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Nonlinear Singular First Order Partial Differential Equations Whose Characteristic Exponent Takes a Positive Integral Value

By

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Abstract

We consider nonlinear singular partial differential equations of the form $(tD_t-\rho(x))u=ta(x)+G_2(x)(t, tD_tu, u, D_1u, \dots, D_nu)$.

It has been proved by Gérard and Tahara that there exists a unique holomorphic solution with $u(0, x)\equiv 0$ if the characteristic exponent $\rho(x)$ avoids positive integral values. In the present paper we consider what happens if $\rho(x)$ takes a positive integral value at $x=0$. Generically, the solution $u(t, x)$ is singular along the analytic set $\{t=0, \rho(x)\in\mathbf{N}^*\}$, $\mathbf{N}^*=\{1, 2, \dots\}$, and we investigate how far it can be analytically continued.

§1. Introduction

In this paper we consider the following type of nonlinear singular partial differential equations:

$$(tD_t-\rho(x))u=ta(x)+G_2(x)(t, tD_tu, u, D_1u, \dots, D_nu), \tag{1}$$

where $(t, x)\in\mathbf{C}_t\times\mathbf{C}_x^n$, $x=(x_1, \dots, x_n)$, $D_t=\partial/\partial t$, $D_i=\partial/\partial x_i$. We assume that $\rho(x)$ and $a(x)$ are holomorphic functions defined in a polydisk D centered at the origin of $\mathbf{C}_x^n$ and

$$G_2(x)(t, z, X_0, X_1, \dots, X_n)=\sum_{p+q+|\alpha|\geq 2} a_{pq\alpha}(x)t^pz^qX_0^{\alpha_0}\cdots X_n^{\alpha_n},\ |\alpha|=\alpha_0+\cdots+\alpha_n.$$

Here $a_{pq\alpha}(x)$ is holomorphic in D and $\sum_{p+q+|\alpha|\geq 2}\sup_{x\in D}|a_{pq\alpha}(x)|t^pz^qX_0^{\alpha_0}\cdots X_n^{\alpha_n}$ is a convergent power series in $(t, z, X_0, \dots, X_n)$.

Now we look for a (necessarily unique) local holomorphic solution $u(t, x)$ with $u(0, x)\equiv 0$. The right hand side of (1) is well-defined because of this

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initial condition.

The following theorem is proved in [1].

Theorem 1 (**Gérard-Tahara**). *Let $\mathring{x} \neq 0$ be a point in D. If $\rho(\mathring{x}) \notin \mathbb{N}^* = \{1, 2, 3, \ldots\}$, then the equation* (1) *has a unique holomorphic solution $u(t, x)$ with $u(0, x) \equiv 0$ in a neighborhood of $(0, \mathring{x}) \in \mathbb{C}_t \times \mathbb{C}_x^n$.*

In this paper we consider *what happens if $\rho(\mathring{x})$ takes a positive integral value.*

First we explain the calculation in [1]. They express $u(t, x)$ as a power series:

$$u(t, x) = \sum_{m \geq 1} u_m(x) t^m. \tag{2}$$

Then $\{u_m(x)\}$ satisfies the following recurrence formula:

$$u_1(x) = \frac{a(x)}{1-\rho(x)}, \tag{3}$$

and for $m \geq 2$

$$\begin{aligned}&(m-\rho(x))u_m(x)\\ &=f_m(u_1(x), 2u_2(x), \ldots, (m-1)u_{m-1}(x), u_1(x), \ldots, u_{m-1}(x),\\ &\qquad D_1u_1, \ldots, D_nu_1, \ldots, D_1u_{m-1}, \ldots, D_nu_{m-1}, \{a_{pq\alpha}(x)\}_{p+q+|\alpha| \leq m}).\end{aligned} \tag{4}$$

Here f_m is a polynomial whose coefficients are 1.

The assumption $\rho(\mathring{x}) \notin \mathbb{N}^*$ guarantees that $u_m(x)$ is holomorphic in a common neighborhood of the origin of $\mathbb{C}_x^n$ for all m. On the other hand, if $\rho(\mathring{x}) \in \mathbb{N}^*$ then $u_m(x)$ may be singular at $x = \mathring{x}$ for some m, and there exists no holomorphic solution with $u(0, x) \equiv 0$ in any neighborhood of $(0, \mathring{x}) \in \mathbb{C}_t \times \mathbb{C}_x^n$. This situation is what we would like to consider in this paper.

