

On the Thomae Formula for \mathbf{Z}_N Curves

Dedicated to Professor Masaki KASHIWARA on his 50th birthday

By

Atsushi NAKAYASHIKI*

Abstract

We shall give an elementary and rigorous proof of the Thomae formula for \mathbf{Z}_N curves which was discovered by Bershadsky and Radul [1, 2]. Instead of using the determinant of the Laplacian we use the traditional variational method which goes back to Riemann, Thomae, Fuchs. In the proof we made explicit the algebraic expression of the chiral Szegő kernels and prove the vanishing of zero values of derivatives of theta functions with \mathbf{Z}_N invariant $1/2N$ characteristics.

§0. Introduction

In [1, 2] Bershadsky and Radul discovered a generalization of Thomae formula for \mathbf{Z}_N curve $s^N = f(z) = \prod_{i=1}^m (z - \lambda_i)$ (Theorem 3 in Section 7 below). The original Thomae formula is the case of hyperelliptic curves $N=2$ and takes the form

$$\theta[e](0)^8 = \left(\frac{\det A}{(2\pi i)^{m-1}} \right)^4 \prod_{k < l} (\lambda_{ik} - \lambda_{il})^2 (\lambda_{jk} - \lambda_{jl})^2,$$

where e is a non-singular even half period corresponding to the partition of the branch points $\{1, \dots, 2m\} = \{i_1 < \dots < i_m\} \sqcup \{j_1 < \dots < j_m\}$, $\{A_i, B_j\}$ a canonical homology basis and $A = (\int_{A_i} z^{j-1} dz/s)_{1 \leq i, j \leq m-1}$. This formula expresses the zero values of the Riemann theta functions with half characteristics as functions of branch points. Thomae formula was used to give generators of the affine ring of the moduli space of hyperelliptic curves with level two structure [12] in terms of theta constants, to give a generalization of the λ function of an elliptic curve [12] (Umemura's appendix). Beside those, F. Smirnov [14] derived a beautiful

Communicated by M. Kashiwara, January 22, 1997.

1991 Mathematics Subject Classification (s): 14H42, 14K25, 81T40

* Graduate School of Mathematics, Kyushu University, 4-2-1, Ropponmatsu, Fukuoka 810-8560, Japan.

theta formula for the solution of the sl_2 Knizhnik-Zamolodchikov equation on level zero using the Thomae formula. For the generalized Thomae formula of \mathbb{Z}_N curves similar results are expected. As for the generalization of λ function for \mathbb{Z}_N curves there are several results [8, 4] based on a different approach. To study the generalization of Smirnov's formula to the case of sl_N is a main motivation for the present work. It will be studied in a forthcoming paper.

Let us comment on the proof of the generalized Thomae formula. Bershadsky and Radul evaluated, in two ways, the determinant of the Laplacian acting on some line bundle on a \mathbb{Z}_N curve and compared them to obtain the formula. However they used a path integral description of the correlation function of conformal fields to identify it with the determinant of the Laplacian. Hence their proof does not seem mathematically rigorous. It may be possible to make their proof rigorous using the theory of determinants and Green functions only, that is, without the path integral.

Instead of going in that manner, here we shall give a rigorous and elementary proof of the formula. Our proof is based on the traditional variational method which goes back to Riemann [13], Thomae [15, 16], L. Fuchs [6, 7]. The role of determinants and path integral is then replaced by Fay's formula [5] relating the Szegő kernel and the canonical symmetric differential. The strategy of the proof itself is similar to that of [1, 2].

The particularity of our proof is to compare the analytic and the algebraic expressions not only in the final formula but also in each step of the proof. As a corollary of those comparison the vanishing of the zero value of the first order derivatives of theta functions with non-singular $1/2N$ characteristics obtains. This result is in turn used to prove the generalized Thomae formula. Hence our proof clarifies some special aspects of theta functions behind the generalized Thomae formula. We also reveal a property of the proportionality constants appeared in the Thomae formula for \mathbb{Z}_N curves which was not treated in [1, 2].

Now the present paper is organized in the following manner. In Section 1 we gather necessary notation and formulas concerning Riemann surfaces and theta functions following the Fay's book [5]. The \mathbb{Z}_N invariant $1/2N$ periods are studied in Section 2. The algebraic expression for the chiral Szegő kernel is given in Section 3. Section 4 is devoted to the explanation of the canonical differential and Fay's formula relating it with the chiral Szegő kernel. In Section 5 the algebraic expression of the canonical differential is studied. The variation of the period matrix is studied in Section 6. In Section 7 the generalized Thomae formula up to moduli independent constants is proved. The property of the proportionality constants is studied in Section 8. In Section 9 the examples of Thomae formula for small N 's are given.

§1. Theta Function

In this paper we mainly follow the notations of the Fay's book [5] which we summarize here. Let τ be the g by g symmetric matrix whose real part is negative definite. Any element $e \in \mathbb{C}^g$ is uniquely expressed as

$$(1) \quad e = \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix}_\tau = 2\pi i \epsilon + \delta \tau,$$

with $\epsilon, \delta \in \mathbb{R}^g$. Here the vectors ϵ, δ etc. are all row vectors. We call ϵ, δ the characteristics of e . The theta function with characteristics is defined by

$$\begin{aligned} \theta \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix} (z) &= \sum_{m \in \mathbb{Z}^g} \exp \left(\frac{1}{2} (m + \delta) \tau (m + \delta)^t + (z + 2\pi i \epsilon) (m + \delta)^t \right) \\ &= \exp \left(\frac{1}{2} \delta \tau \delta^t + (z + 2\pi i \epsilon) \delta^t \right) \theta(z + e), \end{aligned}$$

where

$$\theta \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} (z) = \theta(z),$$

and e is determined by (1). We sometimes use $\theta[e](z)$ instead of $\theta \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix} (z)$.

The transformation property is

$$\begin{aligned} \theta \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix} (z + 2\pi i \lambda + \kappa \tau) \\ = \exp \left(-\frac{1}{2} \kappa \tau \kappa^t - z \kappa^t + 2\pi i (\delta \lambda^t - \epsilon \kappa^t) \right) \theta \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix} (z), \end{aligned}$$

for $\lambda, \kappa \in \mathbb{Z}^g$. We shall list some of the properties which easily follow from the definitions:

$$(2) \quad \begin{aligned} \theta \begin{Bmatrix} \delta + m \\ \epsilon + n \end{Bmatrix} (z) &= \exp(2\pi i n \delta^t) \theta \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix} (z), \\ \theta \begin{Bmatrix} -\delta \\ -\epsilon \end{Bmatrix} (0) &= \theta \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix} (0), \end{aligned}$$

$$\frac{\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (z)}{\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (0)} = \frac{\theta \begin{bmatrix} \delta+m \\ \epsilon+n \end{bmatrix} (z)}{\theta \begin{bmatrix} \delta+m \\ \epsilon+n \end{bmatrix} (0)},$$

for $m, n \in \mathbb{Z}^g$.

Let C be a compact Riemann surface of genus g . Let us fix a marking of C [9] (§1). That means, we fix a canonical basis $\{A_i, B_j\}$ of $\pi_1(C)$, a base point $P_0 \in C$ and a base point in the universal cover \tilde{C} which lies over P_0 . We assume that the tails of A_i, B_j are joined to P_0 . Then we can canonically identify the covering transformation group and the fundamental group $\pi_1(C, P_0)$. We also identify holomorphic 1-forms on C with holomorphic 1-forms on \tilde{C} invariant under the action of $\pi_1(C)$. Let us denote by $\pi : \tilde{C} \rightarrow C$ the projection and by $J(C)$ the Jacobian variety of C which is the set of linear equivalence classes of degree 0 divisors on C . In the following sections we always assume one marking of C .

Let $\{v_j\}$ be the basis of the normalized holomorphic 1-forms. The normalization is

$$\int_{A_j} v_k = 2\pi i \delta_{jk},$$

and set

$$\int_{B_j} v_k = \tau_{jk}.$$

A flat line bundle on C is described by the character of the fundamental group $\chi : \pi_1(C) \rightarrow \mathbf{C}^*$, where \mathbf{C}^* is the multiplicative group of non-zero complex numbers. The two representations χ_1 and χ_2 defines a holomorphically equivalent line bundle if and only if there exists an holomorphic 1-form ω such that

$$\chi_1(\gamma) \chi_2(\gamma)^{-1} = \exp\left(\int_{\gamma} \omega\right)$$

for any $\gamma \in \pi_1(C)$. Let \mathcal{A} and \mathcal{B} be positive divisors of the same degree, say d , and set $\mathcal{A} = \sum_{i=1}^d P_i, \mathcal{B} = \sum_{i=1}^d R_i$. Let us fix points \tilde{P}_i, \tilde{R}_j in \tilde{C} so that they lie over P_i, R_j . Let us set

$$\int_{\mathcal{A}} v_i = \sum_{j=1}^d \int_{\tilde{P}_j}^{\tilde{R}_i} v_i,$$

where the integration in the right hand side is taken in \tilde{C} . Then the flat line bundle corresponding to the degree 0 divisor $\mathcal{B} - \mathcal{A}$ is described by

$$\chi(A_i) = 1, \quad \chi(B_i) = \exp\left(\int_{\mathcal{A}}^{\mathcal{B}} v_i\right).$$

Another choice of \tilde{P}_i, \tilde{R}_j gives an equivalent line bundle.

We say χ is unitary if the image is contained in the unitary group $U(1)$. The following proposition is well known and easily proved.

Proposition 1. *For an isomorphism class of flat line bundles there exists a unique unitary representation χ which defines the line bundle belonging to that class.*

If we take $\delta, \epsilon \in \mathbf{R}^g$ such that

$$\left(\int_{\mathcal{A}}^{\mathcal{B}} v_1, \dots, \int_{\mathcal{A}}^{\mathcal{B}} v_g\right) = \left\{ \begin{matrix} \delta \\ \epsilon \end{matrix} \right\}_{\tau}$$

as a point on the Jacobian variety of C , then the corresponding unitary representation $\tilde{\chi}$ is given by

$$(3) \quad \tilde{\chi}(A_j) = \exp(-2\pi i \delta_j), \quad \tilde{\chi}(B_j) = \exp(2\pi i \epsilon_j).$$

The multiplicative meromorphic function described by $\tilde{\chi}$ is, for example, given by

$$\frac{\theta \left[\begin{matrix} -\delta \\ -\epsilon \end{matrix} \right] \left(\int_{z_0}^x v - \alpha \right)}{\theta \left(\int_{z_0}^x v - \alpha \right)},$$

where v is the vector of normalized holomorphic 1-forms, $x \in \tilde{C}$, $\alpha \in \mathbf{C}^g$ and the integration path is taken in \tilde{C} .

