

Global Existence in Time and Decay Property of Solutions of Boundary Value Problems for Semilinear Hyperbolic Equations of Second Order in the Interior Domain

Dedicated to the memory of Professor Yukiyoishi Ebihara

By

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Introduction

Consider the following boundary value problem for a semilinear hyperbolic equations of second order.

$$\left. \begin{aligned} (0.1) \quad P_u[u] &= f(t, x) && \text{in } (t, x) \in [T, \infty) \times \Omega \\ (0.2) \quad u &= 0 && \text{on } [T, \infty) \times \partial\Omega \end{aligned} \right\} \quad (\mathbf{B.V.})$$

where $T \geq 0$, Ω is assumed to be a bounded domain in \mathbf{R}^n with a smooth boundary $\partial\Omega$,

$$(0.3) \quad P_u[\cdot] = \partial_t^2 \cdot - \sum_{i,j=1}^n \partial_i(a_{ij}(t, x) \partial_j \cdot) + A(t, x; u, \Lambda u)[u]$$

$$(0.4) \quad A(t, x; u, \Lambda u)[\cdot] = \sum_{|\alpha|, |\beta| \leq 1} a_{\alpha\beta}(t, x; u, \Lambda u) \Lambda^\alpha \cdot \Lambda^\beta \cdot,$$

$$\partial_i = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad \Lambda u = (\partial_t u, \partial_1 u, \dots, \partial_n u),$$

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n), \quad \beta = (\beta_0, \beta_1, \dots, \beta_n) \text{ are multi-indices,}$$

$$\text{and } \Lambda^\alpha u = (\partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u; \alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)).$$

We next make following assumptions on P_u .

(A-I) $\sum_{i,j=1}^n \partial_i(a_{ij}(t, x) \partial_j)$ is an elliptic operator satisfying for all $\eta \in \mathbf{R}^n$

$$(0.5) \quad \sum_{i,j=1}^n a_{ij} \eta_i \eta_j \geq C_0 \sum_{i=1}^n \eta_i^2 \quad (C_0 > 0),$$

$$a_{ij}(t, x) = a_{ji}(t, x) \quad (i, j = 1, \dots, n) \text{ for all } (t, x) \in [0, \infty) \times \bar{\Omega}.$$

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(A-II) All the coefficients and data are real valued and $a_{ij}(t, x)$ ($i, j=1, \dots, n$), $a_{\alpha\beta}(t, x; \xi)$ ($|\alpha|, |\beta| \leq 1$) are in $\mathcal{B}^\infty([0, \infty) \times \bar{\mathcal{D}} \times \mathbb{B}_0)$ where $\mathbb{B}_0 = \{\xi \in \mathbb{R}^{n+2} \mid |\xi| < \rho_0\}$ for a positive constant ρ_0 and $\xi = (\xi^*, \xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R} \times \mathbb{R}^{n+1}$.

Define a positive constant c_0 by

$$(0.6) \quad \sum_{i,j=1}^n \sup_{t \geq 0, x \in \mathcal{D}} |\partial_t a_{ij}(t, x)| = c_0.$$

For $0 \leq T_1 < T_2 < \infty$ and non-negative integer m $H_m((T_1, T_2) \times \Omega)$ and $H_m(\Omega)$ denote usual Sobolev spaces of order m on $(T_1, T_2) \times \Omega$ and Ω respectively and we put for a function $h(t, x)$ defined in $(t, x) \in (T_1, T_2) \times \Omega$

$$\|h\|_{m, (T_1, T_2) \times \Omega} = \int_{T_1}^{T_2} \|h\|_{m, \Omega}(t) dt,$$

$$\|h\|_{m, \Omega}(t) = \sum_{|\alpha| \leq m} \|\Lambda^\alpha h\|_{L^2_\omega}(t).$$

In the course of calculations below various constants will be simply denoted by C .

It was shown that the mixed problem for linear hyperbolic equations of second order with (0.2) is L^2 -well posed by Ikawa [6]. Shibata [24] studied the mixed problem for nonlinear hyperbolic equations of second order with a dissipative term with (0.2) and obtained time global classical solutions for small initial data. However nothing is known about time global classical solutions with exponential decay property of mixed problems for hyperbolic equations with nonlinear terms of the type of (0.4). That is to say, it seems to be very difficult to obtain it without dissipative term and appropriate additional condition on nonlinear terms for any given initial data (cf. Ebihara [3], Yamaguchi [25]). In fact, blow-up solutions were obtained by many authors [see [5], [7] - [10], [20)].

On the other hand, in case P_u is a nonlinear operator, time global solutions have been studied by quite a number of articles (see [2] - [5], [7] - [9], [11], [13], [14] - [19], [22], [23], [25], [26] and further references in these papers). In case $P_u[u] = \square u + \gamma u^m$ for an integer $m=3$ and $\gamma=1$, Sather [22] obtained a time global classical solution of a mixed problem for $P_u[u] = 0$ with (0.2). Sattinger [23] introduced the method of 'potential well' to show the existence of a time global weak solution of a mixed problem for wave equations with non-monotonic nonlinear terms (see Lions [11]). In the case of $m=n=3$ and $\gamma = -1$, Ebihara-Nakao-Nambu [3] proved the global existence in time of classical solutions of the mixed problem for $P_u[u] = 0$ with (0.2) (cf. Ebihara [2]). Ebihara [3] considered (B.V.) with initial data without smallness condition when $P_u = \partial_t^2 - \Delta + \mu \partial_t + f(t, x, u, u_t)$ for $\mu \geq 0$ and $T=0$. He proved that there exists a solution such that in the case of $\mu > 0$ it becomes a classical solution with exponential decay property *after finite time*. Also in the case of $\mu = 0$ he obtained 'modified (m)-solution', which is not necessarily a genuine solution of this problem.

On the other hand, recently Wayne [26] studied (B.V.) in case $P_u[u] = \square u - v(x)u + \varepsilon u^3 = 0$ for $v(x) \in L^2(\Omega)$ and a small constant ε and obtained

periodic and quasi-periodic solutions of (B.V.).

The main purpose of this paper is to seek a classical solution of (B.V.) with exponential decay property *after finite time* in case $f(t, x) \equiv 0$. Also we will discuss the existence of a classical solution of (B.V.) with $T = 0$. We shall show our main result.

