(I_\mu)}\le C_{\varepsilon,I_\delta}e^{\varepsilon' t},\quad x\in I_\delta,\quad 0<t<\infty.$$

By the heat kernel method, (cf. Theorem 4.1.), we see that (5.6) and (5.7) yield the unique initial value $\tilde{u}(x,0;y)\equiv\tilde{u}(x,y)\in\mathcal{B}(I_\delta;H_m(I_\mu))$ which denotes the set of $H_m(I_\mu)$-valued hyperfunctions of $x\in I_\delta$ for any fixed $\delta>0$. By the formula (5.2) we have $u(x,y)=\tilde{u}(x,y)+w_\delta(x,y)$, so that we have

$$(5.8)\quad u(x,y)\in\mathcal{B}(I_\delta;H_m(I_\mu)),\quad I_\delta=\{x;\ |x|<\delta\},\quad I_\mu=\{y;\ |y|<\mu\}$$

Now we can apply the results obtained in the papers [21], [22] and [24] on the basis of the investigation in § 3. We can contruct an operator valued

parametrix $G(D_x)$ of

$$Q(D_x) \equiv P(y, D_x, D_y)$$

as an analytic pseudodifferential operator and symbolically we have

$$u(x, y) = G(D)Q(D)u = G(D)f(x, y) = (2\pi)^{-k}\int\int e^{i<x-z,\xi>}G(\xi)f(z, y)dzd\xi$$

which means $u(x, y)$ is a vector $(L_2(I_\mu)$ -) valued analytic function in x in a neighborhood of $x = 0 \in \mathbb{R}^k$ under the condition that $f(x, y)$ is a real analytic function (cf. (3.7)). Then we can easily see (by using (5.2) for example) that $u(x, y)$ is a real analytic function in all the variables (x, y) in a neighborhood of $(0,0) \in \mathbb{R}^{k+n}$. Thus we have completed the proof of Theorem 1.2.

Chapter 2. Gevrey Hypoellipticity

§ 6. The Second Group of the Grushin Operators

As in § 1, we write $(x, y) \in \mathbb{R}^N = \mathbb{R}^{k+n}$. Furthermore, we divide x into two parts such as $x = (x', x'')$ if necessary. Let m be a positive integer and κ and σ be positive rational numbers with κm and $\kappa m/\sigma$ are positive integers. We consider an operator

$$(6.1) \qquad L(x', y, D_x, D_y) = \sum_{\substack{\sigma|\nu|+|\gamma| \le \kappa(m-|\beta|) \\ |\alpha+\beta| \le m}} a_{\alpha\beta\nu\gamma}x'^{\nu}y^{\gamma}D_x^{\alpha}D_y^{\beta}, \quad a_{\alpha\beta\nu\gamma} \in \mathbb{C}.$$

$$\alpha, \nu \in Z_+^k, \quad \beta, \gamma \in Z_+^n.$$

The symbol of the operator L is given by

$$(6.2) \qquad L(x', y, \xi, \eta) = \sum_{\substack{\sigma|\nu|+|\gamma| \le \kappa(m-|\beta|) \\ |\alpha+\beta| \le m}} a_{\alpha\beta\nu\gamma}x'^{\nu}y^{\gamma}\xi^{\alpha}\eta^{\beta}, \quad \xi \in \mathbb{R}^k, \quad \eta \in \mathbb{R}^n.$$

$$\alpha, \nu \in Z_+^k, \beta, \gamma \in Z_+^n.$$

We suppose the following three conditions on L corresponding to those on P given in § 1.

Condition 6.1. (quasihomogeneity) The symbol of the operator L possesses the following property of quasihomogeneity:

$$(6.3) \qquad L(\lambda^{-\sigma}x', \lambda^{-1}y, \lambda^{1+\kappa}\xi', \lambda\xi'', \lambda\eta) = \lambda^m L(x', y, \xi, \eta), \quad \lambda > 0.$$

Condition 6.2. (ellipticity) The operator (6.1) is elliptic for $|x'|+|y|=1$.

