Commuting Squares and the Classification of Finite Depth Inclusions of AFD Type III$_\lambda$ Factors, $\lambda \in (0, 1)$

By

Phan H. LOI[*]

Abstract

We give a new proof of the classification result due to Sorin Popa that a finite depth inclusion of AFD type III$_\lambda$ factors $N \subset M$, $\lambda \in (0, 1)$, with a common discrete decomposition $\{N^\infty \subset M^\infty, \theta\}$ is classified, up to isomorphism, by the type II core $N^\infty \subset M^\infty$ and the standard invariant of θ.

§1. Introduction

Let $\lambda \in (0, 1)$ and $N \subset M$ be an inclusion of AFD type III$_\lambda$ factors with finite index and with a common discrete decomposition $\{N^\infty \subset M^\infty, \theta\}$. It is an interesting problem to classify up to isomorphism inclusions of this kind with the same index.

In [20] Popa has shown that if $N \subset M$ is *strongly amenable* then such an inclusion is classified by the isomorphism class of $N^\infty \subset M^\infty$ and the standard invariant θ^{st} of θ. This result was obtained as a consequence of a powerful classification theorem that properly/strongly outer actions of countable discrete amenable groups on strongly amenable inclusions of type II factors are classified by their standard invariants together with their modules if the factors are of type II$_\infty$. By [18, 19], finite depth inclusions of AFD type II factors are strongly amenable.

In this note, we will give a different proof of the finite depth case of this classification result of Popa by proving directly that the isomorphism class of $N \subset M$ is classified by that of $N^\infty \subset M^\infty$ and the conjugacy class of θ^{st}.

The main arguments of our proof can be summarized as follows. Let $N \subset M$ be a finite depth inclusion of AFD type III$_\lambda$ factors $N \subset M$, $\lambda \in (0, 1)$, with a

---

common discrete decomposition $\{N^\infty \subset M^\infty, \theta\}$. If the standard invariant θ^{st} of θ on $N^\infty \subset M^\infty$ is trivial, then the classification of $N \subset M$ follows from Theorem 6.1 in [12], if θ^{st} is not trivial, then it must have a finite period, say m. We can then show that there exists a finite depth inclusion of type III_{λ^m} factors $Q \subset P$ which is *split*, i.e., isomorphic to $Q^\infty \otimes R_{\lambda^m} \subset P^\infty \otimes R_{\lambda^m}$, where R_{λ^m} is the AFD type III_{λ^m} factor and there exists an automorphism α on $Q \subset P$ having period m and which satisfies: $mod\,(\alpha) \equiv \lambda^{-1}$ (in $\mathbb{R}_*^+/\lambda^{m\mathbb{Z}}$), the standard invariant of α on $N^\infty \subset M^\infty$ is equal to θ^{st} and $N \subset M$ is the simultaneous crossed products of $Q \subset P$ by α. To classify $N \subset M$, it amounts to classifying α. Since $Q \subset P$ is split, to classify α, using Connes' tensoring trick suitably adapted to our setting, we can reduce the problem to the case where α has trivial module and the trivial module case can be further reduced to the case of a periodic and strongly outer α acting on a finite depth inclusion of AFD type II_1 factors $B \subset A$ such that α and α^{st} have order m. To this end, thanks to a result in [1], all we need to show is that the

$$\begin{array}{ccc} A & \subset & A \times_\alpha \mathbb{Z}_m \\ \cup & & \cup \\ B & \subset & B \times_\alpha \mathbb{Z}_m \end{array}$$

higher relative commutants of the commuting square   completely

determines the conjugacy class of α. Thi idea of using the commuting square of the simultaneous crossed product algebras to classify finite group actions on inclusions of factors was suggested in [8].

§2. Notations and Preliminaries

This section is intended as a brief summary of some of the results that will be needed in this paper, further detail can be found in the articles cited therein.

