1-Cocycles for Rotationally Invariant Measures

By

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Abstract

Let H be a separable Hilbert space over $\mathbf{R}$ (dim (H) is finite or infinite), H^a be the algebraic dual space of H, $\mathfrak{B}$ be the cylindrical σ-algebra on H^a and μ be a rotationally invariant probability measure on $(H^a, \mathfrak{B})$. Further let $\theta = \theta(x, U)$ be a 1-cocycle defined on $(x, U) \in H^a \times O(H)$, where $O(H)$ is the rotation group on H. That is,

(c.1) for any fixed $U \in O(H)$, $\theta(x, U)$ *is a* $\mathfrak{B}$*-measurable function of* x,

(c.2) $|\theta(x, U)| \equiv 1$, and

(c.3) for ${}^\vee U_1$, ${}^\vee U_2 \in O(H)$, $\theta(x, U_1) \theta({}^t U_1 x, U_2) = \theta(x, U_1 U_2)$ for μ-a.e. x,

where ${}^t U$ is the algebraic transpose of U. Moreover it is said to be continuous, if the following condition holds for θ.

(c.4) $\theta(x, U) \rightarrow 1$ in μ, if $U \rightarrow \mathrm{Id}$ in the strong operator topology.

Our main result is as follows.

Assume that dim $(H) \neq 3$. Then for any continuous 1-cocycle θ, there exists a $\mathfrak{B}$-measurable function ϕ with modulus 1 such that for any fixed $U \in O(H)$, $\theta(x, U) = \phi({}^t Ux) / \phi(x)$ for μ-a.e. x.

§1. Introduction

Let μ be a rotationally quasi-invariant probability measure on the dual measurable space $(H^a, \mathfrak{B})$. i.e., μ is absolutely continuous with respect to μ_U for each $U \in O(H)$, where $\mu_U = (\mu \circ {}^t U)$ is the image measure of μ by the map ${}^t U^{-1}$. Then a canonical representation $(R_\mu, \mathrm{L}^2_\mu(H^a))$ of $O(H)$ arises such as

$$R_\mu(U): f(x) \in \mathrm{L}^2_\mu(H^a) \mapsto \theta(x, U) \sqrt{\frac{d\mu_U}{d\mu}}(x) f({}^t Ux) \in \mathrm{L}^2_\mu(H^a),$$

where θ is a 1-cocycle defined on $H^a \times O(H)$. Another representation with 1-cocycles appears in the following situation which is analogous to the representations of commutation relations in quantum mechanics.

Suppose that $V(h)$ and $T(U)$ $(h \in H,\ U \in O(H))$ are given as unitary representations on a Hilbert space $\mathcal{H}$ and that they satisfy the following two conditions.

(c.5) V is cyclic and $V(h)$ is continuous on every finite dimensional subspace of H.

(c.6) For $^\forall h \in H$, $^\forall U \in O(H)$, $T(U)V(h) = V(Uh)T(U)$.

(It may be appropriate to say that this pair of representations is a representation of semi-direct product of H and $O(H)$.)

(1) Then the representations are realized on $L^2_\mu(H^a)$ as the following manner with a rotationally quasi-invariant probability measure μ on $(H^a, \mathfrak{B})$ and a 1-cocycle θ.

$$(1.1) \qquad\qquad V(h): f(x) \mapsto \exp(i\langle h, x\rangle)f(x).$$

$$(1.2) \qquad\qquad T(U): f(x) \mapsto \theta(x, U)\sqrt{\frac{d\mu_U}{d\mu}}(x)f(^tUx),$$

where $\langle h, x\rangle$ is the duality bracket for $h \in H$ and $x \in H^a$. Moreover T is continuous, if and only if so is θ.

In fact, take a cyclic vector $e \in \mathcal{H}$ with norm 1 and form a function $\langle V(h)e, e\rangle_{\mathcal{H}}$ of $h \in H$. It is positive definite and continuous in the sense stated in (c.5). In virtue of Bochner's theorem we have a probability measure μ on $(H^a, \mathfrak{B})$ such that

$$(1.3) \qquad\qquad \langle V(h)e, e\rangle = \int_{H^a} \exp(i\langle h, x\rangle)\mu(dx).$$

Hence a map

$$W: \sum_{j=1}^{n} a_j V(h_j)e \in \mathcal{H} \mapsto \sum_{j=1}^{n} a_j \exp(i\langle h_j, x\rangle) \in L^2_\mu(H^a),$$

is well defined (a_j are scalars), it is extended as a unitary operator on the whole space, and

$$(1.4) \qquad\qquad \widetilde{V}(h) := WV(h)W^{-1}: f(x) \mapsto \exp(i\langle h, x\rangle)f(x),$$

$$(1.5) \qquad\qquad \widetilde{T}(U) := WT(U)W^{-1}: f(x) \mapsto \rho_U(x)f(^tUx),$$

where $\rho_U := \widetilde{T}(U)(1)$. It follows from (1.5) that

$$\int_{H^a} \exp(i\langle h, x\rangle)\mu(dx) = \int_{H^a} \exp(i\langle Uh, x\rangle)|\rho_U(x)|^2\mu(dx),$$

and therefore

$$\mu_U(dx) = |\rho_U(x)|^2\mu(dx).$$

Thus μ is $O(H)$-quasi-invariant, and a θ defined by

$$(1.6) \qquad \theta(x, U) := \rho_U(x) \left(\frac{d\mu_U}{d\mu}(x) \right)^{-1/2}$$

is a 1-cocycle. It is easy to see that θ is continuous if and only if T is continuous. (1.5) is now written as

$$(1.7) \qquad \widetilde{T}(U) : f(x) \mapsto \theta(x, U) \sqrt{\frac{d\mu_U}{d\mu}}(x) f({}^t Ux),$$

and the work of realization is complete.

We claim further the following statements without proofs.

(2) For the above pair of representations, (V, T) is irreducible if and only if the corresponding μ is $O(H)$-ergodic.
Here the ergodicity for $O(H)$ means that $\mu(B) = 1$ or 0 provided that $\mu({}^t UB \ominus B) = 0$ for all $U \in O(H)$.

(3) (V, T) is equivalent to (V', T') if and only if $\mu \simeq \mu'$ and θ and θ' are mutually equivalent, where the equivalence for 1-cocycles means that

$$\theta(x, U) = \phi({}^t Ux) / \phi(x) \cdot \theta'(x, U)$$

holds for μ-a.e. x with some $\mathfrak{B}$-measurable function ϕ with modulus 1. Such a 1-cocycle $\phi({}^t U \cdot) / \phi(\cdot)$ is called a 1-coboundary and it will be denoted by θ_ϕ.

By the above, the pair of such representations (V, T) is completely determined by rotationally quasi-invariant measures μ and 1-cocycles θ. We wish to describe their structures. Now the following facts are already obtained.

(4) Any rotationally quasi-invariant measure μ is equivalent to some rotationally invariant measure. (See, [6].)

(5) Any rotationally invariant measure μ is a superposition of $\gamma_c (c \in [0, \infty))$, where γ_c $(c > 0)$ is the uniform measure on the sphere of radious c in the finite dimensional case and is the standard Gaussian measure with mean 0 and variance c^2 in the infinite dimensional case, and γ_0 is the Dirac measure at the origin. (See, [5, 8].)