Condition 6.3. (nonzero eigenvalue) For all $\omega\in\boldsymbol{R}^k$, $|\omega|=1$, the equation

$$(6.4)\qquad L(x', y, \omega, D_y)v(y)=\sum_{\substack{\sigma|\nu|+|\gamma|\leq\kappa(m-|\beta|)\\ |\alpha+\beta|\leq m}} a_{\alpha\beta\nu\gamma}x'^{\nu}y^{\gamma}\omega^{\alpha}D_y^{\beta}v(y)=0 \text{ in } \boldsymbol{R}_y^n$$

has no nonzero solution in $\mathcal{S}(\boldsymbol{R}_y^n)$.

Moreover we assume $1+\kappa>\sigma$. Then we have the following

Theorem 6.1. *Let Ω be an open neighborhood of $(0, 0)\in\boldsymbol{R}^{k+n}$ and consider the equation*

$$(6.5)\qquad L(x, y, D_x, D_y)u(x, y)=f(x, y) \quad \text{in} \quad \Omega,$$

where $u\in\mathcal{D}'^{\{\theta\}}(\Omega)$ and $f\in\mathcal{E}^{\{\theta\}}(\Omega)$ with $\theta=\dfrac{1+\kappa}{1+\kappa-\sigma}$. Then we have $u\in\mathcal{E}^{\{\theta\}}(\Omega)$.

Remark 6.1. We remember $\mathcal{E}^{\{\theta\}}(\Omega)$ the set of Gevrey functions of the index θ defined on Ω and $\mathcal{D}'^{\{\theta\}}(\Omega)$ the set of ultradistributions on Ω with index $\theta>1$ as was mentioned in § 4.

Example 6.1. The operators

$$L=\frac{\partial^2}{\partial y^2}+(x^{2l}+y^{2k})\frac{\partial^2}{\partial x^2}, \qquad l, k=1, 2, \cdots,$$

satisfy all the above conditions with $\kappa=k$, $\sigma=k/l$ and $x=x'$ from which we have $\theta=l(1+k)/\{l(1+k)-k\}$. When $l=k=1$, we have $\theta=2$ which is known the optimal index in this case by the result of G. Métivier, [27]. We note that there is no x''-variable in these cases. If we consider the operators

$$L'=\frac{\partial^2}{\partial y^2}+(x^{2l}+y^{2k})\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}, \qquad l, k=1, 2, \cdots,$$

then we have $x'=x$, $x''=z$ and σ, κ are taken the same as above.

The proof of Theorem 6.1 can be obtained by the similar process as to prove Theorem 1.2 given in § 3, § 4 and § 5. We need to prepare some corresponding steps. First by the conditions 6.1 and 6.2, we can see that L is partially elliptic in the y-direction. Therefore, it will be sufficient to prove Gevrey hypoellipticity for L of the index $\{\theta\}$ in the x-direction.

For each $\xi\in\boldsymbol{R}^k\setminus 0$, we consider

$$L(x', y, \xi, D_y)=\sum_{\substack{\sigma|\nu|+|\gamma|\leq\kappa(m-|\beta|)\\ |\alpha+\beta|\leq m}} a_{\alpha\beta\nu\gamma}x'^{\nu}y^{\gamma}\xi^{\alpha}D_y^{\beta}\in\mathcal{L}(H_{(m,\kappa)}(\boldsymbol{R}_y^n), L_2(\boldsymbol{R}_y^n)).$$

Theorem 6.2. (cf. [10] and Theorem 3.1) *There is a positive constant C such that*

$$\sum_{\substack{\sigma|\nu|+|\gamma|\leq\kappa(m-|\beta|)\\ |\alpha+\beta|\leq m}} \|\{|\xi|^{1/(1+\kappa)}+(|x'|^{\kappa/\sigma}+|y|^{\kappa})|\xi|\}^{m-|\beta|}D_y^\beta u\|\leq \tag{6.6}$$

$$C\|L(x, y, \xi, D_y)u\|, \quad u\in H_{(m,\kappa)}, \quad \xi\in\mathbb{R}^k, \quad |\xi|\geq 1.$$

We denote $L_{(\lambda)}^{(\mu)}(x', y, \xi, D)=\partial_\xi^\mu D_x^\lambda L(x', y, \xi, D)$, $\mu, \lambda\in Z_+^k$, as usual. Then we can derive the following estimates from (6.6):