1) *Commom discrete decomposition.* For $\lambda \in (0, 1)$, let $N \overset{E_N^M}{\subset} M$ be an inclusion of type III_λ factors, where E_N^M is the conditional expectation with minimal index (cf. [6]), $N \subset M$ is said to have a common discrete decomposition with respect to E_N^M if for any generalized trace (also called a λ-trace) φ on N, i.e., φ is a normal faithful semi-finite weight on N and $\sigma_T^\varphi = Id_N$, where $T = |2\pi/\ln \lambda|$, then $\psi = \varphi \cdot E_N^M$ is a generalized trace on M and the pair of type II_∞ factors $N^\infty = N^\varphi \subset M^\infty = M^\psi$ with give rise to a common discrete decomposition of $N \subset M$, i.e., there exist a normal faithful semi-finite trace τ on $N^\infty \subset M^\infty$ and an automorphism θ which acts simultaneously on $N^\infty \subset M^\infty$ satisfying $\tau \cdot \theta = \lambda \tau$ and such that $N \subset M$ is isomorphic to the inclusion of crossed products $N^\infty \times_\theta \mathbb{Z} \subset M^\infty \times_\theta \mathbb{Z}$; $N^\infty \subset M^\infty$ is often called the type II *core* of $N \subset M$. It is important to point out that by a result of Connes, a common discrete decomposition of $N \subset M$ is unique up to conjugacy (cf. [4]).

2) *The standard invariant of an automorphism on an inclusion of factors.* Let $N \subset$

M be an inclusion of factors with finite index and let $\alpha \in Aut(M, N)$, that is, $\alpha \in Aut(M)$ and $\alpha(N) = N$. Suppose that N and M are of type II. Let $\cdots \subset N_{k+1} \subset N_k \subset \cdots \subset N_0 = N \subset M$ be a tunnel of $N \subset M$ (cf. [7]), then it was shown in [12] that α induces an automorphism on the tower of higher relative commutants $\{N_k' \cap N \subset N_k' \cap M; \ k \geq 0\}$ that is known as the *standard invariant* of α and denoted $\Phi_{M,N}(\alpha)$. It was also shown in [12] that $\Phi_{M,N}(\alpha)$ is trace-preserving, fixes all the Jones projections associated with the given tunnel and preserves the inclusion $N_k' \cap N \subset N_k' \cap M$ for each $k \geq 0$. It turns out that $\Phi_{M,N}(\alpha)$ depends only on the choice of the tunnel up to conjugacy and its extension to the closures of $\cup N_k' \cap M$ and $\cup N_k' \cap M$ is denoted α^{st}. For convenience, we will use α^{st} also to represent $\Phi_{M,N}(\alpha)$.

If $N \subset M$ is extremal, i.e., its Jones index is minimal (cf. [6, 16]), then we can construct α^{st} by using the tower of basic constructions as follows. Assuming M is in standard from with canonical involution J, let $N \subset M = M_0 \subset \cdots M_k \subset \cdots M_{k+1} \subset \cdots$, then α can be canonically extended to each M_k by fixing all the Jones projections. We will still use α to denote this unique extension. It is well-known that $N_{k+1} = J M_k' J$ is a tunnel of $N \subset M$, and the standard invariant of α constructed using this tunnel is given by the restriction of $AdJ \cdot \alpha \cdot AdJ$ on each $N_k' \cap N \subset N_k' \cap M$.

Suppose now $N \subset M$ are of type III$_\lambda$ with a common discrete decomposition $\{N^\infty \subset M^\infty, \ \theta\}$ and $\alpha \in Aut(M, N)$. Then as above, we can still define $\Phi_{M,N}(\alpha)$ but it turns out that this will not be sufficient for the classification of α as it does not contain any information about the Connes-Takesaki module of α. Instead the method introduced in [22] has to be used to define a conjugacy invariant for α as follows. First, we choose a unique $\mu \in [1, \ \lambda^{-1})$ such that $mod(\alpha) \equiv \mu$, and let φ be a generalized trace on $N \subset M$ that gives rise to the common discrete decomposition of $N \subset M$ (cf. [5]), then there exists a unitary u in N such that, with $\alpha_0 = Adu \cdot \alpha$, $\varphi \cdot \alpha_0 = \mu^{-1} \varphi$, α_0 commutes with θ and preserves the inclusion of $N^\infty \subset M^\infty$. Hence it makes sense to consider $\Phi_{M^\infty, N^\infty}(\alpha_0)$ and it is straightforward to verify that $\Phi_{M^\infty, N^\infty}(\alpha_0)$ does not depend on the choice of u and is a cocyle conjugacy invariant for α as automorphism on $N \subset M$. For simplicity, $\Phi_{M^\infty, N^\infty}(\alpha_0)$ will be denoted α^{st}.