Thus a factor from measures is completely determined and the ambiguity left for us is a factor from 1-cocycles. It is desirable to classify them with some method. Fortunately we found that they are characterized as 1-coboundaries. (Thus, the equivalence of these representations are reduced to the equivalence of corresponding measures.) This is a main purpose of the present paper.

Here we have something to say about the finite dimensional case in which there are similar arguments with our results by some authors, for example [2]. However they are slightly different from our ones in several points such as cocycle conditions, measurable assumptions, etc. So we will state them in our style for distinction as well as for completeness.

The proofs for the finite dimensional case and for the infinite dimensional case are carried out in Section 2 and in Section 3, respectively. They are

completely independent of each other and each of them is deduced from the characteristic properties of finite and infinite dimensional spaces, respectively.

Besides, in each of the sections we pick up only representations of the type $(R_\mu, L^2_\mu(H^a))$ and study their mutual equivalence.

§2. Finite Dimensional Case

2.1. 1-cocycles on $SO(n)$. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathfrak{B}(\mathbb{R}^n)$ be the natural Borel field on $\mathbb{R}^n$, μ be a rotationally invariant probability measure on $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, and θ be a 1-cocycle. i.e.,
$^\vee U \in SO(n)$, $\theta(x, U)$ is a $\mathfrak{B}(\mathbb{R}^n)$-measurable function of x with modulus 1 and

$$(2.1) \qquad ^\vee U_1, \, ^\vee U_2 \in SO(n), \, \theta(x, U_1)\theta(U_1^{-1}x, U_2) = \theta(x, U_1 U_2)$$

for μ-a.e. x.

Theorem 2.1. *Assume that $n \neq 3$. Then the following statements for θ are all equivalent.*

(1) *θ is a continuous 1-cocycle.*

(2) *There exists a jointly measurable function $\theta'(x, U)$ defined on $\mathbb{R}^n \times SO(n)$ such that $^\vee U \in SO(n)$, $\theta'(x, U) = \theta(x, U)$ for μ-a.e.x.*

(3) *θ is a 1-coboundary.*

Proof. It is obvious that (3) implies (1) and (2). Conversely, under the assumption (1), we have a stochastic version $\zeta(x, U)$ such that $\zeta(x, U)$ is a jointly measurable function of (x, U) with modulus 1 which satisfies for ν-a.e.U, $\theta(x, U) = \zeta(x, U)$ for μ-a.e.x, where ν is the normalized Haar measure on $SO(n)$. Here we will divide this proof "(1), (2)$\Rightarrow$(3)" into two steps.

(I) The case of $\mu(\{0\}) = 0$.

At this time, for the proof it is enough to admit the following lemma.

Lemma 2.2. *Assume that $n \neq 3$, $\mu(\{0\}) = 0$ and that a jointly measurable function $\zeta(x, U)$ with modulus 1 satisfies*

$$\int_{\mathbb{R}^n} \int_{SO(n)} \int_{SO(n)} |\zeta(x, U_1)\zeta(U_1^{-1}x, U_2) - \zeta(x, U_1 U_2)| \mu(dx)\nu(dU_1)\nu(dU_2) = 0.$$

Then there exists a Borel function $\phi(x)$ defined on $\mathbb{R}^n$ with modulus 1 such that

$$^\vee U \in SO(n), \, \zeta(x, U) = \phi(^t Ux)/\phi(x)$$

for μ-a.e. x.

Proof. Let γ be the uniform probability measure on S^{n-1}, and $m(E) := \mu(x \mid \|x\| \in E)$ for all $E \in \mathfrak{B}(\mathbb{R}^+)$, where $\|\cdot\|$ is the norm on $\mathbb{R}^n$. Then we have

$$\mu(B) = \int_{\mathbb{R}^+} \gamma(c^{-1}B) m(dc)$$

for all $B \in \mathfrak{B}(\mathbb{R}^n)$. It follows from the assumption that for $m \times \gamma$-a.e. (c, ω)

$$(2.2) \qquad \zeta(c\omega, U_1)\,\zeta(cU_1^{-1}\omega, U_2) = \zeta(c\omega, U_1 U_2)$$

for $\nu \times \nu$-a.e. (U_1, U_2). From now on we will write $\zeta_c(\omega, U)$ instead of $\zeta(c\omega, U)$. Now take any $e \in S^{n-1}$ and fix it. Then by Fubini's theorem we have for m-a.e.c,

$$(2.3) \qquad \zeta_c(Ue, U_1)\,\zeta_c(U_1^{-1}Ue, U_2) = \zeta_c(Ue, U_1 U_2)$$

for $\nu \times \nu \times \nu$-a.e. (U, U_1, U_2). Let us exchange the variable U_2 to $U_0 := U^{-1}U_1 U_2$. Then it follows from (2.3) that

$$(2.4) \qquad \zeta_c(Ue, U_1) = \zeta_c(Ue, UU_0)/\zeta_c(U_1^{-1}Ue, U_1^{-1}UU_0)$$

for $m \times \nu \times \nu \times \nu$-a.e. (c, U, U_1, U_0). In particular, some U_0 exists such that the above equality holds for $m \times \nu \times \nu$-a.e. (c, U, U_1). We fix it. Put

$$\sigma_c(U) := \zeta_c^{-1}(Ue, UU_0).$$

$\sigma_c(U)$ is a jointly measurable function of (c, U) which satisfies

$$(2.5) \qquad \zeta_c(Ue, U_1) = \sigma_c(U_1^{-1}U)/\sigma_c(U)$$

for $m \times \nu \times \nu$-a.e. (c, U, U_1). Consider the isotoropy subgroup G_e at e, $G_e := \{g \in SO(n) \mid ge = e\}$ and take any g from G_e. Then we have for m-a.e.c,

$$\sigma_c(U_1^{-1}Ug)/\sigma_c(Ug) = \sigma_c(U_1^{-1}U)/\sigma_c(U)$$

for $\nu \times \nu$-a.e. (U, U_1). It follows that for m-a.e. c,

$$\sigma_c(Ug)/\sigma_c(U) = \int_{SO(n)} \sigma_c(U_1^{-1}Ug)/\sigma_c(U_1^{-1}U)\ \mu(dU_1) =: k_c(g)$$

for μ-a.e.U. The measurable function $k_c(g)$ satisfies

$$|k_c(g)| = 1,\ k_c(g_1 g_2) = k_c(g_1)\,k_c(g_2)$$

for all $g, g_1, g_2 \in G_e$. Namely, k_c is a measurable character on G_e, so it is continuous, and in virtue of the classical results for $n \geq 4$ k_c is equal to the identity for m-a.e.c. Consequently we have for m-a.e.c

$$(2.6) \qquad \forall g \in G_e,\ \sigma_c(Ug) = \sigma_c(U)$$

for ν-a.e.U. Again Fubini's theorem implies, for m-a.e.c, for ν-a.e.U, $\sigma_c(Ug) = \sigma_c(U)$ for ν_e-a.e.g, where ν_e is the normalized Haar measure on G_e. Thus we have for m-a.e.c

$$(2.7) \qquad \sigma_c(U) = \int_{G_e} \sigma_c(Ug)\,\nu_e(dg) =: \phi(c, U)$$

for ν-a.e.U. ϕ is a jointly measurable function of (c, U) and it is a function of

(c, Ue).

