Publ. RIMS, Kyoto Univ.
33 (1997), 813-838

Multiple Gamma Functions and Multiple q-Gamma Functions

By

Kimio UENO* and Michitomo NISHIZAWA*

Abstract

We give an asymptotic expansion (*the higher Stirling formula*) and an infinite product representation (*the Weierstrass canonical product representation*) of the Vignéras multiple gamma functions by considering the classical limit of the multiple q-gamma functions.

§1. Introduction

Multiple gamma functions were introduced by Barnes. They are defined to be an infinite product regularized by the multiple Hurwitz zeta functions [2], [3], [4], [5]. After his discovery, many mathematicians have studied this function: Hardy [7], [8] studied this function from his viewpoint of the theory of elliptic functions, and Shintani [20], [21] applied it to the study on the Kronecker limit formula for zeta functions attached to certain algebraic fields.

In the end of 70's, Vignéras [24] redefined multiple gamma functions to be a function satisfying the generalized Bohr-Morellup theorem. Furthermore, Vignéras [24], Voros [25], Vardi [23] and Kurokawa [12], [13], [14], [15] showed that it plays an essential role to express gamma factors of the Selberg zeta functions of compact Riemann surfaces and the determinants of the Laplacians on some Riemannian manifolds.

As we can see from these studies, the multiple gamma functions are fundamental for the analytic number theory: See also [16], [17]. However we do not think that the theory of the multiple gamma functions has been fully explored.

On the other hand, the second author of this paper introduced a q-analogue of the Vignéras multiple gamma functions and showed it to be characterized by

---

Communicated by T. Miwa, April 14, 1997.
1991 Mathematics Subject Classification (s): 33B15, 33D05
* Department of Mathematics, School of Science and Engeneering, Waseda University, Tokyo 169, Japan

a q-analogue of the generalized Bohr-Morellup theorem [24].

In this paper, we will establish an asymptotic expansion formula (*the higher Stirling formula*) and an infinite product representation (*the Weierstrass canonical product representation*) of the Vignéras multiple gamma functions by considering the classical limit of the multiple q-gamma functions. In order to get these results, we will use the method developed in [22]. Namely, by making use of the Euler-MacLaurin summation formula, we derive the Euler-MacLaurin expansion of the multiple q-gamma functions. Taking the classical limit, we are led to the Euler-MacLaurin expansion of the Vignéras multiple gamma functions. The higher Stirling formula and the Weierstrass canonical product representation are immediately deduced from this expansion formula.

This paper is organized as follows. In Section 2, we give a survey of the multiple gamma functions and its q-analogue. In Section 3, we derive the Euler-MacLaurin expansion of the multiple q-gamma functions by using the Euler-MacLaurin summation formula. In Section 4, we consider the classical limit of the multiple q-gamma functions rigorously. In Section 5, we give an asymptotic expansion formula of the Vignéras multiple gamma functions. In Section 6, the Weierstrass canonical product representation of this function is derived.

Acknowledgement

The first author is partially supported by Grant-in-Aid for Scientific Research on Priority Area 231 "Infinite Analysis" and by Waseda University Grant for Special Research Project 95A-257.

§2. A Survey of the Multiple Gamma Function and the Multiple q-Gamma Function

2.1. The Barnes multiple gamma function. We assume that $\omega_1, \omega_2, \cdots \omega_n$ lie on the same side of some straight line through the origin on the complex plane. The Barnes zeta function [5] is defined as

$$\zeta_n(s, z; \boldsymbol{\omega}) := \sum_{k_1, k_2, \cdots, k_n=0}^{\infty} \frac{1}{(z+k_1\omega_1+\cdots+k_n\omega_n)^s},$$

where $\boldsymbol{\omega} := (\omega_1, \omega_2, \cdots, \omega_n)$.

This is a generalization of the Hurwitz zeta function. Barnes [5] introduced his multiple gamma functions through

$$\frac{\Gamma_n(z, \boldsymbol{\omega})}{\rho_n(\boldsymbol{\omega})} := \exp(\zeta_n'(0, z; \boldsymbol{\omega})).$$

where

$$\log \rho_n(\boldsymbol{\omega}) := -\lim_{z \to 0} [\zeta_n'(0, z; \boldsymbol{\omega}) + \log z].$$

It is easy to see that $\Gamma_n(z, \boldsymbol{\omega})$ satisfies the functional relation

$$\frac{\Gamma_n(z, \boldsymbol{\omega})}{\Gamma_n(z+\omega_i, \boldsymbol{\omega})} = \frac{\rho_{n-1}(\boldsymbol{\omega}(i))}{\Gamma_{n-1}(z, \boldsymbol{\omega}(i))},$$

where $\boldsymbol{\omega}(i) := (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_n)$.

2.2. The Vignéras multiple gamma function. As a generalization of the gamma function and the Barnes G-function, Vignéras [24] introduced a hierarchy of functions which she called "multiple gamma functions".

Theorem 2.1. *There exists a unique hierarchy of functions which satisfy*

(1) $G_n(z+1) = G_{n-1}(z) G_n(z)$,

(2) $G_n(1) = 1$,

(3) $\dfrac{d^{n+1}}{dz^{n+1}} \log G_n(z+1) \geq 0$ *for* $z \geq 0$,

(4) $G_0(z) = z$.

Applying Dufresnoy and Pisot's results [6], she showed that these functions satisfying the above properties are uniquely determined and that $G_n(z+1)$ has an infinite product representation

$$(2.1) \qquad G_n(z+1) = \exp\left[-zE_n(1) + \sum_{h=1}^{n-1} \frac{p_h(z)}{h!} (\psi_{n-1}^{(h)}(0) - E_n^{(h)}(1))\right]$$
$$\times \prod_{\mathbf{m} \in \mathbf{N}^{n-1} \times \mathbf{N}^*} \left[\left(1 + \frac{z}{s(\mathbf{m})}\right)^{(-1)^n} \exp\left\{\sum_{l=0}^{n-1} \frac{(-1)^{n-l}}{n-l} \left(\frac{z}{s(\mathbf{m})}\right)^{n-l}\right\}\right],$$

where

$$E_n(z) := \sum_{\mathbf{m} \in \mathbf{N}^{n-1} \times \mathbf{N}^+} \left[\left\{\sum_{l=0}^{n-1} \frac{(-1)^{n-l}}{n-l} \left(\frac{z}{s(\mathbf{m})}\right)^{n-l}\right\} + (-1)^n \log\left(1 + \frac{z}{s(\mathbf{m})}\right)\right],$$

$$\psi_{n-1}(z) := \log G_{n-1}(z+1),$$

$$\frac{e^{tz}-1}{e^z-1} =: 1 + \sum_{k=0}^{\infty} p_k(t) \frac{z^k}{k!},$$

$$s(\mathbf{m}) := m_1 + m_2 + \cdots + m_n \quad \text{for} \quad \mathbf{m} = (m_1, m_2, \cdots, m_n),$$

and $\mathbf{N}^* = \mathbf{N} - \{0\}$.

The Vignéras multiple gamma function can be regarded as a special case of the Barnes multiple gamma function. Namely

$$G_n(z) = \Gamma_n(z, (1, 1\cdots, 1))^{(-1)^{n-1}} \times \text{(the normalization factor)}.$$

In this paper, we will use the word "the multiple gamma function" to refer the Vignéras multiple gamma function.

2.3. The q-gamma function. Throughout this paper, we suppose $0<q<1$. A q-analogue of the gamma function was introduced by Jackson [9], [10].

$$\Gamma(z+1; q) = (1-q)^{-z} \prod_{k=1}^{\infty} \left(\frac{1-q^{z+k}}{1-q^k} \right)^{-1}.$$

Askey [1] pointed out that it satisfies a q-analogue of the Bohr-Morellup theorem. Namely, $\Gamma(z; q)$ is uniquely determined by the following three conditions:

(1) $\Gamma(z+1; q) = [z]_q \Gamma(z; q)$,

(2) $\Gamma(1; q) = 1$,

(3) $\dfrac{d^2}{dz^2} \log \Gamma(z+1; q) \geq 0$ for $z \geq 0$,

where $[z]_q := (1-q^z)/(1-q)$.

As q tends to $1-0$, $\Gamma(z; q)$ converges $\Gamma(z)$ uniformly with respect to z. A rigorous proof of this fact was given firstly by Koornwinder [11].

2.4. The multiple q-gamma function. Recently, one of the authors [19] constructed the function $G_n(z; q)$ which satisfies a q-analogue of the generalized Bohr-Morellup theorem:

Theorem 2.2. *There exists a unique hierarchy of functions which satisfy*

(1) $G_n(z+1; q) = G_{n-1}(z; q) G_n(z; q)$,

(2) $G_n(1; q) = 1$,

(3) $\dfrac{d^{n+1}}{dz^{n+1}} \log G_n(z+1; q) \geq 0$ for $z \geq 0$,

(4) $G_0(z; q) = [z]_q$.