§3. Main Results

For the definition and the basic properities of strongly outer and strongly free automorphisms, we refer the reader to [1, 20, 10, 23]. Our first lemma is a direct corollary of the results in [1, 8, 19].

Lemma 1. *Let $B \subset A$ be a finite depth inclusion of AFD type II$_1$ factors. Let α be a strongly outer action of $\mathbf{Z}_m$ on $B \subset A$ such that α^{st} has period m. Then α is*

classified up to conjugacy by α^{st}.

Proof. Set $M_{00}=B$, $M_{11}=A \times_\alpha \mathbb{Z}_m$, $M_{10}=A$, $M_{01}=B \times_\alpha \mathbb{Z}_m$ and consider the commuting square:
$$\begin{array}{ccc} M_{10} & \subset & M_{11} \\ \cup & & \cup \\ M_{00} & \subset & M_{10} \end{array}$$
. Without any loss of generality, we may assume that M_{00} is in standard form with canonical conjugation J. The downward basic construction of the above commuting square of factors gives rise to another commuting square
$$\begin{array}{ccc} M_{0-1} & \subset & M_{00} \\ \cup & & \cup \\ M_{-1-1} & \subset & M_{-10} \end{array}$$
satisfying $JM'_{10}J=M_{-10}$ and $JM'_{01}J=M_{0-1}$.

By [21], $M_{-1-1} \subset M_{00}$ has finite depth and and thus by [18], there is a tunnel $\{M_{-j-j}; j \geq 0\}$ of $M_{-1-1} \subset M_{00}$ such that $M'_{-j-j} \cap M_{-1-1} \uparrow M_{-1-1}$ and $M'_{-j-j} \cap M_{00} \uparrow M_{00}$ and as pointed out in [8], it follows that $M'_{-j-j} \cap M_{0-1} \uparrow M_{0-1}$ and $M'_{-j-j} \cap M_{-10} \uparrow M_{-10}$. Let $M_{jj}=JM'_{-j-j}J$, then $\{M_{jj}\}$ is isomorphic to the tower of basic constructions of $M_{00} \subset M_{11}$ and by [16] $J \cdot J$ is a trace-preserving anti-isomorphism that clearly maps the commuting squares
$$\begin{array}{ccccccc} M_{0-1} \cap M'_{-j-j} & \subset & M_{00} \cap M'_{-j-j} & & M'_{01} \cap M_{jj} & \subset & M'_{00} \cap M_{jj} \\ \cup & & \cup & \text{onto} & \cup & & \cup \\ M_{-1-1} \cap M'_{-j-j} & \subset & M_{-10} \cap M'_{-j-j} & & M'_{11} \cap M_{jj} & \subset & M'_{10} \cap M_{jj} \end{array}$$
, the latter commuting squares have been computed in Theorem 9 of [1] and they depend on the isomorphism class of $B \subset A$, the conjugacy class of α^{st} and the group $\mathbb{Z}_m$. Therefore the commuting square of factors formed by $B \subset A$ with their simultaneous crossed products by α is isomorphic to the one constructed using α^{st}. Standard arguments then imply that α is conjugate to some power of α^{st} and hence there is a family of Jones projections $\{e_{-j}; j \geq 0\}$ that is associated with a generating tunnel of $B \subset A$ and for which $\alpha(e_{-j})=e_{-j}$ for all j.

Let β be another strongly outer action of $\mathbb{Z}_m$ on $B \subset A$ such that β^{st} has period m and that α^{st} and β^{st} are conjugate. The preceding arguments applied to β show that there is another family of Jones projections $\{f_{-j}; j \geq 0\}$ associated with some generating tunnel of $B \subset A$ such that $\beta(f_{-j})=f_{-j}$ for all j. It follows that if σ is the automorphism on $B \subset A$ such that $\sigma(f_{-j})=f_{-j}$, then $\sigma \cdot \alpha \cdot \sigma^{-1}(f_{-j})=f_{-j}$ for all j, i.e., $\sigma \cdot \alpha \cdot \sigma^{-1}$ is standard with respect to the tunnel determined by $\{f_{-j}; j \geq 0\}$ and is hence conjugate to β by assumption. Thus we have shown that α and β are conjugate.   Q.E.D.