Example. The equation

$$(tD_t - (1-x^g))u = tx^h + G_2(x)(t, tD_tu, u, D_1u, \ldots, D_nu), \quad g, h \in \mathbb{N}^*$$

has a (unique) holomorphic solution $u(t, x) = \sum_{m \geq 1} u_m(x) t^m$ if and only if $g \leq h$.

Remark. It sometimes happens, as is shown in the example above, that $u_m(x)$ determined by (3) and (4) is holomorphic for all m even when $\rho(0) \in \mathbb{N}^*$. In [1] it is proved that $u(t, x) = \sum_{m \geq 1} u_m(x)$ is convergent in a neighborhood of the origin in such a case.

Now we assume the following:

$$\rho(0) \in \mathbb{N}^* = \{1, 2, 3, \ldots\}, \quad \rho(x) \not\equiv \rho(0). \tag{5}$$

Under this assumption, the set $V = \{\rho(x) = \rho(0)\} \subset \mathbb{C}_x^n$ is an analytic set of codimension 1. The equation (1) has a unique holomorphic solution $u(t, x)$

with $u(0, x) \equiv 0$ outside V, and there may be no such solution near any point in V. Now we put

$$d(x) = \mathrm{dist}(x, V \cup \partial D) = \mathrm{dist}(x, V),$$

where dist (x, Z) is the distance from x to a subset $Z \subset \mathbf{C}^n_x$. The second equality holds if x belongs to a sufficiently small neighborhood of the origin. We claim that the solution $u(t, x)$ is holomorphic in an open set of the following form:

$$|t| < Cd(x)^p, \quad x \text{ is sufficiently close to the origin},$$

where p and C are positive constants. A strinking feature of p is that it is completely determined by $\rho(x)$ and nothing else. A more precise statement will be given later.

Assume that $\rho(x) - \rho(0)$ has a zero exactly of order g at $x = 0$. Then we have the following estimate:

$$\left| \frac{1}{\rho(x) - \rho(0)} \right| \leq C' d(x)^{-g}, \tag{6}$$

where C' and g are positive constants. The proof of this estimate will be given in Appendix.

Now we announce

Theorem 2 (Main Theorem).

(i) *If $\rho(0) \geq g + 2$, then the solution $u(t, x)$ of (1) with $u(0, x) \equiv 0$ is holomorphic in a domain of the form*

$$|t| < Cd(x).$$

(ii) *If $\rho(0) < g + 2$, then the solution $u(t, x)$ of (1) with $u(0, x) \equiv 0$ is holomorphic in a domain of the form*

$$|t| < Cd(x)^{\frac{g+2}{\rho(0)}}.$$

In both cases C is a constant > 0 determined by $\rho(x)$, $a(x)$ and $G_2(t, z, X_0, X_1, \ldots, X_n)$.

§2. Proof of the Main Theorem

We express the solution $u(t, x)$ in the form of a power series in t:

$$u(t, x) = \sum_{m=1}^{\infty} u_m(x) t^m.$$

Then $\{u_m(x)\}$ satisfies the following recursive formula:

$$u_1(x) = \frac{a(x)}{1 - \rho(x)}, \tag{7}$$

and for $m\geq 2$

$$\begin{aligned}&(m-\rho(x))u_m(x)\\&=f_m(u_1(x), 2u_2(x), \dots, (m-1)u_{m-1}(x), u_1(x), \dots, u_{m-1}(x),\\&\quad D_1u_1, \dots, D_nu_1, \dots, D_1u_{m-1}, \dots, D_nu_{m-1}, \{a_{pq\alpha}(x)\}_{p+q+|\alpha|\leq m}).\end{aligned} \tag{8}$$

Here f_m is a polynomial whose coefficients are 1. Put $\rho(0)=M\in\mathbb{N}^*=\{1, 2, \dots\}$. Then $u_m(x)$ $(m\geq M)$ may be singular along $V=\{x\in\mathbb{C}^n; \rho(x)=M\}$. It is easy to see that we have the following type of estimate:

$$|u_m(x)|\leq C_m d(x)^{-s_m} \quad (m\geq M) \tag{9}$$

in a common neighborhood of the origin, where C_m is a positive constant and s_m $(m\geq M)$ is a positive integer. Obviously we can take $s_M=g$. (The first $M-1$ terms $s_1, \dots, s_{M-1}$ will be given later in a technical fashion).