Theorem 1. *Assume that (A-I) and (A-II) hold and that $e^{(\sigma+1/2)t}f(t, x) \in H_L((0, \infty) \times \Omega)$ for any integer $L \geq [(n+1)/2] + 3$ and a constant $\sigma > 0$. There exists a sufficiently large constant T and sufficiently small c_0 such that the problem*

(0.1) - (0.2) admits a solution $u(t, x) \in \bigcap_{l=0}^{L-1} C^l([T, \infty); H_{L-1-l}(\Omega)) \cap H_L([T, \infty) \times \Omega)$ having the property:

$$(0.7) \quad \|e^{\sigma t}u\|_{L([T, \infty) \times \Omega)} < +\infty, \quad \sup_{t \in [T, \infty)} e^{\sigma t}\|u\|_{L^{-1, \Omega}}(t) < +\infty,$$

and $\lim_{t \rightarrow \infty} e^{\sigma t}\|u\|_{L^{-1, \Omega}}(t) = 0.$

Remark. i) Based on Theorem 1, we consider the existence of a time global classical solution of (B.V.) with $T = 0$. Put $\zeta_L = \int_0^\infty \|e^{(\sigma+1/2)t}f\|_{L, \Omega}^2(t) dt$. Then there exists a constant $A > 0$ independent of ζ_L and T such that we may take a constant T in $[\max\{0, \log(A\zeta_L)\}, +\infty)$ arbitrarily in Theorem 1. Hence taking ζ_L small enough we obtain a solution in $[0, \infty) \times \Omega$ in Theorem 1. Especially in case $f(t, x) \equiv 0$, our solution of (B.V.) obtained in Theorem 1 is the trivial solution. These arguments will be discussed in detail in §3.

ii) We consider the case where $a_{ij}(t, x)$, $i, j = 1, \dots, n$ and $a_{\alpha\beta}(t, x; \xi)$ ($|\alpha|, |\beta| \leq 1$) are independent of t . Set $u(t + T_0, x) = V(t, x)$ for a solution $u(t, x)$ obtained in Theorem 1 where we denote T decided in Theorem 1 by T_0 . Then it is seen that $V(t, x)$ is a solution of the following problem.

$$\begin{cases} P_V[V(t, x)] = f(t + T_0, x) & \text{in } [0, \infty) \times \Omega, \\ V(t, x) = 0 & \text{on } [0, \infty) \times \partial\Omega. \end{cases}$$

On the other hand, as stated in *i)*, T_0 is decided essentially only by ζ_L for a fixed constant A . Therefore we can solve a problem $P_u[u] = f(t - T_0, x)$ with (0.2) in $[T_0, \infty) \times \Omega$ by the same argument as in Theorem 1 and we write a solution of this problem by $\dot{u}(t, x)$. It is easily seen that $V(t, x) = \dot{u}(t + T_0, x)$ is a solution of (B.V.) in $[0, \infty) \times \Omega$.

iii) When σ is sufficiently large, we obtain the same result as Theorem 1 without the smallness condition on c_0 . It will be discussed in subsection 2.1 of §2.

Now, we discuss the proof of our main result and the contents of the remainder of this paper. It seems to be very difficult to obtain a time global and classical solution of (B.V.) by solving the problem directly. To overcome this difficulty we reduce our problem (B.V.) to the following mixed problem by the

new time variable $s=e^{-t}$.

$$(\text{S.M.P.}) \begin{cases} Q_u[u]=f(s, x) & \text{in} & (s, x) \in (0, S] \times \Omega \\ u=0 & \text{on} & (0, S] \times \partial\Omega \\ s^{-\sigma}u=0, s^{1-\sigma}\partial_s u=0 & \text{at} & s=0 \end{cases}$$

where σ is a positive constant, $S=e^{-T}$ and Q_u is, the so called, a singular hyperbolic operator obtained by replacing ∂_t by $s\partial_s$ in P_u , which will be specified in §1 and we write $f(e^{-t}, x)$ by $f(s, x)$ again. In the below, when there is no danger of confusion, for any function $h(t, x)$ we write $h(e^{-t}, x)$ by $h(s, x)$ again.

That is to say, instead of solving (B.V.) directly, we will seek the time local smooth solution of (S.M.P.) with zero Cauchy data, which gives a classical solution of (B.V.) with exponential decay property.

In case Q_u is a linear operator (S.M.P.) has been studied by Sakamoto [21] in a more general framework. She obtained the existence of smooth solutions of (S.M.P.) for sufficiently large $\sigma > 0$. In this paper we obtain the time local existence theorem of smooth solution of a mixed problem for linear singular hyperbolic equation corresponding to (S.M.P.) for any $\sigma > 0$ and apply it to our problem.

This paper is organized as follows. In the first section, we introduce all notations in what follows and prepare some lemmas to prove our theorems. In Section 2, we derive the energy inequality of iteration scheme of (S.M.P.) at each step. In Section 3, we prove the time local existence of smooth solution of (S.M.P.) by the usual Picard's method. Going back to the original time variable, it will be easily shown that our solution of (S.M.P.) solves (B.V.) classically. Finally we discuss the case where non-homogenous Dirichlet boundary condition is given instead of (0.2). Also it will be shown that assumptions on coefficients of P_u in Theorem 1 can be relaxed.

§1. Preliminaries

By the new time variable $s=e^{-t}$, P_u is reduced to the following operator.

$$Q_u[\cdot] = (s\partial_s)^2 \cdot - \sum_{i,j=1}^n \partial_t(a_{ij}(s, x)\partial_j \cdot) + A(s, x; u, Du)[u]$$

where $Du = (s\partial_{su}, \partial_{1u}, \dots, \partial_{nu})$, $D^\alpha u = ((s\partial_s)^{\alpha_0} \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} u : \alpha = (\alpha_0, \alpha_1, \dots, \alpha_n))$. By this change of time variable, (A-III) and (0.5) are rewritten by (A-III)' and (1.0) respectively in the following.

(A-III)' All the coefficients and data are real valued and $D^j a_{ij}(s, x)$ ($i, j=1, \dots, n$), $D^j D_\xi^\omega a_{\alpha\beta}(s, x; \xi)$ ($|\alpha|, |\beta| \leq 1$), $|\gamma|, |\omega| \geq 0$, are bounded in $[0, 1] \times \bar{\Omega} \times \bar{\mathbb{B}}_0$ where $D_\xi^\omega = \partial_{\xi^*}^{\omega^*} \partial_{\xi_0}^{\omega_0} \dots \partial_{\xi_n}^{\omega_n}$ for a multi-index ω .

$$(1.0) \quad \sum_{i,j=1}^n \sup_{0 < s \leq 1, x \in \Omega} |s\partial_s a_{ij}(s, x)| = c_0.$$

If coefficients of P_u satisfy **(A-I)** and **(A-II)**, coefficients of Q_u satisfy **(A-I)** and **(A-II)'**. Then we consider the following mixed problem corresponding to **(B.V.)**.

$$\left. \begin{aligned} (1.1) \quad & Q_u[u] = f(s, x) && \text{in} && (s, x) \in (0, S] \times \Omega \\ (1.2) \quad & u = 0 && \text{on} && (0, S] \times \partial\Omega \\ (1.3) \quad & s^{-\sigma}u = 0, \quad s^{1-\sigma}\partial_{s^2}u = 0 && \text{at} && s = 0 \end{aligned} \right\} \quad \text{(S.M.P.)}$$

where σ is a positive constant and $S = e^{-T} \leq 1$.