We denote by Δ the Riemann divisor for our choice of the canonical homology basis which satisfies

$$2\Delta \equiv K_C.$$

Here K_C is the divisor class of the canonical bundle of C and \equiv means the linear equivalence. Let L_0 be the degree $g-1$ line bundle corresponding to Δ . For a divisor α with degree 0 let us denote by \mathcal{L}_α the corresponding flat line bundle and set $L_\alpha = \mathcal{L}_\alpha \otimes L_0$. For a non-singular odd half period α let h_α be the section of L_α which satisfies

$$h_\alpha^2(x) = \sum_{j=1}^g \frac{\partial \theta[\alpha]}{\partial z_j} (0) v_j(x).$$

Then the prime form is defined by

$$E(x, y) = \frac{\theta[\alpha](y-x)}{h_\alpha(x)h_\alpha(y)},$$

$$y-x = \int_x^y v,$$

where $x, y \in \tilde{C}$ and $v = (v_1(x), \dots, v_g(x))$. Let π_j be the projection from $C \times C$ to the j -th component and $\delta: C \times C \rightarrow J(C)$ the map $(x, y) \mapsto y-x$. Then $E(x, y)$ can be considered as a section of the line bundle $\pi_1^* L_0^{-1} \otimes \pi_2^* L_0^{-1} \otimes \delta^* \Theta$, where Θ is the line bundle on $J(C)$ defined by the theta divisor. Let us fix the transformation property of the half differential on \tilde{C} under the action of $\pi_1(C)$ so that the section of $\pi^* L_0$ is invariant. This means, in particular, that $E(x, y)$ transforms under the action of A_i, B_i in y as

$$E(x, y+A_i) = E(x, y), \quad E(x, y+B_i) = \exp\left(-\frac{\tau_{ii}}{2} - \int_x^y v_i\right) E(x, y).$$

Here we denote the action of A_i, B_i in an additive manner. The prime form has the nice expansion as follows. Let u be a local coordinate around $P \in \tilde{C}$. Then the expansion of $E(x, y)$ in $u(y)$ at $u(x)$ takes the form

$$(4) \quad E(x, y) \sqrt{du(x)} \sqrt{du(x)} = u(y) - u(x) + O((u(y) - u(x))^3).$$

Since the expansion is of local nature we sometimes use the way of saying that $P \in C$, the local coordinate u around P and the expansion in $u(y)$ at $u(x)$ etc.

§2. Z_N Curve and $\frac{1}{2N}$ Period

Let us consider the plane algebraic curve $s^N = f(z) = \prod_{j=1}^{Nm} (z - \lambda_j)$. We compactify it by adding N infinity point $\infty^{(1)}, \dots, \infty^{(N)}$ and denote the compact Riemann surface by C . The genus g of C is $g = 1/2 (N - 1) (Nm - 2)$. The N -cyclic automorphism ϕ of C is defined by $\phi : (z, s) \mapsto (z, \omega s)$, where ω is the N -th primitive root of unity. There are Nm branch points Q_1, \dots, Q_{Nm} whose projection to z coordinate are $\lambda_1, \dots, \lambda_{Nm}$.

The basis of holomorphic 1-forms on C is given by

$$w_\beta^{(\alpha)} = \frac{z^{\beta-1} dz}{s^\alpha} \quad 1 \leq \alpha \leq N-1, \quad 1 \leq \beta \leq \alpha m - 1.$$

Let us describe the divisors which we need and their relations. The following lemma is easily proved.

Lemma 1. *For any $P \in C$ the linear equivalence class $P + \phi(P) + \dots + \phi^{N-1}(P)$ does not depend on the point P .*

We set

$$D \equiv P + \phi(P) + \dots + \phi^{N-1}(P).$$

The following lemma is easily proved.

Lemma 2. *The following relations hold.*

1. $D \equiv NQ_i \equiv \infty^{(1)} + \dots + \infty^{(N)}$ for any i .
2. $K_C \equiv (L-1)D$, where $L = (N-1)m - 1$.
3. $\sum_{j=1}^{Nm} Q_j \equiv mD$.

Following Bershadsky-Radul [2] we shall describe the important object of our study, the \mathbb{Z}_N invariant $1/N$ or $1/2N$ periods. Let us consider an ordered partition $\Lambda = (\Lambda_0, \dots, \Lambda_{N-1})$ of $\{1, 2, \dots, Nm\}$ such that the number $|\Lambda_i|$ of elements of Λ_i is equal to m for any i . With each Λ we associate the divisor class e_Λ by

$$e_\Lambda \equiv \Lambda_1 + 2\Lambda_2 + \dots + (N-1)\Lambda_{N-1} - D - \Delta,$$

where for a subset S of $\{1, 2, \dots, Nm\}$ we set

$$S = \sum_{j \in S} Q_j.$$

For a given Λ we denote by $\Lambda(j)$ the ordered partition

$$\Lambda(j) = (\Lambda_j, \Lambda_{j+1}, \dots, \Lambda_{j+N-1}),$$

where we consider the index of Λ_j by modulo N . Then

Proposition 2. *For any ordered partition Λ we have*

1. $Ne_\Lambda \equiv 0$ for N being even and $2Ne_\Lambda \equiv 0$ for N being odd.
2. $e_\Lambda \equiv e_{\Lambda(2)} \equiv \dots \equiv e_{\Lambda(N)}$.
3. $-e_\Lambda \equiv \Lambda_{N-1} + 2\Lambda_{N-2} + \dots + (N-1)\Lambda_1 - D - \Delta$.

This proposition is easily proved using Lemma 2. For $\Lambda = (\Lambda_0, \dots, \Lambda_{N-1})$ we set

$$\Lambda^- = (\Lambda_0^-, \dots, \Lambda_{N-1}^-) = (\Lambda_0, \Lambda_{N-1}, \dots, \Lambda_1),$$

and $\Lambda = \Lambda^+$. Let $\theta(z)$ be the theta function associated with our choice of canonical homology basis. Then

Proposition 3. *The $1/2N$ period e_Λ is non-singular, that means*

$$\theta(e_\Lambda) \neq 0.$$

This proposition was proved in [2]. One can find another proof in [8]

which is similar to that of [12] in the hyperelliptic case.

§3. Chiral Szegő Kernel

Definition 1. For $e \in \mathbb{C}^g$ satisfying $\theta(e) \neq 0$ the chiral Szegő kernel $R(x, y|e)$ is defined by

$$R(x, y|e) = \frac{\theta[e](y-x)}{\theta[e](0)E(x, y)} \quad x, y \in \tilde{C}.$$

We remark that $R(*, *|e)$ depends only on the image of e to the Jacobian variety $J(C)$. Let π_i be the projection to the i -th component of $C \times C$. Then $R(x, y|e)$ can be considered as a meromorphic section of the line bundle $\pi_1^*L_e \otimes \pi_2^*L_{-e}$ and is holomorphic outside the diagonal set $\{(x, x) | x \in C\}$.

We shall give an algebraic expression for $R(x, y|e_\Lambda)$. Let us set

$$(5) \quad \mathcal{L} = \left\{ -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \dots, \frac{N-1}{2} \right\},$$

$$(6) \quad q_l(i) = \frac{1-N}{2N} + \left\{ \frac{l+i+\frac{N-1}{2}}{N} \right\},$$

for $l \in \mathcal{L}$ and $i \in \mathbb{Z}$. Here $\{a\} = a - [a]$ is the fractional part of a , $[a]$ being the Gauss symbol. For an ordered partition $\Lambda = (\Lambda_0, \dots, \Lambda_{N-1})$ we define the number $k_i, i = 1, \dots, Nm$ by

$$(7) \quad i \in \Lambda_j \quad \text{if and only if } k_i = j.$$

For each $l \in \mathcal{L}$ we set

$$f_l(x, \Lambda) = \prod_{i=1}^{Nm} (z(x) - \lambda_i)^{q_l(k_i)} \sqrt{dz(x)}.$$

The following proposition was found in [2].

Proposition 4. $f_l(x, \Lambda)$ is a meromorphic section of L_{e_Λ} whose divisor is

$$(8) \quad \text{div } f_l = \Lambda_{1-j} + 2\Lambda_{2-j} + \dots + (N-1)\Lambda_{1-j} - \sum_{k=1}^N \infty^{(k)},$$

where $l = -(N-1)/2 + j$.

Let us set

$$F(x, y|\Lambda) = \frac{1}{N} \frac{\sum_{l \in \mathcal{L}} f_l(x, \Lambda) f_{-l}(y, \Lambda^-)}{z(y) - z(x)}.$$

Here the choice of the branch of $f_l(x, \Lambda)$ should be specified as in (12) and (13). Note that both $F(x, y|\Lambda)$ and $R(x, y|e_\Lambda)$ can be considered as meromorphic

sections of $\pi_1^* L_{e_A} \otimes \pi_2^* L_{-e_A}$. Then we have

Theorem 1. For an ordered partition Λ we have

$$R(x, y|e_A) = F(x, y|\Lambda).$$

As a corollary of this theorem we have the vanishing of the theta derivative constants.

Corollary 1. For any ordered partition Λ and any i we have

$$(9) \quad \frac{\partial \theta[e_A]}{\partial z_i}(0) = 0.$$

Note that whether the left hand side of (9) vanishes or not depends only on the divisor class of e_A by (2). Hence the statement has unambiguously a sense. This curious result is a natural generalization of the hyperelliptic case where e_A is a non-singular even half period and the corollary is obvious. A theta function $\theta[e_A](z)$ is even if and only if e_A is an even half period. For $N > 2$ it is not even in general.

Lemma 3. The following properties hold.

1. $q_l(i) = q_{l'}(i')$ if $i+l = i'+l'$.
2. $q_l(i+N) = q_l(i)$ for any i .
3. $\sum_{i=0}^{N-1} q_l(i) = 0$.