Theorem 6.3. (cf. [10]) *There are constants C_0 and C_1 such that*

$$\|L_{(\lambda)}^{(\mu)}(x', y, \xi, D)u\|\leq C_0C_1^{|\mu+\lambda|}\mu!\lambda!|\xi|^{-|\mu|+\delta|\lambda|}\|L(x', y, \xi, D)u\|, \tag{6.7}$$

$$u\in H_{(m,\kappa)}(\mathbb{R}_y^n), \quad \xi\in\mathbb{R}^k, \quad |\xi|\geq 1, \quad \delta=\frac{\sigma}{1+\kappa}.$$

Proof. The estimates with respect to $\partial_\xi^\mu L=L^{(\mu)}$ are obvious, so we shall consider the estimates with respect to $\partial_x^\lambda L$. $(\partial_x^\lambda L)u$ may be considered as a linear sum of terms

$$x^{\nu-\lambda}y^\gamma\xi^\alpha D_y^\beta u(y), \quad \nu\geq\lambda.$$

Then comparing the quasihomogeneous order of $x^{\nu-\lambda}y^\gamma\xi^\alpha$ with that of the right hand side of (6.7), we easily obtain the estimates of type (6.7).

Theorem 6.4. (cf. Theorem 3.3 and Theorem 3.4) (i) *There is an inverse $G(x', \xi)\in\mathcal{L}(L_2(\mathbb{R}^n), H_{(m,\kappa)}(\mathbb{R}^n))$ of L such that*

$$G(x', \xi)L(x', y, \xi, D)=I \quad \text{in} \quad H_{(m,\kappa)}(\mathbb{R}_y^n), \tag{6.8}$$

$$L(x', y, \xi, D)G(x', \xi)=I \quad \text{in} \quad L_2(\mathbb{R}_y^n). \tag{6.9}$$

(ii) *There are constants C_0 and C_1 such that*

$$\|G_{(\lambda)}^{(\mu)}(x, \xi)\|_{(m)}\leq C_0C_1^{|\mu+\lambda|}\mu!\lambda!|\xi|^{-|\mu|+\delta|\lambda|}, \quad \mu, \lambda\in Z_+^k, \tag{6.10}$$

$$\xi\in\mathbb{R}^k, \quad |\xi|\geq 1, \quad \delta=\frac{\sigma}{1+\kappa}.$$

The proof of the above theorem can be completed easily by using (6.7) and the following lemma.

Lemma 6.1. *Let $n=1$ and $\Pi(x', \xi)$ be the orthogonal projection on the null space of ${}^tL(x', y, \xi, D_y)$ in $L_2(\mathbb{R}_y)$. Then $\Pi(x', \xi)\equiv 0$, i.e. null space of ${}^tL(x', y, \xi, D)$ in $L_2(\mathbb{R}^n)$ is reduced to $\{0\}$.*

Sketch of the proof of Lemma 6.1.

We remember that $\Pi(x', \xi)$ is given by a Dunford integral:

$$\Pi(x', \xi) = \frac{1}{2\pi i}\int_{\Gamma} \frac{dz}{z - {}^tL(x', \xi, y, D)}, \tag{6.11}$$

where Γ is a circumference of sufficiently small radius $|z| = \varepsilon$ depending on x', (cf. [5]), from where we can see that $\Pi(x', \xi)$ is analytic in x' and ξ. On the other hand, concerning with ${}^tL$, we have the inequality of the kind (6.6) replacing the right-hand side with

$$C\{\| {}^tL(x', y, \xi, D)u\| + \|u\|\}, \quad u \in H_{(m,k)}, \quad \xi \in \boldsymbol{R}^k, \quad |\xi| \geq 1.$$

Now let $u(x', y) \in \mathscr{A}(x', L_2(\boldsymbol{R}_y))$ be a solution of the equation

$${}^tL(x', y, \omega, D_y)u(x', y) = 0, \quad |\omega| = 1. \tag{6.12}$$

Then by iterating the above inequality, we have the estimates of the following type:

$$\| |x'|^j u(x', y)\|_{L_2(\boldsymbol{R}_y)} \leq C' \| u(x', y)\|_{L_2(\boldsymbol{R}_y)}, \quad j = 1, 2, \cdots \tag{6.13}$$

This yields that $u(x', y) \equiv 0$ since $u(x', y)$ is analytic in $x' \in \boldsymbol{R}^{k'}$. Hence we have $\Pi(x', \omega) = 0$ and by the quasihomogeneity we have $\Pi(x', \xi) = 0$.