Let us explain our basic notations required to treat with **(S.M.P.)**. $u(s, x) \in H_{\lambda, m}((0, S) \times \Omega)$ for non-negative integer m and $\lambda > 0$, if and only if $s^{-\lambda}D^\alpha u(s, x) \in L^2((0, S) \times \Omega)$ for $|\alpha| \leq m$. Put for $s \in (0, S]$

$$\begin{aligned} \|u\|_{m, \Omega}(s) &= \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2(\Omega)}(s), \\ \|u\|_{m, (0, S) \times \Omega} &= \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2((0, S) \times \Omega)}. \end{aligned}$$

Sometimes we denote $(\cdot, \cdot)_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ by (\cdot, \cdot) and $\|\cdot\|$ simply. Denote $\|h(s, x)\|_\infty^{(m)} = \sum_{|\alpha|+|\omega| \leq m} \sup_{\substack{(s, x) \in (0, 1) \\ \xi \in Y}} |D^\alpha D^\omega h(s, x; \xi)|$ where Y is an open set in \mathbb{R}^{n+2} . We

define $W_\infty^{(m)}((0, S) \times \Omega; Y)$ to consist of functions $h(s, x; \xi)$ satisfying $\|h\|_\infty^{(m)} < +\infty$.

Lemma 1.0. For $u(t, x) \in H_m((T_1, T_2) \times \Omega)$, there exists $\tilde{u}(t, x) \in H_m(\mathbb{R} \times \mathbb{R}^n)$ and a constant $C > 0$ such that $\tilde{u}(t, x) = u(t, x)$ in $(T_1, T_2) \times \Omega$ and support of \tilde{u} lies an arbitrary bounded open set in \mathbb{R}^{n+1} whose interior contains $[T_1, T_2] \times \bar{\Omega}$ and that it holds

$$(1.4) \quad \|\tilde{u}\|_{m, (-\infty, \infty) \times \mathbb{R}^n} \leq C \|u\|_{m, (T_1, T_2) \times \Omega}.$$

Proof. Applying Proposition 3.4 and Theorem 3.13 in Mizohata [13] to $u(t, x)$, we extend $u(t, x)$ to $\tilde{u}(t, x) \in H_m(\mathbb{R} \times \mathbb{R}^n)$ and (1.4) holds.

Lemma 1.1. Assuming that $|\alpha| \leq m, |\beta| \leq M, m \leq M, (n+1)/2 < m + M - |\alpha| - |\beta|$, for $q(s, x)$ and $r(s, x) \in H_{\sigma+1/2, M}((0, S) \times \Omega)$, there exists a constant $C > 0$ such that it holds that

$$(1.5) \quad \|s^{-\sigma-1/2}D^\alpha q D^\beta r\|_{L^2((0, S) \times \Omega)} \leq C \|s^{-\sigma-1/2}q\|_{m, (0, S) \times \Omega} \|s^{-1/2}r\|_{M, (0, S) \times \Omega},$$

$$(1.6) \quad \|s^{-\sigma-1/2}D^m(qr)\|_{L^2((0, S) \times \Omega)} \leq C \|s^{-\sigma-1/2}q\|_{m, (0, S) \times \Omega} \|s^{-1/2}r\|_{M, (0, S) \times \Omega}.$$

Proof. Going back to the original time variable $t = -\log s$, we write $q(e^{-t}, x)$ and $r(e^{-t}, x)$ by $Q(t, x)$ $R(t, x)$ for $t \in [-\log S, \infty)$ respectively. According to Lemma 1.0, we have extensions $\tilde{Q}(t, x)$ and $\tilde{R}(t, x)$ of $Q(t, x)$ and $R(t, x)$ respectively. First we will prove (1.5). We have

$$\|s^{-\sigma-1/2}D^\alpha q D^\beta r\|_{L^2((0, S) \times \Omega)} \leq C \sum_{\gamma \leq \alpha} \|s^{-1/2}D^\gamma (s^{-\sigma}q) D^\beta r\|_{L^2((0, S) \times \Omega)}$$

changing time variable from s to t

$$= C \sum_{\gamma \leq \alpha} \|A^\gamma (e^{\sigma t}Q) A^\beta R\|_{L^2((-\infty, T) \times \Omega)} \leq C \sum_{\gamma \leq \alpha} \|A^\gamma (e^{\sigma t}Q) A^\beta \tilde{R}\|_{L^2((-\infty, \infty) \times \mathbb{R}^n)}$$

applying Dionne [1, Theorem 6.3]

$$\leq C \|e^{-\sigma t} Q\|_{m, \mathbf{R}^{n+1}} \|\tilde{R}\|_{M, \mathbf{R}^{n+1}}$$

by using Lemma 1.0

$$\leq C \|e^{-\sigma t} Q\|_{m, (T, \infty) \times \Omega} \|R\|_{M, (T, \infty) \times \Omega} = C \|s^{-\sigma-1/2} q\|_{m, (0, S) \times \Omega} \|s^{-1/2} r\|_{M, (0, S) \times \Omega}.$$

In the same manner we have (1.6). Hence we omit the proof of (1.6). \square

We define for $L_0 = [(n+1)/2] + 2$ and a positive constant ρ_1

$$(1.7) \quad \Pi_1 = \{h(s, x) \in H_{\sigma+1/2, L_0}((0, S) \times \Omega) \mid \|s^{-\sigma-1/2} h\|_{L_0, (0, S) \times \Omega} < \rho_1\}.$$

From now on, ρ_1 is assumed to be so small that $(h, Dh) \in \mathbb{B}_0$ for any $h(s, x) \in \Pi_1$. In fact, we have by the change of time variable

$$\sum_{|\alpha| \leq 1} |D^\alpha h(s, x)| = \sum_{|\alpha| \leq 1} |A^\alpha h(t, x)| \leq \sum_{|\alpha| \leq 1} \sup_{(t, x) \in \mathbf{R}^+ \times \mathbf{R}^n} |A^\alpha \tilde{h}(t, x)|$$

by Sobolev's lemma and Lemma 1.0

$$\leq C \|\tilde{h}(t, x)\|_{L_0, \mathbf{R}^{n+1}} \leq C \|h(t, x)\|_{L_0, (T, \infty) \times \Omega} = C \|s^{-1/2} h(s, x)\|_{L_0, (0, S) \times \Omega}.$$

Lemma 1.2. *Assume that $q(s, x; \xi) \in W_\infty^{(M)}((0, S) \times \Omega; \mathbb{B}_0)$ and that $u(s, x) \in H_{\sigma+1/2, M+1}((0, S) \times \Omega)$ for $m \leq M$ and $(n+1)/2 < M-1$ and $u(s, x) \in \Pi_1$. There exists a constant $C > 0$ such that it holds*

$$\|q(s, x; u, Du)\|_{m, (0, S) \times \Omega} \leq C \|q(s, x; \xi)\|_\infty^{(m)} \times (1 + \|s^{-1/2} u\|_{M+1, (0, S) \times \Omega})^m.$$

Proof. We consider

$$(1.8) \quad (D_\xi^\gamma q)(s, x; u, Du) D^\alpha \{(D^\beta u, DD^\beta u)^\gamma\}, \quad |\alpha| + |\gamma| \leq m, \quad |\beta| = 1.$$