We have

Proposition 1. *We can choose $s_m=m+g-M$ $(m\geq M)$ if $M\geq g+2$. On the other hand, we can choose $s_{lM+k}=l(g+2)+k-2$ $(l\geq 1, 0\leq k\leq M-1)$ if $M<g+2$.*

Proof.
Obviously we have

$$|D_ku_m(x)|\leq C_m' d(x)^{-(s_m+1)}, \quad m\geq M, k=1, \dots, n \tag{10}$$

for some positive constant C_m'. Hence we may set, for $m\geq M+1$,

$$s_m=\max\left[\left\{s_{m_1}+1; 1\leq m_1\leq m-1\right\}\right.$$
$$\left.\cup\left\{\sum_{j'=1}^{j}(s_{m_{j'}}+1); 2\leq j\leq m, m_{j'}\geq 1(1\leq j'\leq j), \sum_{j'=1}^{j}m_{j'}\leq m\right\}\right]. \tag{11}$$

Here we set $s_m=-1$ $(1\leq m\leq M-1)$. This technical choice is made in order to deal with the exceptional cases $m=1, \dots, M-1$ where u_m is holomorphic and so u_m and its derivatives are bounded. In these cases we don't need the kind of estimate like (10), in which the term $+1$ was necessary because of the singularity of $u_m(m\geq M)$. The quantity $s_{m_1}+1$ comes from G_2's terms which are linear in $u_1, \dots, u_{m-1}, D_1u_1, \dots, D_nu_1, \dots, D_1u_{m-1}, \dots, D_nu_{m-1}$ and $\sum_{j'=1}^{j}(s_{m_{j'}}+1)$ comes from G_2's j-th degree terms in $u_1, \dots, u_{m-1}, D_1u_1, \dots, D_nu_1, \dots, D_1u_{m-1}, \dots, D_nu_{m-1}$.

We can simplify (11) slightly. Since $s_m\geq s_{m-1}+1$ $(m\geq M+1)$ follows immediately from (11), we have $s_m\geq s_{m-1}(m\geq 2)$. Hence we obtain

$$s_m=\max\left[\left\{s_{m-1}+1\right\}\cup\left\{\sum_{j'=1}^{j}(s_{m_{j'}}+1); 2\leq j\leq m, m_{j'}\geq 1(1\leq j'\leq j), \sum_{j'=1}^{j}m_{j'}=m\right\}\right].$$

Moreover, by using the fact

$$\{s_{m-1}+1\} \subset \{(s_{m_1}+1)+(s_{m_2}+1);\ m_1 \geq 1,\ m_2 \geq 1,\ m_1+m_2=m\},$$

we obtain

$$s_m = \max_{j=2,\dots,m} \left\{ \sum_{j'=1}^{j} (s_{m_{j'}}+1);\ m_1 \geq 1\ ,\dots,\ m_j \geq 1,\ \sum_{j'=1}^{j} m_{j'}=m \right\}. \tag{12}$$

Now we prove the case $M \geq g+2$ by induction on m. The desired formula is obviously true for $m=M$. Assume that the formula is true for $s_1, \dots, s_{m-1}$, where $m \geq M+1$. Then we have

$$s_{m-1}+1=\{(m-1)+g-M\}+1=m+g-M.$$

The proof will end when we prove that $s_{m-1}+1$ attains the maximum in the right hand side of (12) by using the following iequality:

$$\sum_{j'=1}^{j} (s_{m_{j'}}+1) \leq \sum_{j' \in A} m_{j'} + (\mathrm{card} A)(g-M+1), \tag{13}$$
$$A \overset{\text{def}}{=} \{j';\ m_{j'} \geq M\} \subset \{1, \dots, j\}.$$

If $A=\phi$, then $\sum_{j'=1}^{j} (s_{m_{j'}}+1)=0 \leq m+g-M$, where the last inequality follows from the assumption $m \geq M+1$.