We call it "the multiple q-gamma function". It is given by the following infinite product representation [19]

$$(2.2) \qquad G_n(z+1; q) := (1-q)^{-\binom{z}{n}} \prod_{k=1}^{\infty} \left\{ \left(\frac{1-q^{z+k}}{1-q^k} \right)^{\binom{-k}{n-1}} (1-q^k)^{g_n(z,k)} \right\}$$

for $n \geq 1$, where

$$g_n(z, u) = \binom{z-u}{n-1} - \binom{-u}{n-1}.$$

In the next section, we will derive a representation of the multiple q-gamma function called *Euler-MacLaurin expansion* and consider its classical limit. This limit formula gives some important properties of the multiple gamma functions.

§3. The Euler-MacLaurin Expansion of $G_n(z+1; q)$

By means of the Euler-MacLaurin summation formula

$$\sum_{r=M}^{N-1} f(r) = \int_M^N f(t)\,dt + \sum_{k=1}^{n} \frac{B_k}{k!}\left\{f^{(k-1)}(N) - f^{(k-1)}(M)\right\}$$
$$+(-1)^{n-1}\int_M^N \frac{\bar{B}_n(t)}{n!} f^{(n)}(t)\,dt \quad (\text{for } f\in C^n[M, N]),$$

we give an expansion formula of the multiple q-gamma functions which we call *the Euler-MacLaurin expansion* [22]. This formula plays an important role in the following sections.

Proposition 3.1. *Suppose* $\Re z > -1$ *and* $m > n$, *then*

$$\log G_n(z+1; q)$$
$$= \left\{\binom{z+1}{n} + \sum_{r=1}^{n} \frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\log\left(\frac{1-q^{z+1}}{1-q}\right)$$
$$+\sum_{r=1}^{n}\left\{\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\times\int_1^{z+1}\frac{\xi^r}{r!}\frac{q^{\xi}\log q}{1-q^{\xi}}d\xi$$
$$+\sum_{j=0}^{n-1} G_{n,j}(z)C_j(q) + \sum_{r=1}^{m}\frac{B_r}{r!}F_{n,r-1}(z; q) - R_{n,m}(z; q),$$

where

$$F_{n,r-1}(z; q) := \left[\frac{d^{r-1}}{dt^{r-1}}\left\{\binom{-t}{n-1}\log\left(\frac{1-q^{z+t}}{1-q^{z+1}}\right)\right\}\right]_{t=1},$$

$$C_j(q) := -\sum_{r=1}^{n+1}\frac{B_r}{r!}f_{j+1,r-1}(q)$$
$$+\frac{(-1)^n}{(n+1)!}\int_1^{\infty}\left[\bar{B}_{n+1}(t)\left\{\frac{d^{n+1}}{dt^{n+1}}\left\{t^j\log\left(\frac{1-q^t}{1-q}\right)\right\}\right\}\right]dt,$$

$$f_{j+1,r-1}(q) := \left[\frac{d^{r-1}}{dt^{r-1}}\left\{t^j\log\left(\frac{1-q^t}{1-q}\right)\right\}\right]_{t=1},$$

$$R_{n,m}(z; q) := \frac{(-1)^{m-1}}{m!}\int_1^{\infty}\left[\bar{B}_m(t)\left\{\frac{d^m}{dt^m}\left\{\binom{-t}{n-1}\log\left(\frac{1-q^{z+t}}{1-q^{z+1}}\right)\right\}\right\}\right]dt.$$

The polynomial $G_{n,j}(z)$ *is introduced through*

$$\binom{z-u}{n-1}=\sum_{j=0}^{n-1}G_{n,j}(z)u^j.$$

Proof. From (2.2) and the definition of $G_{n,j}(z)$, we obtain

$$\log G_n(z+1;q)=-\binom{z}{n}\log(1-q)-\sum_{k=1}^{\infty}\binom{-k}{n-1}\log(1-q^{z+k}) \\ +\sum_{j=0}^{n-1}G_{n,j}(z)\{\sum_{k=1}^{\infty}k^j\log(1-q^k)\}. \tag{3.1}$$

Let $L_r(z)$ and L_r be

$$L_r(z):=\frac{Li_r(q^z)}{\log^{r-1}q},\quad L_1(z):=-\log(1-q^z),\quad L_r:=L_r(q),$$

where $Li_r(z)$ is Euler's polylogarithm

$$Li_r(z):=\sum_{k=1}^{\infty}\frac{z^k}{k^r},$$

and denote by ${}_nS_j$, the Stirling number of the first kind;

$$\sum_{j=0}^{n}{}_nS_ju^j=[u]_n,$$

where $[u]_n=u(u-1)\cdots(u-n+1)$.

By making use of the Euler-MacLaurin summation formula, we have

$$\sum_{k=1}^{\infty}\binom{-k}{n-1}\log(1-q^{z+k}) \\ =\sum_{j=0}^{n-1}\frac{(-1)^j{}_{n-1}S_j}{(n-1)!}\sum_{r=0}^{j}\frac{(-1)^rj!}{(j-r)!}L_{r+2}(z+1) \\ +\sum_{r=1}^{m}\frac{B_r}{r!}\left\{\left(\frac{d}{dt}\right)^{r-1}\binom{-t}{n-1}\right\}\Bigg|_{t=1}L_1-\sum_{r=1}^{m}\frac{B_r}{r!}F_{n,r-1}(z;q)+R_{n,m}(z;q), \tag{3.2}$$

and

$$\sum_{k=1}^{\infty}k^j\log(1-q^k) \\ =\sum_{r=0}^{j}\frac{(-1)^rj!}{(j-r)!}L_{r+2}+\sum_{r=1}^{j}\frac{B_r}{r!}\frac{j!}{(j+1-r)!}L_1+C_j(q). \tag{3.3}$$

$L_r(z)$ and L_r cause divergence as $q \to 1-0$, but we can prove that these divergent terms vanish. In order to show it, we express $L_r(z)$ as the sum of L_r and convergent terms. By partial integration, we can show

$$(3.4) \qquad L_{l+1}(z) = \frac{z^l}{l!}\log\left(\frac{1-q^z}{1-q}\right) + \sum_{r=0}^{l}\frac{(z-1)^{l-r}}{(l-r)!}L_{r+1} + \sum_{r=1}^{l}\frac{(-1)^l z^{l-r}}{(l-r)!}T_r(z),$$

where

$$T_r(z) = \int_1^z \frac{\xi^r}{r!}\frac{q^\xi \log q}{1-q^\xi}d\xi.$$

Substituting these formula into (3.1), we obtain

$$\begin{aligned}
(3.5) \qquad & \log G_n(z+1;q) \\
&= -\sum_{j=0}^{n-1}\frac{(-1)^j {}_{n-1}S_j}{(n-1)!}\sum_{l=1}^{j}\frac{(-1)^l j!}{(j-l)!}\sum_{r=0}^{l}\frac{z^{l-r}}{(l-r)!}L_{r+2} \\
&\quad + \sum_{j=0}^{n-1}G_{n,j}(z)\sum_{r=0}^{j}\frac{(-1)^r j!}{(j-r)!}L_{r+2} \\
&\quad + \left\{\binom{z}{n} - \sum_{r=1}^{m}\frac{B_r}{r!}\left(\frac{d}{dt}\right)^{r-1}\binom{-t}{n-1}\bigg|_{t=1}\right. \\
&\qquad - \sum_{j=0}^{n-1}\frac{(-1)^j {}_{n-1}S_j}{(n-1)!}\sum_{l=0}^{j}\frac{(-1)^l j!}{(j-l)!}\frac{z^{l+1}}{(l+1)!} \\
&\qquad\quad \left. + \sum_{j=0}^{n-1}G_{n,j}(z)\sum_{r=0}^{j}\frac{B_r}{r!}\frac{j!}{(j+1-r)!}\right\}L_1 \\
&\quad + \left\{-\sum_{j=0}^{n-1}\frac{(-1)^l {}_{n-1}S_j}{(n-1)!}\sum_{l=0}^{j}\frac{(-1)^l j!}{(j-l)!}\frac{(z+1)^{l+1}}{(l+1)}\right. \\
&\qquad \left. + \sum_{j=0}^{n-1}\frac{B_{j+1}}{(j+1)!}\left(\frac{d}{dt}\right)^j\binom{-t}{n-1}\bigg|_{t=1}\right\}\times\log\left(\frac{1-q^{z+1}}{1-q}\right) \\
&\quad - \sum_{j=0}^{n-1}\frac{(-1)^j {}_{n-1}S_j}{(n-1)!}\sum_{l=0}^{j}\frac{(-1)^l j!}{(j-l)!}\sum_{r=0}^{l}\frac{(-1)^r(z+1)^{l-r}}{(l-r)!}T_{r+1}(z+1) \\
&\quad + \sum_{r=0}^{m}\frac{B_r}{r!}F_{n,r-1}(z;q) + \sum_{j=1}^{n-1}G_{n,j}(z)C_j(q) \\
&\quad - R_{n,m}(z;q).
\end{aligned}$$

We prove that the divergent terms in (3.5) vanish. From the definition of $G_{n,j}$ and of the Stirling numbers, we can easily show that terms involving L_{r+2} in (3.5) are canceled out.