In case $|\gamma| \leq 1$ it is easily seen that

$$(1.9) \quad \begin{aligned} & \| (D_\xi^\gamma q)(s, x; u, Du) D^\alpha \{(D^\beta u, DD^\beta u)^\gamma\} \|_{L^\gamma, (0, S) \times \Omega} \\ & \leq C \|q(s, x; \xi)\|_\infty^{(1)} (1 + \|u\|_{M+1, (0, S) \times \Omega}). \end{aligned}$$

In case $|\gamma| \geq 2$, by using Lemma 1.1 we have

$$(1.10) \quad \begin{aligned} & \| (D_\xi^\gamma q)(s, x; u, Du) D^\alpha \{(D^\beta u, DD^\beta u)^\gamma\} \|_{L^\gamma, (0, S) \times \Omega} \\ & \leq C \|q(s, x; \xi)\|_\infty^{(m)} \left(\sum_{i=2}^m \|s^{-1/2} u\|_{M+1, (0, S) \times \Omega}^i \right). \end{aligned}$$

Essentially, $D^\alpha q(s, x; u, Du)$ for $|\alpha| \leq m$ is reduced to the form of (1.8). Hence by (1.9) and (1.10) we prove Lemma 1.2. \square

Lemma 1.3. *Suppose that $V(s, x)$ and $z(s, x) \in H_{\sigma+1/2, M+1}((0, S) \times \Omega)$ satisfying $V, V + \tau z \in \Pi_1$ for $\tau \in [0, 1]$. For $m \leq M$ and $(n+1)/2 \leq M-1$ it holds that for $q(s, x; \xi) \in W_\infty^{(M)}((0, S) \times \Omega; \mathbb{B}_0)$ and a constant $C > 0$*

$$(1.11) \quad \begin{aligned} & \|s^{-\sigma-1/2} \{q(s, x; V+z, D(V+z)) - q(s, x; V, DV)\} \|_{m-1, (0, S) \times \Omega} \\ & \leq C \|s^{-\sigma-1/2} z\|_{M, (0, S) \times \Omega}, \end{aligned}$$

where C depends on $\|q(s, x, \xi)\|_{\infty}^{(m)} \times \{1 + (\|s^{-1/2}z\|_{M+1, (0, S) \times \Omega} + \|s^{-1/2}V\|_{M+1, (0, S) \times \Omega})\}^{m-1}$.

Proof. In the same way as in the proof of Dionne [1, Theorem 6.4] we arrive at (1.11) using (1.9) and (1.10).

From Shibata [24, Theorem 4.5] we have the following lemma.

Lemma 1.4. *For any positive integer k there exists a constant M_k such that for any $\phi(x) \in H_{k+2}(\Omega) \cap \dot{H}_1(\Omega)$ it holds*

$$\sum_{|\alpha'|=k+2} \|D^{\alpha'} \phi\| \leq M_k \left\{ \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j \phi) \right\}_{k, \Omega} + \kappa_{k+1}(t) \|\phi\|_{1, \Omega}$$

where $\kappa_k(t) = \sum_{i,j=1}^n \sum_{|\alpha'| \leq k} \text{ess sup}_{x \in \Omega} |D^{\alpha'} a_{ij}(t, x)|$ and $\alpha' = (0, \alpha_1, \dots, \alpha_n)$ and $\dot{H}_1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H_1(\Omega)$.

§2. Energy Inequalities

We suppose that $w(s, x)$ is a known function satisfying for a constant $\mu > 0$

$$(2.1) \quad \|s^{-\sigma-1/2}w\|_{L^{+1, (0, 1)} \times \Omega}^2 < \mu.$$

We take μ small so that it holds $w \in \Pi_1$ in the rest of this paper. Let consider the following mixed problem for a linear singular hyperbolic equation for a positive constant σ .

$$(S.M.P.)_w \begin{cases} Q_w[u] = f(s, x) & \text{in } (s, x) \in (0, S] \times \Omega \\ u = 0 & \text{on } (0, S] \times \partial\Omega \\ s^{-\sigma}u = 0, s^{1-\sigma}\partial_s u = 0 & \text{at } s = 0. \end{cases}$$

2.1. Basic Estimate

Put

$$(2.2) \quad \phi[u](s) = \|s \partial_s u\|^2(s) + \sum_{i,j=1}^n (a_{ij} \partial_i u, \partial_j u)(s).$$

Then define $\phi_k[u](s) = \sum_{|\beta| \leq k} \phi[D^\beta u](s)$ for a non-negative integer k .

Lemma 2.1. *Assume that (A-I) and (A-II)' hold. For a positive constant σ , there exist a constant C and sufficiently small $S \leq 1$ and c_0 such that we have for $u(s, x) \in \mathcal{D}((0, S] \times \Omega)$, $\theta(S) = S^{2\sigma}$ and $s \in (0, S]$*

$$(2.3) \quad \begin{aligned} \phi[s^{-\sigma}u](s) + \sigma \int_0^s \tau^{-2\sigma-1} \phi[u](\tau) d\tau &\leq C \int_0^s \tau^{-2\sigma-1} \|Q_w[u]\|^2(\tau) d\tau \\ &+ C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2. \end{aligned}$$

Proof. Multiplying $Q_w[u]$ by $\partial_s u$ and integrating by parts we have

$$(2.4) \quad \partial_s \{ \|s \partial_s u\|^2 (s) + \sum_{i,j=1}^n (a_{ij} \partial_i u, \partial_j u) (s) \} = \\ \sum_{i,j=1}^n (a_{ij} \partial_i u, \partial_j u) (s) - 2 (A (s, x; w, Dw) [w], \partial_s u) (s) + 2 (Q_w [u], \partial_s u) (s).$$

Term by term multiplying by $s^{-2\sigma}$ and integrating over $(0, s)$

$$(2.5) \quad s^{-2\sigma} \phi [u] (s) + 2\sigma \int_0^s \tau^{-2\sigma-1} \phi [u] (\tau) d\tau = (s^{-2\sigma} \phi [u]) (0) + \int_0^s \left(2 (Q_w [u], \tau^{-2\sigma} \partial_\tau u) (\tau) \right. \\ \left. + \sum_{i,j=1}^n (a_{ij} \tau \partial_i u, \tau^{-2\sigma} \partial_j u) (\tau) - 2 \sum_{|\alpha|, |\beta| \leq 1} (a_{\alpha\beta} D^\alpha w D^\beta w, \tau^{-2\sigma} \partial_\tau u) (\tau) \right) d\tau.$$

On the other hand, from (0.5) it follows that for a positive constant C_0

$$\sum_{i,j=1}^n (a_{ij} \partial_i u, \partial_j u) (s) \geq C_0 \sum_{i=1}^n \|\partial_i u\|^2 (s).$$

For $c_1 = \min \{C_0, 1\}$ we have

$$(2.6) \quad \phi [u] (s) \geq c_1 (\|s \partial_s u\|^2 (s) + \sum_{i=1}^n \|\partial_i u\|^2 (s)).$$