Proof. The properties 1 and 2 are obvious. Let us prove 3. Using 1 and 2 we have

$$\sum_{i=0}^{N-1} q_l(i) = \sum_{i=0}^{N-1} q_{-N/2}(i) = \sum_{i=0}^{N-1} \left(\frac{N-1}{2N} + \frac{i}{2} \right) = 0.$$

□

Proof of Proposition 4. Let us consider $f_l(x, \Lambda)^2$. It is considered as a multi-valued 1-form on C . We show that it can be considered as a (usual) meromorphic section of $L_{e_A}^{\otimes 2} \simeq \mathcal{L}_{2e_A} \otimes K_C$. The meromorphy at points except the branch points and $\infty^{(k)}$ is obvious. We can take $t = (z - \lambda_i)^{1/N}$ as a local coordinate around Q_i . Then

$$(z - \lambda_i)^{2q_l(k_i)} dz = t^{N-1+2Nq_l(k_i)} N dt.$$

If we write $l = -(N-1/2) + j$ ($0 \leq j \leq N-1$), we have

$$N-1+2Nq_l(k_i) = 2N \left\{ \frac{k_i+j}{N} \right\}.$$

At $\infty^{(k)}$ we can take $t = 1/z$ as a local coordinate and we have

$$f_l(x, \Lambda)^2 = \frac{-1}{t^2} dt (1 + O(t))$$

by the property 3 of Lemma 3. Hence, for each $p \in C$ and a sufficiently small neighborhood U_p of p , f_l^2 defines a meromorphic function on U_p with the divisor

$$(10) \quad 2 \left(\sum_{k=1}^{N-1} k \Lambda_{k-j} - \sum_{k=1}^N \infty^{(k)} \right).$$

Since the divisor (10) is linearly equivalent to $2(e_{\Lambda(1-j)} + \Delta) \equiv 2(e_{\Lambda} + \Delta)$ by 2 of Proposition 2, $f_l(x, \Lambda)^2$ can be considered as an element of

$$H^0 \left(C, L_{e_{\Lambda}}^{\otimes 2} \left(-2 \sum_{k=1}^{N-1} k \Lambda_{k-j} + 2 \sum_{k=1}^N \infty^{(k)} \right) \right).$$

Note that

$$(11) \quad H^0 \left(C, L_{e_{\Lambda}} \left(- \sum_{k=1}^{N-1} k \Lambda_{k-j} + \sum_{k=1}^N \infty^{(k)} \right) \right)$$

is one dimensional. Hence $f_l(x, \Lambda)$ can be considered as an element of (11). Thus Proposition 4 is proved. \square

Lemma 4. *The following expression holds:*

$$f_{-l}(y, \Lambda^-) = \prod_{i=1}^{Nm} (z(y) - \lambda_i)^{-q_l(k_i)} \sqrt{dz(y)}.$$

Proof. Recall that

$$\Lambda^- = (\Lambda_0^-, \dots, \Lambda_{N-1}^-), \quad \Lambda_j^- = \Lambda_{N-j}.$$

Then we have

$$f_{-l}(y, \Lambda^-) = \prod_{i=1}^{Nm} (z(y) - \lambda_i)^{q_{-l}(N-k_i)} \sqrt{dz(y)}.$$

Hence it is sufficient to prove

$$q_{-l}(N-i) = -q_l(i),$$

for any l and i . This can be easily checked. \square

Lemma 5. *$F(x, y|\Lambda)$ is regular outside the diagonal set $\{x=y\}$.*

Proof. A priori we know that $F(x, y|\Lambda)$ has poles at most at $z(x) = z(y)$. Hence it is sufficient to prove that $F(x, y|\Lambda)$ is regular at $z(x) = z(y)$ and $x \neq y$. If we write $l = -(N-1)/2 + j$ we have

$$q_l(k_i) - q_{-\frac{N-1}{2}}(k_i) = q_{-\frac{N-1}{2}}(j+k_i) - q_{-\frac{N-1}{2}}(k_i) = \frac{j}{N} \pmod{\mathbf{Z}}.$$

Therefore we can set

$$q_l(k_i) - q_{-\frac{N-1}{2}}(k_i) = \frac{j}{N} + r_{ij}, \quad r_{ij} \in \mathbf{Z}.$$

Let us choose the branch of $f_i(x, \Lambda), f_i(x, \Lambda^-)$ such that the following equations hold:

$$(12) \quad f_i(x, \Lambda) = (z(x) - \lambda_i)^{r_{ij}} s(x)^j f_{-\frac{N-1}{2}}(x, \Lambda),$$

$$(13) \quad f_{-l}(x, \Lambda^-) = (z(x) - \lambda_i)^{-r_{ij}} s(x)^{-j} f_{-\frac{N-1}{2}}(x, \Lambda^-).$$

Then we have

$$\begin{aligned} &F(x, y|\Lambda) \\ &= f_{-\frac{N-1}{2}}(x, \Lambda) f_{-\frac{N-1}{2}}(y, \Lambda^-) \sum_{j=0}^{N-1} \prod_{i=1}^{Nm} \left(\frac{z(x) - \lambda_i}{z(y) - \lambda_i} \right)^{r_{ij}} \left(\frac{s(x)}{s(y)} \right)^j. \end{aligned}$$

Now in the limit

$$z(x) \longrightarrow z(y), \quad s(x) \longrightarrow \omega^r s(y), \quad 1 \leq r \leq N-1,$$

with $\omega = \exp 2\pi i/N$ we have

$$F(x, y|\Lambda) \longrightarrow f_{-\frac{N-1}{2}}(y^{(r)}, \Lambda) f_{-\frac{N-1}{2}}(y, \Lambda^-) \sum_{j=0}^{N-1} \omega^r = 0,$$

where $y^{(r)} = (z(y), \omega^r s(y))$. \square

The following lemma is proved by a direct calculation.

Lemma 6. *Let $P \in C$ be a non-branch point. We can take z to be a local coordinate around p . Then the expansion in $z(y)$ at $z(x)$ takes the form*

$$\begin{aligned} &F(x, y|\Lambda^\pm) \\ &= \frac{\sqrt{dz(x)} \sqrt{dz(y)}}{z(y) - z(x)} \left[1 + \frac{1}{2N} \sum_{i,j=1}^{Nm} \frac{q(k_i, k_j)}{(z(x) - \lambda_i)(z(x) - \lambda_j)} (z(y) - z(x))^2 + \dots \right], \end{aligned}$$

where $q(i, j) = \sum_{l \in \mathcal{I}} q_l(i) q_l(j)$.

The following lemma is a consequence of the expansion (4) of the prime form $E(x, y)$.

Lemma 7. *Under the same conditions of Lemma 6 we have*

$$\begin{aligned} &R(x, y|e_\Lambda) \\ &= \frac{\sqrt{dz(x)} \sqrt{dz(y)}}{z(y) - z(x)} \left[1 + \sum_{i=1}^g \frac{\partial \log \theta[e_\Lambda]}{\partial z_i}(0) v_i(x) (z(y) - z(x)) + \dots \right], \end{aligned}$$

where $v_i(x)$ means the coefficient of $dz(x)$ in $v_i(x)$.

Proof of Theorem 1. Let χ be the unitary representation corresponding to \mathcal{L}_{e_A} . If we write

$$e_A = \begin{Bmatrix} \delta \\ \epsilon \end{Bmatrix}_\tau,$$

then $\chi(A_i)$ and $\chi(B_i)$ are given by (3) in section 1. The transformation property of $R(x, y|e_A)$ is

$$R(x + \gamma_1, y + \gamma_2|e_A) = \chi(\gamma_1) \chi(\gamma_2)^{-1} R(x, y|e_A),$$

for $\gamma_1, \gamma_2 \in \pi_1(C)$. On the other hand if we pull $F(x, y|A)$ back to $\tilde{C} \times \tilde{C}$ then

$$F(x + \gamma_1, y + \gamma_2|A) = \chi_1(\gamma_1) \chi_2(\gamma_2) F(x, y|A),$$

for some unitary representation χ_1 and χ_2 . In fact if x rounds a cycle of C $f_i(x, A)$ is multiplied by an appropriate $2N$ the root of unity. The same is true for y .

Let us set

$$\tilde{F}(x, y|A) = \frac{R(x, y|e_A)}{F(x, y|A)}.$$

Then $\tilde{F}(x, y|A)$ is the section of the trivial line bundle and obeys the tensor product of unitary representations of $\pi_1(C) \times \pi_1(C)$. Hence $\tilde{F}(x, y|A)$ is invariant under the action of $\pi_1(C) \times \pi_1(C)$. This means that $R(x, y|e_A)$ and $F(x, y|A)$ have the same transformation property. Therefore the function

$$I(x, y) = F(x, y|A) - R(x, y|e_A)$$

can be considered as a section of the line bundle $\pi_1^* L_{e_A} \otimes \pi_2^* L_{-e_A}$. By Lemma 6 and 7 we know that $I(x, y)$ is holomorphic at least except $\bigcup_{i=1}^{2N} \{Q_i\} \times \{Q_i\}$. Since $I(x, y)$ is meromorphic on $C \times C$, $I(x, y)$ has no singularity. By Proposition 3, e_A is non-singular which means

$$H^0(C, L_{e_A}) = 0.$$

Hence

$$H^0(C \times C, \pi_1^* L_{e_A} \otimes \pi_2^* L_{-e_A}) = \pi_1^* H^0(C, L_{e_A}) \otimes \pi_2^* H^0(C, L_{-e_A}) = 0.$$

Thus $I(x, y) = 0$. \square

Proof of Corollary 1. This is a direct consequence of Theorem 1, Lemma 6 and 7. \square

Corollary 2. *Under the same conditions and notations as in Lemma 6, then we have*

$$R(x, y|e_\Lambda)R(x, y|-e_\Lambda) = \frac{dz(x)dz(y)}{(z(y) - z(x))^2} \left[1 + \frac{1}{N} \sum_{i,j=1}^{Nm} \frac{q(k_i, k_j)}{(z(x) - \lambda_i)(z(x) - \lambda_j)} (z(y) - z(x))^2 + \dots \right],$$

§4. Canonical Symmetric Differential

The canonical symmetric differential $\omega(x, y)$ is defined by the following properties.