Remark. We would like to point out that Lemma 1 will still hold if $B \subset A$ are of type II_∞ and $\alpha \in Aut(A, B)$ is trace preserving because in this case with a little more work one can show that there exist type II_1 factors $B_0 \subset A_0$ and an automorphism $\alpha_0 \in Aut(A_0, B_0)$ such that $B \subset A \simeq B_0 \otimes \mathfrak{B}(\mathfrak{H}) \subset A_0 \otimes \mathfrak{B}(\mathfrak{H})$ and α is conjugate to $\alpha_0 \otimes Id_{\mathfrak{B}(\mathfrak{H})}$.

To continue, we recall that an inclusion of type III_λ factors $Q \subset P$ is said to

be *split* if there exist type II factors $B \subset A$ and a type III_λ factors M such that $Q \subset P$ is isomorphic to $B \otimes M \subset A \otimes M$.

Lemma 2. *Let $Q \subset P$ be a split inclusion of AFD type III_λ factors with finite depth. Let α be a strongly free action of $\mathbf{Z}_m$ on $Q \subset P$ with trivial module and such that α^{st} has period m. Then α is classified up to conjugacy by α^{st}.*

Proof. Let $\{Q^\infty \subset P^\infty, \theta\}$ be a common discrete decomposition of $Q \subset P$. By Lemma 2 of [14], there exists a conjugate α_0 of α such that α_0 commutes with θ and $\alpha_0(Q^\infty) = Q^\infty$, $\alpha_0(P^\infty) = P^\infty$ and
$$\begin{array}{ccc} P & \subset & P \times_\alpha \mathbf{Z}_m \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z}_m \end{array}$$
is isomorphic to

$$\begin{array}{ccc} P^\infty \times_\theta \mathbf{Z} & \subset & (P^\infty \times_{\alpha_0} \mathbf{Z}_m) \times_\theta \mathbf{Z} \\ \cup & & \cup \\ Q^\infty \times_\theta \mathbf{Z} & \subset & (Q^\infty \times_{\alpha_0} \mathbf{Z}_m) \times_\theta \mathbf{Z} \end{array}$$. As
$$\begin{array}{ccc} P^\infty & \subset & P^\infty \times_{\alpha_0} \mathbf{Z}_m \\ \cup & & \cup \\ Q^\infty & \subset & Q^\infty \times_{\alpha_0} \mathbf{Z}_m \end{array}$$
is isomorphic to

$$\begin{array}{ccc} P^\infty \otimes R_{0,1} & \subset & (P^\infty \times_{\alpha_0} \mathbf{Z}_m) \otimes R_{0,1} \\ \cup & & \cup \\ Q^\infty \otimes R_{0,1} & \subset & (Q^\infty \times_{\alpha_0} \mathbf{Z}_m) \otimes R_{0,1} \end{array}$$, where $R_{0,1}$ is the AFD type II_∞ factor, we can

consider the automorphism $\theta_0 = \theta \otimes \theta_\lambda^{-1}$, where $mod(\theta_\lambda) = \lambda$.

Since $Q \subset P$ is split by assumption, the standard invariant of θ on $Q^\infty \subset P^\infty$ is trivial by [12] and so the same is true for θ_0. Since $mod(\theta_0) = 1$, it follows that θ_0 is approximately inner by [12], i.e., $\theta_0(x) = \lim_{n \to \infty} Adu_n(x)$ for some sequence of unitaries u_n in $Q^\infty \otimes R_{0,1}$ and for all $x \in P^\infty \otimes R_{0,1}$. On the other hand, since α_0 commutes with θ, we have: $\lim_{n \to \infty} Adu_n = \theta_0 = (\alpha_0 \otimes Id_{R_{0,1}}) \cdot \theta_0 \cdot (\alpha_0 \otimes Id_{R_{0,1}})^{-1}$