Now take a Borel section $\mathcal{N} : S^{n-1} \mapsto SO(n)$ of the map $\mathcal{M} : U \in SO(n) \mapsto Ue \in S^{n-1}$. That is, $\mathcal{MN} = I$ on S^{n-1}. Let us put

$$\phi'(c, \omega) := \phi(c, \mathcal{N}\omega)$$

for $(c, \omega) \in \mathbb{R}^+ \times S^{n-1}$. Then ϕ' is jointly measurable and satisfies for m-a.e.c,

(2.8) $\sigma_c(U) = \phi'(c, Ue)$

for ν-a.e.U. In this way, we have for m-a.e.c,

$$\zeta(cUe, U_1) = \phi'(c, U_1^{-1}Ue) / \phi'(c, Ue)$$

for $\nu \times \nu$-a.e.(U, U_1). Finally we put

$$\phi(x) := \phi'(\|x\|, \|x\|^{-1}x).$$

Then ϕ is a Borel function and we have

$$\zeta(x, U) = \phi(U^{-1}x) / \phi(x)$$

for $\mu \times \nu$-a.e.(x, U). The rest of the proof is a standard argument. Let

$$G := \{U \in SO(n) \,|\, \zeta(x, U) = \phi(U^{-1}x) / \phi(x) \text{ for } \mu\text{-a.e. } x\}.$$

Then G is a measurable subgroup and $\nu(G) = 1$ by what we have proceeded. Thereby $G = SO(n)$. $\square$

Hence (2) implies (3). As for the implication " $(1) \Rightarrow (3)$ ", note that for ν-a.e.U, $\int_{\mathbb{R}^n} |\theta(x, U) - \phi(U^{-1}x)/\phi(x)| \mu(dx) = 0$, and use the continuity of θ.

(II) General case

We write down $\mu = \alpha \gamma_0 + \beta \mu'$, with $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, where γ_0 is the Dirac measure at 0 and $\mu'(\{0\}) = 0$. By the property (2.1),

$$\theta(0, U_1) \theta(0, U_2) = \theta(0, U_1 U_2)$$

for all $U_1, U_2 \in SO(n)$. Moreover if (1) in Theorem 2.1 holds, then $\theta(0, U)$ is a continuous function of $U \in SO(n)$ and if (2) in Theorem 2.1 holds, $\theta'(0, U)$ is a measurable homomorphism, so it is continuous. Hence for $n \geq 4$ we have

$$\theta(0, U) \equiv 1 \text{ and } \theta'(0, U) \equiv 1$$

under the assumption (1) and (2) respectively. On the other hand, it follows from the result of (I) that there exists a Borel function $\phi'(x)$ such that

$$^\forall U \in SO(n), \ \theta(x, U) = \phi'(U^{-1}x) / \phi'(x)$$

for μ'-a.e.x. So a Borel function defined by

$$\phi(x) := \phi'(x), \text{ if } x \neq 0, \text{ and } \phi(0) := 1$$

satisfies

$$\forall U \in SO(n), \ \theta(x, U) = \phi(U^{-1}x)/\phi(x)$$

for μ-a.e. x. There is nothing to prove in case of $n = 1$. The case of $n = 2$ is easy.
□

Remark 2.1. If the cocycle condition (c.3) holds for every $x \in S^{n-1}$ and $U \in SO(n)$ without exceptional points, and $\theta(x, U)$ is a measurable function of U for each fixed $x \in S^{n-1}$, then it follows easily that there exists a measurable function ζ on S^{n-1} such that $\theta(x, U) = \dfrac{\zeta(U^{-1}x)}{\zeta(x)}$ for $\forall x \in S^{n-1}$ and for $\forall U \in SO(n)$ in the case of $n \neq 3$. (cf. Proposition 1.2 in [2]).

For the proof take a Borel cross section $\mathcal{N}$ from S^{n-1} to $SO(n)$ such that $\mathcal{N}(x)e_0 = x$, where $e_0 = (0, 0, ..., 0, 1) \in S^{n-1}$, and put $\Delta(U) := \mathcal{N}(Ue_0)^{-1}U$. Further putting $\chi(V) := \theta(e_0, V)$ for $\forall V \in SO(n-1)$ and $\zeta(x) := \theta(e_0, \mathcal{N}(x)^{-1})$, we can easily checked that

$$\theta(x, U) = \frac{\zeta(U^{-1}x)}{\zeta(x)} \chi(\Delta(U^{-1}\mathcal{N}(x)))^{-1}.$$

Since χ is a measurable character of $SO(n-1)$ and $n \geq 4$, so $\chi \equiv 1$ and the conclusion follows.

However the above situation is different from the assumption of Theorem 2.1 in various points such as the cocycle conditions, measurabilities and supports of measures.

Remark 2.2. For $n = 3$, this assertion does not hold in general as is seen by Proposition 1.7 in [2]. The typical counter example is, using the above notation,

$$\theta_\chi(x, U) = \chi(\Delta(U^{-1}\mathcal{N}(x))),$$

where χ is any non trivial continuous character on $SO(2)$. These θ_χ are all continuous.

2.2. 1-cocycles on $O(n)$. Next we consider actions of $O(n)$. Take a Borel function $Q(c)$ defined on $[0, \infty)$ such that $Q^2(c) = 1$, and define a function θ^Q by

$$(2.9) \qquad \theta^Q(x, U) := Q(\|x\|) \frac{1 - \det U}{2} + \frac{1 + \det U}{2}.$$

It is a continuous 1-cocycle on $O(n)$ as is easily seen.

Theorem 2.3. *Let μ be a rotationally invariant measure on $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ and θ be a 1-cocycle on $O(n)$. Then the followings are equivalent.*

(1) *θ is continuous.*

(2) *There exists a Borel function $\phi(x)$ with modulus 1 and a such Q as above which satisfy*

$$^\forall U \in O(n), \quad \theta(x, U) = \theta^Q(x, U) \cdot \theta_\phi(x, U)$$

for μ-a.e. x.

Proof. It is obvious "$(2) \Rightarrow (1)$". We prove the converse relation. To begin with, let us restrict θ to $SO(n)$. Then some function ϕ exists such that

$$^\forall U \in SO(n), \quad \theta(x, U) = \phi(U^{-1}x)/\phi(x)$$

for μ-a.e. x. This equality is true even in the exceptional case for $n = 3$, because the proof of Lemma 2.2 works validly in this case, though we omit the detail of it. Next take any $T_0 \in O(n) \backslash SO(n)$ and fix it. We have for any $U \in SO(n)$,

$$\begin{aligned}
\theta(x, T_0 U) &= \theta(x, T_0) \theta(T_0^{-1}x, U) \\
&= \theta(x, T_0) \phi((T_0 U)^{-1}x) \overline{\phi(T_0^{-1}x)}.
\end{aligned}$$

Thus,

$$(2.10) \qquad \theta(x, T_0 U) = \eta(x) \phi((T_0 U)^{-1}x) \overline{\phi(x)}$$

for μ-a.e. x, where $\eta(x) := \theta(x, T_0) \phi(x) \overline{\phi(T_0^{-1}x)}$. It follows that for all $U_1, U_2 \in SO(n)$

$$\begin{aligned}
\theta(x, U_1 T_0) \theta((U_1 T_0)^{-1}x, U_2 T_0) &= \eta(x) \phi((U_1 T_0)^{-1}x) \overline{\phi(x)} \\
&\cdot \eta((U_1 T_0)^{-1}x) \phi((U_2 T_0)^{-1}(U_1 T_0)^{-1}x) \overline{\phi((U_1 T_0)^{-1}x)},
\end{aligned}$$

and

$$\theta(x, U_1 T_0 U_2 T_0) = \phi((U_1 T_0 U_2 T_0)^{-1}x) \overline{\phi(x)}.$$