1. $\omega(x, y)$ is a meromorphic section of $\pi_1^* \Omega_C^1 \otimes \pi_2^* \Omega_C^1$ on $C \times C$, where π_i is the projection to the i -th component of $C \times C$.
2. $\omega(x, y)$ is holomorphic except the diagonal set $\{x = y\}$ where it has a double pole. For $p \in C$ if we take a local coordinate u around p then the expansion in $u(x)$ at $u(y)$ takes the form

$$\omega(x, y) = \left(\frac{1}{(u(x) - u(y))^2} + \text{regular} \right) du(x)du(y).$$

3. The A_j period in x variable is zero for any j :

$$\int_{A_j} \omega(x, y) = 0.$$

4. $\omega(x, y) = \omega(y, x)$.

The following proposition is well known.

Proposition 5. *The canonical differential exists and is unique.*

In fact there is an analytical description of $\omega(x, y)$ in terms of the theta function (see for example [5] p26, Corollary 2.6):

$$\omega(x, y) = dx dy \log E(x, y) = - \sum_{i,j=1}^g \frac{\partial^2 \log \theta}{\partial z_i \partial z_j} (y - x - f) v_i(x) v_j(y),$$

for any non-singular point $f \in (\Theta)$, where $(\Theta) = (\theta(z) = 0)$. The uniqueness can be easily proved using $H^0(C \times C, \pi_1^* \Omega_C^1 \otimes \pi_2^* \Omega_C^1) = \pi_1^* H^0(C, \Omega_C^1) \otimes \pi_2^* H^0(C, \Omega_C^1)$.

There is a remarkable identity due to Fay [5] (Corollary 2.12) connecting the chiral Szegő kernel and the canonical symmetric differential. The formula is

$$(14) \quad R(x, y|e)R(x, y|-e) = \omega(x, y) + \sum_{i,j=1}^g \frac{\partial^2 \log \theta[e]}{\partial z_i \partial z_j} (0) v_i(x) v_j(y),$$

for any $e \in \mathbb{C}^g$ such that $\theta(e) \neq 0$.

For a non-branch point $P \in C$ we can take z as a local coordinate around P . Let us define

$$G_z(z) = \lim_{y \rightarrow x} \left[\omega(x, y) - \frac{dz(x) dz(y)}{(z(y) - z(x))^2} \right].$$

It is known [5, 10] that $6G_z$ is the projective connection which satisfies

$$6G_t(t) dt^2 = 6G_z(z) dz^2 + \{z, t\} dt^2$$

for another local coordinate t around P , where $\{z, t\}$ is the Schwarzian differential defined by

$$\{z, t\} = \frac{z'''}{z'} - \frac{3}{2} \left(\frac{z''}{z'} \right)^2.$$

By Corollary 2 and (14) we have

Proposition 6.

$$G_z(z) = \frac{1}{N} \sum_{i,j=1}^{Nm} \frac{q(k_i, k_j) dz(x)^2}{(z(x) - \lambda_i)(z(x) - \lambda_j)} - \sum_{i,j=1}^g \frac{\partial^2 \log \theta[e_\Lambda]}{\partial z_i \partial z_j} (0) v_i(x) v_j(x).$$

As a corollary of this expression we have

Corollary 3. *Let $t = (z - \lambda_i)^{1/N}$ be the local coordinate around the branch point Q_i . Then the coefficient of $t^{N-2} dt$ in the Laurent expansion of $G_z(z)$ in t is*

$$2N \sum_{j \neq i} \frac{q(k_i, k_j)}{\lambda_i - \lambda_j} - \frac{1}{(N-2)!} \sum_{r,s=1}^g \sum_{\alpha=0}^{N-2} \binom{N-2}{\alpha} \frac{\partial^2 \log \theta[e_\Lambda]}{\partial z_r \partial z_s} (0) v_r^{(\alpha)}(Q_i) v_s^{(N-2-\alpha)}(Q_i),$$

where $v_r^{(\alpha)}(Q_i)$ is the coefficient of $t^\alpha dt$ in the expansion of $v_r(x)$ in t .

§5. Another Description of Canonical Differential

Let $P_l^{(l)}(z, w)$ be a polynomial satisfying the conditions

1. $P_l^{(l)}(z, w) = \sum_{j=0}^{lm} P_{lj}^{(l)}(w) (z-w)^j$ with

$$P_{l0}^{(l)}(w) = f(w), \quad P_{l1}^{(l)}(w) = \frac{l}{N} \sum_{i=1}^{Nm} \frac{f(w)}{w - \lambda_i}.$$

2. $\deg_w P_l^{(l)}(z, w) \leq (N-l)m$.

The following lemma can be easily proved.

Lemma 8. *The polynomial $P_l^{(l)}(z, w)$ satisfying the above conditions 1,2 exists for $l=1, \dots, N-1$.*

We set

$$\xi^{(0)}(x, y) = \frac{dz(x) dz(y)}{(z(x) - z(y))^2},$$

$$\xi^{(l)}(x, y) = \frac{P_l^{(l)}(z(x), z(y)) dz(x) dz(y)}{s(x)^l s(y)^{N-l} (z(x) - z(y))^2} \text{ for } l=1, \dots, N-1,$$

$$\xi(x, y) = \frac{1}{N} \sum_{l=0}^{N-1} \xi^{(l)}(x, y).$$

The condition 1,2 implies that $\xi^{(l)}(x, y)$ is regular on $C \times C$ at least except $\{z(x) = z(y)\}$.

- Proposition 7.** 1. $\xi(x, y)$ is holomorphic outside the diagonal set $\{x = y\}$.
 2. For a non-branch point $P \in C$, we take z as a local coordinate around P . Then the expansion in $z(x)$ at $z(y)$ is

$$\xi(x, y) = \frac{dz(x) dz(y)}{(z(x) - z(y))^2} + O((z(x) - z(y))^0).$$

Proof. For $y \in C$ let $y^{(r)} = (z(y), \omega^r s(y))$. Suppose that y is not a branch point. Then we can take z as a local coordinate around $y^{(r)}$. By calculation we have the expansion of $\xi^{(l)}(x, y)$ in $z(x)$ at $y^{(r)}$ as

(15)

$$\begin{aligned} \xi^{(l)}(x, y) = \omega^{-rl} & \left[\frac{1}{(z(x) - z(y))^2} - \frac{l^2}{2N^2} \left(\frac{d}{dz} \log f(z(y)) \right)^2 - \frac{l}{2N} \frac{d^2}{dz^2} \log f(z(y)) \right. \\ & \left. + \frac{P_{l2}^{(l)}(z(y))}{f(z(y))} + O((z(x) - z(y))^1) \right] dz(x) dz(y). \end{aligned}$$

By definition $\xi(x, y)$ is regular at least except $z(x) = z(y)$. In order to prove the property 1 of the proposition it is sufficient to prove that $\xi(x, y)$ has no singularity at $x = y^{(r)}$ for $1 \leq r \leq N-1$. Note that if $y = Q_i$ for some i , then $z(x) = z(y)$ is equivalent to $x = y = Q_i$. Hence by the expansion (15), $\xi(x, y)$ is regular at $x = y^{(r)}$. The property 2 is also obvious from (15) above. \square

Corollary 4. $\omega(x, y) - \xi(x, y)$ is holomorphic on $C \times C$.

Proof. By Proposition 7, $\omega(x, y) - \xi(x, y)$ is regular at least except $\bigcup_{i=1}^{Nm} (Q_i, Q_i)$. Hence $\omega(x, y) - \xi(x, y)$ is regular everywhere on $C \times C$, since $\omega(x, y) - \xi(x, y)$ is meromorphic on $C \times C$. \square

By this corollary there exists a set of polynomials $P_k^{(l)}(z, w)$ such that

$$\omega(x, y) - \xi(x, y) = \sum_{l=1}^{N-1} \sum_{k=1, k \neq l}^{N-1} \frac{P_k^{(l)}(z(x), z(y)) dz(x) dz(y)}{s(x)^k s(y)^{N-l}},$$

where by changing the definition of $P_k^{(l)}(z, w)$ the $k = l$ term can be excluded. The condition for the right hand side to be regular at $z(x) = \infty$ and $z(y) = \infty$ is

$$\deg_z P_k^{(l)}(z, w) \leq km - 2, \quad \deg_w P_k^{(l)}(z, w) \leq (N-l)m - 2.$$

Hence we can write

$$P_k^{(l)}(z, w) = \sum_{j=0}^{km-2} P_{kj}^{(l)}(w) (z-w)^j,$$

for some polynomials $P_{kj}^{(l)}(w)$. Now by the condition that the A period of $\omega(x, y)$ is zero we have

Proposition 8. *The relation*

$$\sum_{l=1}^{N-1} P_{l2}^{(l)}(\lambda_i) = -f'(\lambda_i) \frac{\partial}{\partial \lambda_i} \log \det A$$

holds, where A is the $g \times g$ period matrix of non-normalized form:

$$A = \left(\int_{A_i} w_j^{(\alpha)} \right).$$

Proof. Let us take $t = (z - \lambda_i)^{1/N}$ as a local coordinate around Q_i . Then we have

$$\begin{aligned} \frac{dz(y)}{s(y)^{N-l}} &= \frac{N}{\prod_{j \neq i} (\lambda_i - \lambda_j)^{(N-l)/N}} t^{l-1} dt (1 + O(t^N)) \quad 1 \leq l \leq N-1, \\ \frac{dz(y)}{(z(x) - z(y))^2} &= \frac{N}{(z(x) - \lambda_i)^2} t^{N-1} dt (1 + O(t^N)). \end{aligned}$$

Therefore if we set

$$\omega^{(l)}(x) = \frac{1}{N} \frac{P_l^{(l)}(z(x), \lambda_i) dz(x)}{s(x)^l (z(x) - \lambda_i)^2} + \sum_{k=1, k \neq l}^{N-1} \frac{P_k^{(l)}(z(x), \lambda_i) dz(x)}{s(x)^k},$$

then the condition that the coefficients of $dt, tdt, \dots, t^{N-2}dt$ in the expansion of $\int_A \omega(x, y)$ vanish is equivalent to

$$(16) \quad \int_A \omega^{(l)}(x) = 0 \quad 1 \leq l \leq N-1.$$

Noting that

$$P_l^{(l)}(z, \lambda_i) = \frac{l}{N} f'(\lambda_i) (z - \lambda_i) + \sum_{j=0}^{lm-2} P_{l,j+2}^{(l)}(\lambda_i) (z - \lambda_i)^{j+2},$$

$$\frac{\partial}{\partial \lambda_i} \frac{dz}{s^l} = \frac{l}{N} \frac{dz}{s^l(z-\lambda_i)},$$

we see that (16) is equivalent to

$$(17) \quad \frac{f'(\lambda_i)}{N} \frac{\partial}{\partial \lambda_i} \int_{A_i} \frac{dz}{s^l} + \frac{1}{N} \sum_{j=0}^{l_m-2} P_{i,j+2}^{(l)}(\lambda_i) \int_{A_i} \frac{(z-\lambda_i)^j dz}{s^l} \\ + \sum_{k=1, k \neq l}^{N-1} \sum_{j=0}^{k_m-2} P_{k,j}^{(l)}(\lambda_i) \int_{A_i} \frac{(z-\lambda_i)^j dz}{s^k} = 0.$$