Next, we prove the coefficient of L_1 vanishes. Since

$$\sum_{j=0}^{n-1}\frac{(-1)^j{}_{n-1}S_j}{(n-1)!}\sum_{l=0}^{j}\frac{(-1)^l j!}{(j-l)!}\frac{z^{l+1}}{(l+1)!}=\int_0^z\binom{t-1}{n-1}dt$$

and

$$\sum_{j=0}^{n-1}G_{n,j}(z)\sum_{r=0}^{j}\frac{B_r}{r!}\frac{j!}{(j+1-r)!}=\sum_{r=0}^{n-1}\frac{B_r}{r!}\left\{\left(-\frac{d}{dz}\right)^{r-1}\binom{z-1}{n-1}\right\},$$

we have

$$\begin{aligned}&(\text{the coefficient of } L_1)\\ &=\binom{z}{n}-\int_0^z\binom{t-1}{n-1}dt+\sum_{r=0}^{m}\frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}\binom{z-1}{n-1}\\ &\quad-\sum_{r=0}^{m}\frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}\binom{z-1}{n-1}\Bigg|_{z=0}.\end{aligned}$$

We prove that the right hand side of the above formula is equal to zero. In a formal sense,

$$(e^{-\frac{d}{dz}}-1)^{-1}+\left(\frac{d}{dz}\right)^{-1}=\sum_{r=1}^{\infty}\frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}.$$

Imposing the boundary condition at $z=0$, we make the both sides above act on $\binom{z-1}{n-1}$. Then,

$$(e^{-\frac{d}{dz}}-1)^{-1}\binom{z-1}{n-1}+\int_0^z\binom{t-1}{n-1}dt+\sum_{r=1}^{n}\frac{B_r}{r!}\left[\left(-\frac{d}{dt}\right)^{r-1}\binom{t-1}{n-1}\right]_{t=0}^{t=z}=0$$

because $\binom{t-1}{n-1}$ is a polynomial of $(n-1)$-degree. Since $F(z):=-\binom{z}{n}$ satisfies

$$F(0)=0,\quad \binom{z-1}{n-1}=F(z-1)-F(z),$$

it can be seen that

$$(e^{-\frac{d}{dz}}-1)^{-1}\binom{z-1}{n-1}=-\binom{z}{n}.$$

Thus we have the formula

$$-\binom{z}{n}+\int_0^z\binom{t-1}{n-1}dt+\sum_{r=1}^{n}\frac{B_r}{r!}\left[\left(-\frac{d}{dt}\right)^{r-1}\binom{t-1}{n-1}\right]_{t=0}^{t=z}=0, \tag{3.6}$$

which shows that the coefficient of L_1 is equal to zero. Hence we have proved that the all coefficients of L_r vanish in (3.5).

Finally, we calculate the coefficients of $\log\left(\frac{1-q^{z+1}}{1-q}\right)$ and $T_r(z+1)$. Using the formula (3.6), we have

$$\begin{aligned}&\left(\text{the coefficient of } \log\left(\frac{1-q^{z+1}}{1-q}\right) \text{ in } (3.5)\right)\\ &=\binom{z+1}{n}+\sum_{r=1}^{n}\frac{B_r}{r!}\left(-\frac{d}{dt}\right)^{r-1}\binom{z}{n-1}.\end{aligned} \tag{3.7}$$

In order to calculate the coefficients of $T_r(z+1)$ in (3.5), we note that

$$\binom{z-u}{n-1}=\sum_{j=0}^{n-1}\frac{(-1)^j{}_{n-1}S_j}{(n-1)!}\sum_{l=0}^{j}\frac{(-1)^l j!}{(j-l)!}\sum_{r=0}^{l}\frac{(-1)^r(z+1)^{l-r}}{(l-r)!}\frac{u^r}{r!}.$$

Using this identity, we have

$$(\text{the coefficient of } T_r(z+1) \text{ in } (3.5))=\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}. \tag{3.8}$$

Substituting (3.7) and (3.8) in (3.5), we obtain Proposition 3.1. □

§4. The Classical Limit of $G_n(z+1;q)$

In this section, we study the classical limit of $G_n(z+1;q)$ using the Euler-MacLaurin expansion. We will see that this limit formula gives an asymptotic expansion for the multiple gamma functions, which is a generalization of the Stirling formula for the gamma function. First, we consider the classical limit of the Euler-MacLaurin expansion in the domain $\{z\in\mathbf{C}|\Re z>-1\}$.

Proposition 4.1. *Suppose $\Re z>-1$ and $m>n$.*

$$\lim_{q\to 1-0}\log G_n(z+1;q)$$

$$
\begin{aligned}
&=\left\{\binom{z+1}{n}+\sum_{r=1}^{n}\frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\log(z+1)\\
&-\sum_{r=1}^{n}\left\{\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\frac{1}{r!r}\{(z+1)^r-1\}\\
&-\sum_{j=0}^{n-1}G_{n,j}(z)C_j+\sum_{r=1}^{m}\frac{B_r}{r!}F_{n,r-1}(z)\\
&-R_{n,m}(z),
\end{aligned}
$$

where

$$
\begin{aligned}
C_j:=&-\sum_{r=1}^{n+1}\frac{B_r}{r!}\left(\frac{d}{dt}\right)^{r-1}\{t^j\log t\}\Big|_{t=1}\\
&+\frac{(-1)^n}{(n+1)!}\int_1^\infty \bar{B}_{n+1}(t)\left(\frac{d}{dt}\right)^{n+1}\{t^j\log t\}dt
\end{aligned}
$$

$$
F_{n,r-1}:=\left(\frac{d}{dt}\right)^{r-1}\left\{\binom{-t}{n-1}\log\left(\frac{z+t}{z+1}\right)\right\}\Big|_{t=1},
$$

$$
R_{n,m}(z):=\frac{(-1)^{m-1}}{m!}\int_1^\infty \bar{B}_m(t)\left(\frac{d}{dt}\right)^m\left\{\binom{-t}{n-1}\log\left(\frac{z+t}{z+1}\right)\right\}dt.
$$

Furthermore this convergence is uniform on any compact set in $\{z\in\mathbb{C}|\Re z>-1\}$.

Proof. Taking Proposition 3.1 into acount, we must show that

$$\lim_{q\to 1-0}\log\left(\frac{1-q^{z+1}}{1-q}\right)=\log(z+1), \tag{4.1}$$

$$\lim_{q\to 1-0}\int_1^{z+1}\frac{\xi^r}{r!}\frac{q^\xi\log q}{1-q^\xi}d\xi=-\int_1^\infty\frac{\xi^{r-1}}{r!}d\xi=-\frac{1}{r!r}\{(z+1)^r-1\}, \tag{4.2}$$

$$\lim_{q\to 1-0}F_{n,r-1}(z;q)=F_{n,r-1}(z), \tag{4.3}$$

$$\lim_{q\to 1-0}R_{n,m}(z;q)=R_{n,m}(z), \tag{4.4}$$

$$\lim_{q\to 1-0}C_j(q)=C_j, \tag{4.5}$$

and further have to show that this convergence is uniform. Here we prove only (4.4). The other formulas can be verified in a similar way.