Compairing the right hand side of the above two equalities, we get for all $U \in SO(n)$,

$$(2.11) \qquad \eta(T_0^{-1}U^{-1}x) \eta(x) = 1$$

for μ-a.e. x, and therefore for μ-a.e. x,

$$(2.12) \qquad \eta(x) = \int_{SO(n)} \overline{\eta(T_0^{-1}U^{-1}x)} \nu(dU).$$

So $\eta(x)$ depends on only the norm of x. Namely, $\eta(x) \equiv Q(\|x\|)$ for some measurable function Q defined on $[0, \infty)$. It follows from (2.15) that $Q^2(\|x\|)$

$=1$ for μ-a.e. x. Consequently, we have for any $U \in SO(n)$,

$$\theta(x, T_0 U) = Q(\|x\|) \phi((T_0 U)^{-1}x) \overline{\phi(x)}$$

for μ-a.e. x, and the proof is now complete. $\square$

Remark 2.3. As for the uniqueness of ϕ and Q in Theorem 2.1 and Theorem 2.3, ϕ is determined up to a norm-dependent function with modulus 1 and Q is determined uniquely.

2.3. Equivalence of representations. Let us consider the following representation of $SO(n)$ or $O(n)$,

$$R^\theta(U): f(x) \in L_\mu^2(\mathbf{R}^n) \mapsto \theta(x, U)f(U^{-1}x) \in L_\mu^2(\mathbf{R}^n).$$

In case of $SO(n)$ $(n \neq 3)$, R^θ is equivalent to R^I $(\theta \equiv 1)$, by what we have considered and in case of $O(n)$, R^θ is equivalent to $R^{\theta Q} =: R^Q$ by virtue of Theorem 2.3. We state below a study of the mutual equivalence of R^Q.

Theorem 2.4. *Put* $\Gamma := \{c \geq 0 \mid Q(c) = 1\}$ *and* $\Gamma' := \{c \geq 0 \mid Q'(c) = 1\}$. *Then in order that*

$$(R^Q, L_\mu^2(\mathbf{R}^n)) \simeq (R^{Q'}, L_\mu^2(\mathbf{R}^n)),$$

it is necessary and sufficient that

    (1) $\dim(L_m^2(\Gamma)) = \dim(L_m^2(\Gamma'))$ *and* $\dim(L_m^2(\Gamma^c)) = \dim(L_m^2(\Gamma'^c))$, *if* $m(\{0\}) = 0$ *or if* $m(\{0\}) > 0$ *and* $Q(0) = Q'(0)$, *and that*
    (2) $\dim(L_m^2(\Gamma)) = \dim(L_m^2(\Gamma')) = \dim(L_m^2(\Gamma^c)) = \dim(L_m^2(\Gamma'^c)) = \infty$, *if* $m(\{0\}) > 0$ *and* $Q(0) \neq Q'(0)$.

Proof. (Necessity) Let A be an intertwining unitary operator from $(R^Q, L_\mu^2(\mathbf{R}^n))$ to $(R^{Q'}, L_\mu^2(\mathbf{R}^n))$. Restricting the representations to $SO(n)$, we see that

$$\forall f \in L_\mu^2(\mathbf{R}^n), \ A(f(U^{-1}\cdot))(x) = (Af)(U^{-1}x).$$

Thus Af is rotationally invariant, if so is f, and hence a unitary operator $\tilde{A}$ on $L_m^2[0, \infty)$ induced from A satisfies

(2.13) $\tilde{A}(Q \cdot \tilde{f}) = Q' \cdot \widetilde{Af}$

for all $\tilde{f} \in L_m^2[0, \infty)$. It follows that

(2.14) $\tilde{A} \circ P_\Gamma = P_{\Gamma'} \circ \tilde{A}$,

where P_Γ is the projection from $L_m^2[0, \infty)$ to $L_m^2(\Gamma)$. So we have

$$\dim\left(L_m^2\left(\Gamma\right)\right)=\dim\left(L_m^2\left(\Gamma'\right)\right)\ \text{and}\ \dim\left(L_m^2\left(\Gamma^c\right)\right)=\dim\left(L_m^2\left(\Gamma'^c\right)\right).$$

Here let us consider the second case : $m\left(\{0\}\right)>0$ and $Q\left(0\right)\neq Q'\left(0\right)$, and suppose that $\dim\left(L_m^2\left(\Gamma\right)\right)$ or $\dim\left(L_m^2\left(\Gamma^c\right)\right)$ would be finite, say $\dim\left(L_m^2\left(\Gamma\right)\right)<\infty$. 0 belongs to only one of Γ or Γ', so we assume that $0\in\Gamma\setminus\Gamma'$.

By the way, the representation $\left(R_r,\ L_r^2\left(S^{n-1}\right)\right)$,

$$R_r:f\left(\omega\right)\mapsto f\left(U^{-1}\,\omega\right)\ \ \left(U\in O\left(n\right)\right),$$

has irreducible decompositions $\left(R_r,\ \mathcal{H}_l\right)$ with mulutiplicity 1:

$$L_r^2\left(S^{n-1}\right)=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_l\oplus\cdots,$$

where $\mathcal{H}_0=\{const.fun.\}$, $\mathcal{H}_1=\{\langle h,\ \omega\rangle\}_{h\in\mathbb{R}^n},\cdots$ and so on. In general $\mathcal{H}_l$ are obtained by orthogonalizing polynomials of degree l on S^{n-1} succesively. (See, [9].) Now according to the following two decompositions,

$$L_\mu^2\left(\mathbb{R}^n\right)=L_\mu^2\left(\{0\}\right)\oplus L_m^2\left(\Gamma\setminus\{0\}\right)\otimes L_r^2\left(S^{n-1}\right)\oplus L_m^2\left(\Gamma^c\right)\otimes L_r^2\left(S^{n-1}\right)$$
$$L_\mu^2\left(\mathbb{R}^n\right)=L_m^2\left(\Gamma'\right)\otimes L_r^2\left(S^{n-1}\right)\oplus L_\mu^2\left(\{0\}\right)\oplus L_m^2\left(\Gamma'^c\setminus\{0\}\right)\otimes L_r^2\left(S^{n-1}\right),$$

we consider an element $k\left(c\right)h_e\left(\omega\right)\in L_m^2\left(\Gamma'\right)\otimes\mathcal{H}_1$, where $h_e\left(\omega\right):=\langle e,\ \omega\rangle$, and $e\in\mathbb{R}^n$ is any fixed vector. Then some calculations derive that for μ-a.e. x

$$A^{-1}\left(k\otimes h_e\right)\left(x\right)\neq 0\Rightarrow Q\left(\|x\|\right)=+1,$$

and hence for any $f\in L_m^2\left(\Gamma^c\right)\otimes L_r^2\left(S^{n-1}\right)$ $\ A^{-1}\left(k\otimes h_e\right)\left(x\right)\overline{f\left(x\right)}=0$ for μ-a.e. x. This shows that

$$A^{-1}\left(L_m^2\left(\Gamma'\right)\otimes\mathcal{H}_1\right)\subset L_\mu^2\left(\{0\}\right)\oplus L_m^2\left(\Gamma\setminus\{0\}\right)\otimes L_r^2\left(S^{n-1}\right).$$

It follows from considerations for irreducible components that

$$A^{-1}\left(L_m^2\left(\Gamma'\right)\otimes\mathcal{H}_1\right)\subset L_m^2\left(\Gamma\setminus\{0\}\right)\otimes\mathcal{H}_1,$$

and therefore there exists a one to one operator U from $L_m^2\left(\Gamma'\right)$ to $L_m^2\left(\Gamma\setminus\{0\}\right)$ such that $A^{-1}=U\otimes\mathrm{Id}$. So we should have $\dim\left(L_m^2\left(\Gamma\right)\right)=\dim\left(L_m^2\left(\Gamma'\right)\le\dim\left(L_m^2\left(\Gamma\setminus\{0\}\right)\right)$, which contradicts to the assumption.

