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Representations of Hermitian Kernels by Means of Kreĭn Spaces

By

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Abstract

Hermitian kernels are studied as generalizations of kernels of positive type. The main tool is the axiomatic concept of induced Kreĭn space. The existence of Kolmogorov decompositions of a hermitian kernel and their uniqueness, modulo unitary equivalence, are characterized. The existence of reproducing kernel Kreĭn spaces is shown to be equivalent to the existence of Kolmogorov decompositions. Applications Applications to the Naĭmark dilations of Toeplitz hermitian kernels on the set of integers and to the uniqueness of the Kreĭn space completions of nondegenerate inner product spaces are included.

§1. Introduction

The theory of positive definite kernels and their reproducing kernel representations is well established and has significant applications in various domains as seen in [2], [4], [6], [26], [15], [29]. L. Schwartz [30] has considered a parallel theory, of Hilbert spaces continuously embedded in quasi-complete locally convex spaces, and he showed the equivalence with the abstract reproducing kernel Hilbert space theory of N. Aronszajn and S. Bergman. In addition, "not as a monstrosity, but as an interesting novelty", to quote his own words, L. Schwartz generalized this theory to hermitian kernels, and hence to genuine Kreĭn spaces, apparently independent of the theory of operators on Kreĭn or Pontryagin spaces, which was already developed at that time. Among other pathologies that appear in this theory, of special concern is the lack of uniqueness of the associated reproducing kernel spaces. From a slightly different point of view, P. Sorjonen [31] considered reproducing kernel

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Pontryagin spaces, which are uniquely determined.

Historically, two parallel main constructions appear to be used: the first one is based on a quotient-completion process, cf. A. N. Kolmogorov [18], M. A. Naĭmark [24], [25], while the second one uses image-spaces, cf. N. Aronszajn [2], L. de Branges [4], L. Schwartz [30], etc. The quotient-completion pattern, applied to certain hermitian kernels with a finite number of negative squares, was followed also by M. G. Kreĭn and H. Langer in generalizing to Pontryagin spaces much of the corresponding theory on Hilbert spaces, cf. [20], [21], [22]. In the positive definite case these two approaches turned out to be perfectly equivalent, see e.g. [15]. Due to the nonuniqueness, in the hermitian indefinite case the relation between the two constructions in less obvious.

Our first aim is to investigate the interplay between these two approaches and to show their equivalence. We look at hermitian kernels through the Kolmogorov decompositions and investigate their properties. In this context, we address the uniqueness problem for Kolmogorov decompositions, modulo unitary equivalence, and then for reproducing kernel Kreĭn spaces. Finally, we show some applications to the Naĭmark dilations of hermitian Toeplitz kernels and to Kreĭn space completions of abstract indefinite inner product spaces.

The paper is organized as follows. In Section 2 we present an axiomatic study of Kreĭn spaces induced by selfadjoint operators. Induced Kreĭn spaces, obtained by a renorming process as in Example 2.1, appeared since the early stages of operator theory on indefinite inner product spaces. We indicate another equivalent construction (the Kreĭn space $\mathscr{H}_A$, cf. Example 2.3) in connection with applications in dilation theory, lifting and extension of operators in Kreĭn spaces in [9], [8]. We show that the axiomatic approach of induced Kreĭn spaces contains also the de Branges construction based on operator ranges as in [5], cf. Example 2.5. The characterization of the uniqueness of induced Kreĭn spaces is obtained in Theorem 2.8, whose proof is inspired by our paper [8]. As a consequence of this result, we can obtain the result of T. Hara [17] on the uniqueness of de Branges spaces, cf. Corollary 2.9. Section 3 is devoted to the existence problem of Kolmogorov decompositions for hermitian kernels. The main result of this section is Theorem 3.1 which combines characterizations obtained by L. Schwartz, in the case of Kreĭn subspaces continuously contained in quasi-complete locally convex spaces, with the renorming procedure. Here, the use of induced Kreĭn spaces plays a key role. We also show that a multiplicative structure of the Kolmogorov decomposition for hermitian kernels on the set $\boldsymbol{Z}$ of integers can be obtained, see Theorem 3.3.

The main result in Section 4 is Theorem 4.1, which gives a characterization of the hermitian kernels addmitting unique Kolmogorov decomposition, in terms of an associated family of selfadjoint operators. Theorem 5.1 shows that, even for a general hermitian kernel K, the existence of

Kolmogorov decompositions of K is equivalent to the existence of Kreĭn spaces with reproducing kernel K. As a consequence of this result and of Theorem 4.1, the uniqueness of the reproducing kernel Kreĭn space associated to a given hermitian kernel is characterized. This may be viewed as an answer to a question left open in the paper [30] of L. Schwartz, see also [1].

Section 6 gives an application to Naĭmark dilations of hermitian Toeplitz kernels. In connection with the existence problem of Naĭmark dilations it appears as natural to introduce an intermediate class, what we call "hermitian kernels of bounded shift type", which provides a tool for the solution of this problem, cf. Theorem 6.8. Then we address the question whether the Naĭmark dilation can be characterized only within the class of hermitian Toeplitz kernels and we prove in Theorem 6.10 that this can be done only for the subclass admitting fundamentally reducible Naĭmark dilations.

In the last section we sketch how similar arguments can be used to provide an answer for a characterization of the uniqueness of Kreĭn spaces associated with abstract indefinite inner product spaces (see, M. Tomita [32] and F. Hansen [16] for related problems).

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§2. Induced Kreĭn Spaces

Let $(\mathscr{H}, [\cdot, \cdot])$ be a *Kreĭn space*, that is, an indefinite inner product space for which there exists a positive definite inner product $\langle\cdot, \cdot\rangle$ turning $(\mathscr{H}, \langle\cdot, \cdot\rangle)$ into a Hilbert space, and such that there exists a symmetry J on $\mathscr{H}$, that is $J^* = J^{-1} = J$, with $[x, y] = \langle Jx, y\rangle$ for all $x, y \in \mathscr{H}$. Such a symmetry is called a *fundamentel symmetry* and its spectral decomposition, denoted $\mathscr{K} = \mathscr{K}^- [+] \mathscr{K}^+$, is called a *fundamental decomposition*.

We consider a bounded selfadjoint operator A on $\mathscr{H}$, that is, $A^\# = A$, where we always denote by $\#$ the involution determined by the indefinite inner product. On $\mathscr{H}$ define a new inner product $[\cdot, \cdot]_A$ by the formula

$$[x, y]_A = [Ax, y], \quad x, y \in \mathscr{H} \tag{2.1}$$

The goal of this section is to investigate the properties of some Kreĭn spaces associated with the inner product $[\cdot, \cdot]_A$. By definition, a *Kreĭn space induced* by A is a pair $(\mathscr{K}, \Pi)$, where $\mathscr{K}$ is a Kreĭn space and $\Pi \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is an operator with dense range and satisfying

$$[\Pi x, \Pi y,] = [Ax, y], \; x, y \in \mathscr{H}. \tag{2.2}$$

In order to illustrate this concept it is useful to see some examples. We can give three such examples of Kreĭn spaces induced by a given selfadjoint operator A, all of them related by certain unitary equivalence. More precisely, two Kreĭn spaces $(\mathcal{K}_i, \Pi_i)$, $i=1, 2$, induced by the same selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ are *unitarily equivalent* if there exists a *unitary* operator U (that is, $U^{\#} = U^{-1}$) in $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $U\Pi_1 = \Pi_2$.

Example 2.1. Let J be a fundamental symmetry on $\mathcal{H}$ and let JA denote the selfadjoint operator with respect to the Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_J)$. Let $\mathcal{H}_-$ and $\mathcal{H}_+$ be the spectral subspaces of the operator JA corresponding to the semiaxis $(-\infty, 0)$ and, respectively, $(0, +\infty)$. Then we have the decomposition

$$\mathcal{H} = \mathcal{H}_- \oplus \operatorname{Ker} A \oplus \mathcal{H}_+.$$

Note that $(\mathcal{H}_-, -[\cdot, \cdot]_A)$ and $(\mathcal{H}_+, [\cdot, \cdot]_A)$ are positive definite inner product spaces and hence they can be completed to Hilbert spaces $\mathcal{K}^-$ and, respectively, $\mathcal{K}^+$. Then we can define the Kreĭn space $(\mathcal{K}_A, [\cdot, \cdot]_A)$ by letting

$$\mathcal{K}_A = \mathcal{K}^- [+] \mathcal{K}^+, \tag{2.3}$$

where the inner product is the extension by continuity of the inner product $[\cdot, \cdot]_A$. The operator $\Pi_A \in \mathcal{L}(\mathcal{H}, \mathcal{K}_A)$ is, by definition, the composition of the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H} \ominus \operatorname{Ker} A$ with the embedding of $\mathcal{H} \ominus \operatorname{Ker} A$ into $\mathcal{K}_A$. With these definitions, it is readily verified that $(\mathcal{K}_A, \Pi_A)$ is a Kreĭn space induced by A.

We now take a closer look at the strong topology of the Kreĭn space $\mathcal{K}_A$. Consider the seminorm $\mathcal{H} \ni x \mapsto \| |JA|^{1/2} x \|$. The kernel of this seminorm is exactly $\operatorname{Ker} A$ and the completion of $\mathcal{H} \ominus \operatorname{Ker} A$ with respect to this norm is exactly the space $\mathcal{K}_A$. Moreover, the strong topology of $\mathcal{K}_A$ is induced by the extension of this seminorm. The positive definite inner product associated with the norm $\| |JA|^{1/2} \cdot \|$ is $\langle |JA| \cdot, \cdot \rangle_J$ and hence, letting $JA = S_{JA} |JA|$ be the polar decomposition of JA and S_{JA} denote the corresponding selfadjoint partial isometry, it follows that S_{JA} can be extended by continuity to $\mathcal{K}_A$ and this is exactly the fundamental symmetry of $\mathcal{K}_A$ corresponding to $\langle |JA| \cdot, \cdot \rangle_A$.

We finally note that the construction of the space $\mathcal{K}_A$ does depend on the fundamental symmetry J, but it is easy to prove, that all the spaces constructed in this way are unitarily equivalent. ∎

The Kreĭn spaces of type $\mathcal{K}_A$ appeared since the early stages of operator theory in Kreĭn spaces. A very useful result is a lifting one, originally proved by M. G. Kreĭn [19] and rediscovered by W. J. Reid [28], P. D. Lax [23], and J. Dieudonné [10]. The indefinite variant below was proved by A. Dijksma, H. Langer, and H. de Snoo [12].

Lemma 2.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Kreĭn spaces and let $A \in \mathcal{L}(\mathcal{H}_1)$, $A = A^{\#}$, $B \in \mathcal{L}(\mathcal{H}_2)$, $B = B^{\#}$, $T_1 \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, and $T_2 \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be such that*

$$[T_1 x, y]_B = [x, T_2 y]_A, \quad x \in \mathcal{H}_1, \quad y \in \mathcal{H}_2,$$

or, equivalently, $T_2^\sharp A = BT_1$. *Then there exist unique operators* $\widetilde{T}_1 \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ *and* $\widetilde{T}_2 \in \mathcal{L}(\mathcal{K}_B, \mathcal{K}_A)$ *such that*

$$\widetilde{T}_1 \Pi_A = \Pi_B T_1, \quad \widetilde{T}_2 \Pi_B = \Pi_A T_2,$$

and

$$[\widetilde{T}_1 x, y]_B = [x, \widetilde{T}_2 y]_A, \quad x \in \mathcal{K}_A, y \in \mathcal{K}_B.$$

Another example of an induced Kreĭn spcace can be described as follows.

Example 2.3. Let again J be a fundamental symmetry on $\mathcal{H}$ and consider the polar decomposition

$$JA = S_{JA}|JA|, \tag{2.4}$$

of the selfadjoint operator JA on the Hilbert space $(\mathcal{H}, \langle\cdot, \cdot\rangle_J)$. The operator S_{JA} is a selfadjoint partial isometry on this Hilbert space. Define $\mathcal{H}_A = \mathrm{cl}\mathcal{R}(JA) = \mathcal{H} \ominus \mathrm{Ker}\, A$ and note that this subspace is invariant under S_{JA}. The restriction of S_{JA} to $\mathcal{H}_A$ is a symmetry, that is, it is unitary and selfadjoint, and let us consider the (indefinite) inner product $[\cdot, \cdot]S_{JA}$ by

$$[x, y]_{S_{JA}} = \langle S_{JA}\, x, y\rangle_J, \quad x, y \in \mathcal{H}_A. \tag{2.5}$$

We consider the Kreĭn space $(\mathcal{H}_A, [\cdot, \cdot]_{S_{JA}})$ and define the operator $\pi_A \in \mathcal{L}(\mathcal{H}, \mathcal{H}_A)$ by

$$\pi_A x = |JA|^{1/2} x, \quad x \in \mathcal{H}. \tag{2.6}$$

Note that

$$[\pi_A x, \pi_A y]_{S_{JA}} = \langle S_{JA}|JA|^{1/2}x, |JA|^{1/2}y\rangle_J = [Ax, y], \quad x, y \in \mathcal{H},$$

and hence $(\mathcal{H}_A, \pi_A)$ is a Kreĭn space induced by A. ∎

We note the following result connecting the constructions in Examples 2.1 and 2.3.

Lemma 2.4. *The Kreĭn spaces* $(\mathcal{K}_A, \Pi_A)$ *and* $(\mathcal{H}_A, \pi_A)$ *are unitarily equivalent, more precisely, the linear mapping*

$$\mathcal{H} \ominus \mathrm{Ker}\, A \ni x \mapsto |JA|^{1/2} x \in \mathcal{H}_A,$$

has a unique extension to a unitary operator $U \in \mathcal{L}(\mathcal{K}_A, \mathcal{H}_A)$ *such that* $U\Pi_A = \pi_A$.

As a consequence of Lemma 2.4, we can see that the construction of $(\mathcal{H}_A, \pi_A)$ does not depend on the fundamental symmetry J, modulo unitary equivalence.

We consider now a third example of an induced Kreĭn space.

Example 2.5. Let again J be a fundamental symmetry on $\mathcal{H}$ and consider the polar decomposition of JA as in (2.4). The space $\mathcal{B}_A = \mathcal{R}(|JA|^{1/2})$ endowed with the positive definite inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}_A}$ is defined by

$$\langle |JA|^{1/2}x,\ |JA|^{1/2}y\rangle_{\mathcal{B}_A}=\langle P_{\mathcal{H}\ominus\operatorname{Ker} A}x,\ y\rangle_J,\qquad x,y\in\mathcal{H}. \tag{2.7}$$

This positive definite inner product is correctly defined and $(\mathcal{B}_A,\langle\cdot,\cdot\rangle_{\mathcal{B}_A})$ is a Hilbert space. To see this, just note that we have made the operator $|JA|^{1/2}$: $\mathcal{H}\ominus\operatorname{Ker} A\to\mathcal{B}_A$ a Hilbert space unitary operator.