Next if $\mathrm{card} A=1$, we have, since each $m_{j'} \geq 1$,

$$\sum_{j' \in A} m_{j'} \leq m-\mathrm{card} A^c = m-j+1.$$

Here we have used the notation $A^c=\{1, 2, \dots, j\} \setminus A$. So it follows from (13)

$$\begin{aligned} \sum_{j'=1}^{j} (s_{m_{j'}}+1) &\leq m-j+1+(g-M+1) \\ &= m+g-M-j+2 \\ &\leq m+g-M, \end{aligned}$$

because $j \geq 2$.

Last, if $\mathrm{card} A \geq 2$,

$$\begin{aligned} \sum_{j' \in A} m_{j'} + (\mathrm{card} A)(g-M+1) &\leq m+2(g-M+1) \\ &\leq m+g-M, \end{aligned}$$

because $g-M \leq -2$.

The case $M \geq g+2$ has now been proved.

Next, we prove the case $M<g+2$. First we assume that $l=1$. The case $k=0$ obviously holds. It is easy to see that $s_m=s_{m-1}+1$, $M+1 \leq m \leq 2M-1$. Hence the case $l=1$ is proved.

Now suppose that the claim has been shown for $s_1, s_2, \dots, s_{lM+k-1}$, $l \geq 2$, $0 \leq k \leq M-1$. Then we have

$$\begin{aligned} (s_{(l-1)M}+1)+(s_{M+k}+1) &= \{(l-1)(g+2)-1\}+\{(g+2)+k-1\} \\ &= l(g+2)+k-2. \end{aligned}$$

We are going to prove that this attains the maximum of the right hand side of (12). When $l_1+\cdots+l_j=l-l'$, $k_1+\cdots+k_j=Ml'+k$,

$$\begin{aligned}(s_{l_1M+k_1}+1)+\cdots+(s_{l_jM+k_j}+1) &= \sum_{j'\in A}\{l_{j'}(g+2)+k_{j'}-1\}\\ &= (g+2)\sum_{j'\in A} l_{j'}+\sum_{j'\in A} k_{j'}-\mathrm{card}A\end{aligned}$$

for $A=\{j'; l_{j'}\geq 1\}\subset\{1, 2, \ldots, j\}$. Remark that $\sum_{j'\in A} l_{j'}=\sum_{j'=1}^{j} l_{j'}=l-l'$ and that $k_{j'}\geq 1$ if $l_{j'}=0$. If $A=\phi$, then the right hand side is equal to $0\leq l(g+2)+k-2$. The claim obviously holds in this case.

Next if $A\neq\phi$, then

$$\begin{aligned}&(s_{l_1M+k_1}+1)+\cdots+(s_{l_jM+k_j}+1)\\ &=(l-l')(g+2)+\sum_{j'\in A} k_{j'}-\mathrm{card}A.\end{aligned}\tag{14}$$

Let us estimate the second term in the right hand side. If $j'\notin A$ then $l_{j'}=0$ and $k_{j'}\geq 1$. So it follows that

$$\begin{aligned}\sum_{j'\in A} k_{j'} &= (Ml'+k)-\sum_{j'\notin A} k_{j'}\\ &\leq (Ml'+k)-\mathrm{card}A^c\\ &=(Ml'+k)-(j-\mathrm{card}A).\end{aligned}\tag{15}$$

By combining (14) and (15) we obtain

$$\begin{aligned}&(s_{l_1M+k_1}+1)+\cdots+(s_{l_jM+k_j}+1)\\ &\leq (l-l')(g+2)+(Ml'+k)-j\\ &=l(g+2)+l'\{-(g+2)+M\}+k-j\\ &\leq l(g+2)+k-j\\ &\leq l(g+2)+k-2.\end{aligned}$$

□

Thus we have proved that $s_{lM+k}=l(g+2)+k-2$.