$= \lim_{n \to \infty} Ad(\alpha_0 \otimes Id_{R_{0,1}})(u_n)$. It follows that $\{u_n^*(\alpha_0 \otimes Id_{R_{0,1}})(u_n)\}$ is a centralizing sequence for $Q^\infty \otimes R_{0,1} \subset P^\infty \otimes R_{0,1}$. As $\alpha_0 \otimes Id_{R_{0,1}}$ is strongly, outer, it is centrally free by [20] and hence by [4], there is a centralizing sequence $\{v_n\}$ in $Q^\infty \otimes R_{0,1}$ such that $\lim_{n \to \infty}(u_n^*(\alpha_0 \otimes Id_{R_{0,1}})(u_n) - v_n^*(\alpha_0 \otimes Id_{R_{0,1}})(v_n)) = 0$, hence $\theta_0 = \lim_{n \to \infty} Ad(u_n v_n^*)$ as automorphism on $Q^\infty \otimes R_{0,1} \subset P^\infty \otimes R_{0,1} \times_{\alpha_0 \otimes Id_{R_{0,1}}} \mathbf{Z}_m$. In other words, θ_0 is approximately inner on $Q^\infty \otimes R_{0,1} \subset P^\infty \otimes R_{0,1} \times_{\alpha_0 \otimes Id_{R_0}} \mathbf{Z}_m$ and hence the corresponding standard invariant of θ_0 is trivial and this in turn implies that the standard invariant of θ on $Q^\infty \subset P^\infty \times_\alpha \mathbf{Z}_m$ is trivial. Since $Q \subset P \times_\alpha \mathbf{Z}_m$ has finite depth by [1, 21], Theorem 6.1 in [12] implies that $Q \subset P \times_\alpha \mathbf{Z}_m$ is a split inclusion and a simple argument will show that the commuting square
$$\begin{array}{ccc} P & \subset & P \times_\alpha \mathbf{Z}_m \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z}_m \end{array}$$
is isomorphic to the commuting square of tensor products

$$\begin{array}{ccccc} P^\infty \otimes R_\lambda & \subset & (P^\infty \times_{\alpha_0} \mathbf{Z}_m) \otimes R_\lambda & = & (P^\infty \otimes R_\lambda) \times_{\alpha_0 \otimes Id_{R_\lambda}} \mathbf{Z}_m \\ \cup & & \cup & & \\ Q^\infty \otimes R_\lambda & \subset & (Q^\infty \times_{\alpha_0} \mathbf{Z}_m) \otimes R_\lambda & = & (Q^\infty \otimes R_\lambda) \times_{\alpha_0 \otimes Id_{R_\lambda}} \mathbf{Z}_m \end{array}$$, where R_λ is the AFD

factor of type III$_\lambda$. Therefore α is conjugate to an automorphism of the form $\alpha_0 \otimes Id_{R_\lambda}$, where α_0 is strongly outer on $Q^\infty \subset P^\infty$ and $\alpha_0^{st} = \alpha^{st}$. By Lemma 1, α_0 is classified by its standard invariant and so the same is true for α.   Q.E.D.

Lemma 3. *Let $Q \subset P$ be a split inclusion of type* III$_\lambda$ *factors. Let $\alpha \in Aut\,(P,$ $Q)$ be such that α has period m and mod $(\alpha) \equiv \lambda^{-1/m}$. Then $\alpha \simeq \alpha \otimes Id_{R_\lambda}$.*

Proof. Let $N = Q \times_\alpha \mathbb{Z}_m$ and $M = P \times_\alpha \mathbb{Z}_m$, then N and M are of type III$_{\lambda^{1/m}}$. By the arguments in Case 1 of Theorem 1 in [14], there is a generalized trace ψ on $N \subset M$ such that $\hat{\alpha} = \sigma_T^\psi$, where $T = |2\pi/\ln \lambda|$. We claim that $N \subset M$ is isomorphic to $N \otimes R_\lambda \subset M \otimes R_\lambda$. Indeed, as we have $N \times_{\hat\alpha} \mathbb{Z}_m \subset M \times_{\hat\alpha} \mathbb{Z}_m \simeq Q \subset P$ and $Q \subset P \simeq Q \otimes R_\lambda \subset P \otimes R_\lambda$ by assumption and as $\hat{\alpha} = \sigma_T^\psi$, $\hat{\alpha} \in \overline{Int}\,(M, N)$ by [22], the subfactor version of Corollary II.3 in [4] (with an action of $\mathbb{Z}_m$ instead of $\mathbb{Z}$) can then be used to infer that $N \subset M \simeq N \otimes R_\lambda \subset M \otimes R_\lambda$.

Now if φ is a generalized trace on R_λ such that $\sigma_T^\varphi = Id$ then $\sigma_T^\psi \simeq \sigma_T^{\psi \otimes \varphi} = \sigma_T^\psi \otimes Id_{R_\lambda}$. By the Takesaki Duality Theorem, it follows that $\alpha \simeq \alpha \otimes Id_{R_\lambda}$ as desired.
                                                                Q.E.D.