(Sufficiency)

(I) The case of $m\left(\{0\}\right)=0$. From the assumption there exists a unitary operator $\widetilde{A}$ on $L_m^2\left[0,\ \infty\right)$ such that

$$\widetilde{A}\left(Q\cdot\widetilde{f}\right)=Q'\cdot\widetilde{A}\widetilde{f},$$

for all $\widetilde{f}\in L_m^2\left[0,\ \infty\right)$. Since $L_\mu^2\left(\mathbb{R}^n\right)$ is regarded as $L_m^2\left(0,\ \infty\right)\otimes L_r^2\left(S^{n-1}\right)$, so

$$A : \sum_{j=1}^{n} f_j(c) g_j(\omega) \mapsto \sum_{j=1}^{n} (\widetilde{Af_j})(c) g_j(\omega)$$

is a desired one of intertwining unitary operators.

(II) The case of $m(\{0\}) > 0$ and $Q(0) = Q'(0)$. If the common value of Q is 1, then we have

$$\dim(L^2_m(\Gamma \setminus \{0\})) = \dim(L^2_m(\Gamma' \setminus \{0\})), \text{ and } \dim(L^2_m(\Gamma^c)) = \dim(L^2_m(\Gamma'^c)).$$

Set $m_+ = m|\mathbf{R}^+$ and take a unitary operator $\widetilde{A}$ on $L^2_{m_+}(\mathbf{R}^+)$ such that

$$\widetilde{A}(Q \cdot \widetilde{f}) = Q' \cdot \widetilde{Af}$$

for all $\widetilde{f} \in L^2_{m_+}(\mathbf{R}^+)$. Then in this case regarding $L^2_\mu(\mathbf{R}^n)$ as $L^2_\mu(\{0\}) \oplus L^2_{m_+}(\mathbf{R}^+) \otimes L^2_r(S^{n-1})$, we define A on $L^2_\mu(\mathbf{R}^n)$ such that

$$A : r \oplus \widetilde{f}(c) g(\omega) \mapsto r \oplus (\widetilde{Af})(c) g(\omega).$$

This is a desired one of intertwining unitary operators. The case of $Q(0) = Q'(0) = -1$ is treated similarly.

(III) The case of $m(\{0\}) > 0$ and $Q(0) \neq Q'(0)$. Without loss of generality, we may assume that $Q(0) = 1$, $Q'(0) = -1$. Put $\delta(x) := m(\{0\})^{-1/2} \chi_{\{0\}}(\|x\|)$, (in general, χ_E stands for the indicator function of the set E), and take c.o.n.s's : $e_1, \ldots, e_n, \ldots$ on $L^2_m(\Gamma \setminus \{0\})$, $e'_1, \ldots, e'_n, \ldots$ on $L^2_m(\Gamma')$, $f_1, \ldots, f_n, \ldots$ on $L^2_m(\Gamma^c)$, $f'_1, \ldots, f'_n, \ldots$ on $L^2_m(\Gamma'^c \setminus \{0\})$, respectively. We define a unitary operator A_0 as follows.

$$A_0 : \delta \mapsto e'_1 \otimes 1, \ e_1 \otimes 1 \mapsto e'_2 \otimes 1, \ldots, e_n \otimes 1 \mapsto e'_{n+1} \otimes 1, \ldots$$
$$f_1 \otimes 1 \mapsto \delta, \ f_2 \otimes 1 \mapsto f'_1 \otimes 1, \ldots, f_{n+1} \otimes 1 \mapsto f'_n \otimes 1, \ldots.$$

A_0 is defined on

$$L^2_\mu(\{0\}) \oplus L^2_m(\Gamma \setminus \{0\}) \otimes \mathcal{H}_0 \oplus L^2_m(\Gamma^c) \otimes \mathcal{H}_0$$

and maps the space to

$$L^2_m(\Gamma') \otimes \mathcal{H}_0 \oplus L^2_\mu(\{0\}) \oplus L^2_m(\Gamma'^c \setminus \{0\}) \otimes \mathcal{H}_0,$$

and it gives an intertwining operator from R^Q to $R^{Q'}$. Now let us take any unitary operator $\widetilde{A}_1$ on $L^2_m(0, \infty)$ which gives one to one correspondence between $L^2_m(\Gamma \setminus \{0\})$ and $L^2_m(\Gamma')$ and between $L^2_m(\Gamma^c)$ and $L^2_m(\Gamma'^c \setminus \{0\})$, and define A_1 on $L^2_m(0, \infty) \otimes \mathcal{H}_0^\perp$ such that

$$A_1 : \widetilde{f}(c) g(\omega) \mapsto \widetilde{A}_1 \widetilde{f}(c) g(\omega).$$

Then $A := A_0 \oplus A_1$ is a desired one of intertwining unitary operators. $\square$

Corollary 2.5. $(R^Q, \mathrm{L}^2_\mu(\mathbb{R}^n)) \simeq (R^I, \mathrm{L}^2_\mu(\mathbb{R}^n))$ *if and only if* $Q(c) = 1$ *for m-a.e.c.*

§3. Infinite Dimensional Case

3.1. Continuous 1-cocycles for $O(H)$. In this section we assume that $\dim(H) = \infty$.

Let μ be a rotationally invariant probability measure on $(H^a, \mathfrak{B})$ and θ be a continuous 1-cocycle. i.e.,

$$(3.1) \qquad\qquad\qquad |\theta(x, U)| \equiv 1.$$

$$(3.2) \qquad {}^\forall U_1, {}^\forall U_2 \in O(H), \ \theta(x, U_1)\theta({}^t U_1 x, U_2) = \theta(x, U_1 U_2),$$

for μ-a.e. x,

$$(3.3) \qquad\qquad \int_{H^a} |1 - \theta(x, U)|\mu(dx) \to 0, \quad \text{if } U \to \mathrm{Id},$$

in the strong topology.

Theorem 3.1. *Any continuous 1-cocycle θ is a 1-coboundary. Namely, there exists a $\mathfrak{B}$-measurable function ϕ with modulus 1 such that*

$$ {}^\forall U \in O(H), \ \theta(x, U) = \phi({}^t U x)/\phi(x) $$

for μ-a.e. x.

Proof. We shall divide the proof into 4-steps.

(I) First we prove the following lemma.

