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Quantum Toroidal Algebras and Their Vertex Representations

By

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Abstract

We construct the vertex representations of the quantum toroidal algebras $U_q(\mathfrak{sl}_{n+1,tor})$. In the classical case the vertex representations are not irreducible. However in the quantum case they are irreducible.

§1. Introduction

The classical toroidal algebras have been studied by many authors [MEY], [S], [Y], etc. Here "classical" means $q = 1$. The definition of the quantum toroidal algebras is given in [GKV]. They gave a geometric realization of the quantum toroidal algebras without any results on their representation theory. Recently Varagnolo and Vasserot [VV] proved Schur-type duality between representations of the quantum toroidal algebras and the double affine Hecke algebra introduced by Cherednik [C]. This is an analogue of the duality between the quantum affine algebras and the affine Hecke algebras given by Chari and Pressly [CP]. In [VV] only the representations of "trivial central charge" was studied. It is known that there are two subalgebras $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ and $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$ of $U_q(\mathfrak{sl}_{n+1,tor})$ such that there are surjective algebra homomorphisms $U_q(\widehat{\mathfrak{sl}}_{n+1}) \to U_q^{(i)}(\widehat{\mathfrak{sl}}_{n+1})$ for $i=1,2$. In this paper we say that M has a level $(0,0)$ instead of the trivial central charge. The first 0 means that M has a level 0 as a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module and the second 0 means that M has a level 0 as a $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$-module. This notation is an analogue of level 0 representations of the affine quantum algebras.

In this paper we try to consider an analogue of "integrable representations"

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in the toroidal case. Let us recall the integrability of quantum Kac-Moody modules. Let $U_q(\mathfrak{g})$ be a quantum Kac-Moody algebra and V a $U_q(\mathfrak{g})$-module. We say V is integrable if V has a weight space decomposition and locally nilpotent actions of the Chevalley generators of $U_q(\mathfrak{g})$. Therefore the definition of "integrability" needs Chevalley-type generators. The toroidal algebras is defined through Drinfeld type of generators and its Chevalley-type generators are not known. Therefore we are not able to define the integrability at this moment. However, in the affine case, Frenkel-Jing [FJ] realized the integrable representations with level 1 by the vertex representations. Thus if there are "vertex representations" of quantum toroidal algebras, they must be interesting example of the integrable representations still not defined. In the $q=1$ case, vertex representations of the toroidal algebras have been already considered by Moody-Eswara Rao-Yokonuma [MEY]. In this paper we construct the q-analogue of the representations defined by them for $\mathfrak{g}=\mathfrak{sl}_{n+1}$ with level (1,0) and (1,1). Therefore we give a new class of the representations of the quantum toroidal algebras. In the $q=1$ case, the Fock modules are not irreducible over the Heisenberg algebra and the vertex representations are not irreducible over the toroidal algebra. In the quantum case, it is not the case: the Fock modules are irreducible and also the vertex representations with level (1,0) are irreducible.

The algebra $U_q(\mathfrak{g}_{tor})$ has infinitely many generators satisfying infinitely many relations (See §2). It is preferable that $U_q(\mathfrak{g}_{tor})$ is written by finitely many generators with finitely many relations. According to [GKV] there are finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$ but the relations among these generators are highly non-trivial. In this paper we give an explicit form of finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$ and closed relations of them (See §4). They coincide with the generators by Vasserot [V].

§2. Definition of Quantum Toroidal Algebras

2.1. Notations. Let $\mathfrak{g}$ be a complex semisimple Lie algebra of type A_n and $\hat{\mathfrak{g}}$ an affine Kac-Moody Lie algebra of type $A_n^{(1)}$. We denote their Cartan subalgebras by $\mathfrak{h}$ and $\hat{\mathfrak{h}}$ respectively. We denote by $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ the simple roots of $\mathfrak{g}$, by $\bar{h}_1, \ldots, \bar{h}_n$ the simple coroots of $\mathfrak{g}$, by $\bar{\Lambda}_1, \ldots, \bar{\Lambda}_n$ the fundamental weights of $\mathfrak{g}$, by $\alpha_0, \ldots, \alpha_n$ the simple roots of $\hat{\mathfrak{g}}$, by $h_0, \ldots, h_n$ the simple coroots of $\hat{\mathfrak{g}}$ and $\Lambda_0, \ldots, \Lambda_n$ the fundamental weights of $\hat{\mathfrak{g}}$. Let $\bar{Q}=\oplus_{i=1}^n \mathbb{Z}\bar{\alpha}_i$ be the root lattice of $\mathfrak{g}$, $\bar{P}=\oplus_{i=1}^n \mathbb{Z}\bar{\Lambda}_i$ the weight lattice of $\mathfrak{g}$, $Q=\oplus_{i=0}^n \mathbb{Z}\alpha_i$ the root lattice of $\hat{\mathfrak{g}}$ and $P=\oplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ the weight lattice of $\hat{\mathfrak{g}}$. Here δ is the null root.