Sketch of the proof of Theorem 6.1.

The proof will be carried out by the similar method as in the proof of Theorem 1.2 given in § 5. We suppose that u and $f \in \mathscr{E}^{\{\theta\}'}(\Omega)$ and $f \in \mathscr{E}^{(\theta)}$ in a neighborhood of $(0, 0) \in \Omega$, $\theta = \frac{1+\kappa}{1+k-\sigma}$. We can use the local expression formula (5.2). Then by the inequality (4.16) with $s = \theta$, the estimate (5.3) is replaced by

$$|F(x, y, \beta_1, \beta_2, \xi, \eta)| \leq Ce^{-c|\eta|^{1/\theta}}, \quad |\beta_1| + |\beta_2| \leq 2\delta, \quad |\eta| \geq c'|\xi|. \tag{6.14}$$

We note that the estimate (5.4) is replaced by

$$|F(x, y, \beta_1, \beta_2, \xi, \eta)| \leq C_\varepsilon e^{-\varepsilon(|\xi|^{1/\theta} + |\eta|^{1/\theta})}, \quad (\xi, \eta) \in \boldsymbol{R}^N, \quad \varepsilon > 0 \tag{6.15}$$

which is obtained by (4.3).

Let $\tilde{u}(x, t; y)$ be the same as in (5.5). Then $\tilde{u}$ satisfies the heat equation (5.6) and by using (6.14) and (6.15), we have the estimate of the form

$$\|\tilde{u}(x, t; y)\|_{Hm(I_\mu)} \leq C_{\varepsilon, I_\mu} \exp[(\varepsilon/t)^{1/(2\theta-1)}], \quad x \in I_\delta, \quad 0 < t < \infty \tag{6.16}$$

By Theorem 4.1, (6.16) means that $\tilde{u}(x, 0; y) \equiv \tilde{u}(x, y) \in \mathscr{D}^{\{\theta\}'}(I_\delta; H_m(I_\mu))$ for any fixed $\delta > 0$. By the formula (5.2), we have $u(x, y) = \tilde{u}(x, y) + w_\delta(x, y)$ so that we have

(6.17) $$u(x, y) \in \mathcal{D}^{\{\theta\}'}(I_\delta; H_m(I_\mu)).$$

Now we can apply directly the results in the paper [20] to obtain the Gevrey hypoellipticity in the x-direction by constructing an operator valued left parametrix of L starting with $G(x', D_x)$ and finally we get the fact $u(x, y) \in \mathcal{E}^{\{\theta\}}$ in a neighborhood of $(0, 0) \in \mathbb{R}^{k+n}$. We can take $\theta = \frac{1+k}{1+k-\sigma} = \frac{1}{1-\sigma/(1+k)}$ according to the result of [20], (cf. (6.10)).

§ 7. The Third Group of the Grushin Operators

The third group looks similar to the first group in appearance, but the condition of quasihomogeneity is different. We write $(x, y) \in \mathbb{R}^{k+n} = \mathbb{R}^N$ and $x = (x', x'')$. In this case there are always the variables x' and x'' so that $N \geq 3$. Let m be a positive integer and κ be a rational number with κm a positive integer. We consider the operator

(7.1) $$M(y, D_x, D_y) = \sum_{\substack{|\gamma| \leq \kappa(m-|\beta|) \\ |\alpha+\beta| \leq m}} a_{\alpha\beta\gamma} y^\gamma D_x^\alpha D_y^\beta, \quad a_{\alpha\beta\gamma} \in \mathbb{C}.$$

We suppose the following three conditions on M.

Condition 7.1. (quasihomogeneity) The symbol of the operator M possesses the following property of quasihomogeneity:

(7.2) $$M(\lambda^{-1}y, \lambda^{1+\kappa}\xi', \lambda\xi'', \lambda\eta) = \lambda^m M(y, \xi, \eta), \quad \lambda > 0.$$

Condition 7.2. (ellipticity) The operator M in (7.1) is elliptic for $|y| = 1$.