Lemma 3.2. *Let $h_1, \cdots, h_n, \cdots$ be a c.o.n.s. in H. Take any $U_n \in O(H)$ such that $U_n(\mathrm{Sp}\{h_1, \cdots, h_n\}) \perp \mathrm{Sp}\{h_1, \cdots, h_n\}$ for each n. Then if there exists some $\varphi \in \mathrm{L}^2_\mu(H^a)$ such that $\{\theta(x, U_n)\varphi({}^t U_n x)\}_n$ has a non zero weak limit point, then some $\mathfrak{B}$-measurable non zero function $\phi \in \mathrm{L}^2_\mu(H^a)$ exists such that $T(U)\phi = \phi$ for all $U \in O(H)$, where $T(U)\phi(x) := \theta(x, U)\phi({}^t U x)$.*

Proof. To begin with, we claim that φ can be taken as a bounded function, say ψ. In fact, for $\epsilon := \|w\|/2$, where w is the non zero weak limit point of $\{\theta(x, U_n)\varphi({}^t U_n x)\}_n$, there exists a step function ψ such that $\|\varphi - \psi\| \leq \epsilon$. So there exists a weak limit point of $\{\theta(x, U_n)\psi({}^t U_n x)\}_n$ in the ϵ-neighbourhood of w. If necessary, taking a subsequence we may assume that $\theta(x, U_n)\psi({}^t U_n x) \to \phi(x)$ in the weak topology. Nothing that for any $U \in O(k) := \{U \in O(H) | U = \mathrm{Id} \text{ on } \mathrm{Sp}\{h_1, \cdots, h_k\}^\perp\}$, $U_n^{-1} U U_n \to \mathrm{Id}$ in the strong topology, we have

$$\int_{H^a} |\theta(x, U U_n) - \theta(x, U_n)|^2 \mu(dx) = \int_{H^a} |\theta({}^t U_n x, U_n^{-1} U U_n) - 1|^2 \mu(dx)$$

$$= \int_{H^a} |\theta(x, U_n^{-1}UU_n) - 1|^2 \mu(dx) \to 0, \quad (n \to \infty).$$

Further,

$$\int_{H^a} |\phi({}^tU_n x) - \phi({}^tU_n{}^tU x)|^2 \mu(dx) = \int_{H^a} |\phi(x) - \phi({}^tU_n{}^tU^tU_n^{-1}x)|^2 \mu(dx) \to 0 \quad (n \to \infty).$$

Therefore using $M := \sup_x |\phi(x)|$, we get

$$\int_{H^a} |\theta(x, U_n) \phi({}^tU_n x) - \theta(x, U) \theta({}^tU x, U_n) \phi({}^tU_n{}^tU x)|^2 \mu(dx)$$

$$\leq 2M^2 \int_{H^a} |\theta(x, U_n) - \theta(x, UU_n)|^2 \mu(dx)$$

$$+ 2 \int_{H^a} |\phi({}^tU_n x) - \phi({}^tU_n{}^tU x)|^2 \mu(dx) \to 0 \quad (n \to \infty).$$

It follows that by the well known property of weak convergence,

$$^\vee U \in O(k), \ \theta(x, U) \phi({}^tU x) = \phi(x)$$

for μ-a.e. x. Since $O_0(H) := \cup_{k=1}^\infty O(k)$ is dense in $O(H)$, so we have for all $U \in O(H)$,

$$(3.4) \qquad\qquad \theta(x, U) \phi({}^tU x) = \phi(x).$$

for μ-a.e. x. $\quad\square$

(II) Take any $\epsilon \in \mathbf{R}$ such that $0 < \epsilon < 1$. Then by the continuity of θ, there exists some $d \in \mathbf{N}$ such that

$$\int_{H^a} |1 - \theta(x, U)|^2 \mu(dx) \leq \epsilon^2,$$

for all $U \in O(H)$ such that $Uh_j = h_j \, (1 \leq j \leq d)$. Here we take V_n for each n such that

$$V_n h_j = h_j, \ (1 \leq j \leq d), \ V_n h_{d+j} = h_{d+n+j}, \ V_n h_{d+n+j} = h_{d+j} \ (1 \leq j \leq n)$$
$$\text{and } V_n h_j = h_j \ (j > 2n + d).$$

Since $\|\theta(\cdot, V_n) - 1\|_2 \leq \varepsilon$, so any weak limit point λ of $\{\theta(\cdot, V_n)\}_n$ is non zero. Put

$$O(d)^\perp := \{U \in O(H) \,|\, Uh_j = h_j \text{ for } 1 \leq j \leq d\}.$$

Then $V_n^{-1}UV_n$ converges strongly to Id for all $U \in O(d)^\perp \cap O_0(H)$, so repeating a similar argument in (I) we have $\theta(x, U) \lambda({}^tU x) = \lambda(x)$ for μ-a.e. x. As $O(d)^\perp \cap O_0(H)$ is dense in $O(d)^\perp$, so we get

$$(3.5) \qquad\qquad {}^\vee U \in O(d)^\perp, \ \theta(x, U) \lambda({}^tU x) = \lambda(x)$$

for μ-a.e. x.

(III) Take any $n \in \mathbf{N}$ and set U_n for each n as follows.

$$U_n h_j = h_{n+j}, \ U_n h_{n+j} = h_j \ (1 \leq j \leq n) \ U_n h_j = h_j \ (j > 2n).$$

Moreover we take an isometric operator S on H such that $Sh_j = h_{d+j} \ (j=1, \cdots)$. Then we have

(3.6) $V_n S = S U_n \ (n=1, \cdots)$.

Now we extend $T(\cdot)$ for an isometric operator R. Take a sequence $\{R_n\}_n \subset O(H)$ such that $\lim_n R_n = R$ in the strong topology. Then

$$\int_{H^a} |\theta(x, R_n) - \theta(x, R_m)|^2 \mu(dx) = \int_{H^a} |1 - \theta({}^t R_n x, R_n^{-1} R_m)|^2 \mu(dx)$$
$$= \int_{H^a} |1 - \theta(x, R_n^{-1} R_m)|^2 \mu(dx).$$

Since $R_n^{-1} R_m \to \mathrm{Id} \ (n, m \to \infty)$, so $\{\theta(\cdot, R_n)\}_n \subset L_\mu^2(H^a)$ forms a Cauchy sequence. We denote the limit by $\theta(x, R)$. Of course $\theta(x, R)$ does not depend on a particular choice of $\{R_n\}_n$. Further it is easy to see that

(3.7) $\int_{H^a} |f({}^t R_n x) - f({}^t R x)|^2 \mu(dx) \to 0 \ (n \to \infty)$

for all $f \in L_\mu^2(H^a)$, because μ is ${}^t R$-invariant. We define $T(R)$ as follows.

$$T(R) : f(x) \mapsto \theta(x, R) f({}^t R x)$$

for all $f \in L_\mu^2(H^a)$. $T(R)$ is an isometric operator on $L_\mu^2(H^a)$ and $T(R_n)$ converges strongly to $T(R)$. It follows that for all $U \in O(H)$,

$$T(UR) = \lim_n T(UR_n) = \lim_n T(U) T(R_n) = T(U) T(R),$$
$$T(RU) = \lim_n T(R_n U) = \lim_n T(R_n) T(U) = T(R) T(U).$$

In particular, we have from (3.6)

$$T(V_n) T(S) = T(S) T(U_n).$$

Taking the adjoints and noting that $U_n^{-1} = U_n$, $V_n^{-1} = V_n$,

(3.8) $T(S)^* T(V_n) = T(U_n) T(S)^*.$

Therefore for all $f, g \in L_\mu^2(H^a)$

(3.9) $\langle T(V_n)f, T(S) T(S)^* g \rangle_2 = \langle T(U_n) T(S)^* f, T(S)^* g \rangle_2.$

By the way, $T(S) T(S)^*$ is the projection from $L_\mu^2(H^a)$ to $\mathrm{Im}(T(S))$ and

$\mathrm{Im}(T(S)) = \{\theta(x, S) f_d(x) | f_d$ is a square summable $\mathfrak{B}_d$-measurable function$\}$,

where the Borel field $\mathfrak{B}_d$ is a minimal σ-algebra with which all the functions x

$\mapsto \langle h_j, x \rangle$ $(j=d+1,\cdots, n,\cdots)$ are measurable.