With the factorization property established in Lemma 3, we can apply Connes' tensoring trick to obtain following.

Lemma 4. *Let $Q \subset P$ be a split inclusion of AFD type* III$_\lambda$ *factors with finite depth. Let $\alpha \in Aut\,(P, Q)$ be such that mod $(\alpha) \equiv \lambda^{-1/m}$, and α and α^{st} both have order m. Then α is classified up to conjugacy by α^{st}.*

Proof. As $Q \subset P$ is split, $Q \subset P$ is isomorphic to $Q \otimes R_\lambda \otimes R_\lambda \subset P \otimes R_\lambda \otimes R_\lambda$ and so we can consider the automorphism $\alpha \otimes \sigma^{-1}$, where σ is an automorphism on R_λ having period m and module $\lambda^{-1/m}$.

Since $\sigma^{-1} \otimes \sigma$ has trivial module, $\sigma^{-1} \otimes \sigma$ is conjugate to $s_m \otimes Id_{R_\lambda}$ by [2], where s_m is the model outer automorphism of period m on the AFD type II$_1$ factor R_0 and we have used the isomorphism between $R_\lambda \otimes R_\lambda$ and $R_0 \otimes R_\lambda$ to define $s_m \otimes Id_{R_\lambda}$. It follows that $\alpha \otimes \sigma^{-1} \otimes \sigma \simeq \alpha \otimes s_m \otimes Id_{R_\lambda} \simeq \alpha \otimes Id_{R_\lambda} \simeq \alpha$ because $\alpha \otimes s_m \simeq \alpha$ by [15, 23], and $\alpha \otimes Id_{R_\lambda} \simeq \alpha$ by Lemma 4. By [23], $\alpha \otimes \sigma^{-1}$ is still strongly free hence by Lemma 2, $\alpha \otimes \sigma^{-1}$ is classified up to conjugacy by its standard invariant which is given by α^{st}, it follows that α is also classified up to conjugacy by α^{st}.   Q.E.D.

We can now proceed to prove the following classification result originally due to Sorin Popa.

Theorem 1. *Let $N \subset M$ be a finite depth inclusion of AFD type* III$_\lambda$ *factors with a common discrete decomposition $\{N^\infty \subset M^\infty, \theta\}$. Then $N \subset M$ is classified up to isomorphism by $N^\infty \subset M^\infty$ and θ^{st}.*

f.

Proof. By Connes' uniqueness theorem of the discrete decomposition and by the definition of the standard invariant in [12], it is clear that $\{N^\infty \subset M^\infty, \theta^{st}\}$ is a conjugacy invariant of $N \subset M$.

Conversely, we will show that $N \subset M$ is determined up to isomorphism by $N^\infty \subset M^\infty$ and θ^{st}. First of all, if θ^{st} is trivial, then the result follows from Theorem 6.1 of [12]. Thus we may assume that θ^{st} is not the identity.

Since the higher relative commutants for $N \subset M$ are given by the fixed point algebras of those for $N^\infty \subset M^\infty$ under θ^{st} by [12], and since $N \subset M$ and hence $N^\infty \subset M^\infty$, have finite depth (cf. [12]), Lemma 2.2 in [13] implies that θ^{st} has period m for some $m > 1$ (see also [10]). Let φ be a generalized trace on N such that $N^\infty = N^\varphi$ and $M^\infty = M^\psi$, where $\psi = \varphi \cdot E_N^M$. Let $Q \subset P$ be the simultaneous fixed point algebras of $N \subset M$ under the modular automorphism $\sigma_{T/m}^\psi$, where $T = |2\pi/\ln\lambda|$. Then $Q = N^\infty \times_{\theta^m} \mathbf{Z}$ and $P = M^\infty \times_{\theta^m} \mathbf{Z}$ (cf. [11]) and hence Q and P are of type III$_{\lambda^m}$. Since $(\theta^m)^{st} = (\theta^{st})^m = Id$ and $Q \subset P$ has finite depth, Theorem 6.1 in [12] implies that $Q \subset P$ is split and so $Q \subset P$ is classified by its type II core which is given by $N^\infty \subset M^\infty$. On the other hand, by Takesaki Duality, there exists an automorphism α on $Q \subset P$ of order m and such that $\hat\alpha = \sigma_{T/m}^\psi$, and $N = Q \times_\alpha \mathbf{Z}_m$ and $M = P \times_\alpha \mathbf{Z}_m$. It follows that $mod\,(\alpha) \equiv \lambda^{-1}$ (in $\mathbf{R}_*^+/\lambda^{m\mathbf{Z}}$) and $\alpha^{st} = \theta^{st}$ (cf. Theorem 1 in [13]). By Lemma 4, α is classified up to conjugacy by α^{st}.