We denote the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$ (*resp.* $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}^*$) by $\langle , \rangle$. The invariant bilinear form on P is given by $(\alpha_i|\alpha_j)=-\delta_{i,j-1}+2\delta_{ij}-\delta_{i,j+1}$ and $(\delta|\delta)=0$. The projection form P to $\bar{P}$ is given by $\bar{\Lambda}_i=\Lambda_i-\Lambda_0$ and $\bar{\delta}=0$.

2.2. We will give the definition of the quantum toroidal algebra $U_q(\mathfrak{g}_{tor})$.

Definition 2.2.1. Let $M=(m_{ij})_{0\le i,j\le n}$ be a skew-symmetric $(n+1)\times(n+1)$-matrix with integral coefficients and let κ be an element of $\mathbf{Q}(q)^*$. $U_q(\mathfrak{g}_{tor})$ is an associated algebra over $\mathbf{Q}(q)$ with generators:

$$E_{i,k},\ F_{i,k},\ H_{i,l},\ K_i^{\pm},\ q^{\pm\frac{1}{2}c},\ q^{\pm d_1},\ q^{\pm d_2},$$

for $k\in\mathbf{Z}$, $l\in\mathbf{Z}\setminus\{0\}$ *and* $i=0,1,\ldots,n$.

We introduce $K_{i,k}^{\pm}$ *as the Fourier components of the following generating functions*:

$$K_i^+(z)=\sum_{k\ge 0}K_{i,k}^+z^{-k}=K_i^+\exp\Big((q-q^{-1})\sum_{k\ge 1}H_{i,k}z^{-k}\Big),$$

$$K_i^-(z)=\sum_{k\le 0}K_{i,k}^-z^{-k}=K_i^-\exp\Big(-(q-q^{-1})\sum_{k\ge 1}H_{i,-k}z^{k}\Big).$$

The defining relations of $U_q(\mathfrak{g}_{tor})$ *are then written as follows*:

(2.2.1) $q^{\pm\frac{1}{2}c}$ *are central*,

(2.2.2) $$K_i^+K_i^-=K_i^-K_i^+=1,$$

(2.2.3) $$[K_i^{\pm},K_j^{\pm}]=0,$$

(2.2.4) $$[K_i^{\pm},H_{j,l}]=0,$$

(2.2.5) $$[H_{i,k},H_{j,l}]=\delta_{k+l,0}\frac{1}{k}[k\langle h_i,\alpha_j\rangle]\frac{q^{kc}-q^{-kc}}{q-q^{-1}}\kappa^{-km_{ij}},$$

(2.2.6) $$[q^{\pm d_1},K_j^{\pm}]=0,$$

(2.2.7) $$q^{d_1}H_{j,l}q^{-d_1}=q^lH_{j,l},$$

(2.2.8) $$[q^{\pm d_2},H_{j,l}]=0,$$

(2.2.9) $$q^{d_1}E_{j,k}q^{-d_1}=q^kE_{j,k},$$

$$q^{d_1}F_{j,k}q^{-d_1}=q^kF_{j,k},$$

(2.2.10) $$q^{d_2}E_{j,k}q^{-d_2}=q^{\delta_{j0}}E_{j,k},$$

$$q^{d_2}F_{j,k}q^{-d_2}=q^{-\delta_{j0}}F_{j,k},$$

(2.2.11) $$K_i^+E_{j,k}K_i^-=q^{\langle h_i,\alpha_j\rangle}E_{j,k},$$

$$K_i^+F_{j,k}K_i^-=q^{-\langle h_i,\alpha_j\rangle}F_{j,k},$$

(2.2.12) $$[H_{i,k},E_{j,l}]=\frac{1}{k}[k\langle h_i,\alpha_j\rangle]q^{-\frac{1}{2}|k|c}\kappa^{-km_{ij}}E_{j,k+l},$$

$$[H_{i,k}, F_{j,l}] = -\frac{1}{k}[k\langle h_i, \alpha_j\rangle] q^{\frac{1}{2}|k|c} \kappa^{-km_{ij}} F_{j,k+l},$$