Condition 7.3. (nonzero eigenvalue) For all $\omega \in \mathbb{R}^k$, $|\omega| = 1$, the equation

(7.3) $$M(y, \omega, D_y)v(y) = \sum_{\substack{|\gamma| \leq \kappa(m-|\beta|) \\ |\alpha+\beta| \leq m}} a_{\alpha\beta\gamma} y^\gamma \omega^\alpha D_y^\beta v(y) = 0 \quad \text{in} \quad \mathbb{R}_y^n$$

has no nonzero solution in $\mathcal{S}(\mathbb{R}_y^n)$.

Theorem 7.1. *Let Ω be an open neighborhood of $(0, 0) \in \mathbb{R}^{k+n}$ and consider the equation*

(7.4) $$M(y, D_x, D_y)u(x, y) = f(x, y) \quad \text{in} \quad \Omega,$$

where $u \in \mathcal{D}^{\{\theta\}'}(\Omega)$ and $f \in \mathcal{E}^{\{\theta\}}(\Omega)$ with $\theta = 1 + \kappa$. Then we have $u \in \mathcal{E}^{\{\theta\}}(\Omega)$.

Example 7.1. The operators

$$M=\frac{\partial^2}{\partial y^2}+y^{2k}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2},\quad k=1, 2, \cdots,$$

satisfy all the above three conditions with $\kappa=k$ and $x'=x$, $x''=z$. When $k=1$, we have $\theta = 2$ which is known the optimal index by the result of Baouendi-Goulaouic, [2] and also we refer to [15], p.310. For general integer k, it turns out that the index $\theta=1+k$ is also optimal by the result of [11] and [31] as was mentioned in the introduction.

The proof of Theorem 7.1 is obtained by the quite similar steps of those of Theorem 1.2 and Theorem 6.1. We need only the following preparations.

Theorem 7.2. (cf. [10] and Theorem 6.2) *There is a positive constant C such that*

$$\sum_{\substack{|\gamma|\le\kappa(m-|\beta|)\\ |\alpha+\beta|\le m}}\|\{|\xi'|^{1/(1+\kappa)}+|\xi''|+|y|^{\kappa}|\xi'|\}^{m-|\beta|}D_y^{\beta}u(y)\|\le C\|M(y, \xi, D_y)u(y)\|,\quad u\in H_{(m,\kappa)},\quad \xi\in\boldsymbol{R}^k,\ |\xi|\ge 1. \tag{7.5}$$

We also denote by $M^{(\mu)}(y, \xi, D)=\partial_\xi^\mu M(y, \xi, D)$, $\mu\in Z_+^k$. Then we can derive the following estimates from (7.5) by using the quasihomogeneity.

Theorem 7.3. (cf. Theorem 6.3) *There are constants C_0 and C_1 such that*

$$\|M^{(\mu)}(y, \xi, D)u\|\le C_0 C_1^{|\mu|}\mu!|\xi|^{-|\mu|/(1+\kappa)}\|M(y, \xi, D)u\|,\quad u\in H_{(m,\kappa)}(\boldsymbol{R}_y^n),\quad \xi\in\boldsymbol{R}^k,\ |\xi|\ge 1. \tag{7.6}$$

Theorem 7.4. (cf. Theorem 3.4 and Theorem 6.4) (i) *In case $n>1$, there is an inverse $G(\xi)\in\mathcal{L}(L_2(\boldsymbol{R}_y^n), H_{(m,\kappa)}(\boldsymbol{R}_y^n))$ of M such that*

$$G(\xi)M(y, \xi, D)=I\quad in\quad H_{(m,\kappa)}(\boldsymbol{R}_y^n), \tag{7.7}$$

$$M(y, \xi, D)G(\xi)=I\quad in\quad L_2(\boldsymbol{R}_y^n). \tag{7.8}$$

(ii) *In case $n=1$, let $\Pi(\xi)$ be the orthogonal projection on the null space of ${}^tM(y, \xi, D)$ in $L_2(\boldsymbol{R}_y)$. Then there is a pseudoinverse $G(\xi)\in\mathcal{L}(L_2(\boldsymbol{R}_y), H_{(m,\kappa)}(\boldsymbol{R}_y))$ of $M(y, \xi, D)$ such that*

$$G(\xi)M(y, \xi, D)=I\quad in\quad H_{(m,\kappa)}(\boldsymbol{R}_y), \tag{7.9}$$

$$M(y, \xi, D)G(\xi)=I-\Pi(\xi)\quad in\quad L_2(\boldsymbol{R}_y). \tag{7.10}$$

(iii) *In any case $n\ge 1$, there are constants C_0 and C_1 such that*

$$\|G^{(\mu)}(\xi)\|_{(m)}\le C_0 C_1^{|\mu|}\mu!|\xi|^{-|\mu|/(1+\kappa)},\quad \mu\in Z_+^k,\quad |\xi|\ge 1. \tag{7.11}$$