We will make use of the following

Lemma 1. *Let Ω be a domain in $\mathbf{C}_x^n$, $x=(x_1,\ldots, x_n)$, and assume that a holomorphic function $u(x)$ in Ω satisfies*

$$|u(x)|\leq\frac{C(r)}{r^a},\quad a\in\mathbf{N}=\{0, 1, 2, \ldots\},$$

where $r=\mathrm{dist}(x; \partial\Omega)$ is the distance from x to the boundary $\partial\Omega$ of Ω and $C(r)$ is a polynomial in r of degree $\leq a$ with non-negative coefficients. Then we have

$$|D_i u(x)|\leq\frac{e(a+1)C(r)}{r^{a+1}},\quad i=1, \ldots, n.$$

Proof.

We may assume that $i=1$ without loss of generality. Goursat's formula implies that

$$D_1 u(x)=\frac{1}{2\pi\sqrt{-1}}\oint_{\Gamma}\frac{u(y, x_2, \dots, x_n)}{(x_1-y)^2}dy,$$

where $\Gamma=\left\{|y-x_1|=\frac{1}{a+1}r\right\}\subset\mathbf{C}_y$. Since

$$\operatorname{dist}((y, x_2, \dots, x_n); \partial\Omega)\geq r-\frac{r}{a+1}=\frac{a}{a+1}r \quad \text{for} \quad y\in\Gamma,$$

we have by writing $C(r)=\sum_{j=0}^{a}C_j r^j$,

$$\begin{aligned}\sup_{y\in\Gamma}|u(y, x_2, \dots, x_n)| &\leq \sum_{j=0}^{a}C_j\left(\frac{a}{a+1}r\right)^{j-a}\\ &\leq\left(\frac{a}{a+1}\right)^{-a}\sum_{j=0}^{a}C_j r^{j-a}\leq e\frac{C(r)}{r^a}.\end{aligned}$$

Hence we get the following estimate:

$$\begin{aligned}|D_1 u(x)| &\leq \frac{1}{2\pi}\cdot 2\pi\frac{r}{a+1}\cdot\frac{1}{\left(\frac{1}{a+1}r\right)^2}\cdot e\frac{C(r)}{r^a}\\ &=\frac{e(a+1)C(r)}{r^{a+1}}.\end{aligned}$$

□

Now we come back to the proof of the Main Theorem.

In a sufficiently small neighborhood of the origin, we may assume that the following estimates hold:

$$\begin{aligned}&|ju_j(x)|\leq A, \quad |D_i u_j(x)|\leq A, \quad (j=1, \dots, M-1, i=1, \dots, n),\\ &|u_M(x)|\leq Ad(x)^{-g}, \quad |D_i u_M(x)|\leq Ad(x)^{-(g+1)}, \qquad (i=1, \dots, n),\\ &\frac{s_m+1}{Nm}\leq 1\ (m\geq M), \quad |m-\rho(x)|\geq N\sigma m\ (m\geq M+1),\\ &|a_{pq\alpha}(x)|\leq A_{pq\alpha},\end{aligned}$$

where A, N, σ and $A_{pq\alpha}$ are positive constants and $\sum_{p+q+|\alpha|\geq 2} A_{pq\alpha}t^p z^q X_0^{\alpha_0}X_1^{\alpha_1}\cdots X_n^{\alpha_n}$ is a convergent power series.

Consider now the following analytic equation (with parameter $d>0$):

$$\begin{aligned}\sigma Y=&\sigma(At+At^2+\cdots+At^{M-1}+\frac{A}{d^{g+1}}t^M)\\ &+\frac{1}{d}\sum_{p+q+|\alpha|\geq 2}A_{pq\alpha}t^p Y^q Y^{\alpha_0}(eY)^{\alpha_1}\cdots(eY)^{\alpha_n}\\ &-\frac{1}{d}\sum_{m=2}^{M}B_m t^m,\end{aligned}$$

where B_m is the coefficient defined by the following identity:

$$\sum_{m=2}^{\infty} B_m t^m = \sum_{p+q+|\alpha|\geq 2} A_{pq\alpha} e^{\alpha_1+\cdots+\alpha_n} t^p (At+At^2+\cdots+At^{M-1})^{q+|\alpha|},$$