On $\mathcal{B}_A$ we define the (indefinite) inner product $[\cdot,\cdot]_{\mathcal{B}_A}$ by

$$[|JA|^{1/2}x,\ |JA|^{1/2}y]_{\mathcal{B}_A}=\langle S_{JA}x,\ y\rangle_J,\quad x,y\in\mathcal{H}\ominus\operatorname{Ker} A. \tag{2.8}$$

Since the operators $|JA|^{1/2}$ and S_{JA} commute it follows that

$$[a,b]_{\mathcal{B}_A}=\langle S_{JA}a,\ b\rangle_{\mathcal{B}_A},\quad a,b\in\mathcal{B}_A.$$

In the following we prove that the operator $S_{JA}|\mathcal{B}_A$ is a symmetry. Indeed, let $x,y\in\mathcal{H}$ be arbitrary. Since

$$\langle S_{JA}|JA|^{1/2}x,\ |JA|^{1/2}y\rangle_{\mathcal{B}_A}=\langle S_{JA}x,\ y\rangle_J=\langle x,\ S_{JA}y\rangle_J=\langle |JA|^{1/2}x,\ S_{JA}|JA|^{1/2}y\rangle_{\mathcal{B}_A},$$

it follows that S_{JA} is $\langle\cdot,\cdot\rangle_{\mathcal{B}_A}$-symmetric. Moreover,

$$\begin{aligned}\langle S_{JA}|JA|^{1/2}x,\ S_{JA}|JA|^{1/2}y\rangle_{\mathcal{B}_A}&=\langle S_{JA}x,\ S_{JA}y\rangle_J\\ &=\langle P_{\mathcal{H}\ominus\operatorname{Ker} A}x,\ y\rangle_J=\langle |JA|^{1/2}x,\ |JA|^{1/2}y\rangle_{\mathcal{B}_A},\end{aligned}$$

and hence the operator $S_{JA}|\mathcal{B}_A$ is $\langle\cdot,\cdot\rangle_{\mathcal{B}_A}$-isometric. We have thus proved that $(\mathcal{B}_A,[\cdot,\cdot]_{\mathcal{B}_A})$ is a Kreĭn space and that $S_{JA}|\mathcal{B}_A$ is a fundamental symmetry.

We define now a linear operator $\Pi_{\mathcal{B}_A}:\mathcal{H}\to\mathcal{B}_A$ by

$$\Pi_{\mathcal{B}_A}h=|JA|h,\quad h\in\mathcal{H}. \tag{2.9}$$

From $|JA|=|JA|^{1/2}|JA|^{1/2}$ it follows that $\mathcal{R}(\Pi_{\mathcal{B}_A})=\mathcal{R}(|JA|)$ and it is easy to see that $\mathcal{R}(\Pi_{\mathcal{B}_A})$ is dense in $\mathcal{B}_A$. In order to prove that the operator $\Pi_{\mathcal{B}_A}$ is bounded note that

$$\|b\|_{\mathcal{B}_A}=\||JA|^{1/2}b\|_J,\quad b\in\mathcal{B}_A.$$

Then

$$\|\Pi_{\mathcal{B}_A}b\|_{\mathcal{B}_A}=\||JA|^{1/2}b\|_J\leq\||JA|^{1/2}\|_J\cdot\|b\|_J,\quad b\in\mathcal{B}_A.$$

Finally, for arbitrary $x,y\in\mathcal{H}$ we have

$$\begin{aligned}[\Pi_{\mathcal{B}_A}x,\ \Pi_{\mathcal{B}_A}y]_{\mathcal{B}_A}&=[|JA|^{1/2}|JA|^{1/2}x,\ |JA|^{1/2}|JA|^{1/2}y]_{\mathcal{B}_A}\\ &=\langle S_{JA}|JA|^{1/2}x,\ |JA|^{1/2}y\rangle_J=\langle JAx,\ y\rangle_J=[Ax,y],\end{aligned}$$

and hence $(\mathcal{B}_A,[\cdot,\cdot])$ is a Kreĭn space induced by A. ∎

Remark 2.6. The Kreĭn spaces induced by a selfadjoint operator as in Example 2.5 can be characterized in yet another way. More precisely, to any Kreĭn space $\mathcal{K}$ continuously embedded into the Kreĭn space $\mathcal{H}$ one associates a selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ in the following way: let $\iota: \mathcal{K} \hookrightarrow \mathcal{H}$ be the inclusion operator which is supposed bounded and take $A = \iota\iota^{\#} \in \mathcal{L}(\mathcal{H})$. Clearly A is selfadjoint and $(\mathcal{K}, \iota^{\#})$ is a Kreĭn space induced by A. Conversely, from Example 2.5 it is easy to prove that $\mathcal{B}_A$ is a Kreĭn space continuously embedded in $\mathcal{H}$ and such that $\iota\iota^{\#} = JAJ$, where $\iota: \mathcal{B}_A \hookrightarrow \mathcal{H}$ is the canonical embedding. ∎

The Kreĭn spaces of type $\mathcal{B}_A$ appeared explicitly in the work of L. de Branges [5], and, in a slightly different but more general formulation, in the work of L. Schwartz [30]. The connection between the induced Kreĭn spaces $(\mathcal{H}_A, \pi_A)$ and $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ can be also easily established.

Lemma 2.7. *The induced Kreĭn spaces $(\mathcal{H}_A, \pi_A)$ and $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ are unitarily epuivalent, more precisely, the mapping*

$$\mathcal{H}_A \ni x \mapsto |JA|^{1/2} x \in \mathcal{B}_A,$$

extends uniquely to a unitary operator $V \in \mathcal{L}(\mathcal{H}_A, \mathcal{B}_A)$ such that $V_{\pi_A} = \Pi_{\mathcal{B}_A}$.

Note also that, as before, a consequence of Lemma 2.7 is that the construction of the induced Kreĭn space $(\mathcal{B}_A, \Pi_{\mathcal{B}_A})$ does not depend on the fundamental symmetry J, modulo unitary equivalence.

Until now all the examples of induced Kreĭn spaces we have produced turned out to be unitarily equivalent and hence, it appears as natural to ask whether all the possible Kreĭn spaces induced by a fixed selfadjoint operator are unitarily equivalent. The answer is that, in general, this is not true.

Theorem 2.8. *Let A be a bounded selfadjoint operator in the Kreĭn space $\mathcal{H}$. The following statements are equivalent:*

(i) *The Kreĭn space induced by A is unique, modulo unitary equivalence.*

(ii) *For some (equivalently, for any) fundamental symmetry J of $\mathcal{H}$, there exists an $\varepsilon > 0$ such that either $(0, \varepsilon) \subset \rho(JA)$ or $(-\varepsilon, 0) \subset \rho(JA)$.*

(iii) *For some (equivalently, for any) Kreĭn space $\{\mathcal{K}, \Pi\}$ induced by A, the range of Π contains a maximal uniformly definite subspace of $\mathcal{K}$.*

Proof. (i) ⇒ (ii). In order to simplify the notation let us first notice that, without restricting the generality, we can assume that $\mathcal{H}$ is a Hilbert space. Indeed, the statement (ii) is a topological property of the spectrum of JA which either holds for any fundamental symmetry J of $\mathcal{H}$ or it does not hold for any fundamental symmetry and hence, we can replace the Kreĭn space $\mathcal{H}$ with the Hilbert space $(\mathcal{H}, \langle\cdot, \cdot\rangle_J)$, for some arbitrary fundamental symmetry J, and then, instead of A, we consider the operator JA. Moreover, in the following we will denote by $\langle\cdot, \cdot\rangle$ the positive definite inner product of $\mathcal{H}$ and by $\|\cdot\|$ the corresponding norm.

Let us assume that the statement (ii) does not hold. Then there exists a decreasing sequence of values $\{\mu_n\}_{n\geq 1}\subseteq\sigma(A)$, $0<\mu_n<1$ such that $\mu_n\to 0$ $(n\to\infty)$, and there exists a decreasing sequence of values $\{\nu_n\}_{n\geq 1}\subseteq\sigma(-A)$, $0<\nu_n<1$, such that $\nu_n\to 0$ $(n\to\infty)$. Then there exist sequences of vectors $\{e_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$ such that

$$e_n\in E((\mu_{n+1}^2,\mu_n^2])\mathscr{H},\qquad f_n\in E([-\nu_n^2-\nu_{n+1}^2))\mathscr{H},\quad n\geq 1, \tag{2.10}$$

$$[Ae_i,e_j]=\delta_{ij},\quad [Af_i,f_j]=-\delta_{ij},\quad i,j\geq 1, \tag{2.11}$$

where E denotes the spectral measure of A. As a consequence, we also have

$$[Ae_i,f_j]=0,\quad i,j\geq 1. \tag{2.12}$$

We consider the linear manifolds $\mathscr{D}_{0\pm}$

$$\mathscr{D}_{0+}=\operatorname{lin}\{e_n|n\geq 1\},\quad \mathscr{D}_{0-}=\operatorname{lin}\{f_n|n\geq 1\},$$

and note that as linear manifolds in the Kreĭn space $\mathscr{K}_A$, $\mathscr{D}_{0+}$ is uniformly positive and $\mathscr{D}_{0-}$ is uniformly negative. To see this, let $A=A_+-A_-$ be the Jordan decomposition of the selfadjoint operator A and note that $|A|=A_++A_-$ and $|A|^{1/2}=A_+^{1/2}+A_-^{1/2}$. Recalling that the positive definite inner product onto $\mathscr{K}_A$ is $\langle|A|\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the positive definite inner product onto $\mathscr{H}$, from (2.10), (2.11) and (2.12) it follows that $\{e_n,f_n\}_{n\geq 1}$ is orthonormal with respect to $\langle|A|\cdot,\cdot\rangle$ and hence $\mathscr{D}_{0+}$ is uniformly positive and $\mathscr{D}_{0-}$ is uniformly negative. As a consequence, letting

$$\mathscr{D}_0=\mathscr{D}_{0+}\dot{+}\mathscr{D}_{0-},$$

$\mathscr{D}_0$ is a linear manifold in $\mathscr{K}_A$ such that its closure (that is, with respect to the norm $\||A|^{1/2}\cdot\|$) is a regular subspace.

Define the sequence $\{\lambda_n\}_{n\geq 1}$ by

$$\lambda_n=\min\{\sqrt{1-\mu_n^2},\sqrt{1-\nu_n^2}\}.$$

Then $0<\lambda_n<1$, $\lambda_n\uparrow 1\,(n\to\infty)$ and

$$\sup_{n\geq 1}\frac{\max\{\mu_n,\nu_n\}}{\sqrt{1-\lambda_n^2}}\leq 1. \tag{2.13}$$

Let us consider now the sequence $\{\mathscr{S}_n\}_{n\geq 1}$, of subspaces of the Kreĭn space $\mathscr{K}_A$, defined by

$$\mathscr{S}_n=\boldsymbol{C}e_n\dot{+}\boldsymbol{C}f_n,\ n\geq 1,$$

and then define the operators $U_n\in\mathscr{L}(\mathscr{S}_n)$

$$U_n = \frac{1}{\sqrt{1-\lambda_n^2}}\begin{bmatrix} 1 & \lambda_n \\ \lambda_n & 1 \end{bmatrix}, \quad n \geq 1.$$

The operators U_n are unitary in $\mathcal{S}_n$ and the spectral radii

$$|U_n|_{sp} = \frac{1+\lambda_n}{\sqrt{1-\lambda_n^2}} \to \infty \ (n \to \infty).$$

Note that

$$\mathcal{D}_0 = \mathrm{lin}\{\mathcal{S}_k | k \geq 1\} = \cup_{n\geq 1}(\mathcal{S}_1 \dot{+} \mathcal{S}_2 \dot{+} \cdots \dot{+} \mathcal{S}_n).$$

We consider the linear manifold

$$\mathcal{D} = \mathcal{D}_+ \dot{+} \mathcal{D}_- = \mathcal{R}(A)$$

which is, by construction, dense in $\mathcal{K}_A$, where $\mathcal{D}_\pm = \mathcal{R}(A_\pm)$. Also, $\mathcal{D}_{0\pm} \subseteq \mathcal{D}_\pm$ and hence $\mathcal{D}_0 \subseteq \mathcal{D}$. Moreover, the following decomposition holds

$$\mathcal{D} = \mathcal{D}_0 \dot{+} (\mathcal{D} \cap \mathcal{D}_0^{\perp}). \tag{2.14}$$

With respect to (2.14) we define a block-diagonal operator U in $\mathcal{K}_A$, with domain $\mathcal{D}$ and the same range, by $U|\mathcal{S}_n = U_n$, $n \geq 1$ and $U|(\mathcal{D} \cap \mathcal{D}_0^{\perp}) = I|(\mathcal{D} \cap \mathcal{D}_0^{\perp})$. The operator U is isometric, it has dense range as well as dense domain, and it is unbounded since its point spectrum $\sigma_p(U) \supseteq_{n\geq 1} \sigma(U_n)$ is unbounded. Using these elements we define the operator Π from the dense domain $\mathcal{D} \dot{+} \mathrm{Ker} A$ into $\mathcal{K}_A$ by $\Pi = U\Pi_A$. We claim that $\{\mathcal{K}_A, \Pi\}$ is a Kreĭn space induced by A.

Indeed, $\mathcal{R}(\Pi) = \mathcal{D}$ is dense in $\mathcal{K}_A$. Further,

$$[\Pi x, \Pi y] = [U\Pi_A x, U\Pi_A y] = [\Pi_A x, \Pi_A y] = [Ax, y], \quad x, y \in \mathcal{D}.$$

We now prove that Π is bounded. From the discussion in Example 2.1, the Hilbert norm associated to $\mathcal{K}_A$ is given by $\||A|^{1/2}\cdot\|$, where $\|\cdot\|$ is the norm in $\mathcal{H}$. Define the spaces

$$\mathcal{F}_n = E((\mu_{n+1}^2, \mu_n^2])\mathcal{H} \oplus E([-\nu_n^2, -\nu_{n+1}^2))\mathcal{H}, \quad n \geq 1,$$

and note that by the Spectral Theorem and (2.10) we have

$$\|P_{\mathcal{F}_n}|A|^{1/2}P_{\mathcal{F}_n}|A|^{1/2}P_{\mathcal{F}_n}\| = \||A|^{1/2}P_{\mathcal{F}_n}\| \leq \max\{\mu_n, \nu_n\},$$

where $P_{\mathcal{F}_n}$ is the Hilbert space projection of $\mathcal{H}$ onto $\mathcal{F}_n$.

For $x \in \mathcal{D} \cap \mathcal{D}_0^{\perp}$, it follows from the definition of U that

$$\||A|^{1/2}Ux\| = \||A|^{1/2}x\| \leq \|A\|^{1/2}\|x\|.$$

For $x \in S_n$, we have

$$\||A|^{1/2}Ux\| = \||A|^{1/2}U_n x\| = \||A|^{1/2}P_{\mathcal{F}_n}U_n x\|$$
$$\leq \|P_{\mathcal{F}_n}|A|^{1/2}P_{\mathcal{F}_n}\|\|U_n\|\|x\| \leq 4\,(\sup_{n\geq 1}\frac{\max\{\mu_n, \nu_n\}}{\sqrt{1-\lambda_n^2}})\,\|x\| \leq 4\|x\|.$$

In view of the definition of the operator U and taking into account that the closure of $\mathcal{D}_0$ in the Kreĭn space $\mathcal{K}_A$ is a regular subspace of $\mathcal{K}_A$, it follows that there exists a constant $C > 0$ such that

$$\||A|^{1/2}Ux\| \leq C\|x\|, \quad x \in \mathcal{D}.$$

This shows that the operator $\Pi = U\Pi_A$ (recall the definition of Π_A in Example 2.1) is continuous: $\mathcal{D} \dotplus \operatorname{Ker} A \subset \mathcal{H} \to \mathcal{K}_A$. Therefore we can extend it by continuity to a bounded operator $\Pi \in \mathcal{L}(\mathcal{H}, \mathcal{K}_A)$ and hence $\{\mathcal{K}_A, \Pi\}$ is a Kreĭn space induced by A.

Finally, since U is unbounded it follows that $\{\mathcal{K}_A, \Pi_A\}$ is not unitarily equivalent with $\{\mathcal{K}_A, \Pi\}$.