Proof. Only the case $n=1$ need to be verified. The null space of ${}^tM(y, \xi, D_y)$ is finite dimensional, whose functions satisfy the estimate of the type (2.9). For $\xi=(\xi', \xi'')\in\mathbb{R}^k$, we define the quasihomogeneous function

$$|\xi|_\varkappa=\{\xi_1^{2m}+\cdots+\xi_{k'}^{2m}+\xi_{k'+1}^{2m(1+\varkappa)}+\cdots+\xi_k^{2m(1+\varkappa)}\}^{1/2m(1+\varkappa)}. \tag{7.12}$$

Since $\varkappa m$ is a positive integer by assumption, this is analytic outside of the origin. Now for $\omega\in\mathbb{R}^k$ such that $|\omega|_\varkappa=1$ take any function $u(y)$ satisfying the equation

$${}^tM(y, \omega, D_y)u(y)=0 \quad \text{in} \quad \mathbb{R}.$$

Then for $\xi=(\lambda^{1+\varkappa}\omega', \lambda\omega'')$ we can easily verify that

$${}^tM(y, \xi, D_y)u(\lambda u)=0 \quad \text{in} \quad \mathbb{R}.$$

That is to say, we have

$${}^tM(y, \xi, D_y)u(|\xi|_\varkappa y)=0 \quad \text{in} \quad \mathbb{R}. \tag{7.13}$$

Observing that the following type of the estimate

$$|\partial_\xi^\alpha(|\xi|_\varkappa)|\leq C_0C_1^{|\alpha|}\alpha!|\xi|^{1-|\alpha|/(1+\varkappa)}, \quad \alpha\in Z_+^k, \quad |\xi|\geq 1, \tag{7.14}$$

holds (cf. [13]) and by using again the formula of Faà di Bruno as in § 3, we obtain the estimate of the type

$$\|\partial_\xi^\alpha u(|\xi|_\varkappa y)\|\leq C_0C_1^{|\alpha|}\alpha!|\xi|^{1-|\alpha|/(1+\varkappa)}\|u\|, \quad \alpha\in Z_+^k, \tag{7.15}$$

from where we have the estimate of the form

$$\|\partial_\xi^\alpha\Pi(\xi)\|_{(0)}\leq C_0C_1^{|\alpha|}\alpha!|\xi|^{-|\alpha|/(1+\varkappa)}, \quad \alpha\in Z_+^k, \tag{7.16}$$

where $\Pi(\xi)$ denotes the projection operator on the null space of ${}^tM(y, \xi, D_y)$ in $L_2(\mathbb{R}_y)$. The remaining procedure is just the same as in § 3. Q.E.D.

Outline of the proof of Theorem 7.1.

By the above observation, we see that $G(D_x)$ is a vector-valued parametrix of $M(y, D_x, D_y)$ with the pseudolocal property of the Gevrey index θ, $\theta=1+k$, (cf. (7.11) and [12], [20]). Then symbolically we have the equation

$$u=G(D_x)M(y, D_x, D_y)u(x, y)=G(D_x)f(x, y),$$

where the right-hand side is in $\mathcal{E}^{\{\theta\}}$ in a neighborhood of the origin. Hence we have u is in $\mathcal{E}^{\{\theta\}}$ in the x-direction in a neighborhood of the origin. Since u is partially analytic in the y-direction, we may say $u\in\mathcal{E}^{\{\theta\}}$ in the whole variables.