We consider (17) as a linear equation for the g variables $\{P_{k,r}^{(l)}(\lambda_i)\}$. Solving (17) in $P_{i,2}^{(l)}$ by the Cramer's formula and summing up in l we have the statement of the proposition. \square

The idea of deriving equations of the form (17) is due to Bershadsky-Radul [1]. By calculations we have

Corollary 5. *The coefficient of $t^{N-2}dt$ in the expansion of $G_z(z)$ in $t = (z - \lambda_i)^{1/N}$ is*

$$-\mu N \sum_{j=1, j \neq i}^{Nm} \frac{1}{\lambda_i - \lambda_j} - N \frac{\partial}{\partial \lambda_i} \log \det A,$$

where

$$\mu = \frac{(N-1)(2N-1)}{6N}.$$

§6. Variational Formula of Period Matrix

Let us consider the equation

$$s_i^N = (z - \lambda_i - t) \prod_{j=1, j \neq i}^{Nm} (z - \lambda_j)$$

which is a one parameter deformation of the curve C by a small parameter t . We denote the corresponding compact Riemann surface by C_t . Let $\bar{\pi}$ be the projection $\bar{\pi} : C \rightarrow \mathbf{P}^1$ which maps (z,s) to z . We can take a canonical dissection $\{A_i(t), B_j(t)\}$ of C_t such that $\bar{\pi}(A_i(t)), \bar{\pi}(B_j(t))$ do not depend on t for $|t|$ being sufficiently small. The integration of a holomorphic 1-form on C_t along $A_i(t), B_j(t)$ can be considered as the integration of a multi-valued holomorphic 1-form on $\mathbf{P}^1 - \{\lambda_1, \dots, \lambda_{Nm}\}$ along $\bar{\pi}(A_i(t)), \bar{\pi}(B_j(t))$. Hence we can think of the integration cycles $A_i(t), B_j(t)$ as if they are independent of t . Therefore we simply write A_i, B_j instead of $A_i(t), B_j(t)$ in the calculations in this section. Let $\{v_j(x, t)\}$ be the basis of normalized holomorphic 1-forms on C_t

with respect to $\{A_i(t), B_j(t)\}$. We denote by $\tau(t) = (\tau_{kr}(t))$ the period matrix

$$\tau_{kr}(t) = \int_{B_k} v_r(x, t).$$

We set

$$w_{\beta t}^{(\alpha)} = \frac{z^{\beta-1} dz}{s_t^\alpha}.$$

We also use our previous notation $v_j(x) = v_j(x, 0)$, $\tau_{kr} = \tau_{kr}(0)$, $s = s_0$, $w_\beta^{(\alpha)} = w_{\beta 0}^{(\alpha)}$. Our aim in this section is to prove

Theorem 2.

$$\frac{d\tau_{jk}}{dt}(0) = \frac{1}{N(N-2)!} \sum_{\alpha=0}^{N-2} \binom{N-2}{\alpha} v_j^{(\alpha)}(Q_i) v_k^{(N-2-\alpha)}(Q_i).$$

We define the connection matrix σ and c by

$$(18) \quad v_j(x) = \sum_{\alpha, \beta} \sigma_{j(\alpha\beta)} w_\beta^{(\alpha)}(x), \quad w_\beta^{(\alpha)}(x) = \sum_j c_{(\alpha\beta)j} v_j(x).$$

Let $P \in C$ and u be a local coordinate around P . Let $\omega(P; n)$ be the abelian differential of the second kind satisfying the following conditions.

1. $\omega(P; n)$ is holomorphic except the point $P \in C$ where it has a pole of order $n \geq 2$. At P we have the expansion of the form

$$\omega(P; n) = -\frac{n-1}{u^n} du (1 + O(u^n)).$$

2. $\omega(P; n)$ has zero A_j periods for any j :

$$\int_{A_j} \omega(P; n) = 0.$$

The differential $\omega(P; n)$ depends on the choice of the local coordinate u . In our case $P = Q_i$ we always take $u = (z - \lambda_i)^{1/N}$ as a local coordinate around Q_i . In this sense $\omega(P; n)$ is uniquely determined. It is known that the following relation holds

$$(19) \quad \int_{B_j} \omega(P; n) = -\frac{1}{(n-2)!} v_j^{(n-2)}(P),$$

where $v_j^{(n-2)}(P)$ is the coefficient of $u^{n-2} du$ in the expansion of $v_j(x)$ in u .

Lemma 9. *If we expand $v_j(x, t)$ as*

$$(20) \quad v_j(x, t) = v_j(x) + v_{j1}(x)t + \dots,$$

then we have

$$v_{j1}(x) = -\sum_{\alpha,\beta} \frac{\sigma_{j(\alpha\beta)} \lambda_i^{\beta-1}}{\prod_{j \neq i} (\lambda_i - \lambda_j)^{\alpha/N}} \omega(Q_i; \alpha+1).$$

Proof. We have the expansion

$$(21) \quad w_{\beta i}^{(\alpha)}(x) = w_{\beta}^{(\alpha)}(x) + \frac{\alpha z^{\beta-1} dz}{N(z - \lambda_i) s^\alpha} t + O(t^2),$$

and the relation

$$(22) \quad w_{\beta i}^{(\alpha)}(x) = \sum_{j=1}^g \int_{A_j} w_{\beta i}^{(\alpha)} \cdot v_j(x, t).$$

Substituting the expansions (20) and (21) into the equation (22) and comparing the coefficient of t we have

$$(23) \quad \sum_{j=1}^g v_{j1}(x) \int_{A_j} w_{\beta}^{(\alpha)} = \frac{\alpha}{N} \eta_{\beta}^{(\alpha)}(x),$$

$$\eta_{\beta}^{(\alpha)}(x) = \frac{z^{\beta-1} dz}{(z - \lambda_i) s^\alpha} - \sum_{j=1}^g v_j(x) \int_{A_j} \frac{z^{\beta-1} dz}{(z - \lambda_i) s^\alpha}.$$

Then $\eta_{\beta}^{(\alpha)}(x)$ has the following properties:

1. $\int_{A_k} \eta_{\beta}^{(\alpha)}(x) = 0$ for any $k=1, \dots, g$.
2. Taking $u = (z - \lambda_i)^{1/N}$ as a local coordinate around Q_i we have the expansion

$$\eta_{\beta}^{(\alpha)}(x) = \frac{N \lambda_i^{\beta-1}}{f'(\lambda_i)^{\alpha/N}} \frac{du}{u^{\alpha+1}} + O(1).$$

Hence we have

$$(24) \quad \eta_{\beta}^{(\alpha)}(x) = -\frac{N \lambda_i^{\beta-1}}{\alpha f'(\lambda_i)^{\alpha/N}} \omega(Q_i; \alpha+1).$$

Since

$$\int_{A_j} w_{\beta}^{(\alpha)} = c_{(\alpha\beta)j},$$

and σ is the inverse matrix of c , we have the desired result from (23) and (24). □

Now comparing the coefficient of $u^{N-1-\alpha} du$ of the both hand sides of the first equation of (18) we have

$$\sum_{\beta=1}^{\alpha m-1} \sigma_{j(\alpha\beta)} \lambda_i^{\beta-1} = \frac{f'(\lambda_i)^{\alpha/N}}{N(N-1-\alpha)!} v_j^{(N-1-\alpha)}(Q_i),$$

for $1 \leq \alpha \leq N-1$ and thus

$$v_{j1}(x) = -\frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{1}{(N-1-\alpha)!} v_j^{(N-1-\alpha)}(Q_i) \omega(Q_i; \alpha+1).$$

Integrating both hand sides of this equation along the cycle B_k and using the relation (19) we obtain

$$\int_{B_k} v_{j1}(x) = \frac{1}{N(N-2)!} \sum_{\alpha=0}^{N-2} \binom{N-2}{\alpha} v_j^{(N-2-\alpha)}(Q_i) v_k^{(\alpha)}(Q_i).$$

□

§7. Thomae Formula

Now let us prove the generalized Thomae formula.

Theorem 3. For an ordered partition $\Lambda = (\Lambda_0, \dots, \Lambda_{N-1})$ we have

$$\theta[e_\Lambda](0)^{2N} = C_\Lambda (\det A)^N \prod_{i < j} (\lambda_i - \lambda_j)^{2Nq(k_i, k_j) + N\mu},$$

where $k_i = j$ for $i \in \Lambda_j$,

$$q(i, j) = \sum_{i \in \mathcal{L}} q_i(i) q_i(j), \quad \mu = \frac{(N-1)(2N-1)}{6N},$$

and $q_i(i), \mathcal{L}$ are given by (6), (5) in section 3. The complex number C_Λ does not depend on λ_i 's. They satisfy $C_\Lambda^{2N} = C_{\Lambda'}^{2N}$ for any Λ, Λ' .