(2.2.13)
$$\kappa^{m_{ij}} E_{i,k+1} E_{j,l} - q^{\langle h_i, \alpha_j\rangle} \kappa^{m_{ij}} E_{j,l} E_{i,k+1} = q^{\langle h_i, \alpha_j\rangle} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k},$$

$$\kappa^{m_{ij}} F_{i,k+1} F_{j,l} - q^{-\langle h_i, \alpha_j\rangle} \kappa^{m_{ij}} F_{j,l} F_{i,k+1} = q^{-\langle h_i, \alpha_j\rangle} F_{i,k} F_{j,l+1} - F_{j,l+1} F_{i,k},$$

(2.2.14)
$$[E_{i,k}, F_{j,l}] = \delta_{i,j} \frac{1}{q - q^{-1}} \{q^{\frac{1}{2}(k-l)c} K^+_{i,k+l} - q^{\frac{1}{2}(l-k)c} K^-_{i,k+l}\},$$

(2.2.15)
$$\sum_{\sigma \in \mathfrak{S}_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} E_{i,k_{\sigma(1)}} \cdots E_{i,k_{\sigma(r)}} E_{j,l} E_{i,k_{\sigma(r+1)}} \cdots E_{i,k_{\sigma(m)}} = 0,$$

$$\sum_{\sigma \in \mathfrak{S}_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} r \\ m \end{bmatrix} F_{i,k_{\sigma(1)}} \cdots F_{i,k_{\sigma(r)}} F_{j,l} F_{i,k_{\sigma(r+1)}} \cdots F_{i,k_{\sigma(m)}} = 0,$$

for $i \neq j$,

where $m = 1 - \langle h_i, \alpha_j\rangle$.

In these relations we denote $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$, $[n]! = \prod_{k=1}^{n} [k]$, $\begin{bmatrix} m \\ r \end{bmatrix} = \dfrac{[m]!}{[r]![m-r]!}$.

2.3. Let $U_q''(\mathfrak{g}_{tor})$ be the subalgebra of $U_q(\mathfrak{g}_{tor})$ generated by $E_{i,k}$, $F_{i,k}$, $K_i^{\pm}$, $H_{i,l}$, $q^{\pm\frac{1}{2}c}$. Let $U_q^{\phi'}(\mathfrak{g}_{tor})$ (*resp.* $U_q'^{\phi}(\mathfrak{g}_{tor})$) be the subalgebra generated by $U_q''(\mathfrak{g}_{tor})$ and $q^{\pm d_1}$ (*resp.* $q^{\pm d_2}$). Let $U_q^{(1)\prime}(\hat{\mathfrak{sl}}_{n+1})$ be the subalgebra generated by $E_{i,k}$, $F_{i,k}$, $K_i^{\pm}$, $H_{i,l}$, $q^{\pm\frac{1}{2}c}$ $(1 \leq i \leq n,\ k \in \mathbb{Z},\ l \in \mathbb{Z} \setminus \{0\})$ and $U_q^{(1)}(\hat{\mathfrak{sl}}_{n+1})$ the subalgebra generated by $U_q^{(1)\prime}(\hat{\mathfrak{sl}}_{n+1})$ and $q^{\pm d_1}$. Let $U_q^{(2)\prime}(\hat{\mathfrak{sl}}_{n+1})$ the subalgebra generated by $E_{i,0}$, $F_{i,0}$, $K_i^{\pm}$ $(0 \leq i \leq n)$ and $U_q^{(2)}(\hat{\mathfrak{sl}}_{n+1})$ the subalgebra generated by $U_q^{(2)\prime}(\hat{\mathfrak{sl}}_{n+1})$ and $q^{\pm d_2}$. By the definition it is clear that there are surjective homomorphisms $U_q(\hat{\mathfrak{sl}}_{n+1}) \to U_q^{(2)}(\hat{\mathfrak{sl}}_{n+1})$ and $U_q'(\hat{\mathfrak{sl}}_{n+1}) \to U_q'^{(2)\prime}(\hat{\mathfrak{sl}}_{n+1})$.

