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Some Remarks on the Symplectic Pairing on the Moduli Space of Representations of the Fundamental Group of Surfaces

By

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§0. Introduction

This work grew out of an attempt to understand a conjectural remark made by Professor Kyoji Saito to the author about a possible link between the Fox-calculus description of the symplectic structure on the moduli space of representations of the fundamental group of surfaces into a Lie group and pairs of mutually dual sets of generators of the fundamental group. In fact in his paper [3], Prof. Kyoji Saito gives an explicit description of the system of dual generators of the fundamental group.

If S is a compact surface of genus $g \geq 2$ and its fundamental group $\pi_1(S)$ is given by a presentation $\pi_1(S) = \langle A_1, B_1, \ldots, A_g, B_g \mid \prod_{i=1}^{g} [A_i, B_i] = I \rangle$ (where I is the identity element), then the dual set of generators of $\pi_1(S)$ are defined by

$$\alpha_i = C_{i-1} B_i^{-1} C_i^{-1}$$

$$\beta_i = C_i A_i^{-1} C_{i-1}^{-1}$$

for $i=1, \ldots, g$ where $C_i = \prod_{j=1}^{i} [A_j, B_j]$.

It can be checked that $\prod_{i=1}^{g} [\alpha_i, \beta_i] = I$ so that $\pi_1(S)$ (henceforth simply denoted as π) has another presentation $\pi = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] = I \rangle$. Both the presentations are mutually dual since the following holds

$$A_i = \mathscr{C}_{i-1} \beta_i^{-1} \mathscr{C}_i^{-1}$$

$$B_i = \mathscr{C}_i \alpha_i^{-1} \mathscr{C}_{i-1}^{-1}$$

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† We would like to express our deep sorrow. Our friend Krishnamurthi Guruprasad was involved in a fatal car accident and passed away on January 5, 1998. K. Saito (communicator)

for $i=1, \ldots, g$ where $\mathscr{C}_i=\prod_{j=1}^{i}[\alpha_j, \beta_j]$.

Now if G is a compact Lie group together with adjoint invariant inner product on its Lie algebra, then the moduli space $\mathscr{M}=\mathrm{Hom}(\pi, \mathscr{G})/\mathscr{G}$ of conjugacy class of representations of π into G admits a natural symplectic structure. We explore in this paper the relationship between the Fox-calculus description of this symplectic structure and the mutually dual presentations of the fundamental group described above. In fact this paper re-interprets the results of [2] in the framework of mutually dual presentations of the fundamental group.

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§1. Fox-Calculus Description of the Symplectic Structure on the Moduli Space

Goldman ([1]) described explicitly the symplectic structure on the moduli space in terms of Fox-calculus. We recall this description in this section.

Let S be a compact surface of genus $g\geq 2$ and $\pi := \pi_1(S)$ be its fundamental group. Let $\pi=\langle A_1, B_1, \ldots, A_g, B_g \mid \prod_{i=1}^{g}[A_i, B_i]=I\rangle$ be a given presentation where I is the identity element. Let $C_i=\prod_{j=1}^{i}[A_j, B_j]$ and $R=C_g$ so that $R=I$ is the unique relation among the generators of π.

The anti-automorphism $\#$ on the integral group ring $\mathbb{Z}\pi$ is defined by

$$\#\left(\sum n_i a_i\right)=\sum n_i a_i^{-1} \text{ for } n_i\in\mathbb{Z} \text{ and } a_i\in\pi.$$

Using notations from Fox calculus [1,§3] we have

$$\frac{\partial R}{\partial A_i}=C_{i-1}(I-A_iB_iA_i^{-1})=C_{i-1}-C_iB_i$$
$$\frac{\partial R}{\partial B_i}=C_{i-1}(A_i-A_iB_iA_i^{-1}B_i^{-1})=C_{i-1}A_i-C_i$$

so that

$$\#\frac{\partial R}{\partial A_i}=C_{i-1}^{-1}-B_i^{-1}C_i^{-1}$$
$$\#\frac{\partial R}{\partial B_i}=A_i^{-1}C_{i-1}^{-1}-C_i^{-1}.$$