(ii) $\Rightarrow$ (iii). As before, we can assume, without restricting the generality, that $\mathcal{H}$ is a Hilbert space. Let then $A = A_+ - A_-$ be the Jordan decomposition of the operator A. Denoting $\mathcal{H}_\pm = \mathrm{cl}\mathcal{R}(A_\pm)$, the following decomposition holds

$$\mathcal{H} = \mathcal{H}_+ \oplus \operatorname{Ker} A \oplus \mathcal{H}_-.$$

The operators $A_\pm$ restricted to $\mathcal{H}_\pm$ are selfadjoint operators in the Hilbert spaces $\mathcal{H}_\pm$, respectively. As in Example 2.1 it follows that the strong topology of $\mathcal{K}$ is determined by the norms $\mathcal{H}_\pm \ni x \mapsto \|(A_\pm)^{1/2}x\|$.

To make a choice, let us assume that there exists $\varepsilon > 0$ such that $(-\varepsilon, 0) \subseteq \rho(A)$, equivalently A_- has closed range. This implies that $(\mathcal{H}_-, \|(A_-)^{1/2}\cdot\|)$ is a complete normed space and hence, by the definition of the Kreĭn space $\mathcal{K}_A$, $\mathcal{H}_-$ is a maximal uniformly negative subspace of $\mathcal{K}_A$. If $(0, \varepsilon) \subseteq \rho(A)$, then we prove in a similar way that $\mathcal{H}_+$ is a maximal uniformly positive subspace of $\mathcal{K}_A$.

(iii) $\Rightarrow$ (i). Let $\{\mathcal{K}_i, \Pi_i\}$ $i = 1, 2$, be Kreĭn spaces induced by A. The equation $U\Pi_1 x = \Pi_2 x$, $x \in \mathcal{H}$, uniquely determines an isometric operator densely defined in $\mathcal{K}_1$ and with dense range in $\mathcal{K}_2$. If $\mathcal{R}(\Pi_1)$ contains a maximal uniformly definite subspace then by Lemma 2.3 in [8] it follows that U has a unique extension to a bounded unitary operator and hence the two Kreĭn spaces induced by A are unitarily equivalent. Moreover, since bounded unitary operators map maximal uniformly definite subspaces into subspaces with the same property, it follows that if (ii) holds for some Kreĭn space induced by A then it holds for any other Kreĭn space induced by A. ∎

Let $\mathcal{K}_i$ be two Kreĭn space continuously embedded into the Kreĭn space $\mathcal{H}$ and denote by $\iota_i : \mathcal{K}_i \hookrightarrow \mathcal{H}$ the corresponding embedding operators, that is $\iota_i h = h$,

$h\in\mathcal{K}_i$, $i=1, 2$. We say that the Kreĭn spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ correspond to the same selfadjoint operator A in $\mathcal{H}$ if $\iota_1\iota_1^{\#}=\iota_2\iota_2^{\#}=A$. If A is nonnegative, equivalently, the Kreĭn spaces $\mathcal{K}_i$ are actually Hilbert spaces, this implies that $\mathcal{K}_1=\mathcal{K}_2$. The answer of this question for genuine Kreĭn spaces was given by T. Hara [17]. A parallel treatement can be found in [13], [8], with a bridge settled in [14]. We can obtain these results as consequences of Theorem 2.8.

Corollary 2.9. *Given a selfadjoint operator* $A\in\mathcal{L}(\mathcal{H})$, *the following statements are mutually equivalent:*

(*a*) *There exists a unique Kreĭn space* $\mathcal{K}$ *continuously embedded in* $\mathcal{H}$ *and associated to* A.

(*b*) *For some, equivalently, for all fundamental symmetry* J, *there exists* $\varepsilon>0$ *such thet either* $(-\varepsilon, 0)\subset\rho(JA)$ *or* $(0, \varepsilon)\subset\rho(JA)$.

(*b′*) *For some, equivalently, for all fundamental symmetry* J, *there exists* $\varepsilon>0$ *such that either* $(-\varepsilon, 0)\subset\rho(AJ)$ *or* $(0, \varepsilon)\subset\rho(AJ)$.

(*c*) *There exists a Kreĭn space* $\mathcal{K}$ *continuously embedded in* $\mathcal{H}$, $\iota:\mathcal{K}\to\mathcal{H}$ *such that* $\iota\iota^{\#}=A$ *and* $\mathcal{R}(\iota^{\#})$ *contains a maximal uniformly definite subspace of* $\mathcal{K}$.

Proof. Let $\mathcal{K}_i$ be two Kreĭn spaces continuously embedded in $\mathcal{H}$ and let $\iota_i:\mathcal{K}_i\to\mathcal{H}$ be the embedding operators, $i=1, 2$. Assume that the induced Kreĭn spaces $(\mathcal{K}_i, \iota_i^{\#})$ are unitarily equivalent, that is, there exists a unitary operator $U\in\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $U\iota_2=\iota_1$. Then $\iota_1=\iota_2U^{\#}$ and taking into account that ι_i are embeddings, it follows that

$$U^{\#}x=\iota_2U^{\#}x=\iota_1x=x, \quad x\in\mathcal{K}_2,$$

and hence $\mathcal{K}_1=\mathcal{K}_2$ and $\iota_1=\iota_2$. This shows that the two Kreĭn spaces coincide. Taking into account of Remark 2.6, we can now apply Theorem 2.8 and get the equivalence of the statements (a), (b)′, and (c). The equivalence of (b) and (b)′ is clear, since $JA=J(AJ)J$, that is, JA and AJ are unitarily equivalent, and hence their spectra coincide. ∎

§3. Kolmogorov Decompositions of Hermitian Kernels

Let $\mathcal{J}$ be a set of indices and $\mathbf{H}=\{\mathcal{H}_i\}_{i\in\mathcal{J}}$ be a family of Kreĭn spaces with (indefinite) inner products denoted by $[\cdot, \cdot]_{\mathcal{H}_i}$. A mapping K defined on $\mathcal{J}\times\mathcal{J}$ such that $K(i, j)\in\mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$ for all $i, j\in\mathcal{J}$ is called an $\mathbf{H}$-*kernel*. The $\mathbf{H}$-kernel K is called *hermitian* if

$$K(i, j)=K(j,i)^{\#}, \quad i, j\in\mathcal{J}. \tag{3.1}$$

We denote by $\mathcal{F}(\mathbf{H})$ the set of all families $f=\{f_i\}_{i\in\mathcal{J}}$ of vectors such that $f_i\in\mathcal{H}_i$, for all $i\in\mathcal{J}$, and by $\mathcal{F}_0(\mathbf{H})$ we denote the set of all $f\in\mathcal{F}(\mathbf{H})$ of finite support, that is, the set $\operatorname{supp} f=\{i\in\mathcal{J} \mid f_i\neq 0\}$ is finite.

If K is a hermitian $\mathbf{H}$-kernel then one can introduce on $\mathscr{F}_0(\mathbf{H})$ an indefinite inner product $[\cdot, \cdot]_K$ defined by

$$[f, g]_K = \sum_{i,j\in\mathscr{I}} [K(i,j)f(j), g(i)]_{\mathscr{H}_i}, f, \quad g\in\mathscr{F}_0(\mathbf{H}). \tag{3.2}$$

Also, recall that an $\mathbf{H}$-kernel K is called *positive semidefinite* (equivalently, *of positive type*) if

$$\sum_{i,j\in\mathscr{I}} [K(i,j)h(j), h(i)]_{\mathscr{H}_i} \geq 0, \quad h\in\mathscr{F}_0(\mathbf{H}). \tag{3.3}$$

Let us notice that every positive semidefinite $\mathbf{H}$-kernel is hermitian. Also, a hermitian $\mathbf{H}$-kernel is positive semidefinite if and only if the corresponding inner product $[\cdot, \cdot]_A$ in (3.2) is nonegative.

Let us denote by $\mathfrak{K}^b(\mathbf{H})$ the class of all hermitian $\mathbf{H}$-kernels and by $\mathfrak{K}^+(\mathbf{H})$ the subclass of all positive semidefinite $\mathbf{H}$-kernels. On $\mathfrak{K}^h(\mathbf{H})$ we define addition, subtraction and multiplication with real numbers in a natural way. Moreover, on $\mathfrak{K}^h(\mathbf{H})$ we have a natural partial order defined as follows: if A, $B\in\mathfrak{K}^h(\mathbf{H})$ then $A\leq B$ means $[f, f]_A\leq[f, f]_B$, for all $f\in\mathscr{F}_0(\mathbf{H})$. With this definition we have

$$\mathfrak{K}^+(\mathbf{H}) = \{A\in\mathfrak{K}^h(\mathbf{H}) \mid A\geq 0\}, \tag{3.4}$$

and $\mathfrak{K}^+(\mathbf{H})$ is a strict cone of $\mathfrak{K}^h(\mathbf{H})$, that is, it is closed under addition and multiplication with nonnegative numbers, and $\mathfrak{K}^+(\mathbf{H}) \cap -\mathfrak{K}^+(\mathbf{H}) = 0$.

We also note that an inner product $[\cdot, \cdot]$ can be defined for arbitrary f, $g\in\mathscr{F}(\mathbf{H})$, provided at least one of f and g has finite support, by

$$[f, g] = \sum_{i\in\mathscr{I}} [f(i), g(i)]_{\mathscr{H}_i}. \tag{3.5}$$

To each $\mathbf{H}$-kernel K we can associate the *convolution operator*, denoted also by K, and defined by

$$K: \mathscr{F}_0(\mathbf{H}) \to \mathscr{F}(\mathbf{H}), \ (Kf)(i) = \sum_{i\in\mathscr{I}} K(i,j)f(j), \quad f\in\mathscr{F}_0(\mathbf{H}). \tag{3.6}$$

Then the kernel K is nonnegative (hermitian) if and only if the corresponding convolution operator K is nonnegative (hermitian), that is, $[Kf, g]\geq 0$ ($[Kf, g] = [f, Kg]$), for all $f, g\in\mathscr{F}_0(\mathbf{H})$.

By definition, a *Kolmogorov decomposition* of the hermitian $\mathbf{H}$-kernal K is a pair $(V; \mathscr{K})$, where $(\mathscr{K}, [\cdot, \cdot])$ is a Kreĭn space and $V = \{V_i\}_{i\in\mathscr{I}}$ is a family of linear operators, subject to the following conditions:

(a) $V_i\in\mathscr{L}(\mathscr{H}_i, \mathscr{K})$, for all $i\in\mathscr{I}$.

(b) $K = \bigvee_{i\in\mathscr{I}} V_i\mathscr{H}_i$.

(c) $K(i,j) = V_i^{\#}V_j$ for all $i, j\in\mathscr{I}$.

Following L. Schwartz [30], we say that two kernels A, $B\in\mathfrak{K}^+(\mathbf{H})$ are *disjoint* if for any kernel $P\in\mathfrak{K}^+(\mathbf{H})$ such that $P\leq A$ and $P\leq B$ it follows $P=0$.

In the next theorem, the equivalences of the assertions (1), (1)′, (2), and (2)′ are transcripted from similar results obtained by L. Schwartz in [30].

Theorem 3.1. *Let $K\in\mathfrak{K}^h(\mathbf{H})$. The following assertions are equivalent:*

(1) *There exists $L\in\mathfrak{K}^+(\mathbf{H})$ such that $-L\leq K\leq L$.*

(1)′ *There exists $L\in\mathfrak{K}^+(\mathbf{H})$ such that*

$$|[f,g]_K|\leq[f,f]_L^{1/2}[g,g]_L^{1/2},\quad f,g\in\mathscr{F}_0(\mathbf{H}).$$

(2) *$K=K_1-K_2$ with $K_1, K_2\in\mathfrak{K}^+(\mathbf{H})$.*

(2)′ *$K=K_+-K_-$ with $K_\pm\in\mathfrak{K}^+(\mathbf{H})$ and disjoint.*

(3) *There exists a Kolmogorov decomposition $(V;\mathscr{K})$ of K.*

Proof. The equivalence of (1) and (1)′ is obtained as follows. Let $L\in\mathfrak{K}^+(\mathbf{H})$ be such that $-L\leq K\leq L$, that is

$$|[f,f]_K|\leq[f,f]_L,\quad f\in\mathscr{F}_0(\mathbf{H}).$$

Let $f, g\in\mathscr{F}_0(\mathbf{H})$. Since K is hermitian we have

$$4\mathrm{Re}[f,g]_K=[f+g,f+g]_K-[f-g,f-g]_K$$

and hence

$$4|\mathrm{Re}[f,g]_K|\leq[f+g,f+g]_L+[f-g,f-g]_L=2[f,f]_L+2[g,g]_L.$$

Let $\lambda\in\boldsymbol{C}$ be chosen such that $|\lambda|=1$ and $\mathrm{Re}[f,\lambda g]_K=[f,\lambda g]_K$. Then

$$|[f,\lambda g]_K|\leq\frac{1}{2}[f,f]_L+\frac{1}{2}[g,g]_L. \tag{3.7}$$

We distinguish two possible cases. First, assume that either $[f,f]_L=0$ or $[g,g]_L=0$. To make a choice assume $[f,f]_L=0$. Consider the inequality (3.7) with g replaced by tg for $t>0$. Then

$$|[f,g]_K|\leq\frac{t}{2}[g,g]_K.$$

Letting $t\to 0$ we get $[f,g]_K=0$.

Second case, assuming that both $[f,f]_L$ and $[g,g]_L$ are nontrivial, in (3.7) replace f by $[f,f]_L^{-1/2}f$ and g by $[g,g]_L^{-1/2}g$ and get

$$|[f,g]_K|\leq[f,f]_L^{1/2}[g,g]_L^{1/2}.$$

We thus proved that (1) ⇒ (1)′. The converse implication follows by letting $f=g$.

(1)′⇒(2)′. Let $\mathscr{K}_L$ be the quotient-completion of the pre-Hilbert space $(\mathscr{F}_0(\mathbf{H}), [\cdot,\cdot]_L)$ to a Hilbert space. More precisely, letting $\mathscr{N}_L=\{f\in\mathscr{F}_0(\mathbf{H})\mid[f,f]_L=0\}$ denote the isotropic subspace of the positive semidefinite inner product space $(\mathscr{F}_0(\mathbf{H}), [\cdot,\cdot]_L)$ we consider the quotient $\mathscr{F}_0(\mathbf{H})/\mathscr{N}_L$ and complete it to a Hilbert space $\mathcal{K}_L$. The inequality $(1)'$ implies that the isotropic subspace $\mathcal{N}_L$ is contained into the isotropic subspace $\mathcal{N}_K$ of the inner product $(\mathcal{F}_0(\mathbb{H}), [\cdot, \cdot]_K)$. Therefore, $[\cdot, \cdot]_K$ uniquely induces an inner product on $\mathcal{K}_L$, also denoted by $[\cdot, \cdot]_K$ such that the inequality in $(1)'$ still holds for all $f, g \in \mathcal{K}_L$. By the Riesz representation theorem we get a selfadjoint and contractive operator $A \in \mathcal{L}(\mathcal{K}_L)$, $A = A^*$ such that

$$[f, g]_K = [Af, g]_L, \quad f, g \in \mathcal{K}_L. \tag{3.8}$$

Let $A = A_+ - A_-$ be the Jordan decomposition of A in $\mathcal{K}_L$. Then $A_\pm$ are also contractions and hence

$$[A_\pm f, f]_L \leq [f, f]_L, \quad f \in \mathcal{K}_L. \tag{3.9}$$

We now prove that the nonnegative inner products $[A_\pm \cdot, \cdot]$ uniquely induce kernels $\mathcal{K}_\pm \in \mathfrak{K}^+(\mathbb{H})$ such that

$$[f, f]_{K\pm} \leq [f, f]_L, \quad f \in \mathcal{F}_0(\mathbb{H}),$$

and $K = K_+ - K_-$.