Since

$$
\lim_{q\to 1-0}\left(\frac{d}{dt}\right)^{r-1}\left\{\binom{-t}{n-1}\log\left(\frac{1-q^{z+t}}{1-q^{z+1}}\right)\right\}=\left(\frac{d}{dt}\right)^{r-1}\left\{\binom{-t}{n-1}\log\left(\frac{z+t}{z+1}\right)\right\},
$$

in order to show (4.4), it is sufficient to prove that the procedure of taking the classical limit commutes with the integration. Let us introduce polynomials $M_r(x)$ through

$$\frac{d^r}{dz^r}\log(1-q^{z+t}) = -\left(\frac{\log q}{1-q^{z+t}}\right)^r q^{z+t}M_r(q^{z+t})$$

(cf. [18]). They satisfy the recurrence relation.

$$M_1(x)=1, \quad (x^2-x)\frac{d}{dx}M_r(x)+\{(r-1)x+1\}M_r(x)=M_{r+1}(x)$$

and $M_r(1)=(r-1)!$. Using this we have

$$\begin{aligned}&\int_1^\infty \bar{B}_m(t)\left(\frac{d}{dt}\right)^m\binom{-t}{n-1}\log\left(\frac{1-q^{z+t}}{1-q^{z+1}}\right)dt\\ &=\frac{1}{(n-1)!}\sum_{j=1}^{n-1}\left\{(-1)^j{}_{n-1}S_j\sum_{l=0}^{j}\binom{m}{l}[j]_l\right.\\ &\quad\left.\times\int_1^\infty \bar{B}_m(t)t^{j-l}\left(\frac{\log q}{1-q^{t+z}}\right)^{m-l}q^{t+z}\,M_{m-l}(q^{t+z})dt\right\}.\end{aligned}$$

Therefore we have to show

$$\begin{aligned}(4.6)\qquad &\lim_{q\to1-0}\int_1^\infty \bar{B}_m(t)\left\{t^{j-l}\left(\frac{\log q}{1-q^{t+z}}\right)^{m-l}q^{t+z}\,M_{m-l}(q^{t+z})\right\}dt\\ &=\int_1^\infty \lim_{q\to1-0}\bar{B}_m(t)\left\{t^{j-l}\left(\frac{\log q}{1-q^{t+z}}\right)^{m-l}q^{t+z}\,M_{m-l}(q^{t+z})\right\}dt.\end{aligned}$$

This can be proved by means of the Lebesgue convergence theorem and of the following lemma.

Lemma 4.2. *Suppose that $\alpha\in\mathbf{N}$ be fixed and that y_0 and y_1 are fixed constants such that $-1<y_0<y_1$. Then there exists a constant C depending on y_0 and y_1 such that*

$$0<\left(\frac{\log q}{q^{t+y}-1}\right)^\alpha<\frac{C}{(t+y)^\alpha} \quad \textit{for} \quad y\in[y_0,y_1],\quad 1<t<\infty,\quad 0<q<1.$$

Next we show that this convergence is uniform on any compact set in the domain $\{z\in\mathbf{C}|\Re z>-1\}$.

Put

$$\Phi(z,q):=\int_1^\infty \bar{B}_m(t)t^{j-1}\left(\frac{\log q}{1-q^{z+t}}\right)^{m-l}q^{t+z}M_{m-l}(q^{z+t})dt.$$

From the consideration above, it follows that $\{\Phi(z,q)\,|0<q\leq1\}$ is a uniformly bounded family of functions and

$$\Phi(z,q)\to\int_1^\infty \bar{B}_m(t)\frac{t^{j-l}(-1)^{m-l}(m-l-1)!}{(t+z)^{m-l}}dt \quad \text{as} \quad q\to 1-0.$$

Hence, by Vitali's convergence theorem, this convergence is uniform on any compact set in the domain $\{z\in\mathbb{C}|\Re z>-1\}$. $\square$

The constant C_j in Proposition 4.1 can be expressed in terms of the special value of the Riemann zeta function.

Lemma 4.3.

$$C_j=-\zeta'(-j)-\frac{1}{(j+1)^2}.$$

Proof. From the definition of $\zeta(s)$,

$$\zeta'(s)=-\sum_{k=1}^\infty\frac{\log k}{k^s} \quad \text{for} \quad \Re s>1.$$

By the Euler-MacLaurin summation formula, we obtain

$$\begin{aligned}(4.7)\qquad \zeta'(s)=&-\frac{1}{(s-1)^2}+\sum_{r=1}^n\frac{B_r}{r!}\left(\frac{d}{dt}\right)^{r-1}\{t^{-s}\log t\}\Big|_{t=1}\\ &-\frac{1}{n!}\int_1^\infty \bar{B}_n(t)\left(\frac{d}{dt}\right)^n\{t^{-s}\log t\}dt.\end{aligned}$$

Since

$$\left(\frac{d}{dt}\right)^n\{t^{-s}\log t\}=-\frac{\partial}{\partial s}[-s]_n\, t^{-s-n}+[-s]_n\, t^{-s-n}\log t,$$

(4.7) can be analytically continued to $\{z\in\mathbb{C}|\Re z>-n+1\}$. So if we put $s=-j$, $n=j+2$, then the claim is proved. $\square$

Next we prove that the limit function in Proposition 4.1 coincides with the multiple gamma function.

Theorem 4.4. *Suppose $m>n$. Then, as $q\to 1-0$, $G_n(z+1;q)$ converges to $G_n(z+1)$ uniformly on any compact set in the domain $\mathbb{C}\backslash\mathbb{Z}_{<0}$ and*

$$\begin{aligned}(4.8)\qquad &\log G_n(z+1)\\ &=\left\{\binom{z+1}{n}+\sum_{r=1}^n\frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\log(z+1)\\ &-\sum_{r=1}^n\left\{\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\times\frac{1}{r!r}\{(z+1)^r-1\}\end{aligned}$$

$$-\sum_{j=0}^{n-1} G_{n,j}(z)\left\{\zeta'(-j)+\frac{1}{(j+1)^2}\right\}+\sum_{r=1}^{m}\frac{B_r}{r!}F_{n,r-1}(z)$$
$$-R_{n,m}(z).$$

Proof. In the domain $\{z \in \mathbf{C} \mid \Re z > -1\}$, we have already proved the existence of the limit function and uniformity of the convergence. Let us put

$$\tilde{G}_n(z+1) := \lim_{q\to 1-0} G_n(z+1; q).$$

Because of the uniformity of the convergence, we have particularly

$$\lim_{q\to 1-0}\left[\left(\frac{d}{dz}\right)^{n+1}\{\log G_n(z+1; q)\}\right]=\left(\frac{d}{dz}\right)^{n+1}\{\log \tilde{G}_n(z+1)\}$$

so that, from Theorem 2.2, $\tilde{G}_n(z+1)$ satisfies the conditions in Theorem 2.1 (1) ~(4). Since a hierarchy of such functions is uniquely determined, so $G_n(z+1)=\tilde{G}_n(z+1)$ in $\{z\in\mathbf{C}\mid\Re z>-1\}$. Thus the claim of the theorem in the case that $\Re z>-1$ has been proved for $\{z\in\mathbf{C}\mid\Re z>-1\}$.

Next, we show that in $\{z\in\mathbf{C}\mid\Re z\leq -1,\ z\neq -1\}$,

$$G_n(z+1; q)\to G_n(z+1) \quad \text{as} \quad q\to 1-0$$

and that the convergence is uniform on any compact set in this domain. For the proof, we use induction on n.

The case that $n=1$ was considered by Koornwinder [11]. So, we consider the case of n. Let K be a compact set in $\{z\in\mathbf{C}\mid -2<\Re z\leq -1,\ z\neq -1\}$. We assume that

$$G_{n-1}(z+1; q)\to G_{n-1}(z+1) \quad \text{as} \quad q\to 1-0$$

and that the convergence is uniform on K.

From (2.2), we see that, if q is sufficiently close to 1, $G_{n-1}(z+1; q)$ has no poles and no zeros on K, neither has $G_{n-1}(z+1)$ from (2.1). Therefore, as $q\to 1-0$,

$$G_n(z+1; q)=\frac{G_n(z+2; q)}{G_{n-1}(z+1; q)}$$

uniformly converges to

$$\frac{G_n(z+2)}{G_{n-1}(z+1)}=G_n(z+1)$$

on K.

Repeating this procedure, we can verify, for any n, $G_n(z+1; q)$ converge to $G_n(z+1)$ in a compact set in the domain $\{-3<\Re z\leq -2,\ z\neq -2\}$, $\{-4<\Re z$

$\leq -3, z\neq -3\}$, $\cdots$. Thus the claim of the theorem is proved. $\square$

§5. An Asymptotic Expansion of $G_n(z+1)$

Let us call the expression (4.8) *the Euler-MacLaurin expansion of* $G_n(z+1)$. We should note that (4.8) is valid for $z\in\mathbb{C}\backslash\{\Re z\leq -1\}$. We show that it gives an asymptotic expansion of $G_n(z+1)$ as $|z|\to\infty$, *i.e. the higher Stirling formula.*

Theorem 5.1. *Let* $0<\delta<\pi$, *then*

$$\begin{aligned}
&\log G_n(z+1)\\
&\sim\left\{\binom{z+1}{n}+\sum_{r=1}^{n}\frac{B_r}{r!}\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\log(z+1)\\
&-\sum_{r=1}^{n}\left\{\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\times\frac{1}{r!r}\{(z+1)^r-1\}\\
&-\sum_{j=0}^{n-1}G_{n,j}(z)\left\{\zeta'(-j)+\frac{1}{(j+1)^2}\right\}+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}F_{n,2r-1}(z)
\end{aligned}$$

as $|z|\to\infty$ *in the sector* $\{z\in\mathbb{C}||argz|<\pi-\delta\}$.