Since the family $\{C_t\}$ is locally topologically trivial, we can take a canonical dissection $\{A_i(t), B_j(t)\}$ of C_t such that $A_i(t), B_j(t)$ are continuous in t . We assume that $A_i(t), B_j(t)$ does not go through any branch point $Q_i(t)$. We can also define the base points $P_0(t)$ of C_t and $z_0(t)$ of \tilde{C}_t lying over $P_0(t)$ so that they vary continuously in t . We identify $Q_i(t)$ with the corresponding point in the fundamental domain on \tilde{C}_t which contains the base point $z_0(t)$. Let $k^{P_0}(t)$ be the vector in \mathbf{C}^g whose j -th component is defined by

$$k^{P_0}(t)_j = \frac{2\pi i - \tau_{jj}(t)}{2} + \frac{1}{2\pi i} \sum_{i \neq j} \int_{A_i(t)} v_i(x, t) \int_{z_0(t)}^x v_j(x, t).$$

It is known (see [5] (p8) for example) that

$$\Delta - (g-1)P_0(t) = k^{P_0}(t)$$

in $J(C)$. Let us define $e_\Lambda(t)$ as an element of \mathbf{C}^g by

$$e_\Lambda(t) = \sum_{j \in \Lambda_1} \int_{z_0(t)}^{Q_j(t)} v(x, t) + \dots + (N-1) \sum_{j \in \Lambda_{N-1}} \int_{z_0(t)}^{Q_j(t)} v(x, t) - k^{P_0}(t).$$

Then $e_\Lambda(t)$ is continuous in t . Note that the linear isomorphism $\mathbb{C}^g \simeq \mathbb{R}^{2g}$ sending e to its characteristics with respect $\tau(t)$ is analytic in t . Therefore if we write

$$e_\Lambda(t) = \left\{ \begin{array}{c} \delta(t) \\ \epsilon(t) \end{array} \right\}_{\tau(t)},$$

then $\epsilon(t)$ and $\delta(t)$ are continuous in t . Since $\epsilon(t)$ and $\delta(t)$ are in $1/2N\mathbb{Z}^g$, they are constant in t . Therefore we simply write ϵ, δ instead of $\epsilon(t), \delta(t)$. We denote by $\theta_i[e](z)$ the theta function associated with the canonical basis $\{A_i(t), B_j(t)\}$ of C_t . We set $\theta[e](z) = \theta_0[e](z)$. Then the function $\theta_i[e_\Lambda(t)](0)$ depends on t only through the period matrix $\tau_{kr}(t)$ since

$$\theta_i[e_\Lambda(t)](0) = \sum_{m \in \mathbb{Z}^g} \exp\left(\frac{1}{2}(m + \delta)\tau(t)(m + \delta)^t + 2\pi i \epsilon(m + \delta)^t\right).$$

Using the heat equations

$$\frac{\partial^2 \theta[e_\Lambda]}{\partial z_k \partial z_r}(z) = \frac{\partial \theta[e_\Lambda]}{\partial \tau_{kr}}(z) \quad (k \neq r), \quad \frac{\partial^2 \theta[e_\Lambda]}{\partial z_k^2}(z) = 2 \frac{\partial \theta[e_\Lambda]}{\partial \tau_{kk}}(z),$$

and Lemma 1 we have

$$\begin{aligned} (25) \quad \frac{\partial}{\partial \lambda_i} \log \theta[e_\Lambda](0) &= \frac{d}{dt} \log \theta_t[e_\Lambda(t)](0) \Big|_{t=0} \\ &= \sum_{1 \leq k \leq r \leq g} \frac{\partial \log \theta[e_\Lambda]}{\partial \tau_{kr}}(0) \frac{d\tau_{kr}}{dt}(0) \\ &= \frac{1}{2} \sum_{k,r=1}^g \frac{1}{\theta[e_\Lambda](0)} \frac{\partial^2 \theta[e_\Lambda]}{\partial z_k \partial z_r}(0) \frac{d\tau_{kr}}{dt}(0), \\ &= \frac{1}{2} \sum_{k,r=1}^g \frac{\partial^2 \log \theta[e_\Lambda]}{\partial z_k \partial z_r}(0) \frac{d\tau_{kr}}{dt}(0). \end{aligned}$$

On the other hand by Corollary 3, 5 and Theorem 2 we have

$$\begin{aligned} (26) \quad & -\mu N \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} - N \frac{\partial}{\partial \lambda_i} \log \det A \\ &= 2N \sum_{j \neq i} \frac{q(k_i, k_j)}{\lambda_i - \lambda_j} - N \sum_{k,r=1}^g \frac{\partial^2 \log \theta[e_\Lambda]}{\partial z_k \partial z_r}(0) \frac{d\tau_{kr}}{dt}(0). \end{aligned}$$

Substituting (25) into (26) we have

$$\frac{\partial}{\partial \lambda_i} \log \theta[e_\Lambda](0) = \frac{1}{2} \frac{\partial}{\partial \lambda_i} \log \det A + \frac{\mu}{2} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} + \sum_{j \neq i} \frac{q(k_i, k_j)}{\lambda_i - \lambda_j}.$$

Hence we have proved the first part of Theorem 3.

§8. Property of the Constant C_A

Our aim in this section is to prove the remaining part of Theorem 3, that is, for any ordered partitions Λ and Λ'

$$(27) \quad C_A^{2N} = C_{\Lambda'}^{2N}.$$

As in the previous section we identify branch points Q_i with the corresponding points in the fundamental domain in \tilde{C} .

The key for the proof is the formula of Fay ([5], p30, Cor. 2.17):

$$\theta\left(\sum_{k=1}^g x_k - p - \Delta\right) = c \frac{\det(v_i(x_j))}{\prod_{i < j} E(x_i, x_j)} \frac{\sigma(p)}{\prod_{k=1}^g \sigma(x_k)} \prod_{k=1}^g E(x_k, p),$$

for any $p, x_1, \dots, x_g \in \tilde{C}$, where c is independent on p, x_1, \dots, x_g and

$$\sigma(p) = \exp\left(-\frac{1}{2\pi i} \sum_{j=1}^g \int_{A_j} v_j(y) \log E(y, p)\right).$$

Taking ratios for $p=a, b$ equations we have

$$\frac{\theta(\sum_{k=1}^g x_k - a - \Delta)}{\theta(\sum_{k=1}^g x_k - b - \Delta)} = \frac{\sigma(a)}{\sigma(b)} \prod_{k=1}^g \frac{E(x_k, a)}{E(x_k, b)}.$$

Set $a=Q_i, b=Q_j (i \neq j)$ and taking N -th power of the both hand sides we obtain

$$\frac{\theta(\sum_{k=1}^g x_k - Q_i - \Delta)^N}{\theta(\sum_{k=1}^g x_k - Q_j - \Delta)^N} = \left(\frac{\sigma(Q_i)}{\sigma(Q_j)}\right)^N \prod_{k=1}^g \frac{E(x_k, Q_i)^N}{E(x_k, Q_j)^N}.$$

Since NQ_i and NQ_j are linearly equivalent, there exists $\lambda(i, j), \kappa(i, j) \in \mathbf{Z}^g$ such that

$$N \int_{z_0}^{Q_i} v(x) - N \int_{z_0}^{Q_j} v(x) = N \int_{Q_i}^{Q_j} v(x) = 2\pi i \lambda(i, j) + \kappa(i, j) \tau,$$

$$\kappa(j, i) = -\kappa(i, j), \quad \lambda(j, i) = -\lambda(i, j).$$

If we set

$$f(x) = \frac{E(x, Q_i)^N}{E(x, Q_j)^N},$$

then we have

$$f(x + A_k) = f(x)$$

$$f(x + B_k) = \exp\left(-\sum_l \tau_{lk} \kappa(i, j)_l\right) f(x).$$

Hence the function

$$\exp\left(\sum_i \int_{z_0}^x v_i(x) \kappa(i, j)_i\right) f(x)$$

can be considered as a single valued function on C . Its only zeros are of N -th order at Q_i and only poles are of N -th order at Q_j . Therefore there exists a constant c_{ij} such that

$$\exp\left(\int_{z_0}^x v(x) \kappa(i, j)^t\right) \frac{E(x, Q_i)^N}{E(x, Q_j)^N} = c_{ij} \frac{z(x) - \lambda_i}{z(x) - \lambda_j}.$$

By the property of $\kappa(i, j)$, c_{ij} satisfies

$$c_{ij} = c_{ji}^{-1}, \quad c_{ii} = 1.$$

If we set

$$r_{ij} = c_{ij} \left(\frac{\sigma(Q_i)}{\sigma(Q_j)}\right)^N,$$

$$w(x|i, j) = \int_{z_0}^x v(x) \kappa(i, j)^t, \quad w\left(\sum_k x_k|i, j\right) = \sum_k w(x_k|i, j),$$

we have

$$(28) \quad \exp\left(w\left(\sum_{k=1}^g x_k|i, j\right)\right) \frac{\theta\left(\sum_{k=1}^g x_k - Q_i - \Delta\right)^N}{\theta\left(\sum_{k=1}^g x_k - Q_j - \Delta\right)^N} = r_{ij} \prod_{k=1}^g \frac{z(x) - \lambda_i}{z(x) - \lambda_j},$$

$$r_{ij} = r_{ji}^{-1}, \quad r_{ii} = 1.$$

Now let us take an ordered partition $\Lambda = \Lambda^{(1)} = (\Lambda_0^{(1)}, \dots, \Lambda_{N-1}^{(1)})$ with

$$\Lambda_l^{(1)} = \{i_1^l, \dots, i_m^l\}, \quad 0 \leq l \leq N-1.$$

Let us define $\Lambda^{(2)} = (\Lambda_0^{(2)}, \dots, \Lambda_{N-1}^{(2)})$ by

$$\Lambda_0^{(2)} = \{i_1^0, \dots, i_{m-1}^0, i_m^{N-1}\}, \quad \Lambda_{N-1}^{(2)} = \{i_1^{N-1}, \dots, i_{m-1}^{N-1}, i_m^0\},$$

and $\Lambda_l^{(2)} = \Lambda_l^{(1)} (1 \leq l \leq N-2)$.

If we consider Q_i as $\int_{z_0}^{Q_i} v$, we have the vectors in \mathbf{C}^g :

$$e_{\Lambda^{(1)}} = \Lambda_1^{(1)} + 2\Lambda_2^{(1)} + \dots + (N-1) (\Lambda_{N-1}^{(1)} \setminus \{i_m^{N-1}\}) - Q_{i_m^{N-1}} - k^{P_0},$$

$$e_{\Lambda^{(2)}} = \Lambda_1^{(2)} + 2\Lambda_2^{(2)} + \dots + (N-1) (\Lambda_{N-1}^{(2)} \setminus \{i_m^0\}) - Q_{i_m^0} - k^{P_0},$$

$$= \Lambda_1^{(1)} + 2\Lambda_2^{(1)} + \dots + (N-1) (\Lambda_{N-1}^{(1)} \setminus \{i_m^{N-1}\}) - Q_{i_m^0} - k^{P_0}.$$

Putting $i = i_m^{N-1}, j = i_m^0$ and

$$(x_1, \dots, x_g) = (\Lambda_1^{(1)}, 2\Lambda_2^{(1)}, \dots, (N-1) (\Lambda_{N-1}^{(1)} \setminus \{i_m^{N-1}\}))$$

in (28) we have

(29)

$$\exp(U(\Lambda^{(1)}|i_m^{N-1}, i_m^0)) \frac{\theta(e_{\Lambda^{(1)}})^N}{\theta(e_{\Lambda^{(2)}})^N} = r_{i_m^{N-1}, i_m^0} \prod_{r=1}^{N-1} \prod_{s=1}^{m'} \left(\frac{\lambda_{i_r} - \lambda_{i_r^{N-1}}}{\lambda_{i_r} - \lambda_{i_m^0}} \right)^r,$$

$$U(\Lambda^{(1)}|i_m^{N-1}, i_m^0) = w(\Lambda_1^{(1)} + 2\Lambda_2^{(1)} + \dots + (N-1)(\Lambda_{N-1}^{(1)} \setminus \{i_m^{N-1}\}) | i_m^{N-1}, i_m^0).$$

Here \prod' means the product for $(r, s) \neq (N-1, m)$.