The tangent space $T_{[\rho]}(\mathcal{M})$ at an equivalence class $[\rho]\in\mathcal{M}$ is identified with the first group cohomology $H^1(\pi, \mathfrak{H}_\rho)$ where $\mathfrak{H}_\rho$ is the π-module, with the module structure defined by the composition $\pi \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{H})$.
$Z^1(\pi, \mathfrak{H}_\rho)=\{u:\pi\to\mathfrak{H}_\rho | u(xy)=u(x)+\mathrm{Ad}_\rho(x)(u(y))$ for $x, y\in\pi\}$ is the spece of 1-cocycles.

The following composition of the cup-product and $\langle,\rangle$

$$Z^1(\pi, \mathfrak{H}_\rho)\times Z^1(\pi, \mathfrak{H}_\rho)\xrightarrow{\cup} Z^2(\pi, \mathfrak{H}_\rho\otimes\mathfrak{H}_\rho)\xrightarrow{\langle,\rangle} \mathbf{Z}^2(\pi, \mathbf{R})$$

induces the a skew-symmetric pairing on cohomology

$$H^1(\pi, \mathfrak{H}_\rho)\times H^1(\pi, \mathfrak{H}_\rho)\to H^2(\pi, \mathbf{R})\simeq\mathbf{R}.$$

This skew-symmetric pairing induces the symplectic structure on $\mathcal{M}$.

We assume henceforth that ρ is irreducible. For simplicity we write $x\cdot u(y)$ or just $xu(y)$ for $\mathrm{Ad}_\rho(x)(u(y))$. We know from [1,§(3・4)] that the following formula

$$\omega(u, v)=-\sum_{i=1}^{g}\left\{\left\langle u\left(\#\frac{\partial R}{\partial A_i}\right), v(A_i)\right\rangle+\left\langle u\left(\#\frac{\partial R}{\partial B_i}\right), v(B_i)\right\rangle\right\}$$

(for $u, v\in Z^1(\pi, \mathfrak{H}_\rho)$) gives the symplectic form on $H^1(\pi, \mathfrak{H}_\rho)$.

§2. Mutually Dual Presentations of the Fundamental Group

If $\pi=\langle A_1, B_1, \dots, A_g, B_g | \prod_{i=1}^{g}[A_i, B_i]=I\rangle$ is a given presentation then we set

$$\alpha_i=C_{i-1}B_i^{-1}C_i^{-1}$$

$$\beta_i=C_iA_i^{-1}C_{i-1}^{-1}$$

for $i=1, \dots, g$ where $C_i=\prod_{j=1}^{i}[A_j, B_j]$. It can be checked that $\prod_{i=1}^{g}[\alpha_i, \beta_i]=I$ so that we have another presentation

$$\pi=\langle\alpha_1, \beta_1, \dots, \alpha_g, \beta_g | \prod_{i=1}^{g}[\alpha_i, \beta_i]=I\rangle.$$

If we set $\mathscr{C}_i=\prod_{j=1}^{i}[\alpha_j, \beta_j]$, then it can be checked that

$$A_i=\mathscr{C}_{i-1}\beta_i^{-1}\mathscr{C}_i^{-1}$$

$$B_i=\mathscr{C}_i\alpha_i^{-1}\mathscr{C}_{i-1}^{-1}$$

for $i=1, \dots, g$ where $\mathscr{C}_i=\prod_{j=1}^{i}[\alpha_j, \beta_j]$.

Consequently we see that both the above presentations of π are 'dual' to each other.

§3. An Inner Product on $Z^1(\pi, \mathfrak{H}_\rho)$.