Proof. Straightforward calculation shows that

$$F_{n,r-1}(z)=\sum_{l=1}^{r-1}\left(\sum_{k=0}^{n-1}{}_{n-1}S_k(-1)^k[k]_{r-1-l}\right)\frac{(-1)^{l-1}(l-1)!}{(z+1)^l}.$$

Thus, we can see that

$$\frac{F_{n,2r-1}(z)}{F_{n,2r-3}(z)}=O(z^{-2})\quad\text{as}\quad |z|\to\infty. \tag{5.1}$$

Furthermore, noting that

$$\frac{1}{|z+t|}<\frac{1}{|t||\sin\delta|}$$

in the sector $\{z\in\mathbb{C}||argz|<\pi-\delta\}$ and that $|B_{2m}(t)|\leq|B_{2m}|$ for $0\leq t\leq 1$, we have

$$\begin{aligned}
|R_{n,2m}(z)|&=\left|\frac{-1}{(2m)!}\sum_{j=1}^{n-1}(-1)^j{}_{n-1}S_j\sum_{l=0}^{j}[j]_l(m-l-1)!\int_1^\infty\bar{B}_{2m}(t)\frac{t^j}{(t+z)^{2m-j}}dt\right|\\
&\leq\frac{|B_{2m}|}{(2m)!}\sum_{j=1}^{n-1}\sum_{l=0}^{j}{}_{n-1}S_j\binom{n}{l}\frac{[j]_l(2m-l-1)!}{|\sin\delta|^{2m-j}}\int_1^\infty\frac{dt}{t^{2m-j}}.
\end{aligned}$$

Hence, $|R_{n,2m}(z)|=O(z^{-2m-1+n})=o(F_{n,2m-1}(z))$ as $|z|\to\infty$ in the sector. $\square$

Let us exhibit some examples of the higher Stirling formula. In the case that $n=1$, we obtain

$$\begin{aligned}\log G_1(z+1) &= \log \Gamma(z+1)\\ &\sim \left(z+\frac{1}{2}\right)\log(z+1)-(z+1)-\zeta'(0)\\ &\quad+\sum_{r=1}^{\infty}\frac{B_{2r}}{[2r]_2}\frac{1}{(z+1)^{2r-1}}.\end{aligned}$$

This is the Stirling formula since $\zeta'(0)=-\frac{1}{2}\log(2\pi)$.

Furthermore, in the case that $n=2$, we obtain

$$\begin{aligned}&\log G_2(z+1)\\ &\sim\left(\frac{z^2}{2}-\frac{1}{12}\right)\log(z+1)-\frac{3}{4}z^2-\frac{z}{2}+\frac{1}{4}\\ &\quad-z\zeta'(0)+\zeta'(-1)\\ &\quad-\frac{1}{12}\frac{1}{z+1}+\sum_{r=2}^{\infty}\frac{B_{2r}}{[2r]_3}\frac{1}{(z+1)^{2r-1}}(z-2r+1),\end{aligned}$$

which coincides with an asymptotic expansion of the Barnes G-function (see [23], [25]). In the case that $n=3$, 4, and 5, we have the following results.

Proposition 5.2. *The higher Stirling formula for* $n=3$, 4 *and* 5 *are as follows*:

$$\begin{aligned}&\log G_3(z+1)\\ &\sim\left(\frac{z^3}{6}-\frac{z^2}{4}+\frac{1}{24}\right)\log(z+1)-\frac{11}{36}z^3+\frac{5}{24}z^2+\frac{z}{3}-\frac{13}{72}\\ &\quad-\frac{z^2-z}{2}\zeta'(0)+\frac{2z-1}{2}\zeta'(-1)-\frac{1}{2}\zeta'(-2)\\ &\quad+\frac{1}{12}\frac{1}{z+1}+\sum_{r=2}^{\infty}\frac{B_{2r}}{[2r]_4}\frac{1}{(z+1)^{2r-1}}\{z^2-(6r-11)z+(4r^2-16r+16)\}.\end{aligned}$$

$$\begin{aligned}&\log G_4(z+1)\\ &\sim\left(\frac{z^4}{24}-\frac{z^3}{6}+\frac{z^2}{6}-\frac{19}{720}\right)\log(z+1)\\ &\quad-\frac{4}{72}z^4+\frac{2}{9}z^3+\frac{z^2}{8}-\frac{11}{36}z+\frac{31}{144}\\ &\quad-\frac{z^3-3z^2+2z}{6}\zeta'(0)+\frac{3z^2-6z+2}{6}\zeta'(-1)-\frac{z-1}{2}\zeta'(-2)+\frac{1}{6}\zeta'(-3)\\ &\quad-\frac{1}{12}\frac{1}{z+1}+\frac{1}{720}\frac{1}{(z+1)^3}\left(6z^2+\frac{13}{2}z+\frac{5}{2}\right)\end{aligned}$$

$$+\sum_{r=3}^{\infty}\frac{B_{2r}}{[2r]_5}\frac{1}{(z+1)^{2r-1}}\{z^3-(12r-27)z^2+(20r^2-94r+111)z$$
$$-(8r^3-56r^2+134r-109)\Big\}.$$

$$\log G_5(z+1)$$
$$\sim\left(\frac{z^5}{120}-\frac{z^4}{16}+\frac{11}{72}z^3-\frac{z^2}{8}+\frac{3}{160}\right)\log(z+1)$$
$$-\frac{137}{7200}z^5+\frac{39}{320}z^4-\frac{461}{2160}z^3+\frac{z^2}{1440}-\frac{323}{1440}z+\frac{5639}{43200}$$
$$-\frac{z^4-6z^3+11z^2-6z}{24}\zeta'(0)+\frac{4z^3-18z^2+22z-6}{24}\zeta'(-1)$$
$$-\frac{6z^2-18z+11}{24}\zeta'(-2)+\frac{2z-3}{12}\zeta'(-3)-\frac{1}{24}\zeta'(-4)$$
$$+\frac{1}{12}\frac{1}{z+1}-\frac{1}{720}\frac{1}{(z+1)^3}\left(\frac{35}{4}z^2+\frac{45}{4}z+\frac{9}{2}\right)$$
$$+\sum_{r=3}^{\infty}\frac{B_{2r}}{[2r]_6}\frac{1}{(z+1)^{2r-1}}\Big\{z^4-(20r-54)z^3+(70r^2-375r+506)z^2$$
$$-\left(\frac{200}{3}r^3-540r^2+\frac{4420}{3}r-1354\right)z$$
$$+16r^4-\frac{536}{3}r^3+754r^2-\frac{4279}{3}r+1021\Big\}.$$

§6. The Weierstrass Canonical Product Representation for $G_n(z+1)$

By calculating the formula (4.8) in the case of $m=n+1$, we derive the Weierstrass canonical product representation for the multiple gamma functions. Main theorem of this section is the following:

Theorem 6.1. *For $n\in\mathbb{N}$, we have*

$$G_n(z+1)=\exp(F_n(z))\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{-\binom{-k}{n-1}}\exp(\Phi_n(z,k))\right\},$$

where

$$F_n(z):=\sum_{j=0}^{n-1}G_{n,j}(z)Q_j(z)+\sum_{r=0}^{n-2}\left[\frac{1}{r!}\left(\frac{\partial}{\partial u}\right)^r\binom{z-u}{n-1}\right]_{u=0}^{u=z}\times\zeta'(-r)$$
$$-\int_0^z\binom{z-u}{n-1}du\times\gamma,$$

$$\Phi_n(z,k):=\frac{1}{(n-1)!}\sum_{\mu=-1}^{n-2}\left\{\sum_{r=\mu+1}^{n-1}\frac{{}_{n-1}S_r}{r-\mu}z^{r-\mu}\right\}(-1)^{\mu+1}k^{\mu},$$

$$Q_j(z):=P_j(z+1)-\sum_{r=0}^{j}\binom{j}{r}z^rP_{j-r}(1)$$

$$+\frac{1}{j+1}\sum_{r=1}^{j+1}\binom{j+1}{r}B_{j+1-r}(z)\sum_{l=1}^{r}\frac{(-1)^{l-1}z^l}{l},$$

$$P_j(x):=\sum_{r=0}^{j+1}\frac{B_r}{r!}\varphi_{j,r}x^{j-r+1},$$

$$\varphi_{j,r}:=\left(\frac{d}{dt}\right)^r\left\{\frac{t^{j+1}}{j+1}\log t-\frac{t^{j+1}}{(j+1)^2}\right\}\Bigg|_{t=1}.$$

Proof of this theorem will be carried out through the sections $6.1\sim6.4$, and some examples of this representation will be discussed in the section 6.5.