We have for any positive constant ν

$$(2.7) \quad \text{the fourth term of the right handside of (2.5)} \\ = 2 \sum_{|\alpha|, |\beta| \leq 1} \int_0^s (a_{\alpha\beta} \tau^{-\sigma-1/2} D^\alpha w D^\beta w, \tau^{-\sigma+1/2} \partial_\tau u) (\tau) d\tau \\ \leq C \nu^{-1} \int_0^s \sum_{|\alpha|, |\beta| \leq 1} \|a_{\alpha\beta} \tau^{-\sigma-1/2} D^\alpha w D^\beta w\|^2 (\tau) d\tau + C \nu \int_0^s \|\tau^{-\sigma+1/2} \partial_\tau u\|^2 (\tau) d\tau$$

using Lemma 1.1 and (1.10) and (2.6)

$$\leq c_2 (a_{\alpha\beta}) \theta (S) \nu^{-1} \left(\int_0^s \tau^{-2\sigma-1} \phi_L [w] (\tau) d\tau \right)^2 + C \nu \int_0^s \tau^{-2\sigma-1} \phi [u] (\tau) d\tau$$

where a constant c_2 depends on $\sum_{|\alpha|, |\beta| \leq 1} \|a_{\alpha\beta}\|_\infty^{(0)}$. In the same way we have for any positive constant ν

$$(2.8) \quad \text{the second and third terms of right hand side of (2.5)} \\ \leq (c_3 (a_{ij}) + \nu c_1^{-1}) \int_0^s \tau^{-2\sigma-1} \phi [u] (\tau) d\tau + \nu^{-1} \int_0^s \tau^{-2\sigma-1} \|Q_w [u]\|^2 (\tau) d\tau$$

where c_3 depends on $\sum_{i,j=1}^n \sup_{(0,1) \times \Omega} |s \partial_s a_{ij}|$ and we can take c_3 small enough for c_3 to be taken sufficiently small. Therefore we have

$$(2.9) \quad s^{-2\sigma} \phi [u] (s) + 2\sigma \int_0^s \tau^{-2\sigma-1} \phi [u] (\tau) d\tau \leq c_3 \int_0^s \tau^{-2\sigma-1} \phi [u] (\tau) d\tau \\ + \nu^{-1} \int_0^s \tau^{-2\sigma-1} \|Q_w [u]\|^2 (\tau) d\tau + c_4 \nu \int_0^s \tau^{-2\sigma-1} \phi [u] (\tau) d\tau \\ + c_2 \theta (S) \nu^{-1} \left(\int_0^s \tau^{-2\sigma-1} \phi_L [w] (\tau) d\tau \right)^2.$$

On the other hand, taking c_3 and ν so small that it holds

$$(2.10) \quad c_3 + c_4\nu < \sigma,$$

we have

$$\begin{aligned} s^{-2\sigma}\phi[u](s) + \sigma \int_0^s \tau^{-2\sigma-1}\phi[u](\tau) d\tau &\leq C \int_0^s \tau^{-2\sigma-1} \|\mathbb{Q}_w[u]\|^2(\tau) d\tau \\ &\quad + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1}\phi_L[w](\tau) d\tau \right)^2. \end{aligned}$$

Hence Lemma 2.1 is proved. \blacksquare

If σ is taken large enough for (2.10) to hold instead of taking c_3 and ν sufficiently small in the proof of Lemma 2.1, it is easily seen that the smallness condition on c_0 is not necessary to obtain (2.3). Therefore we have Remark–iii).

2.2. Energy Inequality of Higher Order

Lemma 2.2. *Under the same assumptions as in Lemma 2.1, for a positive constant σ , there exists a constant C and sufficiently small $S \leq 1$ and c_0 such that we have for $u(s, x) \in \mathcal{D}((0, S] \times \Omega)$ and $s \in (0, S]$*

$$(2.11) \quad \begin{aligned} \int_0^s \tau^{-2\sigma-1}\phi_L[u](\tau) d\tau &\leq C \left\{ \int_0^s \tau^{-2\sigma-1} \|\mathbb{Q}_w[u]\|_{L^2_{t,\Omega}}^2(\tau) d\tau \right. \\ &\quad \left. + \theta(S) \left(\int_0^s \tau^{-2\sigma-1}\phi_L[w](\tau) d\tau \right)^2 \right\}. \end{aligned}$$

Proof. We derive (2.11) by induction on N . For $0 \leq N \leq L-1$ suppose that

$$(2.12) \quad \begin{aligned} \int_0^s \tau^{-2\sigma-1}\phi_N[u](\tau) d\tau &\leq C \int_0^s \tau^{-2\sigma-1} \|\mathbb{Q}_w[u]\|_{L^2_{t,\Omega}}^2(\tau) d\tau \\ &\quad + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1}\phi_L[w](\tau) d\tau \right)^2. \end{aligned}$$

Next operating $(s\partial_s)^L$ on $\mathbb{Q}_w[u]$, in the same way as in the proof of Lemma 2.1 we have by using Lemma 1.1

$$(2.13) \quad \begin{aligned} s^{-2\sigma}\phi[(s\partial_s)^L u](s) + \sigma \int_0^s \tau^{-2\sigma-1}\phi[(\tau\partial_\tau)^L u](\tau) d\tau \\ \leq C \left\{ \int_0^s \tau^{-2\sigma-1}\phi_{L-1}[u](\tau) d\tau + \int_0^s \tau^{-2\sigma-1} \|(\tau\partial_\tau)^L \mathbb{Q}_w[u]\|^2(\tau) d\tau \right. \\ \left. + \theta(S) \left(\int_0^s \tau^{-2\sigma-1}\phi_L[w](\tau) d\tau \right)^2 \right\}. \end{aligned}$$

Here we used

$$(2.14) \quad \sum_{|\alpha|, |\beta|=1} \int_0^s \tau^{-2\sigma-1} \|(\tau \partial_\tau)^L (a_{\alpha\beta} D^\alpha w D^\beta w)\|^2(\tau) d\tau \\ \leq C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2.$$

In fact, (2.14) is derived by the analogous way as derived Lemma 1.2. By using (2.12) we have

$$(2.15) \quad s^{-2\sigma} \phi[(s \partial_s)^L u](s) + \sigma \int_0^s \tau^{-2\sigma-1} \phi[(\tau \partial_\tau)^L u](\tau) d\tau \\ \leq C \int_0^s \tau^{-2\sigma-1} \|Q_w[u]\|_{L,\Omega}^2(\tau) d\tau + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2.$$

For $r \geq 2$, operating $(s \partial_s)^{L+1-r}$ on $Q_w[u]$ and applying Lemma 1.4 to