Indeed, the inner product $[A_+ \cdot, \cdot]$ restricted to $\mathcal{F}_0(\mathbb{H})/\mathcal{N}_K$ can be extended to an inner product $[\cdot, \cdot]_+$ on $\mathcal{F}_0(\mathbb{H})$ by letting it be null onto $\mathcal{N}_L$ and hence

$$[f, f]_+ \leq [f, f]_L, \quad f \in \mathcal{F}_0(\mathbb{H}). \tag{3.10}$$

Let $i, j \in \mathcal{J}$ be arbitrary and $i \neq j$. Clearly, we can identify the Kreĭn space $\mathcal{H}_i$ $[+]$ $\mathcal{H}_j$ with the subspace of $\mathcal{F}_0(\mathbb{H})$ consisting of those f such that $\operatorname{supp} f \subseteq \{i, j\}$. With this identification we consider the restrictions of the inner products $[\cdot, \cdot]_+$ and $[\cdot, \cdot]_L$ to $\mathcal{H}_i [+] \mathcal{H}_j$. The inner product $[\cdot, \cdot]_L$ is jointly continous with respect to the strong topology of $\mathcal{H}_i [+] \mathcal{H}_j$. By (3.10) and the equivalence of (1) and $(1)'$ we conclude that the inner product $[\cdot, \cdot]_+$ is also jointly continuous with respect to the strong topology of $\mathcal{H}_i [+] \mathcal{H}_j$ and hence, by the Riesz representation theorem, there exists a selfadjoint operator $B \in \mathcal{L}(\mathcal{H}_i [+] \mathcal{H}_j)$ such that

$$[f, g]_+ = [Bf, g]_{\mathcal{H}_i[+]\mathcal{H}_j}, \quad f, g \in \mathcal{H}_i [+] \mathcal{H}_j.$$

We define

$$K_+(i, j) = P_{\mathcal{H}_i}^{\mathcal{H}_i[+]\mathcal{H}_j} B | \mathcal{H}_j, \quad K_+(i, i) = P_{\mathcal{H}_i}^{\mathcal{H}_i[+]\mathcal{H}_j} B | \mathcal{H}_i,$$

$$K_+(j, j) = P_{\mathcal{H}_j}^{\mathcal{H}_i[+]\mathcal{H}_j} B | \mathcal{H}_j, \quad \text{and} \quad K_+(j, i) = P_{\mathcal{H}_j}^{\mathcal{H}_i[+]\mathcal{H}_j} B | \mathcal{H}_i = K_+(i, j)^\#.$$

In this way we have defined a kernel $K_+ \in \mathfrak{K}^h(\mathbf{H})$ such that $K_+ \leq L$ and

$$[f, g]_{K_+} = [f, g]_+, \quad f, g \in \mathcal{F}_0(\mathbf{H}).$$

Since the inner product $[\cdot, \cdot]_+$ is nonnegative it follows that $K_+ \in \mathfrak{K}^h(\mathbf{H})$.

Similarly we construct the kernel $K_- \in \mathfrak{K}^h(\mathbf{H})$ such that $K_- \leq L$ and

$$[f, g]_{K_-} = [f, g]_-, \quad f, g \in \mathscr{F}_0(\mathbf{H}),$$

where the inner product $[f, g]_-$ is the extension of the restriction of the inner product $[A_- f, g]$ to $\mathscr{F}_0(\mathbf{H})/\mathscr{N}_L$, by letting it be null onto $\mathscr{N}_K$.

From $A = A_+ - A_-$, (3.8) and the constructions of the kernels K_+ and K_-, we conclude that $K = K_+ - K_-$.

Let $P \in \mathfrak{K}^+(\mathbf{H})$ be such that $P \leq K_\pm$. Then

$$[f, f]_P \leq [f, f]_L, \quad f \in \mathscr{F}_0(\mathbf{H}). \tag{3.11}$$

As before, $[\cdot, \cdot]_P$ induces a nonnegative inner product $[\cdot, \cdot]_P$ on $\mathscr{K}_L$ such that (3.9) holds for all $f \in \mathscr{K}_L$. From $P \leq K_\pm$ we obtain that

$$[f, f]_P \leq [A_\pm f, f]_L, \quad f \in \mathscr{K}_L,$$

and, since $A_+ A_- = 0$ this impleis $[f, f]_P = 0$ for all $f \in \mathscr{K}_L$. Since by (3.11) we have $\mathscr{N}_L \in \mathscr{N}_P$ this implies that the inner producrt $[\cdot, \cdot]_P$ is null onto the whole $\mathscr{F}_0(\mathbf{H})$ and hence $P = 0$.

(2)$'\Rightarrow$(2). Obvious.

(2)$\Rightarrow$(1). Indeed, if $K = K_1 - K_2$ with $K_1, K_2 \in \mathfrak{K}^+(\mathbf{H})$ then letting $L = K_1 + K_2 \in \mathfrak{K}^+(\mathbf{H})$ we clearly have $-L \leq K \leq L$.

(1)$' \Rightarrow$ (3). As in the proof of (1)$' \Rightarrow$ (2)$'$ we consider the quotient-completion Hilbert space $\mathscr{K}_L$, the representation (3.8) and the Jordan decomposition $A = A_+ - A_-$. The latter yields in a canonical way a Kreĭn space $(\mathscr{K}, [\cdot, \cdot]_K)$. We again consider $\mathscr{N}_L$ and $\mathscr{N}_K$, the isotropic spaces of the inner product spaces $(\mathscr{F}_0(\mathbf{H}), [\cdot, \cdot]_L)$ and, respectively, $(\mathscr{F}_0(\mathbf{H}), [\cdot, \cdot]_K)$. From the inequality (1)$'$ we have $\mathscr{N}_L \subseteq \mathscr{N}_K$.

For every $i \in \mathscr{J}$ and every vector $h \in \mathscr{H}_i$ we consider the function $h \in \mathscr{F}_0(\mathbf{H})$ defined by

$$h(j) = \begin{cases} h, & j = i, \\ 0, & j \neq i. \end{cases} \tag{3.12}$$

This identification of vectors with functions in $\mathscr{F}_0(\mathbf{H})$ yields a natural embedding $\mathscr{H}_i \hookrightarrow \mathscr{F}_0(\mathbf{H})$. With this embedding we define linear operators $V_i: \mathscr{H}_i \to \mathscr{K}$ by

$$V_i h = h + \mathscr{N}_K \in \mathscr{F}_0(\mathbf{H})/\mathscr{N}_K \subseteq \mathscr{K}, \ h \in \mathscr{H}_i.$$

To prove that the operators V_i are bounded, fix an index $i \in \mathscr{J}$ and unitary norms $\|\cdot\|_{\mathscr{K}}$ and $\|\cdot\|_{\mathscr{H}_i}$ onto the Kreĭn spaces $\mathscr{K}$ and, respectively, $\mathscr{H}_i$ and let $\langle \cdot, \cdot \rangle_{\mathscr{H}_i}$ be the corresponding positive definite inner product onto $\mathscr{H}_i$. Then, for arbitrary $h \in \mathscr{H}_i$ we have

$$\begin{aligned}\|V_i h\|_{\mathcal{K}}^2 &= \|h+\mathcal{N}_K\|_{\mathcal{K}}^2 = \|A_+^{\frac{1}{2}}(h+\mathcal{N}_L)\|_{\mathcal{K}_L}^2 + \|A_-^{\frac{1}{2}}(h+\mathcal{N}_L)\|_{\mathcal{K}_L}^2 \\ &= [A_+(h+\mathcal{N}_L), h+\mathcal{N}_L]_{\mathcal{K}_L} + [A_-(h+\mathcal{N}_L), h+\mathcal{N}_L]_{\mathcal{K}_L} \\ &\leq 2\langle L(i,i)h, h\rangle_{\mathcal{H}_i} = 2\|L(i,i)^{1/2}h\|_{\mathcal{H}_i}^2 \leq 2\|L(i,i)\|\,\|h\|_{\mathcal{H}_i}^2.\end{aligned}$$

This proves that the operators V_i are in $\mathcal{L}(\mathcal{H}_i, \mathcal{K})$ for all $i \in \mathcal{J}$.

In order to prove (c), let $h \in \mathcal{H}_i$ and $g \in \mathcal{H}_j$ be arbitrary vectors. With the identification of vectors with functions in $\mathcal{F}_0(\mathbf{H})$ we have

$$\begin{aligned}[V_i^\# V_j g, h]_{\mathcal{H}_i} &= [V_j g, V_i h]_{\mathcal{K}} = [V_j g, V_i h]_{\mathcal{K}} \\ &= [g+\mathcal{N}_K, h+\mathcal{N}_K]_K = [K(i,j)g, h]_{\mathcal{H}_i}.\end{aligned}$$

To prove (b), let us first note that the linear hull generated by $\{V_i\mathcal{H}_i\}_{i\in\mathcal{J}}$ is $\mathcal{F}_0(\mathbf{H})/\mathcal{N}_K$. Therefore, in order to prove (b) it is necessary and suficient to prove that the linear manifold $\mathcal{F}_0(\mathbf{H})/\mathcal{N}_K$ is K-weakly dense in $\mathcal{K}$.

Indeed, any vector in $\mathcal{K}_L$ can be approximated L-weakly by vectors in $\mathcal{F}_0(\mathbf{H})/\mathcal{N}_L$ and any vector in $\mathcal{K}$ can be approximated K-weakly by vectors in $\mathcal{K}_L/\mathcal{N}_K$. But, since $\mathcal{N}_L \subseteq \mathcal{N}_K$ we have

$$\mathcal{F}_0(\mathbf{H})/\mathcal{N}_K = (\mathcal{F}_0(\mathbf{H})/\mathcal{N}_L)/\mathcal{N}_K,$$

and by means of the polarization formula and the inequality in (1) the L-weak topology is stronger than the K-weak topology, hence $\mathcal{F}_0(\mathbf{H})/\mathcal{N}_K$ is K-weakly dense in $\mathcal{K}$.

(3) ⇒ (1). Let $(\mathcal{K}, [\cdot,\cdot])$ be a Kreĭn space and $\{V_i\}_{i\in\mathcal{J}}$ be a family of bounded linear operators $V_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{K})$, $i \in \mathcal{J}$, such that

$$K(i,j) = V_i^\# V_j, \quad i, j \in \mathcal{J}.$$

Fix on $\mathcal{K}$ a fundamental symmetry J and for each $i \in \mathcal{J}$ fix a fundamental symmetry J_i on $\mathcal{H}_i$. Then define the kernel L by

$$L(i,j) = J_i V_i^\# V_j, \quad i, j \in \mathcal{J}.$$

It remains to prove that $L \in \mathfrak{K}^+(\mathbf{H})$ and that $-L \leq K \leq L$.

Indeed, let $f \in \mathcal{F}_0(\mathbf{H})$. Then

$$\begin{aligned}\sum_{i,j\in\mathcal{J}} [L(i,j)f(j), f(i)]_{\mathcal{H}_i} &= \sum_{i,j\in\mathcal{J}} [J_i V_i^\# V_j f(j), f(i)]_{\mathcal{H}_i} \\ &= \sum_{i,j\in\mathcal{J}} \langle V_j f(j), V_i f(i)\rangle_J \geq \Big\|\sum_{i,j\in\mathcal{J}} V_i f(i)\Big\|_J^2 \geq 0.\end{aligned}$$

Further, assume that supp $f = \{i_1, \ldots, i_n\}$ and consider the Kreĭn space $\mathcal{K}^n$, the direct sum of n copies of $\mathcal{K}$. Then we consider the fundamental symmetry $\tilde{J}$, the direct sum of n copies of J, and the identity operator I of the Kreĭn space $\mathcal{K}^n$. Taking into account that $\tilde{J} \leq I$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\tilde{J}}$, we conclude that

$$\sum_{i,j\in\mathscr{I}} [K(i,j)\,f(j),f(i)]_{\mathscr{H}_i} = \sum_{i,j\in\mathscr{I}} [J_i V_i^{\#} V_j f(j),f(i)]_{\mathscr{H}_i} = \sum_{i,j\in\mathscr{I}} \langle JV_j f(i),\, V_i f(i)\rangle_{J_i}$$
$$\leq \sum_{i,j\in\mathscr{I}} \langle V_j f(j),\, V_i f(i)\rangle_J = \sum_{i,j\in\mathscr{I}} [L(i,j)\,f(j),f(i)]_{\mathscr{H}_i}$$

and hence $[f,f]_K \leq [f,f]_L$. Similarly, taking into account that $-I\leq \widetilde{J}$ we conclude that $-[f,f]_L \leq [f,f]_K$. ∎

Remark 3.2. The fact that the family $\mathbb{H}$ consists of Kreĭn spaces is not essential. We can equivalently start with a family $\mathbb{H}$ of Hilbert spaces. Indeed, the first case is reduced to the latter case by fixing on each Kreĭn space $\mathscr{H}_i$ a fundamental symmetry and refer only to the inner products $\langle\cdot,\cdot\rangle_{J_i}$. One good reason to prefer the Kreĭn space formulation is the axiom (c): if the spaces $\mathscr{H}_i$ were Hilbert spaces then it would require the use of fundamental symmetries, more precisely, in this case it would read as follows

(c) $K(i,j)=J_i V_i^* V_j$ for all $i,j\in\mathscr{I}$.

Another stroung reason to prefer the Kreĭn space formulation is related to a subtle question on the existence of elementary rotations in Kreĭn spaces, which is quite different of the Hilbert space situation, as shown in [9]. This will become clear when considering Naĭmark dilations associated to Toeplitz hermitian kernels, see Section 6. ∎

Given an $\mathbb{H}$-kernel K, then rank(K) is by definition the supremum over all rank (K_Δ) taken over all finite subsets $\Delta=\{i_1,\dots,i_n\}\subset\mathscr{I}$, where K_Δ is the restricted kernel $(K(i,j))_{i,j\in\Delta}$. By definition rank(K) is either positive integer or the symbol ∞.

A hermitian $\mathbb{H}$-kernel K has κ negative squares if the inner product space $(\mathscr{F}_0(\mathbb{H}),\,[\cdot,\cdot]_K)$ has negative signature κ, that is, κ is the maximal dimension of all its negative subspaces. It is easy to see that this is equivalent with $K=K_+-K_-$, where $K_\pm\in\mathfrak{K}^+(\mathbb{H})$ are disjoint, such that rank $(K_-)=\kappa$, see e.g. [30]. We are thus entitled to define $\kappa^-(K)=\kappa$, the *number of negative squares* of the kernel K. In particular, hermitian $\mathbb{H}$-kernels with a finite number of negative squares always have Kolmogorov decomposition and for any Kolmogorov decomposition $(V;\mathscr{K})$ of K we have $\kappa^-(\mathscr{K})=\kappa^-(K)<\infty$, hence $\mathscr{K}$ is a Π_κ space, that is, a Pontryagin space with negative signature κ. These kernels have been carefully studied, starting with M.G. Kreĭn and H. Langer [20], [21], [22]. Among the hermitian kernels which were intensively studied there are the Schur, the Carathéodory and the Nevanlinna kernels associated to operator valued analytic functions. These are, respectively

$$S_\Theta(z,\zeta)=\frac{I-\Theta^{\#}(\zeta)\,\Theta(z)}{1-\bar{\zeta}z},\quad C_F(z,\zeta)=\frac{1}{2}\,\frac{F(z)-F^{\#}(\zeta)}{1-\bar{\zeta}z},\quad N_Q(z,\zeta)=\frac{Q(z)-Q^{\#}(\zeta)}{z-\bar{\zeta}}, \tag{3.13}$$

defined for $\mathscr{L}(\mathscr{H})$-valued analytic functions on appropriate domains.