6.1. Rewriting the Euler-MacLaurin expansion of $G_n(z+1)$. In this section we prove the following proposition.

Proposition 6.2.

$$\log G_n(z+1)=\sum_{j=0}^{n-1}G_{n,j}(z)K_j(z) \tag{6.1}$$

where

$$K_j(z):=\frac{B_{j+1}(z+1)}{j+1}\log(z+1)-\zeta'(-j)+P_j(z+1)$$

$$+\sum_{k=1}^{\infty}\left[P_j(z+k+1)-P_j(z+k)+\frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{z+k+1}{z+k}\right)\right].$$

Furthermore the infinite sum in the last term is absolutely convergent.

Proof. In the Euler-MacLaurin expansion of $G_n(z+1)$, we see that

$$\sum_{r=1}^{n}\left\{\left(-\frac{d}{dz}\right)^{r-1}\binom{z}{n-1}\right\}\Bigg|_{z=0}\times\frac{1}{r!r}\{(z+1)^r-1\} \tag{6.2}$$

$$=\sum_{j=0}^{n-1}G_{n,j}(z)\frac{(z+1)^{j+1}-1}{(j+1)^2}.$$

and also that for $1\leq r\leq n-1$,

$$(6.3)\qquad F_{n,r-1}(z)=\sum_{j=0}^{n-1}G_{n,j}(z)\,(z+1)^{j+1-r}\varphi_{j,r}.$$

Moreover, from the properties of the Bernoulli polynomials, we have

$$(6.4)\qquad \binom{z+1}{n-1}+\sum_{r=1}^{n}\frac{B_r}{r!}\left(-\frac{d}{dt}\right)^{r-1}\binom{z}{n-1}=\sum_{j=0}^{n-1}G_{n,j}(z)\frac{B_{j+1}(z+1)}{j+1}.$$

Next we calculate $R_{n,n+1}(z)$. From the definition of $G_{n,r}(z)$, we have

$$(6.5)\qquad R_{n,n+1}(z)=\sum_{j=0}^{n-1}G_{n,j}(z)\Big[\sum_{k=1}^{\infty}\frac{(-1)^n}{(n+1)!}\times\int_0^1 B_{n+1}(t)\left(\frac{d}{dt}\right)^{n+1}\left\{(z+t+k)^j\log\left(\frac{z+t+k}{z+1}\right)\right\}dt\Big].$$

Here we have, for $j\leq n-1$,

$$(6.6)\qquad \int_0^1 B_{n+1}(t)\left(\frac{d}{dt}\right)^{n+1}\left\{(z+t+k)^j\log\left(\frac{z+t+k}{z+1}\right)\right\}dt =O(k^{j-n-1})\quad\text{as}\quad k\to\infty,$$

so that the infinite sum in (6.5) absolutely converges because $j-n-1\leq-2$.
By means of the Euler-MacLaurin summation formula, we have

$$\begin{aligned}(6.7)\qquad &\frac{(-1)^n}{(n+1)!}\int_0^1 B_{n+1}(t)\left(\frac{d}{dt}\right)^{n+1}\left\{(z+t+k)^j\log\left(\frac{z+t+k}{z+1}\right)\right\}dt\\
&=\left\{(z+k)^j+\frac{(z+k)^{j+1}}{j+1}+\sum_{r=1}^{j+1}\frac{B_r}{r!}[j]_{r-1}(z+k)^{j+1-r}\right\}\log(z+k)\\
&-\left\{\frac{(z+k-1)^{j+1}}{j+1}+\sum_{r=1}^{j+1}\frac{B_r}{r!}[j]_{r-1}(z+k+1)^{j+1-r}\right\}\log(z+k+1)\\
&+\Big[-(z+k)^j+\frac{(z+k+1)^{j+1}}{j+1}-\frac{(z+k)^{j+1}}{j+1}\\
&-\sum_{r=1}^{j+1}\frac{B_r}{r!}[j]_{r-1}\left\{(z+k)^{j+1-r}-(z+k+1)^{j+1-r}\right\}\Big]\log(z+1)\\
&-\sum_{r=0}^{n+1}\frac{B_r}{r!}\varphi_{j,r}\left\{(z+k+1)^{j+1-r}-(z+k)^{j+1-r}\right\}.\end{aligned}$$

It is easy to see that the coefficient of $\log(z+1)$ of the above formula vanishes.

Hence we obtain

$$(6.8)\qquad R_{n,n+1}(z)=-\sum_{j=0}^{n-1}G_{n,j}(z)\Big[-\sum_{r=j+2}^{n+1}\frac{B_r}{r!}(z+1)^{j+1-r}\varphi_{j,r}$$
$$+\sum_{k=1}^{\infty}\Big\{\frac{B_{j+1}(z+k+1)}{j+1}\log\Big(\frac{z+k+1}{z+k}\Big)+P_j(z+k+1)-P_j(z+k)\Big\}\Big],$$

and the infinite sum is absolutely convergent since, for $r\geq j+2$, $(z+k+1)^{j+1-r}-(z+k)^{j+1-r}$ decreases more rapidly than k^{-2} as $k\to\infty$.

Substituting (6.2), (6.3), (6.4), (6.8) to (4.8), and noting that

$$-\frac{1}{(j+1)^2}+\sum_{r=1}^{n+1}\frac{B_r}{r!}(z+1)^{j+1-r}-\sum_{r=j+2}^{n+1}\frac{B_r}{r!}(z+1)^{j+1-r}=P_j(z+1),$$

we obtain (6.1). □

In the above proof, we showed that

$$\frac{B_{j+1}(z+k+1)}{j+1}\log\Big(\frac{z+k+1}{z+k}\Big)+P_j(z+k+1)-P_j(z+k)$$
$$=O(k^{-2})\quad\text{as}\quad k\to\infty.$$

We note that the converse is also true.

Lemma 6.3. *There exists a polynomial $A(k,z)$ such that*

$$\frac{B_{j+1}(z+k+1)}{j+1}\log\Big(\frac{z+k+1}{z+k}\Big)+A(k,z)$$
$$=O(k^{-2})\quad as\quad k\to\infty.$$

$A(k,z)$ is equal to $P_j(z+k+1)-P_j(z+k)$.

Proof. Since

$$B_{j+1}(z+k+1)=\sum_{r=0}^{j+1}B_{j+1-r}(1)\,(z+k)^r,$$

we have

$$\frac{B_{j+1}(z+k+1)}{j+1}\log\Big(\frac{z+k+1}{z+k}\Big)$$
$$+\frac{1}{j+1}\sum_{r=0}^{j+1}\binom{j+1}{r}B_{j+1-r}(1)\sum_{l=1}^{r+1}\frac{(-1)^l}{l}(z+k)^{r-l}=O(k^{-2}).$$

By integrating the both sides of

$$B_{j+1}(z+1)=\sum_{r=0}^{j+1} B_{j+1-r}(1) z^r$$

from -1 to 0, we get

$$\frac{1}{j+1}\sum_{r=0}^{j+1}\binom{j+1}{r} B_{j+1-r}(1)\frac{(-1)^r}{r+1}=0. \tag{6.9}$$

This implies the statement. $\square$

6.2. An infinite product representation for the $\zeta'(-j)$. We derive an infinite product representation for $\zeta'(-j)$ in the same way as in the Section 3.1.

Proposition 6.4.

$$\exp(\zeta'(-j))=\exp(P_j(1))\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{B_{j+1}(k+1)}{j+1}}\exp(P_j(k+1)-P_j(k))\right\}$$

and the infinite product absolutely converges.