$$|\alpha| = \alpha_0 + (\alpha_1+\cdots+\alpha_n).$$

By the implicit function theorem, this analytic equation has a unique holomorphic solution Y of the form

$$Y = \sum_{m\geq 1} Y_m(d) t^m.$$

Here $Y_m(d)$ is determined by the following type of recursive formula:

$$Y_1 = \cdots = Y_{M-1} = A, \quad Y_M = \frac{A}{d^{g+1}}$$

and for $m \geq M+1$,

$$\sigma Y_m = \frac{1}{d} F_m (Y_1, \ldots, Y_{m-1}\,;\, eY_1, \ldots, eY_{m-1}\,;\, \{A_{pq\alpha}\}_{p+q+|\alpha|\leq m}),$$

where F_m is a polynomial with positive coefficients.

It is easy to see that $Y_m(d)$ is of the form

$$Y_m(d) = \frac{C_m(d)}{d^{t_m}},$$

where C_m is a polynomial of order $\leq t_m$ with non-negative coefficients. Here $t_m = 0\ (1\leq m\leq M-1)$, $t_M = g+1$ and for $m\geq M+1$

$$t_m = 1 + \max\left[\left\{t_{m_1};\ 1\leq m_1\leq m-1\right\} \cup \left\{\sum_{j'=1}^{j} t_{m_{j'}};\ 2\leq j\leq m,\ m_{j'}\geq 1\ (1\leq j'\leq j),\ \sum_{j'=1}^{j} m_{j'}\leq m\right\}\right]. \tag{16}$$

It is obvious that $t_m = s_m + 1\ (m\geq 1)$. So we have

$$t_m = m+g-M+1\ (m\geq M)\ \text{ if } M\geq g+2,$$

$$t_{lM+k} = l(g+2)+k-1\ (l\geq 1,\ 0\leq k\leq M-1)\ \text{ if } M<g+2.$$

We are going to prove that Y is a majorant power series of u if $d=d(x)$. More precisely, we want to show that for $m\geq 1$

$$|u_m(x)| \leq |mu_m(x)| \leq Y_m(d) \tag{17}$$

$$|D_i u_m(x)| \leq eY_m(d), \quad i=1, 2, \ldots, n. \tag{18}$$

The cases $m=1, 2, \ldots, M$ are obviously true. We will prove the remaining cases by induction on m. Suppose that the above inequalities have been shown to be true for $u_1, u_2, \ldots, u_{m-1}$. Then we have

$$
\begin{aligned}
&|u_m(x)|\\
&\le \frac{1}{|m-\rho(x)|} f_m(|u_1|, 2|u_2|, \ldots, (m-1)|u_{m-1}|, |u_1|, |u_2|, \ldots |u_{m-1}|,\\
&\qquad |D_1u_1|, \ldots, |D_nu_1|, \ldots, |D_1u_{m-1}|, \ldots, |D_nu_{m-1}|; \{|a_{pq\alpha}|\}_{p+q+\alpha\le m})\\
&\le \frac{1}{N\sigma m} f_m(|u_1|, 2|u_2|, \ldots, (m-1)|u_{m-1}|, |u_1|, |u_2|, \ldots |u_{m-1}|,\\
&\qquad |D_1u_1|, \ldots, |D_nu_1|, \ldots, |D_1u_{m-1}|, \ldots, |D_nu_{m-1}|; \{|a_{pq\alpha}|\}_{p+q+\alpha\le m})\\
&\le \frac{1}{N\sigma m} f_m(Y_1, Y_2, \ldots, Y_{m-1}, Y_1, Y_2, \ldots, Y_{m-1},\\
&\qquad eY_1, \ldots, eY_1, \ldots, cY_{m-1}, \ldots, eY_{m-1}; \{A_{pq\alpha}\}_{p+q+\alpha\le m})\\
&= \frac{1}{N\sigma m} F_m(Y_1, \ldots, Y_{m-1}, eY_1, \ldots, eY_{m-1}; \{A_{pq\alpha}\}_{p+q+\alpha\le m}) \quad \text{(It's an equality!)}\\
&= \frac{1}{N\sigma m}\cdot \sigma d Y_m(d) = \frac{d}{Nm} Y_m(d).
\end{aligned}
$$