Let us define the elements of \mathbb{C}^g by

$$-\bar{e}_{\Lambda^{(1)}} = \Lambda_{N-2}^{(1)} + 2\Lambda_{N-3}^{(1)} + \dots + (N-1)(\Lambda_0^{(1)} \setminus \{i_m^0\}) - Q_{i_m^0} - k^{P_0},$$

$$-\bar{e}_{\Lambda^{(2)}} = \Lambda_{N-2}^{(2)} + 2\Lambda_{N-3}^{(2)} + \dots + (N-1)(\Lambda_0^{(2)} \setminus \{i_m^{N-1}\}) - Q_{i_m^{N-1}} - k^{P_0}$$

$$= \Lambda_{N-2}^{(1)} + 2\Lambda_{N-3}^{(1)} + \dots + (N-1)(\Lambda_0^{(1)} \setminus \{i_m^0\}) - Q_{i_m^{N-1}} - k^{P_0},$$

where again Q_i denotes $\int_{z_0}^{Q_i} v$. Then if we set $i = i_m^0, j = i_m^{N-1}$ and

$$(x_1, \dots, x_g) = (\Lambda_{N-2}^{(1)}, 2\Lambda_{N-3}^{(1)}, \dots, (N-1)(\Lambda_0^{(1)} \setminus \{i_m^0\}))$$

in (28) we have

(30)
$$\exp(U'(\Lambda^{(1)}|i_m^0, i_m^{N-1})) \frac{\theta(\bar{e}_{\Lambda^{(1)}})^N}{\theta(\bar{e}_{\Lambda^{(2)}})^N} = r_{i_m^0, i_m^{N-1}} \prod_{r=1}^{N-1} \prod_{s=1}^{m'} \left(\frac{\lambda_{i_r^{N-1-r}} - \lambda_{i_m^0}}{\lambda_{i_r^{N-1-r}} - \lambda_{i_m^{N-1}}} \right)^r,$$

$$U'(\Lambda^{(1)}|i_m^0, i_m^{N-1}) = w(\Lambda_{N-2}^{(1)} + 2\Lambda_{N-3}^{(1)} + \dots + (N-1)(\Lambda_0^{(1)} \setminus \{i_m^0\}) | i_m^0, i_m^{N-1}).$$

Here we have used the property that $\theta(z)$ is an even function of z . Multiplying (29) and (30) we have

(31)
$$\exp\left(w\left(-\sum_{l=0}^{N-1} (N-1-2l)\tilde{\Lambda}_l^{(1)}|i_m^{N-1}, i_m^0\right)\right) \left(\frac{\theta(e_{\Lambda^{(1)}})\theta(\bar{e}_{\Lambda^{(1)}})}{\theta(e_{\Lambda^{(2)}})\theta(\bar{e}_{\Lambda^{(2)}})}\right)^N$$

$$= \prod_{r=1}^{N-1} \prod_{s=1}^{m'} \frac{(\lambda_{i_r} - \lambda_{i_r^{N-1}})^r (\lambda_{i_r^{N-1-r}} - \lambda_{i_m^0})^r}{(\lambda_{i_r} - \lambda_{i_m^0})^r (\lambda_{i_r^{N-1-r}} - \lambda_{i_m^{N-1}})^r},$$

where we set

$$\tilde{\Lambda}_0^{(1)} = \Lambda_0^{(1)} \setminus \{i_m^0\}, \quad \tilde{\Lambda}_{N-1}^{(1)} = \Lambda_{N-1}^{(1)} \setminus \{i_m^{N-1}\}, \quad \tilde{\Lambda}_l^{(1)} = \Lambda_l^{(1)} \quad (l \neq 0, N-1).$$

Since $\bar{e}_{\Lambda^{(k)}} = e_{\Lambda^{(k)}}, k=1, 2$ in $J(C)$ we can set

$$e_{\Lambda^{(k)}} = \left\{ \begin{matrix} \delta^{(k)} \\ \epsilon^{(k)} \end{matrix} \right\}_{\tau} = 2\pi i \epsilon^{(k)} + \delta^{(k)} \tau, \quad \delta^{(k)}, \epsilon^{(k)} \in \frac{1}{2N} \mathbb{Z}^g,$$

$$\bar{e}_{\Lambda^{(k)}} = e_{\Lambda^{(k)}} + 2\pi i m^{(k)} + n^{(k)} \tau, \quad m^{(k)}, n^{(k)} \in \mathbb{Z}^g.$$

Substituting these equations into (31), taking $2N$ -th power of both hand sides

and using the transformation property of theta functions, we get

$$(32) \quad \left(\frac{\theta[e_{\Lambda^{(1)}}](0)}{\theta[e_{\Lambda^{(2)}}](0)} \right)^{4N^2} = B \prod_{r=1}^{N-1} \prod_{s=1}^{m'} \frac{(\lambda_{i_r'} - \lambda_{i_m^{N-1}})^{2Nr} (\lambda_{i_s^{N-1-r}} - \lambda_{i_0^0})^{2Nr}}{(\lambda_{i_s'} - \lambda_{i_0^0})^{2Nr} (\lambda_{i_s^{N-1-r}} - \lambda_{i_m^{N-1}})^{2Nr}},$$

$$B = \exp\left(-2n'\tau\kappa t - N^2 \sum_{k=1}^2 (-1)^k (n^{(k)} \tau n^{(k)t} + 2\delta^{(k)} \tau n^{(k)t} + 2\delta^{(k)} \tau \delta^{(k)t})\right),$$

where we set

$$N\omega\left(-\sum_{l=0}^{N-1} (N-1-2l) \tilde{\Lambda}_l^{(1)} | i_m^{N-1}, i_m^0\right) = (2\pi i m' + n'\tau) \kappa,$$

$$\kappa = \kappa(i_m^{N-1}, i_m^0).$$

We can simplify the right hand side of (32) so that there are no common divisor in the numerator and ther denominator. The result is

$$(33) \quad \left(\frac{\theta[e_{\Lambda^{(1)}}](0)}{\theta[e_{\Lambda^{(2)}}](0)} \right)^{4N^2} = B \prod_{r=1}^{N-2} \prod_{s=1}^m \left(\frac{\lambda_{i_r'} - \lambda_{i_0^0}}{\lambda_{i_r'} - \lambda_{i_m^{N-1}}} \right)^{2N(N-1-2r)} \\ \prod_{s=1}^{m-1} \left(\frac{(\lambda_{i_s^{N-1}} - \lambda_{i_m^{N-1}}) (\lambda_{i_s^0} - \lambda_{i_0^0})}{(\lambda_{i_s^0} - \lambda_{i_m^{N-1}}) (\lambda_{i_s^{N-1}} - \lambda_{i_0^0})} \right)^{2N(N-1)}.$$

Let us compare this equation with those obtained from the proved part of Theorem 3. Let $\{k_i\}$, $\{k'_i\}$ correspond to $\Lambda^{(1)}$, $\Lambda^{(2)}$ respectively as in (7). Then by the proved part of the Thomae formula we have

$$(34) \quad \left(\frac{\theta[e_{\Lambda^{(1)}}](0)}{\theta[e_{\Lambda^{(2)}}](0)} \right)^{4N^2} = C \prod_{i < j} (\lambda_i - \lambda_j)^{4N^2(q(k_i, k_j) - q(k'_i, k'_j))},$$

where $C = (C_{\Lambda^{(1)}}/C_{\Lambda^{(2)}})^{4N^2}$. Note that $4N^2q(k_i, k_j)$ and $4N^2q(k'_i, k'_j)$ are even. Then we have

$$(35) \quad \text{LHS of (34)} \\ = C \prod_{r=1}^{N-1} \prod_{s=1}^m \left(\frac{\lambda_{i_r'} - \lambda_{i_m^{N-1}}}{\lambda_{i_r'} - \lambda_{i_0^0}} \right)^{4N^2q(r, N-1)} \prod_{r=0}^{N-2} \prod_{s=1}^m \left(\frac{\lambda_{i_r'} - \lambda_{i_0^0}}{\lambda_{i_r'} - \lambda_{i_m^{N-1}}} \right)^{4N^2q(r, 0)} \\ \prod_{s=1}^{m-1} \left(\frac{(\lambda_{i_s^{N-1}} - \lambda_{i_0^0}) (\lambda_{i_s^0} - \lambda_{i_m^{N-1}})}{(\lambda_{i_s^{N-1}} - \lambda_{i_m^{N-1}}) (\lambda_{i_s^0} - \lambda_{i_0^0})} \right)^{4N^2q(0, 1)} \\ = C \prod_{r=1}^{N-2} \prod_{s=1}^m \left(\frac{\lambda_{i_r'} - \lambda_{i_0^0}}{\lambda_{i_r'} - \lambda_{i_m^{N-1}}} \right)^{4N^2(q(r, 0) - q(r, N-1))} \\ \prod_{s=1}^{m-1} \left(\frac{(\lambda_{i_s^{N-1}} - \lambda_{i_m^{N-1}}) (\lambda_{i_s^0} - \lambda_{i_0^0})}{(\lambda_{i_s^{N-1}} - \lambda_{i_0^0}) (\lambda_{i_s^0} - \lambda_{i_m^{N-1}})} \right)^{4N^2(q(0, 0) - q(0, 1))}$$

where \prod'' means the product for $(r, s) \neq (0, m)$. Let us calculate $q_l(r, t)$. By the

definition of $q(i, j)$ and Lemma 3, $q(i, j)$ depends only on $|i - j| \pmod N$. Hence using $q_l(0) = N/l$ we have

$$q(r, t) = q(0, t-r) = \sum_{l \in \mathcal{L}} q_l(0) q_l(t-r) = \frac{1}{N} \sum_{l \in \mathcal{L}} l q_l(t-r).$$

The following lemma is obtained by a direct calculation.