Proof. From the proof of Lemma 4.3, we have

$$\begin{aligned}\zeta'(-j)=P_j(1)&+\frac{B_{j+2}}{(j+2)!}\varphi_{j,j+2}\\ &-\frac{(-1)^{j+1}}{(j+2)!}\int_1^{\infty}\bar{B}_{j+2}(t)\left(\frac{d}{dt}\right)^{j+2}\{t^j\log t\}dt.\end{aligned}$$

Here, in the same fashion in the previous section, we have

$$\begin{aligned}&\frac{(-1)^{j+1}}{(j+2)!}\int_1^{\infty}\bar{B}_{j+2}(t)\left(\frac{d}{dt}\right)^{j+2}\{t^j\log t\}dt \\ &=\frac{(-1)^{j+1}}{(j+2)!}\sum_{k=1}^{\infty}\int_0^1 B_{j+2}(t)\left(\frac{d}{dt}\right)^{j+2}\{(t+k)^j\log(t+k)\}dt,\\ &=\sum_{k=1}^{\infty}\left\{-\frac{B_{j+1}(k+1)}{j+1}\log\left(1+\frac{1}{k}\right)+P_j(k)-P_j(k+1)\right.\\ &\qquad\left.+\frac{B_{j+2}}{(j+2)!}\left(\frac{1}{k}-\frac{1}{k+1}\right)\varphi_{j,j+2}\right\}\\ &=\sum_{k=1}^{\infty}\left\{-\frac{B_{j+1}(k+1)}{j+1}\log\left(1+\frac{1}{k}\right)+P_j(k)-P_j(k+1)\right\}+\frac{B_{j+2}}{(j+2)!}\varphi_{j,j+2}.\end{aligned} \tag{6.10}$$

This completes the proof. $\square$

6.3. Good representation for $K_j(z)$. In this section, we give a "good" representation for $K_j(z)$.

Proposition 6.5. *Let k be a positive integer and define*

$$(I)_k := \frac{1}{j+1}\left\{\sum_{r=0}^{j+1}\binom{j+1}{r} B_{j+1-r}(z)\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{l-r}\right\},$$

$$(II)_k := -(I)_{k+1},$$

$$(III)_k := \sum_{r=0}^{j} z^{j-r}\{P_r(k+1)-P_r(k)\} - \frac{1}{k}\frac{z^{j+1}}{j+1},$$

and

$$(IV)_k := \sum_{r=0}^{j}\binom{j}{r} z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l}.$$

Then we have

$$\begin{aligned}
(6.11)\qquad & \frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{z+k+1}{z+k}\right)+P_j(z+k+1)-P_j(z+k) \\
&= \left\{\frac{B_{j+1}(z+k)}{j+1}\log\left(\frac{k}{z+k}\right)+(I)_k\right\} \\
&+ \left\{\frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{z+k+1}{k+1}\right)+(II)_k\right\} \\
&+ \left\{\frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{k+1}{k}\right)+(III)_k\right\} \\
&+ \left\{-(z+k)^j\log\left(1+\frac{z}{k}\right)+(IV)_k\right\} \\
&- \frac{1}{j+1}\left\{\sum_{r=0}^{j+1}\frac{(-1)^r z^{r+1}}{r+1}B_{j+1-r}(z)\right\}\left(\frac{1}{k}-\frac{1}{k+1}\right)
\end{aligned}$$

Furthermore each term in the right hand side decreases like $O(k^{-2})$ as $k\to\infty$.

Proof. It is easy to see that each term in the right hand side is $O(k^{-2})$ as $k\to\infty$. Moreover we can show that $(I)_k\sim(IV)_k$ are polynomials of z and that

$$(I)_k = \frac{1}{j+1}\left\{\sum_{r=0}^{j+1}\binom{j+1}{r}B_{j+1-r}(z)\frac{(-1)^r z^{r+1}}{r+1}\right\}\frac{1}{k}$$
$$+\ (\text{a polynomial of } k),$$

$$(II)_k = -\frac{1}{j+1}\left\{\sum_{r=0}^{j+1}\binom{j+1}{r}B_{j+1-r}(z)\frac{(-1)^r z^{r+1}}{r+1}\right\}\frac{1}{k+1}$$
$$+\ (\text{a polynomial of } k),$$

$$(III)_k = -\frac{z^{j+1}}{j+1}\frac{1}{k}+(\text{a polynomial of } k),$$

$$(IV)_k = \frac{z^{j+1}}{j+1}\frac{1}{k}+(\text{a polynomial of } k).$$

Hence, if we put

$$B(k,z):=(I)_k+(II)_k+(III)_k+(IV)_k$$
$$-\frac{1}{j+1}\left\{\sum_{r=0}^{j+1}\binom{j+1}{r}\frac{(-1)^r z^{r+1}}{r+1}B_{j+1-r}(z)\right\}\left(\frac{1}{k}-\frac{1}{k+1}\right),$$

then, $B(k, z)$ is a polynomial of k, z and

$$\text{(the right hand side of (6.11))}$$
$$=\frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{z+k+1}{z+k}\right)+B(k, z)$$

From Lemma 6.3, $B(k, z)=P_j(z+k+1)-P_j(z+k)$. $\square$

Proposition 6.6.

$$K_j(z)=Q_j(z)+\sum_{r=1}^{j}\binom{j}{r}z^{j-r}\zeta'(-r)-\frac{z^{j+1}}{j+1}\gamma$$
$$+\sum_{k=1}^{\infty}\left\{-(z+k)^k\log\left(1+\frac{z}{k}\right)+\sum_{r=0}^{j}\binom{j}{r}z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l}\right\}$$

and the infinite sum in this formula absolutely converges.

Proof. We have

$$(6.12)\qquad \sum_{k=1}^{\infty}\left\{\frac{B_{j+1}(z+k)}{j+1}\log\left(\frac{k}{z+k}\right)+(I)_k\right\}$$
$$+\sum_{k=1}^{\infty}\left\{\frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{z+k+1}{k+1}\right)+(II)_k\right\}$$
$$=-\frac{B_{j+1}(z+1)}{j+1}\log(z+1)+\frac{1}{j+1}\sum_{r=0}^{j+1}\binom{j+1}{r}B_{j+1-r}(z)\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}.$$

Noting Proposition 6.4 and

$$\gamma=\sum_{k=1}^{\infty}\left\{\log\left(1+\frac{1}{k}\right)-\frac{1}{k}\right\},$$

we get

$$(6.13)\qquad \sum_{k=1}^{\infty}\left\{\frac{B_{j+1}(z+k+1)}{j+1}\log\left(1+\frac{1}{k}\right)+(III)_k\right\}$$
$$=\sum_{r=0}^{j}\binom{j}{r}z^{j-r}\{\zeta'(-r)-P_r(1)\}-\frac{z^{j+1}}{j+1}\gamma.$$

By (6.12) and (6.13), we obtain

$$\sum_{k=1}^{\infty}\left\{\frac{B_{j+1}(z+k+1)}{j+1}\log\left(\frac{z+k+1}{z+k}\right)+P_j(z+k+1)-P_j(z+k)\right\} \tag{6.14}$$
$$=-\frac{B_{j+1}(z+1)}{j+1}\log(z+1)$$
$$+\frac{1}{j+1}\sum_{r=1}^{j+1}\binom{j+1}{r}B_{j+1-r}(z)\sum_{l=1}^{r}\frac{(-1)^{l-1}z^l}{l}$$
$$+\sum_{r=0}^{j}\binom{j}{r}z^{j-r}\{\zeta'(-r)-P_r(1)\}-\frac{z^{j+1}}{j+1}\gamma$$
$$+\sum_{k=1}^{\infty}\left\{-(z+k)^j\log\left(1+\frac{z}{k}\right)+\sum_{r=0}^{j}\binom{j}{r}z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l}\right\}.$$

The proof is completed by substituting (6.14) to the definition of $K_j(z)$ in Proposition 6.2. □

6.4. A proof of main theorem. By Proposition 6.6, we have

$$\log G_n(z+1) \tag{6.15}$$
$$=\sum_{j=1}^{n-1}G_{n,j}(z)\left[Q_j(z)+\sum_{r=0}^{j-1}\binom{j}{r}z^{j-r}\zeta'(-r)-\frac{z^{j+1}}{j+1}\gamma\right.$$
$$\left.+\sum_{k=1}^{\infty}\left\{-(z+k)^j\log\left(1+\frac{z}{k}\right)+\sum_{r=0}^{j}\binom{j}{r}z^{j-r}\sum_{l=0}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l}\right\}\right]$$
$$=\sum_{j=0}^{n-1}G_{n,j}(z)Q_j(z)+\sum_{r=0}^{n-2}\left[\frac{1}{r!}\left(\frac{\partial}{\partial u}\right)^r\binom{z-u}{n-1}\right]_{u=0}^{u=z}\times\zeta'(-r)$$
$$-\int_0^z\binom{z-u}{n-1}du\times\gamma$$
$$-\sum_{k=1}^{\infty}\left[\binom{-k}{n-1}\log\left(1+\frac{z}{k}\right)+\sum_{j=0}^{n-1}G_{n,j}(z)\sum_{r=0}^{j}\binom{j}{r}z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l}\right].$$

Thus, in order to prove Theorem 6.1, it is sufficient to show

$$\sum_{j=0}^{n-1}G_{n,j}(z)\sum_{r=0}^{j}z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l} \tag{6.16}$$
$$=\frac{1}{(n-1)!}\sum_{\mu=-1}^{n-2}\left\{\sum_{r=\mu+1}^{n-1}\frac{{}_{n-1}S_r}{r-\mu}z^{r-\mu}\right\}(-1)^{\mu+1}k^{\mu}.$$

Since

$$(z+k)^j \log\left(1+\frac{z}{k}\right)+\sum_{r=0}^{j}\binom{j}{r} z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^l z^l}{l}k^{r-l}$$
$$=O(k^{-2}) \quad \text{as} \quad k\to\infty,$$

we have

$$(6.17)\qquad -\binom{-k}{n-1}\log\left(1+\frac{z}{k}\right)+\sum_{j=0}^{n-1}G_{n,j}(z)\sum_{r=0}^{j}\binom{j}{r} z^{j-r}\sum_{l=1}^{r+1}\frac{(-1)^{l-1}z^l}{l}k^{r-l}$$
$$=O(k^{-2}) \quad \text{as} \quad k\to\infty,$$

while we obtain

$$(6.18)\qquad -\binom{-k}{n-1}\log\left(1+\frac{z}{k}\right)+\frac{1}{(n-1)!}\sum_{\mu=-1}^{n-2}\left\{\sum_{r=\mu+1}^{n-1}\frac{{}_{n-1}S_r}{r-\mu}z^{r-\mu}\right\}(-1)^{\mu+1}k^{\mu}$$
$$=O(k^{-2}) \quad \text{as} \quad k\to\infty.$$

By the same arguments as in Lemma 6.3, (6.16) is deduced from (6.17) and (6.18). Hence the proof is completed. □

6.5. Examples of the Weierstrass canonical product representation for $G_n(z+1)$. We give some examples of the Weierstrass canonical product representation for the multiple gamma functions.