Under the above preparation, we claim that there exists some $\varphi \in L^2_\mu(H^a)$ which has a non zero weak limit point for $\{T(U_n)\varphi\}_n$. Suppose that it would be false. Then for any $\varphi \in L^2_\mu(H^a)$, $\{T(U_n)\varphi\}_n$ should converge weakly to zero. We take $q((\langle h_1, x \rangle,\cdots, \langle h_d, x \rangle)\lambda(x)$ as $f(x)$, where q is any bounded measurable function on $\mathbb{R}^d$ and λ is a non zero measurable function satisfying (3.5), and put $\varphi := T(S)^*f$. Further we take any $g(x) := \theta(x, S)f_d(x)$ from $\mathrm{Im}(T(S))$. Then

$$(3.10) \qquad \langle f, g \rangle_2 = \langle T(V_n)f, T(S)T(S)^*g \rangle_2$$
$$(3.11) \qquad \qquad = \langle T(U_n)T(S)^*f, T(S)^*g \rangle_2.$$

Thus letting n tend to ∞, we have

$$\langle f, g \rangle_2 = 0$$

from the assumption of reductio ad absurdum. It follows directly that

$$\int_{H^a} q(\langle h_1, x \rangle,\cdots, \langle h_d, x \rangle)\lambda(x)\overline{\theta(x, S)f_d(x)}\mu(dx) = 0,$$

and this shows $\lambda(x)\overline{\theta(x, S)} = 0$ for μ-a.e. x, because $q(\langle h_1, x \rangle,\cdots,\langle h_d, x \rangle)\overline{f_d(x)}$ spans a dense linear subspace. Since $|\theta(x, S)| = 1$ for μ-a.e.x, so $\lambda(x) = 0$ for μ-a.e. x which contradicts to the choice of λ.

(IV) Set

$$\Phi := \{\phi \in L^2_\mu(H^a) | \phi \neq 0, \ ^\forall U \in O(H), \ T(U)\phi = \phi\}.$$

By virtue of the results in (I) and (III), ϕ is not empty. Further we note that if $\phi \in \Phi$, then $^\forall U \in O(H)$, $|\phi(^tUx)| = |\phi(x)|$ for μ-a.e.x. So $\hat{\phi}$ defined by

$$\hat{\phi}(x) := \phi(x)/|\phi(x)|, \quad \text{if } \phi(x) \neq 0, \text{ and } := 0, \text{ otherwise,}$$

again belongs to Φ. Let us put for each $\phi \in \Phi$,

$$(3.12) \qquad S_\phi := \{x \in H^a \mid \phi(x) \neq 0\}, \text{ and } \alpha := \sup_{\phi \in \Phi} \mu(S_\phi).$$

Then we have $\mu(^tUS_\phi \ominus S_\phi) = 0$ for any $U \in O(H)$. We claim that there exists $\omega \in \Phi$ such that $\alpha = \mu(S_\omega)$.

In fact, take a sequence $\{\phi_n\}_n \subset \Phi$ such that $\mu(S_{\phi_n}) \uparrow \alpha$. If necessary, taking $\hat{\phi}_n$ in place of ϕ_n, we may assume that $|\phi_n(x)| \leq 1$. Then, for an $\omega \in L^2_\mu(H^a)$ defined by

$$\omega(x) := \phi_n(x), \text{ if } x \in S_{\phi_n} \setminus \cup_{j=1}^{n-1} S_{\phi_j}, \text{ and } := 0, \text{ otherwise,}$$

we get

$$\int_{H^a}|\theta(x,\,U)\,\omega\,(^tUx)-\omega\,(x)\,|^2\mu\,(dx)\leq\sum_{n=1}^{\infty}\int_{S_{\phi n}}|\theta\,(x,\,U)\,\phi_n\,(^tUx)-\phi_n\,(x)\,|^2\mu\,(dx)=0$$

for all $U\in O\,(H)$. Thus $\omega\in\varPhi$ and $\mu\,(S_\omega)=\alpha$.

Next we claim that $\alpha=1$. If α would be less than 1, then θ is also a continuous 1-cocycle for a new rotationally invariant probability measure ν defined by

$$\nu\,(B):=\mu\,(S_\omega^c)^{-1}\mu\,(B\cap S_w^c)\ \text{ for all }B\in\mathfrak{B}.$$

So applying the above arguments to ν, we have a non zero function $\rho\in\mathrm{L}_\nu^2\,(H^a)$ such that

(3.13) $^\vee U\in O\,(H),\,\theta\,(x,\,U)\,\rho\,(^tUx)=\rho\,(x)$

for ν-a.e.x. Put

$$\phi_0\,(x):=\omega\,(x),\text{ if }x\in S_\omega,\,:=\rho\,(x),\text{ if }x\in S_\rho\backslash S_\omega,\text{ and }:=0,\text{ otherwise.}$$

Then it is easy to see that $\phi_0\in\varPhi$ and $\mu\,(S_{\phi_0})=\mu\,(S_\omega)+\mu\,(S_\rho\backslash S_\omega)>\mu\,(S_\omega)$. By the above, $\omega\,(x)\neq0$ for μ-a.e. x. Thus we have

$$^\vee U\in O\,(H),\,\theta\,(x,\,U)=\phi\,(^tUx)/\phi\,(x)$$

for μ-a.e.x, where ϕ is defined by $\phi\,(x):=|\omega\,(x)|/\omega\,(x)$.   □

Theorem 3.3.   (Uniqueness) *For $\mathfrak{B}$-measurable function ϕ, ϕ' with modulus 1, the followings are equivalent.*
   (1) $\theta_\phi=\theta_{\phi'}$.
   (2) *There exists some Borel function $k\,(c)$ $(c\geq0)$ with modulus 1 such that $\phi\,(x)=\phi'\,(x)\,k\,(p\,(x))$ for μ-a.e.x, where $p\,(x)$ is the average power,*

$$p\,(x)=\left[\overline{\lim}_n\frac{1}{n}\{\langle h_1,\,x\rangle^2+\cdots+\langle h_n,\,x\rangle^2\}\right]^{1/2}.$$