Lemma 10.

$$(36) \quad \sum_{l \in \mathcal{L}} l q_l(r) = \frac{N^2-1}{12} - \frac{1}{2}r(N-r).$$

Thus we have

$$q(r, t) = \frac{1}{N} \left(\frac{N^2-1}{12} - \frac{1}{2}(t-r)(N-t+r) \right).$$

In particular

$$\begin{aligned} 4N^2(q(0,0) - q(0,1)) &= 2N(N-1), \\ 4N^2(q(r,0) - q(r, N-1)) &= 2N(N-1-2r). \end{aligned}$$

Comparing (33) and (35) we have

$$\left(\frac{C_{A^{(1)}}}{C_{A^{(2)}}} \right)^{2N} = B.$$

Let us write

$$B = \exp\left(\sum_{k \leq r} b_{kr} \tau_{kr} \right).$$

Since $C_{A^{(1)}}$ and $C_{A^{(2)}}$ do not depend on τ

$$\frac{\partial}{\partial \tau_{kr}} B = b_{kr} B = 0, \quad \text{for any } k \leq r.$$

Hence $B = 1$. Since any two ordered partitions are transformed to each other by successive exchange of elements of A_i and A_{i+1} , $i = 0, \dots, N-1$, the equation (27) are proved.

§9. Examples

In this section we shall give examples of Thomae formula for small N 's. Recall that

$$\frac{\mu}{2} + q(0, j) = \frac{N-1}{4} - \frac{j(N-j)}{2N}.$$

$$q(i, j) = q(0, |i - j|) = q(0, -|i - j|).$$

We remark that the constants C_A in this section are different from those in Theorem 3 by ± 1 times because of the reordering of the difference products. The properties of the constants remain same.

9.1. $N=2$

We consider the hyperelliptic curve $s^2 = \prod_{j=1}^{2m} (z - \lambda_j)$. Let $A = (A_0, A_1)$ with

$$A_0 = \{i_1 < \dots < i_m\}, \quad A_1 = \{j_1 < \dots < j_m\}.$$

We have

$$q(0,0) = \frac{1}{8}, \quad q(0,1) = -\frac{1}{8}, \quad \mu = \frac{1}{4}.$$

The Thomae formula is

$$\theta[e_A](0)^4 = C_A (\det A)^2 \prod_{k < l} (\lambda_{i_k} - \lambda_{i_l}) (\lambda_{j_k} - \lambda_{j_l}).$$

This is the original Thomae formula in which case $C_A^2 = (2\pi)^{-4(m-1)}$.

9.2. $N=3$

Let $A = (A_0, A_1, A_2)$. We have

$$q(0,0) = \frac{2}{9}, \quad q(0,1) = q(0,2) = -\frac{1}{9}, \quad \mu = \frac{5}{9}.$$

Then

$$\theta[e_A](0)^6 = C_A (\det A)^3 ((A_0 A_0) (A_1 A_1) (A_2 A_2))^3 (A_0 A_1) (A_1 A_2) (A_0 A_2).$$

Here if

$$A_i = \{i_1 < \dots < i_m\}, \quad A_j = \{j_1 < \dots < j_m\},$$

then

$$(A_i A_i) = \prod_{k < l} (\lambda_{i_k} - \lambda_{i_l}), \quad (A_i A_j) = \prod_{k, l=1}^m (\lambda_{i_k} - \lambda_{j_l}).$$

Our result shows that C_A^6 does not depend on A .

9.3. $N=4$

Let $A = (A_0, \dots, A_3)$. We have

$$q(0,0) = \frac{5}{16}, \quad q(0,1) = q(0,3) = -\frac{1}{16}, \quad q(0,2) = -\frac{3}{16}, \quad \mu = \frac{7}{8}.$$

Thomae formula is

$$\theta[e_A](0)^8 = C_A (\det A)^4 ((\Lambda_0 \Lambda_0) (\Lambda_1 \Lambda_1) (\Lambda_2 \Lambda_2) (\Lambda_3 \Lambda_3))^6 \\ ((\Lambda_0 \Lambda_1) (\Lambda_1 \Lambda_2) (\Lambda_2 \Lambda_3) (\Lambda_0 \Lambda_3))^3 ((\Lambda_0 \Lambda_2) (\Lambda_1 \Lambda_3))^2.$$

The constant C_A^8 does not depend on A .

9.4. $N=5$

Let $A = (\Lambda_0, \dots, \Lambda_4)$. We have

$$q(0,0) = \frac{2}{5}, \quad q(0,1) = q(0,4) = 0, \quad q(0,2) = q(0,3) = -\frac{1}{5}, \quad \mu = \frac{6}{5}.$$

Thomae formula is

$$\theta[e_A](0)^{10} = C_A (\det A)^5 ((\Lambda_0 \Lambda_0) (\Lambda_1 \Lambda_1) (\Lambda_2 \Lambda_2) (\Lambda_3 \Lambda_3) (\Lambda_4 \Lambda_4))^{10} \\ ((\Lambda_0 \Lambda_1) (\Lambda_1 \Lambda_2) (\Lambda_2 \Lambda_3) (\Lambda_3 \Lambda_4) (\Lambda_0 \Lambda_4))^6 \\ ((\Lambda_0 \Lambda_2) (\Lambda_1 \Lambda_3) (\Lambda_2 \Lambda_4) (\Lambda_0 \Lambda_3) (\Lambda_1 \Lambda_4))^4.$$

The constant C_A^{10} does not depend on A .

§10. Concluding Remarks

In this paper we have given a rigorous proof of the generalized Thomae formula for \mathbb{Z}_N curves which was previously discovered by Bershadsky and Radul [1, 2] in the study of conformal field theory. Here let us make a comment on the related subjects.

There are several papers ([8, 4] and references therein) studying the generalization of λ function of the elliptic curves to \mathbb{Z}_N curves by studying the cross ratios of four points on a Riemann surface. In those approaches the only ratios of theta constants appear and Thomae type formula is not used. However in the Smirnov's theta formula for the solutions of sl_2 Knizhnik-Zamolodchikov equation on level 0, Thomae formula is needed.

Our strategy to prove the generalized Thomae formula here, which is similar to that of [1, 2], is the comparison of algebraic and analytic expressions of several quantities. In the hyperelliptic case this can be considered as a part of the more general comparison of algebraic and analytic construction of Jacobian varieties due to Mumford [12]. It will be interesting to study the integrable system associated with \mathbb{Z}_N curves and to study the generalization of Thomae type formula for spectral curves.

In fact Thomae [15], Fuchs [6] derived differential equations satisfied by theta constants with respect to branch points in a more general setting and they could integrate them completely only in the case of hyperelliptic curves. The Thomae formula for \mathbb{Z}_N curves provides a new example which is integrable.

The \mathbb{Z}_N curves and the $1/2N$ periods in the generalized Thomae formula are related with the Lie algebra sl_N and the weight zero subspace of the tensor products of the vector representation. Hence it is natural to expect that the Thomae type formula has a good description in terms of Lie algebras and their representations.

For the evaluation of the constant C_A we need to know the explicit description of canonical cycles of \mathbb{Z}_N curve. So far we could describe a canonical basis only in the case of $N=3$ (except $N=2$).

Acknowledgement

We would like to thank Koji Cho, Norio Iwase, Yasuhiko Yamada for the illuminating discussions.

References

- [1] Bershadsky, M. and Radul, A., Conformal field theories with additional Z_N symmetry, *Int. J. Mod. Phys.*, **A2-1**(1987), 165-178.
- [2] ———, Fermionic fields on Z_N curves, *Comm. Math. Phys.*, **116**(1988), 689-700.
- [3] Cho, K. and Yoshida, M., Comparison of (co) homologies of branched covering spaces and twisted ones of base spaces I, *Kyushu J. Math.*, **48-1**(1994), 111-122.
- [4] Farkas, H. M., Generalizations of the λ functions, *Israel Mathematical Conference Proceedings*, **9** (1996), 231-239.
- [5] Fay, J. D., Theta functions on Riemann surfaces, *Lect. Notes in Math.*, **352**(1973), Springer.
- [6] Fuchs, L., Ueber die Form der Argumente der Thetafunktionen und über die Bestimmung von $\mathcal{G}(0, 0, \dots, 0)$ als Function der Klassenmoduln, *Crelle's Journal*, **73**(1871), 305-323.
- [7] ———, Ueber die linearen Differentialgleichungen, welchen die Periodicitätsmoduln der Abelschen Integrale genügen, und über verschiedene Arten von Differentialgleichungen für $\mathcal{G}(0, 0, \dots, 0)$, *Ibid.*, 324-339.
- [8] Gonzales Diez, G., Loci of curves which are prime Galois coverings of \mathbb{P}^1 , *Proc. London Math. Soc.* **62-3**(1991), 469-489.
- [9] Gunning, R. C., Lectures on Riemann Surfaces-Jacobi Varieties, *Math. Notes*, **12** (1972), Princeton Univ. Press.
- [10] Hawley, N. S. and Schiffer, M., Half-order differentials on Riemann surfaces, *Acta. Math.*, **115** (1966), 199-236.
- [11] Knizhnik, V. G., Analytic fields on Riemann surfaces II, *Comm. Math. Phys.*, **112** (1987), 567-590.
- [12] Mumford, D., *Tata lectures on theta II*, Birkhauser, 1983.
- [13] Riemann, B., Theorie der Abelschen Functionen, *Crelle's Journal*, **54**(1857), 115-155.
- [14] Smirnov, F., Form factors, Deformed Knizhnik-Zamolodchikov equations and finite-gap integration, *Comm. Math. Phys.*, **155**(1993), 459-487.
- [15] Thomae, J., Bestimmung von $d\lg \mathcal{G}(0, 0, \dots, 0)$ durch die Classenmoduln, *Crelle's Journal*, **66** (1866), 92-96.
- [16] ———, Beitrag zur Bestimmung von $\mathcal{G}(0, 0, \dots, 0)$ durch die Classenmoduln algebraischer Functionen, *Crelle's Journal*, **71**(1870), 201-222.