Given a presentation $\pi = \langle A_1, B_1, \ldots, A_g, B_g \mid \prod [A_i, B_i] = 1\rangle$, define the symmetric pairing on $Z^1(\pi, \mathfrak{H}_\rho)$ by

$$\langle u, v\rangle_\rho = \sum_{i=1}^{g} \langle u(\alpha_i), v(\alpha_i)\rangle + \langle u(\beta_i), v(\beta_i)\rangle$$

where α_i, β_i are the dual generators defined in §2. $\langle,\rangle_\rho$ defines an inner product on $Z^1(\pi, \mathfrak{H}_\rho)$, since for $u \in Z^1(\pi, \mathfrak{H}_\rho)$

$$\langle u, u\rangle_\rho = 0 \Rightarrow \langle u(\alpha_i), u(\alpha_i)\rangle = 0 = \langle u(\beta_i), u(\beta_i)\rangle \text{ for } i=1, \ldots, g.$$

$$\Rightarrow u(\alpha_i) = u(\beta_i)\rangle = 0 \text{ since } \langle,\rangle \text{ is an inner product on } \mathfrak{H}$$

$$\Rightarrow u \equiv 0 \text{ since } u \text{ vanishes on the generators } \alpha_i, \beta_i \text{ of } \pi.$$

For $\mu \in \mathfrak{H}_\rho$ its co-boundary $\partial\mu \colon \pi \longrightarrow \mathfrak{H}_\rho$ is defined by $(\partial\mu)(x) = x \cdot \mu - \mu$ for $x \in \pi$. The space $B^1(\pi, \mathfrak{H}_\rho)$ is the space of coboundaries. Using the inner-product $\langle,\rangle_\rho$ on $Z^1(\pi, \mathfrak{H}_\rho)$ we can write

$$Z^1(\pi, \mathfrak{H}_\rho) = B^1(\pi, \mathfrak{H}_\rho) \oplus B^1(\pi, \mathfrak{H}_\rho)^\perp.$$

As in [2], we call $B^1(\pi, \mathfrak{H}_\rho)^\perp$ the space of 'harmonic' cocycles.

$$B^1(\pi, \mathfrak{H}_\rho)^\perp = \{u \in Z^1(\pi, \mathfrak{H}_\rho) \mid \langle u, \partial\mu\rangle_\rho = 0\ \forall \mu \in \mathfrak{H}\rho\}.$$

Now $\langle u, \partial\mu\rangle_\rho$

$$= \sum_{i=1}^{g} \{\langle u(\alpha_i), \partial\mu(\alpha_i)\rangle + \langle u(\beta_i), \partial\mu(\beta_i)\rangle\}$$

$$= \sum_{i=1}^{g} \{\langle u(\alpha_i), \alpha_i \cdot \mu - \mu\rangle + \langle u(\beta_i), \beta_i \cdot \mu - \mu\rangle\}$$

$$= \sum_{i=1}^{g} \{\langle \alpha_i^{-1} \cdot u(\alpha_i), \mu\rangle - \langle u(\alpha_i), \mu\rangle + \langle \beta_i^{-1} \cdot u(\beta_i), \mu\rangle - \langle u(\beta_i), \mu\rangle\}$$

$$= -\sum_{i=1}^{g} \langle u(\alpha_i^{-1}) + u(\alpha_i) + u(\beta_i^{-1}) + u(\beta_i), \mu\rangle$$

$$\text{since } \alpha_i^{-1} \cdot u(\alpha_i) = -u(\alpha_i^{-1})$$

$$\beta_i^{-1} \cdot u(\beta_i) = -u(\beta_i^{-1})$$

$$\langle u, \partial\mu\rangle_\rho = 0\ \forall \mu \Rightarrow \langle \sum_{i=1}^{g} \{u(\alpha_i) + u(\alpha_i^{-1}) + u(\beta_i) + u(\beta_i^{-1})\}, \mu\rangle = 0$$

$$\Rightarrow \sum_{i=1}^{g} \{u(\alpha_i) + u(\alpha_i^{-1}) + u(\beta_i) + u(\beta_i^{-1})\} = 0$$

since $\langle,\rangle$ is non-degenerate on $\mathfrak{H}_\rho$.