$\sum_{i,j=1}^n \partial_i (a_{ij} \partial_j (s \partial_s)^{L+1-r} u)$ we have

$$(2.16) \quad \|(s \partial_s)^{L+1-r} u\|_{r,\Omega}^2(s) \leq C \{ \|(s \partial_s)^{L+1-r} Q_w[u]\|_{r-2,\Omega}^2(s) + \|(s \partial_s)^{L+1-r+2} u\|_{r-2,\Omega}^2(s) \} \\ + C \sum_{|\alpha|, |\beta| \leq 1} \|(s \partial_s)^{L+1-r} (a_{\alpha\beta} D^\alpha w D^\beta w)\|_{r-2,\Omega}^2(s) + C\kappa_{r-1}^2(s) \|(s \partial_s)^{L+1-r} u\|_{1,\Omega}^2(s).$$

By the same way as derived (2.14) it is shown that the third term of the right hand side of (2.16), multiplied by $S^{-2\sigma-1}$ and integrated over $(0, S)$, is exceeded by

$$C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2.$$

On the other hand we have

$$\int_0^s \tau^{-2\sigma-1} \kappa_{r-1}^2(\tau) \|(\tau \partial_\tau)^{L+1-r} u\|_{1,\Omega}^2(\tau) d\tau \leq C \int_0^s \tau^{-2\sigma-1} \phi_{L-1}[u](\tau) d\tau$$

using (2.12)

$$(2.17) \quad \leq C \int_0^s \tau^{-2\sigma-1} \|Q_w[u]\|_{L-1,\Omega}^2(\tau) d\tau + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2.$$

Hence from (2.16) it follows that we have

$$(2.18) \quad \int_0^s \tau^{-2\sigma-1} \|(\tau \partial_\tau)^{L+1-r} u\|_r^2(\tau) d\tau \leq C \int_0^s \tau^{-2\sigma-1} \|Q_w[u]\|_{L,\Omega}^2(\tau) d\tau \\ + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2 + C \int_0^s \tau^{-2\sigma-1} \|(\tau \partial_\tau)^{L+3-r} u\|_{r-2,\Omega}^2(\tau) d\tau.$$

Since the third term of the right hand side of (2.18) for $r=2$ and 3 are exceeded by the right hand side of (2.15), summing up (2.18) for $r=2, \dots, L+1$ we have

$$(2.19) \quad \int_0^s \tau^{-2\sigma-1} \phi_L[u](\tau) d\tau \leq C \int_0^s \tau^{-2\sigma-1} \|Q_w[u]\|_{L,\Omega}^2(\tau) d\tau \\ + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1} \phi_L[w](\tau) d\tau \right)^2.$$

This completes the proof of Lemma 2.2. \blacksquare

§3. The Proof of Theorem 1

First we will show the existence of solution of $(\mathbf{S.M.P.})_w$ in this section. Sakamoto [21] proved the existence of a solution of the mixed problem for linear singular hyperbolic equations of higher order with flat Cauchy data. From her result it follows that there exists a smooth solution of $(\mathbf{S.M.P.})_w$ for a sufficiently large $\sigma > 0$. We will study the time local existence of smooth solution of $(\mathbf{S.M.P.})_w$ for any $\sigma > 0$. We have the following result.

Theorem 2. *Assume that (A-I) and (A-II)' hold and that $s^{-\sigma-1}f(s, x) \in H_L((0, 1) \times \Omega)$ for a constant $\sigma > 0$. There exists a sufficiently small constant $S \leq 1$ and c_0 such that $(\mathbf{S.M.P.})_w$ admits a solution $u(s, x)$ in $H_{\sigma+1/2, L+1}((0, S) \times \Omega)$, provided that $w(s, x)$ satisfies (2.1). It holds that*

$$(3.0) \quad s^{-2\sigma}\phi_{L-1}[u](s) + \int_0^s \tau^{-2\sigma-1}\phi_L[u](\tau) d\tau \leq C \int_0^s \tau^{-2\sigma-1} \|f(\tau, x)\|_{L, \Omega}^2(\tau) d\tau + C\theta(S) \left(\int_0^s \tau^{-2\sigma-1}\phi_L[w](\tau) d\tau \right)^2.$$

Proof. Put $F(s, x) = f(s, x) - A(s, x; w; Dw)[w]$. By the analogous way as derived (1.10) we have $A(s, x; w; Dw)[w] \in H_{\sigma+1/2, L}((0, S) \times \Omega)$. Thus there exists sufficiently smooth functions $F_k(s, x)$, $k \in \mathbb{N}$ such that $\text{supp } F_k \subset (0, S] \times \bar{\Omega}$ and $s^{-\sigma-1/2}F_k \rightarrow s^{-\sigma-1/2}F$ strongly in $H_L((0, S] \times \Omega)$ as $k \rightarrow \infty$. Then we consider the following problem.

$$(\mathbf{S.M.P.})_{w, k} \quad \begin{cases} (s\partial_s)^2 u_k - \sum_{i, j=1}^n \partial_i(a_{ij}(t, x)) \partial_j u_k = F_k(s, x) & \text{in } (s, x) \in (0, S] \times \Omega \\ u_k = 0 & \text{on } (0, S] \times \partial\Omega \\ s^{-\sigma} u_k = 0, s^{1-\sigma} \partial_{s_i} u_k = 0 & \text{at } s = 0 \end{cases}.$$

It is well known that there exists a smooth solution $u_k(s, x)$ of $(\mathbf{S.M.P.})_{w, k}$ with $\text{supp } u_k \subset (0, S] \times \bar{\Omega}$ (see Ikawa [6]). Since we have the energy inequality for $(\mathbf{S.M.P.})_{w, k}$ in the same way as in the proof of Lemmas 2.1 and 2.2, we have the following energy inequality of $u_{k+1} - u_k$ for solutions u_k and u_{k+1} of $(\mathbf{S.M.P.})_{w, k}$ and $(\mathbf{S.M.P.})_{w, k+1}$ respectively.

$$(3.1) \quad \int_0^s \tau^{-2\sigma-1}\phi_L[u_{k+1} - u_k](\tau) d\tau \leq C \int_0^s \tau^{-2\sigma-1} \|F_{k+1} - F_k\|_{L, \Omega}^2(\tau) d\tau.$$

On the other hand we have by simple computation

$$(3.2) \quad s^{-2\sigma}\phi_{L-1}[u_k](s) \leq C \int_0^s \tau^{-2\sigma-1}\phi_L[u_k](\tau) d\tau.$$

Hence there exists a function $u(s, x)$ such that

$$(3.3) \quad s^{-\sigma} u_k \rightarrow s^{-\sigma} u \quad \text{strongly in } \bigcap_{l=0}^L C^l((0, S]; H_{L-l}(\Omega))$$

and $s^{-\sigma-1/2}k \rightarrow s^{-\sigma-1/2}u$ strongly in $H_{L+1}((0, S) \times \Omega)$ as $k \rightarrow \infty$.