The problem whether a given hermitian $\mathbb{H}$-kernel does admit or does not admit Kolmogorov decompositions might present some difficulty only for those kernels which have both numbers of positive and negative squares infinite. An example of a hermitian kernel which cannot be written as the difference of two positive kernels, and hence, by Theorem §3.1 does not admit Kolmogorov decomposition, was described by L. Schwartz [30], see also Theorem 2.2 in [1]. On the other hand, for certain domains of analyticity A. Dijksma, H. Langer, and H. de Snoo prove in [11] that the Carathéodory and the Nevanlinna kernels admit "majorants" as in item (1) of Theorem §3.1, and hence, these kernels admit Kolmogorov decompositions.

In the case of hermitian kernels defined on the set $\mathbb{Z}$ of integers, there is also of interest to exhibit a certain multiplicative structure of the Kolmogorov decomposition, since, in the positive definite case applications to realization theory and prediction of nonstationary processes can be obtained using this structure. We have the following result.

Theorem 3.3. *Let $\mathbb{H}=\{\mathcal{H}_i\}_{i\in\mathbb{Z}}$ be a family of Kreĭn spaces and let $K\in\mathfrak{K}^h(\mathbb{H})$ be a hermitian $\mathbb{H}$-kernel snch that $K(n,n)=I_{\mathcal{H}_n}$ for all $n\in\mathbb{Z}$ and admitting a Kolmogorov deeomposition $(V;\mathcal{K})$. Then there exist a family $\{\mathcal{K}_n\}_{n\in\mathbb{Z}}$ of Kreĭn spaces, $\mathcal{H}_n\subseteq\mathcal{K}_n$ for all $n\in\mathbb{Z}$, and a family $\{W_n\}_{n\in\mathbb{Z}}$ of unitary operators such that $W_n\in\mathcal{L}(\mathcal{K}_{n+1},\mathcal{K}_n)$ for all $n\in\mathbb{Z}$ and*

$$V_n=\begin{cases} W_{-1}^{\#}W_{-2}^{\#}\cdots\cdots W_n^{\#}|\mathcal{H}_n, & n<0\\ P_{\mathcal{H}_0}^{\mathcal{K}_0}|\mathcal{H}_0, & n=0\\ W_0W_1\cdots\cdots W_{n-1}|\mathcal{H}_n, & n>0.\end{cases}$$

Proof. Since $V_1^{\#}V_1=I_{\mathcal{H}_1}$ the operator V_1 is isometric and hence there exists a Kreĭn space $\mathcal{D}_1$ and an operator $D_1\in\mathcal{L}(\mathcal{D}_1,\mathcal{K})$ such that

$$W_0=[V_1\ D_1]:\mathcal{H}_n[+]\mathcal{D}_1\to\mathcal{K},$$

is unitary. Denote $\mathcal{K}_0=\mathcal{K}$, $\mathcal{K}_1=\mathcal{H}_1[+]\mathcal{D}_1$ and define the operator $V_{(n)}=W_0^{\#}V_n\in\mathcal{L}(\mathcal{H}_n,\mathcal{K}_1)$ for all $n\in\mathbb{Z}$. Then

$$V_{(i)}^{\#}V_{(j)}=V_i^{\#}W_0W_0^{\#}V_j=V_i^{\#}V_j,\ i,j\in\mathbb{Z}.$$

In particular $V_{(2)}^{\#}V_{(2)}=I_{\mathcal{H}_2}$ and hence, as before, there exists a Kreĭn space $\mathcal{D}_2$ and an operator $D_2\in\mathcal{L}(\mathcal{D}_2,\mathcal{K})$ such that the operator

$$W_1=[V_{(2)}D_2]:\mathcal{H}_2[+]\mathcal{D}_2\to\mathcal{K}_1,$$

is unitary. Denote $\mathcal{K}_1=\mathcal{H}_1[+]\mathcal{D}_1$. We proceed by induction and prove that there exist Kreĭn space $\mathcal{D}_{n+1}$ and operators $D_{n+1}\in\mathcal{L}(\mathcal{D}_{n+1},\mathcal{K})$ such that the operator

$$W_n = [W_0^\# W_1^\# \cdots\cdots W_{n-1}^\# V_{n+1} D_{n+1}] : \mathscr{H}_{n+1} [+] \mathscr{D}_{n+1} \to \mathscr{K}_n,$$

is unitary. Denote $\mathscr{K}_{n+1} = \mathscr{H}_{n+1} [+] \mathscr{D}_{n+1}$. Therefore, for all $n > 0$ we have $V_n = W_0 W_1 \cdots\cdots W_{n-1} | \mathscr{H}_n$. Similarly we obtain unitary operators W_n, $n < 0$, such that $V_n = W_{-1}^\# W_{-2}^\# \cdots\cdots W_n^\# | \mathscr{H}_n$ for all $n < 0$. For $n = 0$ the formula is clear since $\mathscr{K}_0 = \mathscr{K}$ and, from $V_0^\# V_0 = I_{\mathscr{H}_0}$ it follows that V_0 is isometric and hence, modulo the identification of $\mathscr{H}_0$ with $\mathscr{R}(V_0)$ we have $V_0 = P_{\mathscr{H}_0}^{\mathscr{K}_0} | \mathscr{H}_0$. ∎

§4. Uniqueness of the Kolmogorov Decomposition

In the positive definite case the Kolmogorov decomposition is unique, modulo unitary equivalence. In the general hermitian case, the existence of a Kolmogorov decomposition does not imply that it is unique and the question of characterizing those hermitian kernels which possess unique Kolmogorov decomposition is of interest. We first make precise the notion of unitary equivalence of two Kolmogorov decompositions of the same hermitian kernel.

Two Kolmogorov decompositions $(\{V_\iota\}_{\iota \in \mathscr{J}}; \mathscr{K})$ and $(\{U_\iota\}_{\iota \in \mathscr{J}}; \mathscr{H})$ of the same hermitian $\mathbb{H}$-kernel K are *unitarily equivalent* if there exists a unitary operator $\Phi \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ such that for all $i \in \mathscr{J}$ we have $U_\iota = \Phi V_\iota$.

Let K be a hermitian $\mathbb{H}$-kernel. If $L \in \mathfrak{K}^+(\mathbb{H})$ is such that $-L \le K \le L$ then, as in the proof of Theorem 3.1, we denote by $\mathscr{K}_L$ the quotient completion of $(\mathscr{F}_0(\mathbb{H}), [\cdot, \cdot]_L)$ to a Hilbert space and by $K_L \in \mathscr{L}(\mathscr{K}_L)$ the Gram operator of the inner product $[\cdot, \cdot]_K$ with respect to the positive inner product $[\cdot, \cdot]_L$, that is, $[x, y]_K = [K_L x, y]_L$ for all $x, y \in \mathscr{K}_L$.

Theorem 4.1. *Let K be a hermitian $\mathbb{H}$-kernel which has Kolmogorov decomposition. The following assertions are equivalent*:

(i) *The $\mathbb{H}$-kernel K has unique Kolmogorov decomposition, modulo unitary equivalence.*

(ii) *For any positive semidefinite $\mathbb{H}$-kernel L such that $-L \le K \le L$ there exists $\varepsilon > 0$ such that either $(0, \varepsilon) \subset \rho(K_L)$ or $(-\varepsilon, 0) \subset \rho(K_L)$.*

Proof. (i) ⇒ (ii). Assume that there exists a positive semidefinite $\mathbb{H}$-kernel L such that $-L \le K \le L$ and for any $\varepsilon > 0$ we have $(0, \varepsilon) \cap \sigma(K_L) \ne \emptyset$ and $(-\varepsilon, 0) \cap \sigma(K_L) \ne \emptyset$. From Theorem 2.8 it follows that there exists two Kreĭn spaces $(\mathscr{K}, \Pi)$ and $(\mathscr{H}, \Phi)$ induced by the same selfadjoint operator K_L, which are not unitarily equivalent. It is easy to see that the operator $\Psi : \mathscr{R}(\Pi) \to \mathscr{R}(\Phi)$ is correctly defined by

$$\Psi \Pi f = \Phi f, \, f \in \mathscr{K}_L.$$

Then the operator Ψ is isometric, densely defined, with dense range, and it is unbounded since the two induced Kreĭn spaces are not unitarily equivalent. As a consequence, Ψ is closable and its closure, denoted also by Ψ, shares the same properties.

Let $(V;\mathcal{K})$ be the Kolmogorov decomposition of K defined as in the proof of Theorem 3.1, $(1)' \Rightarrow (3)$. We now define a new Kolmogorov decomposition $(U;\mathcal{H})$ and we will prove that it is not unitarily equivalent with $(V;\mathcal{H})$. More precisely, let $U_i = \Psi V_i$ for all $i \in \mathcal{J}$. Since $\mathcal{R}(V_i) \subseteq \mathcal{D}(\Psi)$ and Ψ is closed it follows, via the closed graph principle, that $U_i \in \mathcal{L}(\mathcal{H}_i, \mathcal{K})$ for all $i \in \mathcal{J}$.

Let $i, j \in \mathcal{J}$ be arbitrary and fix vectors $x \in \mathcal{H}_i$ and $y \in \mathcal{H}_j$. Then

$$[U_j y, U_i x] = [\Psi V_j y, \Psi V_i x] = [V_j y, V_i x] = [K(i, j) y, x],$$

and hence $U_i^{\#} U_j = K(i, j)$. Also,

$$\bigvee_{i \in \mathcal{J}} U_j \mathcal{H}_i = \bigvee_{j \in \mathcal{J}} \Psi V_j \mathcal{H}_j = \Psi(\bigvee_{j \in \mathcal{J}} V_j \mathcal{H}_j) = \mathcal{K}.$$

Thus, $(\{U_i\}_{i \in \mathcal{J}}; \mathcal{K})$ is a Kolmogorov decomposition of the $\mathbb{H}$-kernel K. On the other hand, since the operator Ψ is unbounded it follows that the two Kolmogorov decompositions $(\{V_i\}_{i \in \mathcal{J}}; \mathcal{K})$ and $(\{U_i\}_{i \in \mathcal{J}}; \mathcal{K})$ are not unitarily equivalent.

(ii) $\Rightarrow$ (i). Let $(\{V_i\}_{i \in \mathcal{J}}; \mathcal{K})$ and $(\{U_i\}_{i \in \mathcal{J}}; \mathcal{H})$ be two Kolmogorov decompositions of K. Let J and J_i be fundamental symmetries on $\mathcal{K}$ and, respectively, $\mathcal{H}_i$, $i \in \mathcal{J}$. We consider the positive definite $\mathbb{H}$-kernel L_V defined by

$$L_V(i, j) = J_j V_j^* V_i, \; i, j \in \mathcal{J},$$

and as in the proof of Theorem 3.1 it follows that $-L_V \leq K \leq L_V$. We define a linear operator $\Pi_V: \mathcal{F}_0(\mathbb{H}) \to \mathcal{K}$ by

$$\Pi_V(h) = \sum_{i \in \mathcal{J}} V_i h_i, \; h = (h_i)_{i \in \mathcal{J}} \in \mathcal{F}_0(\mathbb{H}). \tag{4.1}$$

Taking into account of the axiom (c) in the definition of a Kolmogorov decomposition we obtain

$$[\Pi_V(h), \Pi_V k]_K = [h, k]_{\mathcal{K}}, \; h, k \in \mathcal{F}_0(\mathbb{H}), \tag{4.2}$$

that is, the operator Π_V is isometric: $(\mathcal{F}_0(\mathbb{H}), [\cdot, \cdot]_K) \to (\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$.

In addition, we claim that Π_V is also isometric when considered as a linear operator $(\mathcal{F}_0(\mathbb{H}), [\cdot, \cdot]_{L_V}) \to (\mathcal{K}, \langle \cdot, \cdot \rangle_J)$. To see this, let $h = (h_i)_{i \in \mathcal{J}}$ be arbitrary in $\mathcal{F}_0(\mathbb{H})$. Then

$$\|\sum_{i \in \mathcal{J}} V_i h_i\|_J^2 = \langle \sum_{i \in \mathcal{J}} V_i h_i, \sum_{i \in \mathcal{J}} V_j h_j \rangle_J = \sum_{i,j \in \mathcal{J}} \langle V_j^* V_i h_i, h_j \rangle_{J_j} = \sum_{i,j \in \mathcal{J}} [J_j V_j^* V_i h_i, h_j]_{\mathcal{H}_j} = \|h\|_{L_V}^2.$$

This proves the claim.

Similarly, considering the positive semidefinite $\mathbf{H}$-kernel L_U defined by

$$L_U(i, j) = J_j U_j^* U_i, \; i, j \in \mathcal{J},$$

we have $-L_U \leq K \leq L_U$ and defining the linear operator $\Pi_U: \mathcal{F}_0(\mathbf{H}) \to \mathcal{H}$ by

$$\Pi_U(h) = \sum_{\iota\in\mathscr{J}} U_\iota h_\iota,\ h = (h_\iota)_{\iota\in\mathscr{J}} \in \mathscr{F}_0(\mathbf{H}), \tag{4.3}$$

obtain

$$[\Pi_U h,\ \Pi_U k]_K = [h,\ k]_{\mathscr{K}},\ h,\ k \in \mathscr{F}_0(\mathbf{H}), \tag{4.4}$$

that is, the operator Π_U is isometric: $(\mathscr{F}_0(\mathbf{H}),\ [\cdot,\ \cdot]_K) \to (\mathscr{K},\ [\cdot,\ \cdot]_{\mathscr{H}})$ and Π_U is also isometric when considered as a linear operator
$(\mathscr{F}_0(\mathbf{H}),\ [\cdot,\ \cdot]_{L_U}) \to (\mathscr{K},\ \langle\cdot,\ \cdot\rangle_J)$.

We now let $L = L_V + L_U$ and clearly we have $-L \le K \le L$. Since $L_U \le L$ it follows that $\mathscr{K}_L$ is contractively embedded into $\mathscr{K}_{L_V}$ and hence Π_V induces a bounded operator $\Pi_V: \mathscr{K}_L \to \mathscr{K}$. From (4.2) it follows

$$\Pi_V^* J \Pi_V = K_L. \tag{4.5}$$

By assumption, there exists $\varepsilon > 0$ such that either $(-\varepsilon,\ 0) \subset \rho(K_L)$ or $(0,\ \varepsilon) \subset \rho(K_L)$ and hence from (4.5), Lemma 2.3 in [8], and taking into account that by the minimality axiom (b) of the Kolmogorov decomposition the operator Π_V has dense range, it follows that there exists a uniquely determined bounded unitary operator Φ_V: $\mathscr{K}_K \to \mathscr{K}$ such that

$$\Phi_V h = \Pi_V h,\ h \in \mathscr{K}_L, \tag{4.6}$$

where $\mathscr{K}_K$ is the Kreĭn space induced by the operator $\mathscr{K}_L$.