Thus we have the following characterisation of harmonic cocycles.

$$B^1(\pi, \mathfrak{H}_\rho)^\perp = \left\{u \in Z^1(\pi, \mathfrak{H}_\rho) \,\middle|\, \sum_{i=1}^{g} \{u(\alpha_i) + u(\alpha_i^{-1}) + u(\beta_i) + u(\beta_i^{-1})\} = 0\right\}$$

§4. The Symplectic Pairing and Its Relationship to Mutually Dual Generators

For $u \in Z^1(\pi, \mathfrak{H}_\rho)$, let $u^\perp$ be its harmonic part i.e $u^\perp$ is the projection of u onto $B^1(\pi, \mathfrak{H}_\rho)^\perp$. As in [2,§1], we define the map ϕ: $Z^1(\pi, \mathfrak{H}_\rho) \to Z^1(\pi, \mathfrak{H}_\rho)$ as

$$\phi(u)(A_i) = u^\perp\left(\# \frac{\partial R}{\partial A_i}\right)$$

$$\phi(u)(B_i) = u^\perp\left(\# \frac{\partial R}{\partial B_i}\right) \text{ for } i=1, \dots, g.$$

We need to check that $\phi(u)$ is a 1-cocycle or equivalently [1, §3·6] that the following identity holds

$$\phi(u)(R) = \sum_{i=1}^{g}\left\{\frac{\partial R}{\partial A_i}\phi(u)(A_i) + \frac{\partial R}{\partial B_i}\phi(u)(B_i)\right\} = 0.$$

Now $\frac{\partial R}{\partial A_i}\phi(u)(A_i)$

$$\begin{aligned}
&= \frac{\partial R}{\partial A_i} u^\perp\left(\# \frac{\partial R}{\partial A_i}\right)\\
&= (C_i - C_i B_i) u^\perp(C_{i-1}^{-1} - B_i^{-1} C_i^{-1})\\
&= C_i u^\perp(C_{i-1}^{-1}) - C_i B_i u^\perp(C_{i-1}^{-1}) - C_{i-1} u^\perp(B_i^{-1} C_i^{-1}) + C_i B_i u^\perp(B_i^{-1} C_i^{-1})\\
&= -\{u^\perp(C_{i-1}) + C_{i-1} u^\perp(B_i^{-1} C_i^{-1})\} - \{C_i B_i u^\perp(C_{i-1}^{-1}) + u(C_i B_i)\}\\
&= -\{u^\perp(C_{i-1} B_i^{-1} C_i^{-1}) + u(C_i B_i C_{i-1}^{-1})\}\\
&= -\{u^\perp(\alpha_i) + u^\perp(\alpha_i^{-1})\}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial R}{\partial B_i}\phi(u)(B_i) &= \frac{\partial R}{\partial B_i} u^\perp\left(\# \frac{\partial R}{\partial B_i}\right)\\
&= (C_i A_i - C_i) u^\perp(A_i^{-1} C_{i-1}^{-1} - C_i^{-1})\\
&= -u^\perp(C_{i-1} A_i) - C_i u^\perp(A_i^{-1} C_{i-1}^{-1}) - C_{i-1} A_i u^\perp(C_i^{-1}) - u^\perp(C_i)\\
&= -\{u^\perp(C_{i-1} A_i C_i^{-1}) + u^\perp(C_i A_i^{-1} C_{i-1}^{-1})\}\\
&= -\{u^\perp(\beta_i) + u^\perp(\beta_i^{-1})\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i=1}^{g}\left\{\frac{\partial R}{\partial B_i}\phi(u)(A_i) + \frac{\partial R}{\partial B_i}\phi(u)(B_i)\right\}\\
&= -\sum_{i=1}^{g}\left\{u^\perp(\alpha_i) + u^\perp(\alpha_i^{-1}) + u^\perp(\beta_i) + u^\perp(\beta_i^{-1})\right\}\\
&= 0
\end{aligned}$$

since $u^{\perp}$ is a harmonic cocycle.