Chapter 3. Generalizations

§ 8. General Grushin Operators with Analytic Coefficients

The differential operators P, L and M in the above three groups are considered to be freezing operators at the origin of more general Grushin operators with analytic coefficients. For example, consider an operator

$$(8.1)\quad P(x, y, D_x, D_y)=\sum_{\substack{|\gamma|\le\kappa(m-|\beta|)\\ |\alpha+\beta|\le m}} a_{\alpha\beta\gamma}(x, y)y^{\gamma}D_x^{\alpha}D_y^{\beta},\quad \alpha\in Z_+^k,\quad \beta, \gamma\in Z_+^n$$

where m is a positive integer and κ is rational number with κm a positive integer fixed as in (1.1). $a_{\alpha\beta\gamma}(x, y)$ are supposed to be real analytic in an open neighborhood Ω of the origin. Furthermore, we suppose the freezing operator

$$(8.2)\qquad P_0(y, D_x, D_y)=\sum_{\substack{|\gamma|\le\kappa(m-|\beta|)\\ |\alpha+\beta|\le m}} a_{\alpha\beta\gamma}(0, 0)y^{\gamma}D_x^{\alpha}D_y^{\beta}$$

satisfies all the three conditions given in § 1. Then we have the same assertion as in Theorem 1.2:

Theorem 8.1. *Let $P(x, y, D_x, D_y)$ be the operator given above. We consider the equation*

$$(8.3)\qquad P(x, y, D_x, D_y)u(x, y)=f(x, y)\quad \text{in}\quad \Omega,$$

where $u(x, y)\in\mathcal{B}(\Omega)$ and $f(x, y)\in\mathcal{A}(\Omega)$. Then $u(x, y)\in\mathcal{A}(\Omega)$.

Sketch of the proof.

Almost the same procedure given at the end of § 5 works well. We remember $I_\delta=\{x\in\mathbb{R}^k;\ |x|<\delta\}$ and $I_\mu=\{y\in\mathbb{R}^n;\ |y|<\mu\}$, where δ and μ are arbitrary positive numbers. Then we may assume $u(x, y)\in\mathcal{B}(I_\delta;\ H_m(I_\mu))$. Now let $G_0(\xi)$ be the pseudoinverse of $P_0(y, \xi, D_y)$ considered as in Theorem 3.3. For $u\in\mathcal{B}(I_\delta;\ H_m(I_\mu))$ we have the equation

$$G_0(\xi)P(x, y, \xi, D)u=u-G_0(P_0-P)u.$$

Symbolically we can write

$$G_0(\xi)P(x, y, \xi, D_y)=I-G_0(P_0-P)\equiv I-K(x, \xi)\quad \text{in}\quad H_m(I_\mu).$$

Here $K\in\mathcal{L}(H_m(I_\mu), H_m(I_\mu))$ whose operator norm is arbitrarily small if we take δ and μ small. Then there is an inverse

$$R(x, \xi)=(I-K(x, \xi))^{-1}\in\mathcal{L}(H_m(I_\mu), H_m(I_\mu)).$$

Thus we have $R(x, \xi)P(x, y, \xi, D_y) = I$. We can easily show that there are constants C_0 and C_1 such that

$$\sup_{x \in I_\delta} \| R_{(\beta)}^{(\alpha)}(x, \xi) \|_{(m)} \leq C_0 C_1^{|\alpha+\beta|} \alpha! \beta! |\xi|^{-|\alpha|}, \quad \alpha, \beta \in Z_+^k, \quad |\xi| \geq 1 \tag{8.4}$$

Starting from $R(x, D)$ we can construct the left parametrix of P, (cf. [20]) and we have finally $u(x, y) \in \mathcal{A}(I_\delta, H_m(I_\mu))$. From this fact we have $u(x, y) \in \mathcal{A}(I_\delta \times I_\mu)$ since u is analytic in y-direction. Q.E.D.

Remark 8.1. We may add to the above operator $P(x, y, D_x, D_y)$ the lower order terms of the form

$$a(x, y) y^\gamma D_x^\alpha D_y^\beta, \quad |\alpha+\beta| \leq m-1, \quad |\gamma| < \kappa(m-|\beta|), \quad a(x, y) \in \mathcal{A}(\Omega).$$

Remark 8.2. For the operators L and M in the second and third groups respectively, the similar discussion as above works well and we obtain the same assertions of Theorem 6.1 and Theorem 7.1 for the corresponding general differential operators with real analytic coefficients.

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