Similarly, performing the same operations with respect to the Kolmogorov decomposition $(U;\ \mathscr{H})$ we get a uniquely determined bounded unitary operator Φ_U: $\mathscr{K}_K \to \mathscr{H}$ such that

$$\Phi_U h = \Pi_U h,\ h \in \mathscr{K}_L. \tag{4.7}$$

We define the bounded unitary operator Φ: $\mathscr{K} \to \mathscr{H}$ by

$$\Phi = \Phi_U \Phi_V^{-1}.$$

Taking into account of (4.6) and (4.7), the definition of the operator Π_V as in (4.1), and the definition of Π_U as in (4.3), it follows that

$$\Phi\Big(\sum_{\iota\in\mathscr{J}} V_\iota h_\iota\Big) = \sum_{\iota\in\mathscr{J}} U_\iota h_\iota,\ (h_\iota)_{\iota\in\mathscr{J}} \in \mathscr{F}_0(\mathbf{H}).$$

This readily implies that for all $i \in \mathscr{J}$ we have $\Phi V_i = U_\iota$ and hence the two Kolmogorov decompositions $(\{V_i\}_{\iota\in\mathscr{J}};\ \mathscr{H})$ and $(\{U_i\}_{\iota\in\mathscr{J}};\ \mathscr{H})$ are unitarily equivalent. ∎

As a consequence of Theorem 4.1 we can obtain a counter-part of a result on nonuniqueness of hermitian kernels of L. Schwartz, cf. Proposition 41 in [30] (the transcription will be clear after Theorem 5.1). Let K and H be two positive semidefinite $\mathbf{H}$-kernels. Then we consider the Hilbert spaces $\mathscr{K}_K$ and $\mathscr{K}_H$, obtained by quotient completion of $(\mathscr{F}_0(\mathbf{H}),\ [\cdot,\ \cdot]_K)$ and, respectively, of

$(\mathcal{F}_0(\mathbb{H}), [\cdot, \cdot]_H)$. If $H \geq K$ then $\mathcal{K}_H$ is contractively embedded into $\mathcal{K}_K$. Following [30], we say that the kernel H is *K-compact* if the embedding $\mathcal{K}_H \hookrightarrow \mathcal{K}_K$ is a compact operator.

Corollary 4.2. *Let $H_+, H_- \in \mathfrak{K}^+(\mathbb{H})$ be two disjoint kernels, both of them of infinite rank. If there exists a kernel $K \in \mathfrak{K}^+(\mathbb{H})$ such that H_+ and H_- are K-compact, then the Kolmogorov decompositions of the kernel $H_+ - H_-$ are not unique, modulo unitary equivalence.*

Proof. Let $H = H_+ - H_-$. Clearly $-K \leq H \leq K$. Let $A_\pm \in \mathcal{L}(\mathcal{K}_K)$ denote the Gram operator of the kernel $H_\pm$. Since H_+ and H_- are disjoint it follows that $A = A_+ - A_-$ is the Gram operator of H. Since $H_\pm$ are of infinite rank and K-compact it follows that $A_\pm$ are compact operators of infinite rank in $\mathcal{L}(\mathcal{K}_K)$ and hence the spectra $\sigma(A_\pm)$ are accumulating to 0. Then the spectrum $\sigma(A)$ is accumulating to 0 from both sides. This clearly contradicts the condition (ii) in Theorem 4.1 and hence the Kolmogorov decompositions of the kernel $H_+ - H_-$ are not unique, modulo unitary equivalence. □

We end this section with another criterion of uniqueness of Kolmogorov decompositions of a hermitian kernel.

Proposition 4.3. *Let K be a hermitian $\mathbb{H}$-kernel such that it has a Kolmogorov decomposition $(\{V_i\}_{i \in \mathcal{J}}; \mathcal{K})$ with the property that there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$ such that either $\mathcal{K}^+$ or $\mathcal{K}^-$ is contained in the linear manifold generated by $V_i \mathscr{H}_i$, $i \in \mathcal{J}$. Then K has unique Kolmogorov decomposition, modulo unitary equivalence.*

Proof. Let $(\{U_i\}_{i \in \mathcal{J}}; \mathscr{H})$ be another Kolmogorov decomposition of K. We define a linear operator Φ by

$$\Phi(\sum_{i \in \mathcal{J}} V_i h_i) = \sum_{i \in \mathcal{J}} U_i h_i, \quad (h_i)_{i \in \mathcal{J}} \in \mathcal{F}_0(\mathbb{H}).$$

The operator Φ is densely defined, with dense range and isometric (in particular, from here follows also that it is correctly defined). Assuming that there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$ such that either $\mathcal{K}^+$ or $\mathcal{K}^-$ is contained in the linear manifold generated by $V_i \mathscr{H}_i$ $i \in \mathcal{J}$, from Lemma 2.3 in [8] it follows that Φ extends uniquely to a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}, \mathscr{H})$ such that $\Phi V_i = U_i$ for all $i \in \mathcal{J}$. □

§5. Kreĭn Spaces with Reproducing Kernel

As in the previous sections let $\mathbb{H} = \{\mathscr{H}_i\}_{i \in \mathcal{J}}$ be a family of Kreĭn spaces and denote by $\mathcal{F}(\mathbb{H})$ the class of functions f defined on $\mathcal{J}$ and such that $f(i) \in \mathscr{H}_i$ for all $i \in \mathcal{J}$. Given $K \in \mathfrak{K}^h(\mathbb{H})$, a *Kreĭn space with reproducing kernel K* is, by definition, a Kreĭn space $(\mathscr{R}, [\cdot, \cdot]_{\mathscr{R}})$ subject to the following conditions:

(1) $\mathscr{R} \subset \mathscr{F}(\mathbb{H})$.

(2) $\mathscr{R} = \bigvee_{i \in \mathscr{J}} K(\cdot, i)\mathscr{H}_i$.

(3) $[f(i), h]_{\mathscr{H}_i} = [f, K(\cdot, j)h]_{\mathscr{R}}$, for all $f \in \mathscr{R}$, $i \in \mathscr{J}$, and $h \in \mathscr{H}_i$.

The next theorem can be obtained from Theorem 3.1 and the characterization of kernels admitting reproducing Kreĭn space representation. We first need a direct connection between Kolmogorov decompositions and reproducing kernel Kreĭn space representation.

Theorem 5.1. *Let $K \in \mathfrak{K}^h(\mathbb{H})$. Then K admits a Kolmogorov decomposition if and only if there exists a Kreĭn space with reproducing kernel K.*

Proof. Let $(V; \mathscr{K})$ be a Kolmogorov decomposition of the $\mathbb{H}$-kernel K and define

$$\mathscr{R} = \{(V_i^{\#} f)_{i \in \mathscr{J}} | f \in \mathscr{K}\}. \tag{5.1}$$

Let $f, g \in \mathscr{K}$ be such that $V_i^{\#} f = V_i^{\#} g$ for all $i \in \mathscr{J}$. Then,

$$[V_i^{\#} f, h_i]_{\mathscr{H}_i} = [V_i^{\#} g, h_i]_{\mathscr{H}_i}, \ i \in \mathscr{J}. \ h_i \in \mathscr{H}_i,$$

or, equivalently,

$$[f - g, V_i h_i] = 0, \ i \in \mathscr{J}, \ h_i \in \mathscr{H}_i.$$

Taking into account of the minimality axiom of the Kolomogorov decomposition, from here we get $f = g$. This shows that the linear mapping

$$\mathscr{K} \ni f \mapsto (V_i^{\#} f)_{i \in \mathscr{J}} \in \mathscr{R}, \tag{5.2}$$

is bijective. We define the inner product $[\cdot, \cdot]_{\mathscr{R}}$ by

$$[(V_i^{\#} f)_{i \in \mathscr{J}}, (V_i^{\#} g)_{i \in \mathscr{J}}]_{\mathscr{R}} = [f, g]_{\mathscr{K}}, \quad f, g \in \mathscr{K}. \tag{5.3}$$

Then endow $\mathscr{R}$ with the strong topology transported by the identification as in (5.2), and hence $(\mathscr{R}, [\cdot, \cdot]_{\mathscr{K}})$ becomes a Kreĭn space unitarily equivalent with $(\mathscr{K}, [\cdot, \cdot]_{\mathscr{K}})$.

By axiom (c) of a Kolomogorov decomposition it follows that

$$K(\cdot, j)h = (V_i^{\#} V_j h)_{i \in \mathscr{J}}, \ j \in \mathscr{J}, \ h \in \mathscr{H}_j. \tag{5.4}$$

This shows that for all $j \in \mathscr{J}$ and all $h \in \mathscr{H}_j$ the functions $K(\cdot, j)h \in \mathscr{R}$. Then, using the axiom (b) of a Kolmogorov decomposition (i.e. the minimality axiom) and the definition of $\mathscr{R}$ as in (5.1) we have

$$\bigvee_{j \in \mathscr{J}} K(\cdot, j)\mathscr{H}_j = \bigvee_{j \in \mathscr{J}} (V_i^{\#} V_j \mathscr{H}_j)_{j \in \mathscr{J}} = \mathscr{R},$$

and hence the minimality axiom in the definition of a reproducing Kreĭn space is satisfied.

Let $f\in\mathcal{R}$ be arbitrary, and hence there exists $g\in\mathcal{K}$ such that $f=(V_i^\# g)_{i\in\mathcal{J}}$. Taking into account of (5.4), for all $i\in\mathcal{J}$ and $h\in\mathcal{H}_i$ we have

$$\begin{aligned}[f(i),h]_{\mathcal{H}_i}&=[V_i^\# g,h]_{\mathcal{H}_i}=[g,V_ih]_{\mathcal{K}}\\ &=[(V_j^\# g)_{j\in\mathcal{J}},(V_j^\# V_ih)_{j\in\mathcal{J}}]_{\mathcal{R}}=[(V_j^\# g)_{j\in\mathcal{J}},K(\cdot,i)h]_{\mathcal{R}}=[f,K(\cdot,i)h]_{\mathcal{R}},\end{aligned}$$

and hence the axiom (c) in the definition of a reproducing kernel Kreĭn space is also satisfied. Thus, $(\mathcal{R},[\cdot,\cdot]_{\mathcal{R}})$ is a Kreĭn space with reproducing kernel K.

Let $(\mathcal{R},[\cdot,\cdot]_{\mathcal{R}})$ be a Kreĭn space with reproducing kernel K. Denote $(\mathcal{K},[\cdot,\cdot]_{\mathcal{K}})=(\mathcal{R},[\cdot,\cdot]_{\mathcal{R}})$ and for arbitrary $j\in\mathcal{J}$ let

$$V_j=K(\cdot,j):\mathcal{H}_j\to\mathcal{K}=\mathcal{R}. \tag{5.5}$$

By axiom (c) of a reproducing kernel Kreĭn space we have

$$[f(i),h]_{\mathcal{H}_i}=[f,V_ih]_{\mathcal{K}},\ f\in\mathcal{R}=\mathcal{K},\ i\in\mathcal{J},\ h\in\mathcal{H}_i.$$

This implies that $V_i^\#$ exists as an everywhere defined operator and coincides with the linear operator

$$\mathcal{K}\ni f\mapsto f(i)\in\mathcal{H}_i.$$

Therefore V_i is a closed operator and since it is defined on the whole Kreĭn space $\mathcal{H}_i$, the closed graph principle implies that $V_i\in\mathcal{L}(\mathcal{H}_i,\mathcal{K})$.

Note that with the definition of the operators V_i as in (5.5) and applying the axiom (c) of the reproducing kernel Kreĭn space to the function $f=K(\cdot,j)h_j$ for arbitrary $i,j\in\mathcal{J}$, $h_i\in\mathcal{H}_i$, and $h_j\in\mathcal{H}_j$ we have

$$\begin{aligned}[V_i^\# V_jh_j,h_i]_{\mathcal{H}_i}&=[V_jh_j,V_ih_i]_{\mathcal{K}}=[K(\cdot,j)h_j,K(\cdot,i)h_i]_{\mathcal{R}}\\ &=[f(i),h_i]_{\mathcal{H}_i}=[K(i,j)h_j,h_i]_{\mathcal{H}_i},\end{aligned}$$

and hence $V_i^\# V_j=K(i,j)$. Thus $(V;\mathcal{K})$ is a Kolmogorov decomposition of the $\mathbb{H}$-kernel K. ∎

It is interesting that for the analytic kernels as in 3.13 there always exist reproducing kernel Kreĭn spaces, cf. D. Alpay [1]. Theorem 5.1 shows that this result is actually equivalent to the realization results of A. Dijksma, H. Langer, and H. de Snoo in [11].

Inspecting the proof of Theorem 5.1 it follows that

Corollary 5.2. *Given a hermitian* $\mathbf{H}$*-kernel* K, *the mapping*

$$(V;\mathcal{K})\mapsto\mathcal{R}(V;\mathcal{K})=\{(V_i^\# f)_{i\in\mathcal{J}}\mid f\in\mathcal{K}\}, \tag{5.6}$$

such that the inner product $[\cdot,\cdot]_{\mathcal{R}(V;\mathcal{K})}$ *on* $\mathcal{R}(V;\mathcal{K})$ *is defined as in* (5.3), *maps the class of all Kolmogorov decompositions of* K *onto the class of all Kreĭn spaces with reproducing* $\mathbf{H}$*-kernel* K.

Corollary 5.3. *In the correspondence* (5.6) *associated to a given hermitian* $\mathbb{H}$*-kernel* K, *two Kolmogorov decompositions* $(V; \mathcal{K})$ *and* $(U; \mathcal{H})$ *of* K *are unitarily equivalent if and only if* $\mathcal{R}(V; \mathcal{K})$ *coincides with* $\mathcal{R}(U; \mathcal{H})$ *as Kreĭn spaces, that is,* $\mathcal{R}(V; \mathcal{K}) = \mathcal{R}(U; \mathcal{H})$ *and* $[\cdot, \cdot]_{\mathcal{R}(V;\mathcal{K})} = [\cdot, \cdot]_{\mathcal{R}(U;\mathcal{H})}$.

Proof. Let $\Phi \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be a unitary operator such that $\Phi U_i = V_i$ for all $i \in \mathcal{J}$. Then

$$\begin{aligned}\mathcal{R}(V; \mathcal{K}) &= \{(V_i^\# f)_{i\in\mathcal{J}} | f \in \mathcal{K}\} = \{(U_i^\# \Phi^\# f)_{i\in\mathcal{J}} | f \in \mathcal{K}\} \\ &= \{(U_i^\# h)_{i\in\mathcal{J}} | h \in \mathcal{H}\} = \mathcal{R}(U; \mathcal{H}).\end{aligned}$$

Moreover, for all $f, g \in \mathcal{K}$, taking into account of (5.3) we have

$$\begin{aligned}[(V_i^\# f)_{i\in\mathcal{J}}, (V_j^\# g)_{j\in\mathcal{J}}]_{\mathcal{R}(V;\mathcal{K})} &= [f, g]_{\mathcal{K}} = [\Phi^\# f, \Phi^\# g]_{\mathcal{H}} \\ &= [(U_i^\# \Phi^\# f)_{i\in\mathcal{J}}, (U_j^\# \Phi^\# g)_{j\in\mathcal{J}}]_{\mathcal{R}(U;\mathcal{H})}.\end{aligned}$$

Conversely, let $(U; \mathcal{H})$ and $(V; \mathcal{K})$ be two Kolmogorov decompositions of the $\mathbb{H}$-kernel K such that $\mathcal{R}(U; \mathcal{H}) = \mathcal{R}(V; \mathcal{K})$ as Kreĭn spaces. Taking into account that the linear mapping (5.2) is bijective, it follows that for all $h \in \mathcal{H}$ there exists a unique $f \in \mathcal{K}$ such that

$$U_i^\# h = V_i^\# f, \; i \in \mathcal{J}.$$

Thus, we can define a linear and bijective mapping $\Phi: \mathcal{H} \to \mathcal{K}$ by

$$\mathcal{H} \ni h \mapsto \Phi h = f \in \mathcal{K}, \; (U_i^\# h)_{i\in\mathcal{J}} = (V_i^\# f)_{i\in\mathcal{J}}. \tag{5.7}$$

Using the fact that $[\cdot, \cdot]_{\mathcal{R}(U;\mathcal{H})} = [\cdot, \cdot]_{\mathcal{R}(V;\mathcal{K})}$ we readily verify that Φ is isometric. Since Φ is bijective this means that $\Phi^\# = \Phi^{-1}$, in particular Φ is closed. An application of the closed graph principle now shows that $\Phi \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and hence Φ is unitary. The way Φ was defined in (5.7) implies that $V_i = \Phi U_i$ for all $i \in \mathcal{J}$ and hence the two Kolmogorov decompositions $(V; \mathcal{K})$ and $(U; \mathcal{H})$ are unitarily equivalent. ∎

As a consequence of Corollary 5.3 and of Theorem 4.1 of characterization of the uniqueness of Kolmogorov decompositions, modulo unitary equivalence, we can formulate a characterization of uniqueness of those $\mathbf{H}$-kernel K admitting unique Kreĭn space with reproducing kernel K.