In the case that $n=1$, we have

$$G_1(z+1)=\Gamma(z+1)=e^{-\gamma z}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{-1}e^{-\frac{z}{k}}\right\}.$$

This is the Weierstrass canonical product representation for the gamma function.

In the case that $n=2$, we have

$$G_2(z+1)=G(z+1)=e^{-z\zeta'(0)-\frac{z^2}{2}\gamma-\frac{z^2+z}{2}}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k\exp\left(-z+\frac{z^2}{2k}\right)\right\}.$$

Since $\zeta'(0)=-\frac{1}{2}\log(2\pi)$, this is the Weierstrass canonical product representation for the Barnes G-function [2].

In the case that $n=3$, 4 and 5, we obtain the following results.

Proposition 6.7. *The Weierstrass canonical product representations in the case that $n=3$, 4 and 5 are as follows:*

$G_3(z+1)$

$$=\exp\left\{-\frac{z^3}{4}+\frac{z^2}{8}+\frac{7}{24}z+\zeta'(-1)-\frac{z(z-1)}{2}\zeta'(0)-\left(\frac{z^3}{6}-\frac{z^2}{4}\gamma\right)\right\}$$
$$\times\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^{-\frac{k(k+1)}{2}}\exp\left\{\left(\frac{z^3}{6}-\frac{z^2}{4}\right)\frac{1}{k}-\left(\frac{z^2}{4}-\frac{z}{2}\right)+\frac{z}{2}k\right\}\right],$$

$G_4(z+1)$

$$=\exp\left\{\frac{61}{144}z^4+\frac{13}{18}z^3+\frac{19}{144}z^2-\frac{5}{24}z\right.$$
$$\left.-\frac{z}{2}\zeta'(-2)+\frac{z^2-2z}{3}\zeta'(-1)-\frac{z^3-3z^2+2z}{6}\zeta'(0)-\frac{z^4-4z^3+4z^2}{24}\gamma\right\}$$
$$\times\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^{\frac{k(k+1)(k+2)}{6}}\exp\left\{\left(\frac{z^4}{24}-\frac{z^3}{6}+\frac{z^2}{6}\right)\frac{1}{k}\right.\right.$$
$$\left.\left.-\left(\frac{z^3}{18}-\frac{z^2}{4}-\frac{z}{3}\right)+\left(\frac{z^2}{12}-\frac{z}{2}\right)k-\frac{z}{6}k^2\right\}\right],$$

$G_5(z+1)$

$$=\exp\left\{-\frac{5}{288}z^5+\frac{7}{64}z^4-\frac{173}{864}z^3-\frac{z^2}{36}+\frac{2827}{17280}z\right.$$
$$+\frac{z}{6}\zeta'(-3)-\frac{z^2-3z}{4}\zeta'(-2)+\frac{2z^3-9z^2+11z}{12}\zeta'(-1)$$
$$\left.-\frac{z^4-6z^3+11z^2-6z}{24}\zeta'(0)-\frac{6z^5-45z^4+110z^3-90z^2}{720}\gamma\right\}$$
$$\times\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^{-\frac{k(k+1)(k+2)(k+3)}{24}}\exp\left\{\left(\frac{z^5}{120}-\frac{z^4}{16}+\frac{11}{72}z^3-\frac{z^2}{8}\right)\frac{1}{k}\right.\right.$$
$$-\left(\frac{z^4}{96}-\frac{z^3}{12}+\frac{11}{48}z-\frac{z}{4}\right)+\left(\frac{z^3}{72}-\frac{z^2}{8}+\frac{11}{24}\right)k$$
$$\left.\left.-\left(\frac{z^2}{24}-\frac{z}{4}\right)k^2+\frac{z}{24}k^3\right\}\right].$$

References

[1] Askey, R., The q-Gamma and q-Beta functions, *Appl. Anal.*, 8(1978), 125-141.

[2] Barnes, E. W. The theory of G-function, *Quat. J. Math.*, **31** (1899), 264-314.

[3] ______, Genesis of the double gamma function, *Proc London Math. Soc*. **31** (1900), 358-381

[4] ______, The theory of the double gamma function, *Phil Trans Royal Soc* (A), **196** (1900), 265-388.

[5] ______, On the theory of the multiple gamma functions, *Trans. Cambridge Phil. Soc.*, **19** (1904), 374-425.

[6] Dufresnoy, J. et Pisot, C., Sur la relation fanctionalle $f(x+1)-f(x)=\phi(x)$, *Bull. Soc. Math. Belgique*, **15** (1963), 259-270

[7] Hardy, G H., On the expression of the double zeta-function and double gamma function in terms of elliptic functions, *Trans. Cambridge. Phil Soc.*, **20** (1905), 395-427.

[8] ______, On double Fourier series and especially these which represent the double zeta-function and incommensurable parameters, *Quart. J. Math*. **37** (1906), 53-79

[9] Jackson, F H, A generalization of the functions $\Gamma(n)$ and x^n, *Proc. Roy. Soc. London*, **74**

(1904), 64-72.

[10] Jackson, F. H., The basic gamma function and the elliptic functions, *Proc. Roy. Soc. London*, **A76** (1905), 127-144.

[11] Koornwinder, T., Jacobi function as limit cases of q-ultraspherical polynomial, *J. Math. Anal. Appl.*, **148** (1990), 44-54.

[12] Kurokawa, N., Multiple sine functions and Selberg zeta functions, *Proc. Japan Acad.*, **67A** (1991), 61-64.

[13] ______, Multiple zeta functions; an example, *Adv. Stud. Pure Math.*, **21** (1992), 219-226.

[14] ______, Gamma factors and Plancherel measures, *Proc. Japan Acad.*, **68A** (1992), 256-260.

[15] ______, On a q-analogues of multiple sine functions, *RIMS, Kokyuroku*, **843** (1992), 1-10.

[16] Kurokawa, N., Lectures delivered at Tokyo Institute of Technology, 1993.

[17] Manin, Yu., Lectures on Zeta Functions and Motives, *Asterisque*, **228** (1995), 121-163.

[18] Moak, D. S., The q-analogue of Stirling Formula, *Rocky Mountain J. Math.*, **14** (1984), 403-413.

[19] Nishizawa, M., On a q-analogue of the multiple gamma functions, *Lett. Math. Phys.*, **37** (1996), 201-209, q-alg/9505086.

[20] Shintani, T., On a Kronecker limit formula for real quadratic fields, *J. Fac. Sci. Univ. Tokyo Sect. 1A*, **24** (1977), 167-199.

[21] ______, A proof of Classical Kronecker limit formula, *Tokyo J. Math.*, **3** (1980), 191-199.

[22] Ueno, K. and Nishizawa, M., Quantum groups and zeta-functions, in: J. Lukierski, Z. Popowicz and J. Sobczyk (eds.) '*Quantum Groups: Formalism and Applications*" Proceedings of the XXX-th Karpacz Winter School. 115-126, Polish Scientific Publishers PWN.

[23] Vardi, I., Determinants of Laplacians and multiple gamma functions, *SIAM. J. Math. Anal.*, **19** (1988), 493-507.

[24] Vignéras, M. F., L'équation fonctionalie de la fonction zeta de Selberg de groupe modulaire *PSL*(2, *Z*), *Asterisque*, **61** (1979), 235-249.

[25] Voros, A., Spectral functions, Special functions and the Selberg zeta functions, *Comm. Math. Phys.*, **110** (1987), 431-465.

[26] Whittacker, E. T. and Watson, G. N., *A Course of Modern Analysis*, Fourth edition, Cambrige Univ. Press, 1927.