Consequently ϕ: $Z^1(\pi, \mathfrak{H}_\rho) \longrightarrow Z^1(\pi, \mathfrak{H}_\rho)$ is well defined. We can identify $B^1(\pi, \mathfrak{H}_\rho)^{\perp}$ with $H^1(\pi, \mathfrak{H}_\rho)$. Now for harmonic cocycles $u, v \in B^1(\pi, \mathfrak{H}_\rho)^{\perp} \approx H^1(\pi, \mathfrak{H}_\rho)$ (i.e. $u=u^{\perp}$, $v=v^{\perp}$) we check the following identity

$$\omega(u, \phi(v)) = -\langle u, v\rangle_\rho.$$

We know from §1 that

$$\omega(u, \phi(v)) = -\sum_{i=1}^{g}\left\{\left\langle u\left(\#\frac{\partial R}{\partial A_i}\right), \phi(v)(A_i)\right\rangle + \left\langle u\left(\#\frac{\partial R}{\partial B_i}\right), \phi(v)(B_i)\right\rangle\right\}.$$

It follows from the definition of dual generators that

$$\#\frac{\partial R}{\partial A_i} = C_{i-1}^{-1} - B_i^{-1}C_i^{-1} = C_{i-1}^{-1} - C_{i-1}^{-1}\alpha_i$$

so that

$$\begin{aligned} u\left(\#\frac{\partial R}{\partial A_i}\right) &= u(C_{i-1}^{-1}) - u(C_{i-1}^{-1}\alpha_i) \\ &= u(C_{i-1}^{-1}) - u(C_{i-1}^{-1}) - C_{i-1}^{-1}u(\alpha_i) \\ &= -C_{i-1}^{-1}u(\alpha_i). \end{aligned}$$

Similarly

$$\#\frac{\partial R}{\partial B_i} = A_i^{-1}C_{i-1}^{-1} - C_i^{-1} = C_i^{-1}\beta_i - C_i^{-1}$$

so that

$$\begin{aligned} u\left(\#\frac{\partial R}{\partial B_i}\right) &= u(C_i^{-1}\beta_i - C_i^{-1}) - u(C_i^{-1}) \\ &= u(C_i^{-1}) - C_i^{-1}u(\beta_i) - u(C_i^{-1}) \\ &= C_i^{-1}u(\beta_i). \end{aligned}$$

We therefore have

$$\begin{aligned} \left\langle u\left(\#\frac{\partial R}{\partial A_i}\right), \phi(v)(A_i)\right\rangle &= \left\langle u\left(\#\frac{\partial R}{\partial A_i}\right), v\left(\#\frac{\partial R}{\partial A_i}\right)\right\rangle \quad \text{since } v=v^{\perp} \\ &= \langle -C_{i-1}^{-1}u(\alpha_i), -C_{i-1}^{-1}v(\alpha_i)\rangle \\ &= \langle u(\alpha_i), v(\alpha_i)\rangle. \end{aligned}$$

Similary

$$\left\langle u\left(\#\frac{\partial R}{\partial B_i}\right), \phi(v)(B_i)\right\rangle = \langle u(\beta_i), v(\beta_i)\rangle.$$

Thus we have

$$\omega(u, \phi(v)) = -\sum_{i=1}^{g}\{\langle u(\alpha_i), v(\alpha_i)\rangle + \langle u(\beta_i), v(\beta_i)\rangle\}$$
$$= -\langle u, v\rangle_{\rho}.$$

The identity $\omega(u, \phi(v)) = -\langle u, v\rangle_{\rho}$ suggests that ϕ may induce a complex structure on $B^1(\pi, \mathfrak{H}_{\rho})^{\perp} \approx H^1(\pi, \mathfrak{H}_{\rho})$ i.e. $\phi^2 = -Id$ on $H^1(\pi, \mathfrak{H}_{\rho})$ and is integrable. But we are unable to prove it. We leave it as a conjecture. Thus the symplectic structure ω on $\mathcal{M} = \mathrm{Hom}(\pi, G)/G$ intertwines the mutually dual presentations of π. The geometry behind this intertwining phenomenon is yet to be explored.

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