Corollary 5.4. *Let* K *be a hermitian* $\mathbf{H}$*-kernel admitting Kolmogorov decompositions, or, equivalently, admitting Kreĭn spaces with reproducing kernel* K. *Then there exists a unique Kreĭn space with reproducing kernel* K *if and only if* K *has unique Kolmogorov decomposition, modulo unitary equivalence, and hence, if and only if the condition* (ii) *in Theorem* 4.1 *holds.*

§6. Toeplitz Hermitian Kernels

Let $\mathscr{H}$ be a Kreĭn space and denote by $\mathfrak{K}(\mathscr{H})$ the set of all functions $K: \mathbb{Z}\times\mathbb{Z}\to\mathscr{L}(\mathscr{H})$. This is consistent with the notation in the previous section if we consider the family $\mathbb{H}=\{\mathscr{H}_n\}_{n\in\mathbf{Z}}$ where the Kreĭn spaces $\mathscr{H}_n$ are copies of the Kreĭn space $\mathscr{H}$. An $\mathscr{H}$-kernel $H\in\mathfrak{K}(\mathscr{H})$ is called a *Toeplitz $\mathscr{H}$-kernel* if there exists an operator valued mapping $T: \mathbb{Z}\to\mathscr{L}(\mathscr{H})$ such that $H(i,j)=T(i-j)$, for all $i,j\in\mathbb{Z}$. We consider the set $\mathfrak{T}(\mathscr{H})$ of all Toeplitz $\mathscr{H}$-kernels. In the following we will be interested mostly in the subclass of Toeplitz hermitian $\mathscr{H}$-kernels $\mathfrak{T}^h(\mathscr{H})$ and the subclass of Toeplitz positive semidefinite $\mathscr{H}$-kernels $\mathfrak{T}^+(\mathscr{H})$. Let us note that $\mathfrak{T}^h(\mathscr{H})$ is closed under addition, subtraction and (left and right) multiplication with bounded operators on $\mathscr{H}$. Also, $\mathfrak{T}^+(\mathscr{H})$ is a strict cone of $\mathfrak{T}^h(\mathscr{H})$.

We consider the complex vector space $\mathscr{F}_0(\mathscr{H})$ of all functions $h: \mathbb{Z}\to\mathscr{H}$ with finite support and for an arbitrary hermitian kernel $H\in\mathfrak{K}^h(\mathscr{H})$ we associate the inner product space $(\mathscr{F}_0(\mathscr{H}),[\cdot,\cdot]_H)$ as in (3.2). On the vector space $\mathscr{F}_0(\mathscr{H})$ we consider two operators, the *forward shift* S_+ defined by $(S_+h)(n)=h(n-1)$, for all $h\in\mathscr{F}_0(\mathscr{H})$ and $n\in\mathbb{Z}$, and the *backward shift* S_- defined by $(S_-h)(n)=h(n+1)$, for all $h\in\mathscr{F}_0(\mathscr{H})$ and all $n\in\mathbb{Z}$.

Lemma 6.1. *Let $H\in\mathfrak{K}^h(\mathscr{H})$. Then H is a Toeplitz kernel if and only if*

$$[S_+h, g]_H=[h, S_-g]_H,\ h, g\in\mathscr{F}_0(\mathscr{H}).$$

Proof. We first notice that an $\mathscr{H}$-kernel H is Toeplitz if and only if $H(n+1,k)=H(n,k-1)$ for all $n,k\in\mathbb{Z}$.

Let $h, g\in\mathscr{F}_0(\mathscr{H})$ be fixed. Then, by the definition of the forward shift S_+ we have

$$\sum_{n,k\in\mathbf{Z}}[H(n,k)(S_+h)(n), g(k)]=\sum_{n,k\in\mathbf{Z}}[H(n+1,k)h(n), g(k)],$$

and similarly, by the definition of the backwardshift S_- we have

$$\sum_{n,k\in\mathbf{Z}}[H(n,k)h(n), (S_-g)(k)]=\sum_{n,k\in\mathbf{Z}}[H(n,k-1)h(n), g(k)].$$

It remains to take into account that the representation of hermitian kernels as inner products on $\mathscr{F}_0(\mathscr{H})$ is faithful. ∎

Remark 6.2. If H is a hermitian Toeplitz $\mathscr{H}$-kernel then both S_+ and S_- are isometric with respect to the inner product $[\cdot,\cdot]_H$, that is, for all $h, g\in\mathscr{F}_0(\mathscr{H})$ we have $[S_+h, S_+g]_H=[h,g]_H$ and $[S_-h, S_-g]_H=[h,g]_H$. The converse is also true, if either S_+ or S_- is isometric with respect to the hermitian $\mathscr{H}$-kernel H then H is Toeplitz. The proof is similar with that of Lemma 6.1. ∎

Let H be a Toeplitz hermitian $\mathcal{H}$-kernel. A *Naĭmark dilation* of H is, by definition, a triple $(U, Q; \mathcal{K})$ subject to the following conditions:

(a) $\mathcal{K}$ is a Kreĭn space, $U \in \mathcal{L}(\mathcal{K})$ is a unitary operator, and $Q \in \mathcal{L}(\mathcal{H}, \mathcal{K})$.

(b) $\mathcal{K} = \bigvee_{n \in \mathbf{Z}} U^n Q \mathcal{H}$.

(c) $H(i, j) = Q^{\#} U^{i-j} Q$, $i, j \in \mathbb{Z}$.

Remark 6.3. If the Toeplitz hermitian $\mathcal{H}$-kernel H has a Naĭmark dilation $(U, Q; \mathcal{K})$ then the equivalent assertions in Theorem 3.1 also hold. Indeed, letting $V(n) = U^n Q \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ it is readily verified that the pair $(V; \mathcal{K})$ is a Kolmogorov decomposition of H. □

Remark 6.4. Let H be a Toeplitz hermitian $\mathcal{H}$-kernel and assume that $H(0, 0) = I$, the identity operator on $\mathcal{H}$. Then each Naĭmark dilation $(U, Q; \mathcal{K})$ of H can be viewed as a pair $(U; \mathcal{K})$ subject to the following conditions:

(a)′ $\mathcal{K}$ is Kreĭn space extension of $\mathcal{H}$ and $U \in \mathcal{L}(\mathcal{K})$ is a unitary operator.

(b)′ $\mathcal{K} = \bigvee_{n \in \mathbf{Z}} U^n \mathcal{H}$.

(c) $H(i, j) = P_{\mathcal{H}} U^{i-j} | \mathcal{H}$, $i, j \in \mathbb{Z}$.

Indeed, if $(U, Q; \mathcal{K})$ is a Naĭmark dilation of $\mathcal{H}$ then $Q^{\#} Q = H(0, 0) = I$ and hence Q is a bounded isometric operator from $\mathcal{H}$ into $\mathcal{K}$. Identifying $\mathcal{H}$ with $Q\mathcal{H}$, the operator Q becomes the embedding $\mathcal{H} \hookrightarrow \mathcal{K}$ and hence $\mathcal{K}$ is a Kreĭn space extension of $\mathcal{H}$. The axioms (a)′ and (b)′ are transcriptions of the corresponding axioms of the Naĭmark dilation. The converse implication is clear. □

Example 6.5. Let $\mathcal{H}$ be a Kreĭn space and $T \in \mathcal{L}(\mathcal{H})$. Then the operator T has a *minimal unitary dilation* $(U; \mathcal{K})$ (see [9]), that is, the following hold:

(α) $\mathcal{K}$ is Kreĭn space extension of $\mathcal{H}$ and $U \in \mathcal{L}(\mathcal{K})$ is unitary;

(β) $\bigvee_{n \in \mathbf{Z}} U^n \mathcal{H} = \mathcal{K}$;

(γ) $P_{\mathcal{H}} U^n | \mathcal{H} = T^n$, for all $n \geq 0$.

Consider now the Toeplitz hermitian $\mathcal{H}$-kernel H defined by

$$H(i, j) = \begin{cases} T^{i-j}, & i > j \\ I, & i = j, \\ T^{\# j-i}, & j > i. \end{cases} \tag{6.1}$$

Then $(U; \mathcal{K})$ is a Naĭmark dilation of H (cf. Remark 6.4). ■

We are now interested in describing the Toeplitz hermitian $\mathcal{H}$-kernels which admit Naĭmark dilations. First we need to introduce some other classes of positive semidefinite $\mathcal{H}$-kernels.

A positive semidefinite $\mathcal{H}$-kernel $L \in \mathfrak{K}^{+}(\mathcal{H})$ is called of *forward shift* *bounded type* if the forward shift S_+ is bounded with respect to the seminorm $\mathcal{F}_0(\mathcal{H}) \ni h \mapsto [h, h]_L^{1/2}$, that is,

$$[S_+h, S_+h]_L \leq C[h, h]_L, \; h \in \mathcal{F}_0(\mathcal{H}),$$

for some constant $C \geq 0$. We denote by $\mathfrak{K}_+^+(\mathcal{H})$ the class of all kernels $L \in \mathfrak{K}^+(\mathcal{H})$ of forward shift bounded type.

Similarly, a positive semidefinite $\mathcal{H}$-kernel $L \in \mathfrak{K}^+(\mathcal{H})$ is called of *backward shift bounded type* if the backward shift S_- is bounded with respect to the seminorm $\mathcal{F}_0(\mathcal{H}) \ni h \mapsto [h, h]_L^{1/2}$ and denote by $\mathfrak{K}_-^+(\mathcal{H})$ the corresponding class. Also, let $\mathfrak{K}_0^+(\mathcal{H}) = \mathfrak{K}_+^+(\mathcal{H}) \cap \mathfrak{K}_-^+(\mathcal{H})$ be the class of (positive semidefinite) $\mathcal{H}$-kernels of *shift bounded type*.

By Remark 6.2 it follows that $\mathfrak{T}^+(\mathcal{H}) \subset \mathfrak{K}_0^+(\mathcal{H})$, hence the latter is a sufficiently rich class. The fact that these two classes do not coincide is proved by the following example.

Example 6.6. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $Q: \mathcal{H} \to \mathcal{K}$ be a bounded operator and let $T: \mathcal{K} \to \mathcal{K}$ be a bounded invertible operator. Consider the $\mathcal{H}$-kernel L defined by $L(i, j) = Q^* T^{*j} T^i Q$, $i, j \in \mathbb{Z}$. Clearly $L \in \mathfrak{K}^+(\mathcal{H})$. We prove that $L \in \mathfrak{K}_0^+(\mathcal{H})$.

Indeed, fix $f \in \mathcal{F}_0(\mathcal{H})$ and denoting $x = \sum_{n \in \mathbb{Z}} T^n Q f(n) \in \mathcal{K}$ notice that $[f, f]_L = \|x\|^2$. Then

$$[S_+f, S_+f]_L = \langle (T^*T)x, x \rangle \leq \|T\|^2 \|x\|^2 = \|T\|^2 [f, f]_L,$$

and hence the forward shift is bounded with respect to the seminorm associated to L. Also

$$[S_-f, S_-f]_L = \langle (T^*T)^{-1}x, x \rangle \leq \|T^{-1}\|^2 \|x\|^2 = \|T^{-1}\|^2 [f, f]_L,$$

and hence the backward shift is bounded, too. ∎

Lemma 6.7. *Let $L \in \mathfrak{K}^+(\mathcal{H})$. The following assertions are equivalent:*

(a) $L \in \mathfrak{K}_0^+(\mathcal{H})$.

(b) *There exists $C > 0$ such that $\frac{1}{C}[f, f]_L \leq [S_+f, S_+f]_L \leq C[f, f]_L$, $f \in \mathcal{F}_0(\mathcal{H})$.*

(c) *There exists $C > 0$ such that $\frac{1}{C}[f, f]_L \leq [S_-f, S_-f]_L \leq C[f, f]_L$, $f \in \mathcal{F}_0(\mathcal{H})$.*

Proof. These equivalences follow from the observation that $S_+S_- = S_-S_+ =$ the identity operator on $\mathcal{F}_0(\mathcal{H})$. ∎

We are now in a position to characterize those Toeplitz hermitian $\mathcal{H}$-kernels which admit Naĭmark dilations.

Theorem 6.8. *Let $H \in \mathfrak{T}^h(\mathcal{H})$. The following assertions are equivalent:*

(1) *There exists $L \in \mathfrak{K}_0^+(\mathcal{H})$ such that $-L \leq H \leq L$.*

(2) *H has a Naĭmark dilation* $\{U, Q; \mathcal{K}\}$.

Proof. (1) ⇒ (2). As in the proof of the similar implication in Theorem 3.1 we consider the Hilbert space $\mathcal{K}_L$ and let $A \in \mathcal{L}(\mathcal{K}_L)$ be the selfadjoint contraction associated to the inner product $[\cdot, \cdot]_H$ as in (3.8). Since the shift operators S_+ and S_- are bounded in the seminorm associated to L it follows that they induce bounded operators, also denoted by $S_\pm$, on $\mathcal{K}_L$. Taking into account of Lemma 6.1 it follows that $AS_+ = S_+^* A$. We now use Lemma 2.2 and get that S_+ induces an operator $U \in \mathcal{L}(\mathcal{K}_H)$. Using Remark 6.2 it follows that U is isometric and since S_+ is surjective on $\mathcal{F}_0(\mathcal{H})$ it follows that U has dense range, hence U is unitary on the Kreĭn space $\mathcal{K}_H$. Note that S_- induces the operator $U^\# = U^{-1}$ on $\mathcal{K}_H$.

Let us define the bounded operator $Q: \mathcal{H} \to \mathcal{K}_H$ by $Qh = h + \mathcal{N}_H$, $h \in \mathcal{H}$. We now prove that $(U, Q; \mathcal{K}_H)$ is a Naĭmark dilation of H.

Indeed, the axiom (a) of the Naĭmark dilation holds and the minimality axiom (b) can be verified similarly as in the proof of Theorem 3.1. As for the axiom (c), we identify any vector h with the function $h \in \mathcal{F}_0(\mathcal{H})$ defined by

$$h(n) = \begin{cases} h, & n=0, \\ 0, & n \neq 0. \end{cases}$$

Then, for arbitray $i, j \in \mathbb{Z}$ we have

$$\begin{aligned}[Q^\# U^{i-j} Qh, g]_{\mathcal{H}} &= [U^i Qh, U^j Qg]_H = [U^i (h + \mathcal{N}_H), U^j (g + \mathcal{N}_H)]_H \\ &= [S_+^i h, S_+^j g]_H = \sum_{n,k \in \mathbb{Z}} [H(n,k) S_+^i h, S_+^j g]_{\mathcal{H}} = [H(i,j) h, g]_{\mathcal{H}},\end{aligned}$$

and hence the axiom (c) is also verified.