Therefore we obtain

$$|mu_m(x)| \le \frac{d}{N} Y_m(d) \le Y_m(d).$$

Here we assume that x is in a ball of radius $<N$ centered at the origin. Hence $0<d<N$. Moreover, since

$$|u_m(x)| \le \frac{1}{Nm} d\, Y_m(d) = \frac{1}{Nm}\frac{C_m(d)}{d^{t_m-1}},$$

we deduce by using the lemma that

$$|D_i u_m(x)| \le \frac{t_m}{Nm} e \frac{C_m(d)}{d^{t_m}} \le e\frac{C_m(d)}{d^{t_m}} = eY_m(d).$$

So induction proceeds. We have proved that $u \ll Y$.

Our next investigation is about the convergence of $Y = \sum_{m\ge1} Y_m(d)\, t^m$ and $u(t, x) = \sum_{m\ge1} u_m(x)\, t^m$.

Fix some sufficiently small $d_0>0$. Then for some $T>0$, the series $\sum_{m\ge1} Y_m(d_0)\, T^m$ is convergent by the implicit function theorem.

Let us consider the case $M \ge g+2$, where $t_m = m+g-M+1$ $(m\ge M)$. We have

$$\infty > \sum_{m\ge M} Y_m(d_0)\, T^m = \sum_{m\ge M} \frac{C_m(d_0)}{d_0^{m+g-M+1}} T^m = \frac{1}{d_0^{g-M+1}} \sum_{m\ge M} C_m(d_0)\left(\frac{T}{d_0}\right)^m.$$

and if $|t/d| < |T/d_0|$ and $0<d\le d_0$, then

$$
\begin{aligned}
u \ll Y &= \sum_{m\ge1} Y_m(d)\, t^m\\
&= \sum_{1\le m\le M-1} Y_m(d)\, t^m + \sum_{m\ge M} \frac{C_m(d)}{d^{m+g-M+1}} t^m\\
&= \sum_{1\le m\le M-1} Y_m(d)\, t^m + \frac{1}{d^{g-M+1}} \sum_{m\ge M} C_m(d)\left(\frac{t}{d}\right)^m,
\end{aligned}
$$

$$|u| \leqq \sum_{1 \leq m \leq M-1} Y_m(d)|t|^m + \frac{1}{d^{g-M+1}} \sum_{m \geq M} C_m(d_0)\left(\frac{T}{d_0}\right)^m < \infty.$$

So for x sufficiently close to the origin, there exists a positive constant C such that $u(t, x)$ is holomorphic in $|t| < Cd(x)$. (In fact $C = |T/d_0|$.)

Next, we consider the case $M < g+2$, where $t_{lM+k} = l(g+2) + k - 1$ ($l \geq 1$, $0 \leq k \leq M-1$). We have

$$\infty > \sum_{m \geq M} Y_m(d_0) T^m = \sum_{k=0}^{M-1} \sum_{l=1}^{\infty} \frac{C_{lM+k}(d_0)}{d_0^{l(g+2)+k-1}} T^{lM+k}$$

$$= \sum_{k=0}^{M-1} \frac{T^k}{d_0^{k-1}} \sum_{l=1}^{\infty} C_{lM+k}(d_0)\left(\frac{T^M}{d_0^{g+2}}\right)^l.$$

Hence if $|t^M/d^{g+2}| < |T^M/d_0^{g+2}|$ and $0 < d \leq d_0$, then

$$\begin{aligned} u &\ll Y \\ &= \sum_{m \geq 1} Y_m(d) t^m \\ &= \sum_{1 \leq m \leq M-1} Y_m(d) t^m + \sum_{k=0}^{M-1} \frac{t^k}{d^{k-1}} \sum_{l=1}^{\infty} C_{lM+k}(d)\left(\frac{t^M}{d^{g+2}}\right)^l, \end{aligned}$$

$$|u| \leqq \sum_{1 \leq m \leq M-1} Y_m(d)|t|^m + \sum_{k=0}^{M-1} \frac{t^k}{d^{k-1}} \sum_{l=1}^{\infty} C_{lM+k}(d_0)\left(\frac{T^M}{d_0^{g+2}}\right)^l < \infty.$$

So for x sufficiently close to the origin, there exists a positive constant C such that $u(t, x)$ is holomorphic in $|t^M| < Cd(x)^{g+2}$.