From the energy inequality for $(\mathbf{S.M.P.})_{w,k}$, $k=1,2,\dots$ and (3.2) it follows that

$$(3.4) \quad s^{-2\sigma}\phi_{L-1}[u](s) + \int_0^s \tau^{-2\sigma-1}\phi_L[u](\tau) d\tau \leq C \int_0^s \tau^{-2\sigma-1} \|F(\tau, x)\|_{L,\Omega}^2(\tau) d\tau$$

(3.4) implies that (3.0) holds. Hence $u(s, x)$ is a solution of $(\mathbf{S.M.P.})_w$ and Theorem 3 is proved. \square

We see easily that it holds for a constant $C_2 > 0$

$$(3.5) \quad \text{the right hand side of (3.0)} \leq C_2(S^{2\sigma}\mu^2 + S\zeta_L).$$

In fact, we have for $s \in (0, S]$

$$\int_0^s \tau^{-2\sigma-1} \|f\|_{L,\Omega}^2(\tau) d\tau \leq S \int_0^s \tau^{-2\sigma-2} \|f\|_{L,\Omega}^2(\tau) d\tau$$

by the change of time variable from s to the original one

$$= e^{-T} \int_T^\infty e^{(2\sigma+1)\tau} \|f\|_{L,\Omega}^2(\tau) d\tau \leq S\zeta_L$$

where $S = e^{-T}$. Through the rest of this paper we take S small enough in (3.5) so that it holds for a solution $u(s, x)$ obtained in Theorem 2 and a constant $C_3 > 0$

$$(3.6) \quad \|s^{-\sigma-1/2}u\|_{L+1,(0,S) \times \Omega}^2 \leq C_3(S^{2\sigma}\mu^2 + S\zeta_L) < \mu.$$

Note that (3.6) is necessary for the energy inequality of the type of (3.0) to hold at each step of the iteration scheme of $(\mathbf{S.M.P.})$.

Proof of Theorem 1. Let us consider the following iteration scheme of $(\mathbf{S.M.P.})$, $j=1,2,\dots$

$$(\mathbf{S.M.P.})_j \quad \begin{cases} P_{u_{j-1}}[u_j] = f(s, x) & \text{in } (0, S) \times \Omega \\ u_j = 0 & \text{on } (0, S) \times \partial\Omega \\ s^{-\sigma}u_j = 0, s^{1-\sigma}\partial_s u_j = 0 & \text{at } s = 0 \end{cases}$$

where $u_0 \equiv 0$.

Theorem 2 implies that there exists a solution u_j of $(\mathbf{S.M.P.})_j$. In fact, since (3.6) is valid for u_1 and S appeared in (3.6), it is seen that $u_{j-1} \in \Pi_1$ and (3.0) holds for u_j in $(\mathbf{S.M.P.})_j$, $j=1,2,\dots$. Then (3.0) implies that $\{u_j\}$ are uniformly bounded in $H_{\sigma+1/2, L+1}((0, S) \times \Omega)$. From $(\mathbf{S.M.P.})_j$ and $(\mathbf{S.M.P.})_{j+1}$ it follows that $u_{j+1} - u_j$, $j=1,2,\dots$ satisfy

$$\begin{cases} P_{u_j}[u_{j+1} - u_j] + (P_{u_j} - P_{u_{j-1}})[u_j] = 0 & \text{in } (0, S) \times \Omega \\ u_{j+1} - u_j = 0 & \text{on } (0, S) \times \partial\Omega \\ s^{-\sigma}(u_{j+1} - u_j) = 0, s^{1-\sigma}\partial_s(u_{j+1} - u_j) = 0 & \text{at } s = 0. \end{cases}$$

In the same way as derived (3.0) we have by using (1.10) and Lemma 1.3

$$(3.7) \quad \begin{aligned} & s^{-2\sigma}\phi_{L-2}[u_{j+1}-u_j](s) + \int_0^s \tau^{-2\sigma-1}\phi_{L-1}[u_{j+1}-u_j](\tau) d\tau \\ & \leq C\mu\theta(S) \int_0^s \tau^{-2\sigma-1}\phi_{L-1}[u_j-u_{j-1}](\tau) d\tau. \end{aligned}$$

Taking μ further sufficiently small, it is seen that $\{u_j\}$ is a strongly convergent sequence in $H_{\sigma+1/2L}((0, S] \times \Omega)$ and $\{s^{-\sigma}u_j\}$ strongly converge in $\bigcap_{i=0}^{L-1} C^i((0, S]; H_{L-1-i}(\Omega))$. Hence there exists a solution $u(s, x)$ of **(S.M.P.)** such that $u(s, x) \in \bigcap_{i=0}^{L-1} C^i((0, S]; H_{L-1-i}(\Omega)) \cap H_L((0, S] \times \Omega)$ satisfying

$$\begin{aligned} & s^{-\sigma}u_j \rightarrow s^{-\sigma}u \quad \text{strongly in } \bigcap_{i=0}^{L-1} C^i((0, S]; H_{L-1-i}(\Omega)) \\ & \text{and } s^{-\sigma-1/2}u_j \rightarrow s^{-\sigma-1/2}u \quad \text{strongly in } H_L((0, S] \times \Omega) \text{ as } j \rightarrow \infty. \end{aligned}$$

Put $u(e^{-t}, x) = U(t, x)$. Since $\|s^{-\sigma}u_j\|_{L-1, \Omega}(0) = 0, j=1, 2, \dots$, it is easily seen that $U(t, x)$ satisfies (0.1), (0.2) and (0.7) in $[T, \infty) \times \Omega$ for $T = -\log S$. Thus this completes the proof of Theorem 1. \square

To obtain Theorem 1 we have to take μ small in the above so that it holds $u_j \in \Pi_1$ for a solution u_j of **(S.M.P.)**, $j=1, 2, \dots$ and that $\{u_j\}$ strongly converge. We can take μ further small so that $C_3(\mu^2 + S\zeta_L) \leq \mu$ in case $S\zeta_L$ is small enough. In fact, for a positive number $x < 1/(2C_3)$ there exists a sufficiently small constant $\delta > 0$ such that $C_3x^2 - x + C_3\delta < 0$. For such μ and δ there exists a constant A independent of S and ζ_L such that

$$(3.8) \quad A = \delta^{-1}, S\zeta_L \leq A^{-1} \text{ and } C_3A^{-1} < \mu - C_3\mu^2$$

hold. On the other hand, even if S is any fixed constant in $(0, 1]$, we may take ζ_L small instead of S so that (3.8) holds. Therefore (3.8) implies that Remark-i) holds. Especially in case $f(t, x) \equiv 0$ our solution of **(B.V.)** obtained in Theorem 1 is the trivial solution since by Lemma 2.2 we have the solution of **(S.M.P.)**, $u_j(s, x) \equiv 0, j=1, 2, \dots$.

We consider the case where the following non-homogeneous boundary condition is imposed on **(B.V.)**:

$$(3.9) \quad u(t, x) = g(t, x) \quad \text{on } [T, \infty) \times \partial\Omega$$

where $g(t, x)$ is the trace of a function $G(t, x)$ satisfying $e^{(\sigma+1/2)t}G(t, x) \in H_{L+2}((0, \infty) \times \Omega)$. Then we have the same result as obtained in Theorem 1.