(2) ⇒ (1). We let $L(i,j) = JQ^* U^{*j} U^i Q$ for all $i, j \in \mathbb{Z}$. As in the proof of Theorem 3.1 we verify that $L \in \mathfrak{K}^+(\mathcal{H})$ and $-L \leq H \leq L$. From Example 6.6 we obtain that the $\mathcal{H}$-kernel L is of shift bounded type. ∎

Remark 6.9. If the hermitian Toeplitz kernel H can be written as a difference $H = H_1 - H_2$ where $H_1, H_2 \in \mathfrak{K}_0^+(\mathcal{H})$ then letting $L = H_1 + H_2$ we have $L \in \mathfrak{K}_0^+(\mathcal{H})$ and $-L \leq H \leq L$, and hence H has Naĭmark dilation. Apparently the converse might be also true, but we do not have a proof. ∎

In the more general case of a hermitian Toeplitz kernel on a group G, a different characterization of the existence of a Naĭmark dilation was recently obtained by F. Pelaez [27], see also M. Cotlar and C. Sadosky [7].

Theorem 6.8 makes natural to ask whether, in order to have Naĭmark dilations of a Toeplitz hermitian $\mathcal{H}$-kernel H, we can choose a Toeplitz $\mathcal{H}$-kernel L such that $-L \leq H \leq L$, and whether we can choose Toeplitz positive semidefinite $\mathcal{H}$-kernels H_+ and H_- such that $H = H_+ - H_-$. The next result and example show that, in general, we cannot.

An operator $T \in \mathscr{L}(\mathscr{K})$ is called *fundamentally reducible* if there exists a fundamental decomposition $\mathscr{K} = \mathscr{K}^+[+]\mathscr{K}^-$ which reduces the operator T, that is, both $\mathscr{K}^+$ and $\mathscr{K}^-$ are invariant under T.

Theorem 6.10. *Let $H \in \mathfrak{T}^+(\mathscr{H})$. The following assertions are equivalent:*

(1) *There exists $L \in \mathfrak{T}^+(\mathscr{H})$ such that $-L \leq H \leq L$.*

(2) *$H = H_1 - H_2$, with $H_1, H_2 \in \mathfrak{T}^+(\mathscr{H})$.*

(2)′ *$H = H_+ - H_-$, with $H_\pm \in \mathfrak{T}^+(\mathscr{H})$ and disjoint.*

(3) *H admits a Naĭmark dilation $\{U, Q; \mathscr{K}\}$ such that U is fundamentally reducible.*

(3)′ *H admits a Naĭmark dilation, uniquely determined modulo unitary equivalence, $\{U, Q; \mathscr{K}\}$ such that U is fundamentally reducible.*

Proof. (1)⇒(2)′. Let $L \in \mathfrak{T}^+(\mathscr{H})$ be such that $-L \leq H \leq L$. By Theorem 3.1 we have

$$|[f, g]_H| \leq [f, f]_L^{1/2} [g, g]_L^{1/2}, \quad f, g \in \mathscr{F}_0(\mathscr{H}),$$

that is, considering the quotient completion Hilbert space $\mathscr{K}_L$ of the inner product space $(\mathscr{F}_0(\mathscr{H}), [\cdot, \cdot]_L)$, the inner product $[\cdot, \cdot]_H$ extends uniquely to a jointly continuous inner product on the whole $\mathscr{K}_L$. Therefore, there exists a selfadjoint operator $A \in \mathscr{L}(\mathscr{K}_L)$ such that

$$[h, g]_H = [Ah, g]_L, \quad h, g \in \mathscr{K}_L.$$

Since L is Toeplitz, both S_+ and S_- are isometric in the positive semidefinite inner product $[\cdot, \cdot]_L$ and hence they can be extended to unitary operators V and W in $\mathscr{L}(\mathscr{K}_L)$. By Lemma 6.1 it follows that $W = V^* = V^{-1}$. Using again Lemma 6.1 applied to the kernel H we obtain $VA = AV$. Therefore, considering the Jordan decomposition $A = A_+ - A_-$, notice that, by the spectral properties of this decomposition, both A_+ and A_- commute with V. Finally, we invoke the construction of the hermitian $\mathscr{H}$-kernels H_+ and H_- as in the proof of Theorem 3.1 (1)⇒(2)′ and note that from $A_\pm V = VA_\pm$ we obtain

$$[S_+ h, g]_{H_\pm} = [h, S_- g]_{H_\pm}, \quad f, g \in \mathscr{F}_0(\mathscr{H}).$$

Thus, again by Lemma 6.1, we have that $H_\pm$ are Toeplitz $\mathscr{H}$-kernels.

(2)′⇒(2). Obvious.

(2)⇒(1). Take $L = H_1 + H_2$ and note that $L \in \mathfrak{T}^+(\mathscr{H})$ and $-L \leq H \leq L$.

(2)⇒(3). Let $H = H_+ - H_-$ with $H_\pm \in \mathfrak{T}^+(\mathscr{H})$. By applying the original Naĭmark dilation theorem, we have two dilations $(U_\pm, Q_\pm; \mathscr{K}_\pm)$ of the Toeplitz positive semidefinite $\mathscr{H}$-kernels, where both $\mathscr{K}_\pm$ are Hilbert spaces. Define $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-$ and on the Hilbert space $\mathscr{K}$ consider the symmetry G,

$$G=\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \text{w.r.t. } \mathcal{K}=\mathcal{K}_+\oplus\mathcal{K}_-.$$

This turns $\mathcal{K}$ into a Kreĭn space such that G is a fundamental symmetry and $\mathcal{K}=\mathcal{K}_+[+]\mathcal{K}_-$ is the associated fundamental decomposition. With respect to this fundamental decomposition we consider the operators $U\in\mathcal{L}(\mathcal{K})$ and $Q\in\mathcal{L}(\mathcal{H},\mathcal{K})$

$$U=\begin{bmatrix} U_+ & 0 \\ 0 & U_- \end{bmatrix}, \quad Q=\begin{bmatrix} Q_+ \\ Q_- \end{bmatrix}. \tag{6.2}$$

We claim that $(U, Q; \mathcal{K})$ is a Naĭmark dilation of $\mathcal{H}$.

Indeed, let J be a fundamental symmetry on $\mathcal{H}$. Then, for arbitrary $i, j\in\mathbb{Z}$ we have

$$Q^{\#}U^{i-j}Q=JQ^*GU^{i-j}Q=JQ_+^*U_+^{i-j}Q_+-JQ_+^*U_-^{i-j}Q_-=H_+(i,j)-H_-(i,j)=H(i,j).$$

As for the minimality condition we have

$$\bigvee_{n\in\mathbf{Z}} U^nQ\mathcal{H}=\bigvee_{n\in\mathbf{Z}} U_+^nQ_+\mathcal{H}\vee\bigvee_{n\in\mathbf{Z}} U_-^nQ_-\mathcal{H}=\mathcal{K}_++\mathcal{K}_-=\mathcal{K}.$$

(3) $\Rightarrow$ (2). Since the unitary operator U is fundamentally reducible, it follows that there exists a fundamental symmetry G on $\mathcal{K}$ which commutes with U and let $\mathcal{K}=\mathcal{K}_+[+]\mathcal{K}_-$ be the associated fundamental decomposition. Then U has the diagonal matrix representation as in (6.2) and the operator Q splits as in (6.2). We consider Hilbert spaces $\mathcal{K}_\pm$, unitary operators $U_\pm$ and operators $Q_\pm$ and, for a fixed fundamental symmetry on $\mathcal{H}$, define the kernels $H_\pm$ by $H_\pm(i,j)=JQ_\pm^*U_\pm^{i-j}Q, \ i, j\in\mathbb{Z}$. Since $U_\pm$ are unitary operators on Hilbert spaces it follows that $H_\pm\in\mathfrak{T}^+(\mathcal{H})$ and we obtain that $H=H_+-H_-$.

Finally, the equivalence of (3) and (3)′ can be obtained by noticing that two Naĭmark dilations $\{U_i, Q_i; \mathcal{K}_i\}$, $i=1, 2$, of H, both of them with fundamentally reducible unitary operators U_i are unitarily equivalent. This statement can be proved easily taking into account that, in this case, the corresponding Naĭmark dilations behave like the direct sum of two Naĭmark dilations on Hilbert space, for which the uniqueness hold. □

We present now an example of a Toeplitz hermitian kernel which admits Naĭmark dilations but none of the assertions of Theorem 6.10 holds.

Example 6.11. *Let* $T\in\mathcal{L}(\mathcal{H})$ be an operator such that for some, equivalently, for all, unitary norm $\|\cdot\|$ on $\mathcal{H}$, $\sup_{n\geq 0}\|T^n\|=\infty$. We consider the Toeplitz hermitian $\mathcal{H}$-kernel as in Example 6.5, in particular it has Naĭmark dilations and let $(U; \mathcal{K})$ (see Remark 6.4) be an arbitrary one. For any unitary norm $\|\cdot\|$ on $\mathcal{K}$ we have

$$\sup_{n\in\boldsymbol{Z}}\|U^n\|\geq\sup_{n\in\boldsymbol{Z}}\|P_{\mathscr{H}}U^n|\mathscr{H}\|=\sup_{n\in\boldsymbol{Z}}\|T^n\|=\infty. \tag{6.3}$$

Since any unitary operator which is fundamentally reducible is similar with a unitary operator in a Hilbert space, in particular it is uniformly bounded, from (6.3) it follows that U is not fundamentally reducible. ∎

We end this section with a result concerning the uniqueness of Naĭmark dilations. Two Naĭmark dilations $(U, Q; \mathcal{K})$ and $(U', Q'; \mathcal{K}')$ of the same Toeplitz $\mathscr{H}$-kernel H are called *unitarily equivalent* if there exists a unitary operator $\Phi\in\mathscr{L}(\mathcal{K}, \mathcal{K}')$ such that $\Phi Q=Q'$ and $\Phi U=U'\Phi$.

Theorem 6.12. *Let $H\in\mathfrak{T}^h(\mathscr{H})$. The following assertions are equivalent:*

(1) *The kernel H has unique Naĭmark dilations, modulo unitary equivalence.*

(2) *For all $L\in\mathfrak{K}_0^+(\mathscr{H})$ such that $-L\leq H\leq L$, considering the Gram operator $H_L\in\mathscr{L}(\mathcal{K}_L)$, there exists $\varepsilon>0$ such that either $(-\varepsilon, 0)\subset\rho(H_L)$ or $(0, \varepsilon)\subset\rho(H_L)$.*

Proof. Indeed, (1)⇒(2) follows from Theorem 4.1. To prove the converse implication we follow the lines of the proof of the implication (ii) ⇒ (i) in Theorem 4.1 and note that, considering two Naĭmark dilations $(U, Q; \mathcal{K})$ and $(U', Q'; \mathcal{K}')$ of H, we associate to them two positive semidefinite $\mathscr{H}$-kernels L_U and $L_{U'}$ of bounded shift type and then $L=L_U+L_{U'}$ is also of bounded shift type. ∎

§7. Pre-Kreĭn Spaces

Unlike the case of positive definite inner product space, for which a canonical way of constructing Hilbert spaces exists, in the indefinite case it might happen that no Kreĭn space can be associated in a natural way, or, if it can be, then it might happen that it is not unique, modulo unitary equivalence. Based on the results obtained until now, in this section we propose an abstract unifying framework for both these cases. Since most of the proofs can be easily completed from what was proved until now, we will omit the details.

Let $(\mathscr{X}, [\cdot, \cdot])$ be an indefinite inner product space. If $\mathscr{X}^0=\{x\in\mathscr{X}\mid[x, y]=0, y\in\mathscr{X}\}$ denotes the isotropic part of $\mathscr{X}$, then there is a naturally defined inner product $[\cdot, \cdot]$ onto the quotient space $\mathscr{X}/\mathscr{X}^0$ which becomes nondegenerate. Thus, without restricting the generality we always can assume that the inner product space $(\mathscr{X}, [\cdot, \cdot])$ is nondegenerate.

Given a nondegenerate inner product space $(\mathscr{X}, [\cdot, \cdot])$, a pair $(\mathcal{K}, \Pi)$ is called a *Kreĭn space induced* by this inner product space, if $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}})$ is a Kreĭn space and $\Pi\colon\mathscr{X}\to\mathcal{K}$ is an injective linear mapping with dense range, such that $[\Pi x, \Pi y]_{\mathcal{K}}=[x, y]$ for all $x, y\in\mathscr{X}$. The existence of Kreĭn spaces induced by abstract inner product spaces is known to be related to the existence of "Hilbert majorant topologies", cf. [3] and the bibliography cited there. The following theorem is simply an abstract transcription of Theorem 3.1.

Theorem 7.1. *If* $(\mathcal{X}, [\cdot, \cdot])$ *denotes a nondegenerate inner product space, then the following assertions are equivalent*:

(1) *There exists a positive definite inner product* $\langle\cdot, \cdot\rangle$ *on* $\mathcal{X}$ *such that* $-\langle x, x\rangle \leq [x, x] \leq \langle x, x\rangle$ *for all* $x \in \mathcal{X}$.

(1)′ *There exists a positive definite inner product* $\langle\cdot, \cdot\rangle$ *on* $\mathcal{X}$ *such that for all* $x, y \in \mathcal{X}$ *it holds* $|[x, y]| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$.

(2) *There exist two nonnegative inner products* $\langle\cdot, \cdot\rangle_{\pm}$ *on* $\mathcal{X}$ *such that* $[x, y] = \langle x, y\rangle_{+} - \langle x, y\rangle_{-}$ *for all* $x, y \in \mathcal{X}$.

(2)′ *There exist two nonnegative inner products* $\langle\cdot, \cdot\rangle_{\pm}$ *on* $\mathcal{X}$ *such that* $[x, y] = \langle x, y\rangle_{+} - \langle x, y\rangle_{-}$ *for all* $x, y \in \mathcal{X}$, *and, in addition, if* $\langle\cdot, \cdot\rangle$ *is a nonnegative inner product on* $\mathcal{X}$ *such that* $\langle x, x\rangle \leq \langle x, x\rangle_{\pm}$ *for all* $x \in \mathcal{X}$, *then* $\langle x, x\rangle = 0$ *for all* $x \in \mathcal{X}$.

(3) *There exists a Kreĭn space induced by* $(\mathcal{X}, [\cdot, \cdot])$.

Two Kreĭn spaces $(\mathcal{K}_i, \Pi_i)$, $i = 1, 2$, induced by the same nondegenerate inner product space $(\mathcal{X}, [\cdot, \cdot])$, are *unitarily equivalent* if there exists a unitary operator $\Phi \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that $\Phi\Pi_1 = \Pi_2$. The following theorem is an abstract transcription of Theorem 4.1. For a positive definite inner product $\langle\cdot, \cdot\rangle$ on $\mathcal{X}$ such that $|[x, y]| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$ for all $x, y \in \mathcal{X}$, we denote by $\mathcal{H}$ the unique Hilbert space induced by $(\mathcal{X}, \langle\cdot, \cdot\rangle)$. It follows from the Riesz representation theorem that there is a uniquely determined operator $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$ such that $[x, y] = \langle Ax, y\rangle$, $x, y \in \mathcal{X}$. We call A the *Gram operator* associated to the positive definite inner product $\langle\cdot, \cdot\rangle$ on $\mathcal{X}$.

Theorem 7.2. *Let* $(\mathcal{X}, [\cdot, \cdot])$ *be a nondegenerate inner product space admitting an induced Kreĭn space. The following assertions are equivalent*:

(1) *The Kreĭn space induced by* $(\mathcal{X}, [\cdot, \cdot])$ *is unique, modulo unitary equivalence.*

(2) *For every positive definite inner product* $\langle\cdot, \cdot\rangle$ *on* $\mathcal{X}$ *with the property that* $|[x, y]| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}$ *for all* $x, y \in \mathcal{X}$, *and with Gram operator* A, *there exists* $\varepsilon > 0$ *such that either* $(-\varepsilon, 0) \subset \rho(A)$ *or* $(0, \varepsilon) \subset \rho(A)$.

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