$$\begin{array}{ccc} P & \subset & P \times_\alpha \mathbf{Z}_m = M \\ \cup & & \cup \\ Q & \subset & Q \times_\alpha \mathbf{Z}_m = N \end{array}$$

Therefore the isomorphism class of the commuting square is determined by $N^\infty \subset M^\infty$ and θ^{st}. In particular, we can conclude that $N \subset M$ is classified up to isomorphism by $N^\infty \subset M^\infty$ and θ^{st}. Q.E.D.

References

[1] Choda, M. and Kosaki, H., Strongly outer actions for an inclusion of factors. *J. Func. Anal.,* **122** (1994), 315-332.

[2] Connes, A., Outer conjugacy classes of automorphisms of factors, *Ann. Sc. Ec. Norm. Sup.,* **8** (1975), 383-420.

[3] _____. Periodic automorphisms of the hyperfinite factor of type II$_1$, *Acta Sci.,* **39** (1977), 39-66.

[4] _____, Une classification des facteurs de type III. *Ann. Sci. Ec. Norm. Sup.,* **6**, (1973), 133-252.

[5] _____. Factors of type III$_1$, property L_λ' and closure of inner automorphisms, *J. Op. Theory,* **14** (1985), 189-211.

[6] Hiai, F., Minimizing indices of conditional expectations onto a subfactor. *Publ. RIMS Kyoto Univ.,* **24** (1988), 673-678.

[7] Jones, V. F. R., Index for subfactors, *Invent. Math.,* **72** (1983), 1-25.

[8] Kawahigashi, Y., Classification of paragroup actions on subfactors, *Publ. RIMS Kyoto Univ.,* **31**

(1995), 481-518.

[9] Kosaki, H., Extension of Jones' theory on index to arbitrary factors, *J. Func. Anal.*, **66** (1986), 123-140.

[10] _____, *Automorphisms in irreducible decompositions of sectors in Quantum and non-commutative Analysis*, Math. Phys. Stud., **16** (1993), 305-316.

[11] Loi, P. H., On the theory of index and type III factors, *J. Operator Theory*, **28** (1992), 251-265.

[12] _____, On automorphisms of subfactors, *J. Func. Anal.*, **141** (1996), 275-293.

[13] _____, Remarks on automorphisms of subfactors, *Proc. AMS*, **121** (1994), 523-531.

[14] _____, Periodic and strongly free automorphisms on inclusions of type III$_\lambda$ factors, $0 < \lambda < 1$, *Inter. Jour. Math.*, **8** (1997), 83-96.

[15] Ocneanu, A., Actions of discrete amenable groups on the hyperfinite type II$_1$ factor, *Springer Verlag Lecture Notes Series*, **1138** (1986).

[16] Pimsner, M. and Popa, S., Entropy and index for subfactors, *Ann. Sc. Ec. Norm. Sup.*, **19** (1986), 57-106.   .

[17] _____, Iterating the basic construction, *Trans. Amer. Math. Soc.*, **310** (1988), 127-133.

[18] Popa, S., Classification of subfactors: the reduction to commuting squares, *Inv. Math.*, **101** (1990), 19-43.

[19] _____, Classification of amenable subfactors of type II, *Acta Math.*, **172** (1994), 352-445.

[20] _____, Classification of actions of discrete amenable groups on amenable factors of type II, *preprint* (1992).

[21] Wiezbicki, J., An estimate of the depth from an intermediate subfactor, *Publ. RIMS, Kyoto Univ.*, **30** (1994), 1139-1144.

[22] Winslow, C., Approximately inner automorphisms on inclusions of type III$_\lambda$ factors, *Pac. J. Math.*, **166** (1994), 385-400.

[23] _____, Strongly free actions on subfactors, *Int. J. Math.*, **4** (1993), 675-688.