The proof of the Main Theorem has now been completed.

§3. Appendix

We give the proof due to T. Oaku of the estimate (6) in the Introduction. The author is very much grateful for him.

Proposition 2. *Let $f(x)$ be a holomorphic function defined in a neighborhood Ω of the origin of $\mathbb{C}_x^n$, $x = (x_1, \ldots, x_n)$. Assume that $f(x)$ has a zero exactly of order $g \in \mathbb{N}^*$ at the origin. Denote by V the set $\{x \in \Omega;\ f(x) = 0\}$, and by $d(x)$ the distance from $x \in \Omega$ to V. Then there exist a neighborhood $\Omega' \subset \Omega$ of the origin and a positive constant $C > 0$ such that $|f(x)| \geq Cd(x)^g$, $x \in \Omega'$.*

Proof.

Let the Maclaurin expansion of $f(x)$ be $f(x) = \sum_{|\alpha| \geq g} f_\alpha x^\alpha$. Set $f_g(x) = \sum_{|\alpha| = g} f_\alpha x^\alpha$, which is a nonzero homogeneous polynomial. By applying a suitable linear change of coordinates if necessary, we may assume that $f_{(g,0,\ldots,0)} \neq 0$. The preparation theorem of Weierstrass implies that $f(x)$ can be written in the following form:

$$f(x)=c(x)\,(x_1^g+a_1(x')x_1^{g-1}+\cdots+a_g(x')),\quad c(0)\neq 0,\ a_1(0')=\cdots=a_g(0')=0,$$

where $c(x)$ is a holomorphic function defined near the origin 0 of $\mathbf{C}^n$ and each $a_i(x')$ $(i=1, 2, \ldots, g)$ is holomorphic near the origin $0'$ of $\mathbf{C}^{n-1}_{x'}$, $x'=(x_2, x_3, \ldots, x_n)$. So there exist functions $\phi_1(x'),\ldots, \phi_g(x')$ (which are not necessarily holomorphic) such that

$$f(x)=c(x)\prod_{j=1}^{g}(x_1-\phi_j(x')),\quad \phi_1(0')=\cdots=\phi_g(0')=0.$$

If x is sufficiently close to the origin, we have

$$\begin{aligned}|f(x)|&\geq\frac{1}{2}|c(0)|\prod_{j=1}^{g}|x_1-\phi_j(x')|\\&\geq\frac{1}{2}|c(0)|d(x)^g.\end{aligned}\qquad\square$$

This kind of estimate does not hold in the real analytic case. We give a counterexample.

Set $f(x_1, x_2)=-x_1^3+x_2^2$, $(x_1, x_2)\in\mathbf{R}^2$. The function f has a zero of order 2 at $(0, 0)$. Set $V^{\mathbf{R}}=\{(x_1, x_2)\in\mathbf{R}^2; f(x_1, x_2)=0\}$. If $d>0$ then the distance to $V^{\mathbf{R}}$ from the point $(-d, 0)$ is d. It is attained by $(0, 0)$. Note that $f(-d, 0)=d^3$.

This is not a paradox. There are some points in $V^{\mathbf{C}}=\{(z_1, z_2)\in\mathbf{C}^2; f(z_1, z_2)=0\}$ that are closer to $(-d, 0)$ than the origin is. Such points are found, for example, on the intersection of $V^{\mathbf{C}}$ and $\mathbf{R}_{x_1}\times i\mathbf{R}_{y_2}\subset\mathbf{C}^2_{(x_1+iy_1,x_2+iy_2)}=\mathbf{C}^2_{(z_1,z_2)}$. In fact, the equation $f(x_1, iy_2)=-x_1^3-y_2^2=0$ defines a curve in $\mathbf{R}^2_{(x_1,y_2)}\ni(-d, 0)$ which is very close to the point $(-d, 0)$ if $d>0$ is sufficiently small.

The reader is referred to [5] for estimates of real analytic functions from below.

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