Proof. (1)$\Rightarrow$(2). Since

(3.14) $\mu=\int_{[0,\infty)}\gamma_c\,m\,(dc),$

where $m\,(E):=\mu\,(x\mid p\,(x)\in E)$, so for all $U\in O\,(H)$

$$\int_{H^a}\left|\frac{\phi\,(^tUx)}{\phi'\,(^tUx)}-\frac{\phi\,(x)}{\phi'\,(x)}\right|\mu\,(dx)=\int_{(0,\infty)}\int_{H^a}\left|\frac{\phi\,(c^tUx)}{\phi'\,(c^tUx)}-\frac{\phi\,(cx)}{\phi'\,(cx)}\right|\gamma\,(dx)\,m\,(dc)=0,$$

where $\gamma:=\gamma_1$. It follows from the separability of $O\,(H)$ that there exists a m-null set N such that for any $c\in N^c$,

$$(3.15) \qquad \forall U \in O(H), \ \int_{H^a} \left| \frac{\phi(c^t U x)}{\phi'(c^t U x)} - \frac{\phi(cx)}{\phi'(cx)} \right| \gamma(dx) = 0.$$

As γ is rotationally ergodic (See, [10].), so (3.15) implies that there exists some constant $k(c)$ such that $\phi(cx) = \phi'(cx) k(c)$ for γ-a.e.x. Of course $k(c)$ is a Borel function of c. In case of $m(\{0\}) > 0$, we make up for the definition by putting $k(0) := \phi(0)/\phi'(0)$. Then it is easy to see that

$$\phi(x) = \phi'(x) k(p(x))$$

for μ-a.e.x. The converse assertion directly follows from the above equation, because we have for all $U \in O(H)$, $p({}^t U x) = p(x)$ for μ-a.e.x. $\square$

3.2. Standard representation of $O(H)$. Let μ be a rotationally invariant probability measure, which is denoted by (3.14). In this subsection we consider the structure of the standard representation,

$$R_\mu(U) : f(x) \in L^2_\mu(H^a) \mapsto f({}^t U x) \in L^2_\mu(H^a).$$

As we have already seen, general representations with 1-cocycles are all equivalent to these $R_\mu(U)$. Put

$$(3.16) \qquad \beta := m(\{0\}) \quad \text{and} \quad \mu_1 := (1-\beta)^{-1} \int_{(0,\infty)} \gamma_c \, m(dc).$$

Then for any $f \in L^2_\mu(H^a)$ we have

$$(3.17) \qquad \|f\|^2_\mu = \beta |f(0)|^2 + (1-\beta) \|f\|^2_{\mu_1}.$$

If $\beta \neq 0$, the representation $(R_\mu, L^2_\mu(H^a))$ is equivalent to $(\mathrm{Id} \oplus R_{\mu_1}, \mathbf{C} \oplus L^2_{\mu_1}(H^a))$ by a map:

$$f \in L^2_\mu(H^a) \mapsto (\sqrt{\beta} f(0), \sqrt{1-\beta} f) \in \mathbf{C} \oplus L^2_{\mu_1}(H^a).$$

So the problem is reduced to the case $m(\{0\}) = 0$.

Theorem 3.4. *Assume that $\mu(\{0\}) = 0$. Then the representation $(R_\mu, L^2_\mu(H^a))$ is equivalent to $(\mathrm{Id} \otimes R_\gamma, L^2_m(\mathbf{R}^+) \otimes L^2_\gamma(H^a))$.*

Proof. For $\alpha_n \in \mathbf{C}$, $k_n \in L^2_m(0, \infty)$, $f_n \in L^2_\gamma(H^a)$ $(n = 1, \cdots, N)$, we have

$$\int_{H^a} \left| \sum_{n=1}^N \alpha_n k_n(p(x)) f_n\left(\frac{x}{p(x)}\right) \right|^2 \mu(dx) = \sum_{n,n'=1}^N \alpha_n \overline{\alpha_{n'}} \int_{H^a} k_n(p(x)) \overline{k_{n'}(p(x))} f_n\left(\frac{x}{p(x)}\right) \overline{f_{n'}\left(\frac{x}{p(x)}\right)} \mu(dx)$$

$$= \sum_{n,n'=1}^N \alpha_n \overline{\alpha_{n'}} \int_{\mathbf{R}^+} \int_{H^a} k_n(c) \overline{k_{n'}(c)} f_n(x/c) \overline{f_{n'}(x/c)} \gamma_c(dx) m(dc)$$

$$= \sum_{n, n'=1}^{N} \alpha_n \overline{\alpha_{n'}} \int_{\mathbf{R}^+} \int_{H^a} k_n(c) \overline{k_{n'}(c)} f_n(x) \overline{f_{n'}(x)} \gamma(dx) m(dc)$$

$$= \int_{\mathbf{R}^+} \int_{H^a} \left| \sum_{n=1}^{N} \alpha_n k_n(c) f_n(x) \right|^2 \gamma(dx) m(dc).$$

Thus a map

$$W: \sum_{n=1}^{N} \alpha_n k_n(c) f_n(x) \longrightarrow \sum_{n=1}^{N} \alpha_n k_n(p(x)) f_n\left(\frac{x}{p(x)}\right)$$

is isometric and it is extended to the whole space as a unitary operator. It is easily checked that W is an intertwining operator.   □

By the way, the representation $(R_\gamma, L^2_\gamma(H^a))$ has the following irreducible decomposition with multiple integral $\mathscr{H}_n$,

(3.18) $L^2_\gamma(H^a) = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_n \cdots,$

where each irreducible representation $(R_\gamma, \mathscr{H}_n)$ appears only one time. (See, [3,4].) It follows from Theorem 3.4 and from the preceding discussions before it that for a general μ, $(R_\gamma, \mathscr{H}_n)$ $(n \geq 1)$ appears in $(R_\mu, L^2_\mu(H^a))$, $\dim(L^2_m(\mathbf{R}^+))$ times and $(R_\gamma, \mathscr{H}_0)$ appears $\dim(L^2_m(\mathbf{R}^+))$ times or $\dim(L^2_m(\mathbf{R}^+)) + 1$ times according to $m(\{0\}) = 0$ or $m(\{0\}) > 0$.

Now if $(R_\mu, L^2_\mu(H^a))$ and $(R_{\mu'}, L^2_{\mu'}(H^a))$ are equivalent, then the intertwining operator maps rotationally invariant functions to rotationally invariant functions. It follows that $\dim(L^2_m[0, \infty)) = \dim(L^2_{m'}[0, \infty))$.

Conversely, if these two dimensions are equal, then the above two representations are mutually equivalent as far as $m(\{0\}) = m'(\{0\}) = 0$ or $m(\{0\})$, $m'(\{0\}) > 0$. If one of $m(\{0\})$, $m'(\{0\})$ is zero and the other one is positive, then the representations are mutually equivalent if and only if $\dim(L^2_m[0, \infty)) = \dim(L^2_{m'}[0, \infty)) = \infty$. We settle these arguments as follows.

Theorem 3.5. *Let μ, μ' be rotationally invariant probability measures. Then in oder that $(R_\mu, L^2_\mu(H^a))$ is equivalent to $(R_{\mu'}, L^2_{\mu'}(H^a))$, it is necessary that $\dim(L^2_m[0, \infty)) = \dim(L^2_{m'}[0, \infty))$.*
Further it is also sufficient under the following additional conditions.

   (1) $\dim(L^2_m[0, \infty)) = \infty$, *or*

   (2) $m(\{0\}) = m'(\{0\}) = 0$, *or $m(\{0\})$, $m'(\{0\}) > 0$ unless $\dim(L^2_m[0, \infty)) = \infty$.*

(If one of them is zero and the other is positive, then the representations are non equivalent.)

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