Corollary 1. *Assume that $e^{(\sigma+1/2)t}G(t, x) \in H_{L+2}((0, \infty) \times \Omega)$ for a constant $\sigma > 0$ and $G(t, x) \in \Pi_2$ where Π_2 is defined by replacing ρ_1 by a constant $\rho_2 < \rho_1$ in the definition of Π_1 . Under the same assumptions as in the Theorem 1, there exist sufficiently small c_0 and $\rho_2 > 0$ and sufficiently large $T > 0$ such that (0.1) and (3.9) admit a solution $u(t, x)$ having the same regularity and properties as in Theorem 1.*

Proof. Put $v(t, x) = u(t, x) - G(t, x)$. Then v satisfies

$$\begin{cases} \partial_t^2 v - \sum_{i,j=1}^n \partial_i (a_{ij}(t, x) \partial_j v) + A(t, x; v+G, \Lambda(v+G)) [v] = f(t, x) - \partial_t^2 G + \sum_{i,j=1}^n \partial_i (a_{ij}(t, x) \partial_j G) \\ \quad + A(t, x; v+G, \Lambda(v+G)) [v] - A(t, x; v+G, \Lambda(v+G)) [G+v] \\ v|_{\partial\Omega} = 0. \end{cases}$$

By the same manner as in the proof of Theorem 1, we can arrive at the desired result. Therefore we omit it. \square

We show that we obtain the similar result as in Theorem 1, even if $|a_{\alpha\beta}(t, x; \xi)| \rightarrow \infty$ ($t \rightarrow \infty$) for any $x \in \bar{\Omega}$ and $\xi \in \bar{\mathbb{B}}_0$. We make the following assumption.

(A-III) All the coefficients and data are real valued and $a_{ij}(t, x)$, $i, j = 1, \dots, n$, $e^{-\varepsilon t} a_{\alpha\beta}(t, x; \xi)$, $|\alpha|, |\beta| \leq 1$, satisfy (A-II) for a positive constant $\varepsilon < \sigma$.

Then we obtain the following result.

Corollary 2. Assume that (A-I) and (A-III) hold for a constant $\sigma > 0$. If $e^{(\sigma+1/2)t} f(t, x) \in H_L((0, \infty) \times \Omega)$, there exists sufficiently small c_0 and sufficiently large $T > 0$ such that we have the same result as in Theorem 1.

Proof. By Lemma 1.1 and (1.10) we have for $u(s, x) \in \mathcal{D}((0, 1] \times \Omega)$ satisfying (2.1)

$$\|s^{-\sigma-1/2} a_{\alpha\beta}(s, x; u, Du) Du^{\alpha} u Du^{\beta} u\|_{L^1(0, S) \times \Omega} \leq CS^{\sigma-\varepsilon} \|s^{\varepsilon} a_{\alpha\beta}\|_{\infty}^{(L)} \|s^{-\sigma-1/2} u\|_{L^2(0, S) \times \Omega}^2.$$

Hence for sufficiently small S and c_0 it is easily seen that we obtain Lemma 2.1 and Lemma 2.2, considering into $0 < \sigma - \varepsilon$. By the same way as in the proof of Theorem 1 we arrive at desired result. Hence we omit it. \square

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References

- [1] Dionne, P. Sur les problèmes de Cauchy hyperboliques bien posés. *J. Anal. Math.*, **10** (1962), 1-90
- [2] Ebihara, Y. On the equation $u_{tt} - u_{xx} = F(x, t, u, u_t, u_x)$. *Funkcial. Ekvac.*, **20** (1977), 77-95.
- [3] _____, On solutions of semilinear wave equations. *Nonlinear Anal.*, **6-5** (1982), 467-486.
- [4] Ebihara, Y. and Nakao, M. and Nambu, T., On the existence of global classical solution of initial boundary value problem for $\square u - u^3 = f$. *Pacific J. Math.*, **60** (1975), 63-70.
- [5] Georgiev, V. and Todorova, G., Existence of a solution of the wave equation with nonlinear damping and source terms. *J. Diff. Equations*, **109** (1994), 295-308.
- [6] Ikawa, M., Mixed problem for hyperbolic equations of second order. *J. Math. Soc. Japan*, **20-4**

- (1968), 580-608.
- [7] Ikehata, R., A note on the global solvability of solutions to some nonlinear wave equations with dissipative term, *Diff. Integral Equations*, **8**-3(1995), 607-616.
- [8] _____, Some remarks on the wave equation with nonlinear damping and source terms, *Nonlinear Anal., T.M.A.*, **27**-10(1996), 1165-1175.
- [9] Ikehata, R. and Suzuki, T., Stable and unstable sets for evolution equations of parabolic and hyperbolic type, *Hiroshima Math. J.*, **26**-3(1996), 475-491.
- [10] John, F., Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, **28**(1979), 235-268.
- [11] Lions, J. L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [12] Mizohata, S., *The Theory of Partial Differential Equations*, Cambridge Univ Press, London, 1973.
- [13] Matumura, A., Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with the first order dissipation, *Publ. RIMS, Kyoto Univ.*, **13**(1977), 349-379.
- [14] Nakao, M., Convergence of solutions of the wave equation with a nonlinear dissipative term to the steady state, *Mem. Fac. Sci. Kyushu Univ. Ser. A.*, **30**-2(1976), 257-265.
- [15] _____, Decay of classical solutions of a semilinear wave equation, *Math. Rep. Kyushu Univ.*, **11**-1(1977), 39-45.
- [16] _____, A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc Japan*, **30**(1978), 747-762.
- [17] _____, Global existence of classical solutions to the initial-boundary value problem of the semilinear wave equations with a degenerate dissipative term, *Nonlinear Anal.*, **15**(1990), 115-140.
- [18] _____, Remarks on the existence and uniqueness of global decaying solutions of the nonlinear dissipative wave equations, *Math. Z.*, **206**(1991), 265-276
- [19] _____, Existence of global smooth solutions to the initial-boundary value problem for the quasilinear hyperbolic equation with a degenerate dissipative term, *J. Diff. Equations*, **98**(1992), 299-327.
- [20] Ohta, M., Blowup of solutions of dissipative nonlinear wave equations, *Hokkaido J.*, to appear.
- [21] Sakamoto, R., Mixed problems for degenerate hyperbolic equations, *J. Math Kyoto Univ.*, **23**-3(1983), 563-597.
- [22] Sather J., The existence of a global classical solution of the initial-boundary value problem for $\square u + u^3 = f$, *Arch Rational Mech. Anal.*, **22**(1966), 292-307
- [23] Sattinger, D. H., On global solution of nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.*, **30**(1968), 148-172.
- [24] Shibata, Y., On the global existence of classical solutions of mixed problem for some second order non-linear hyperbolic operators with dissipative term in the interior domain, *Funkcial Ekvac.*, **25**(1982), 303-345.
- [25] Yamaguchi, M., Existence and stability of global bounded classical solutions of initial boundary value problem for semilinear wave equations, *Funkcial. Ekvac.*, **23**(1980), 289-308.
- [26] Wayne, C. E., Periodic and quasi-periodic solutions of nonlinear wave equations via KAM Theory, *Comm. Math. Phys.*, **127**(1990), 479-528.