The following are straightforward.

Lemma 2.3.1. *For* $1 \leq i \leq n$ *let* $\overline{E_{i,k}} = E_{i,k}\kappa^{\sum_{j=1}^{i} m_{j-1,j}}$, $\overline{F_{i,k}} = F_{i,k}\kappa^{\sum_{j=1}^{i} m_{j-1,j}}$, $\overline{H_{i,l}} = H_{i,l}\kappa^{\sum_{j=1}^{i} m_{j-1,j}}$. *Then the relations between* $\overline{E_{i,k}}, \overline{F_{i,k}}, \overline{H_{i,l}}$ *and* $K_i^{\pm}$ *are precisely the relations of Drinfeld generators of* $U_q(\hat{\mathfrak{sl}}_{n+1})$. *That is, there are surjective homomorphisms* $U_q(\hat{\mathfrak{sl}}_{n+1}) \to U_q^{(1)}(\hat{\mathfrak{sl}}_{n+1})$ *and* $U_q'(\hat{\mathfrak{sl}}_{n+1}) \to U_q^{(1)\prime}(\hat{\mathfrak{sl}}_{n+1})$.

Lemma 2.3.2. *Let* $K_\delta^{\pm} = \prod_{i=0}^{n} K_i^{\pm}$. *Then* $K_\delta^{\pm}$ *are central elements of* $U_q(\mathfrak{g}_{tor})$.

Note that $q^{\pm c}$ is the central elements of $U_q^{(1)}(\hat{\mathfrak{sl}}_{n+1})$ and $K_\delta^{\pm}$ the central

elements of $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$.

§3. Vertex Representations

3.1. Heisenberg algebras. In this section we shall give the vertex representations of $U_q(\mathfrak{g}_{tor})$. We assume $c=1$.

Consider a $\mathbf{Q}(q)$-algebra S_n generated by $H_{i,l}$ $(0 \leq i \leq n,\ l \in \mathbf{Z} \setminus \{0\})$ satisfying:

$$[H_{i,k},H_{j,l}] = \delta_{k+l,0}\frac{1}{k}[k\langle h_i,\alpha_j\rangle]\frac{q^k-q^{-k}}{q-q^{-1}}\kappa^{-km_{ij}}. \tag{3.1.1}$$

We call S_n the Heisenberg algebra.

Let S_n^+ (*resp.* S_n^-) be the subalgebra of S_n generated by $H_{i,l}$ $(0 \leq i \leq n, l>0)$ (*resp.* $0 \leq i \leq n$, $l<0$).

We introduce the Fock space

$$\mathcal{F}_n = S_n v_0$$

with the defining relations:

$$H_{i,l}v_0 = 0, \qquad \text{for } l>0, \tag{3.1.2}$$

$$q^{\frac{1}{2}c}v_0 = q^{\frac{1}{2}}v_0. \tag{3.1.3}$$

Note that $\mathcal{F}_n$ is a free S_n^--module of rank 1.

Let $\mathbf{F}$ be a field of characteristic zero and let $\mathfrak{a}$ be an associative $\mathbf{F}$-algebra generated by x_p, y_p $(p \in \mathbf{Z}_{>0})$, z and its inverse z^{-1} with the following relations:

$$[x_p, z] = [y_p, z] = 0,$$

$$[x_p, x_r] = [y_p, y_r] = 0,$$

$$[x_p, y_r] = \delta_{pr}z.$$

Let $\mathfrak{a}^+$ (*resp.* $\mathfrak{a}^-$) be the subalgebra of $\mathfrak{a}$ generated by x_p (*resp.* y_p). We set $\mathfrak{b} = \mathfrak{a}^+ \otimes \mathbf{F}[z,z^{-1}]$. This is a maximal abelian subalgebra of $\mathfrak{a}$. Fix a nonzero scalar $\lambda \in \mathbf{F}^\times$. Let $\mathbf{F}_\lambda$ be the one-dimensional space $\mathbf{F}$ viewed as a $\mathfrak{b}$-module by:

$$z \cdot 1 = \lambda, \qquad \mathfrak{a}^+ \cdot 1 = 0.$$

Let $F(\lambda)$ be the induced $\mathfrak{a}$-module

$$F(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}} \mathbf{F}_\lambda = \mathfrak{a} \otimes_{\mathfrak{b}} \mathbf{F}_\lambda.$$

By the defining relations of $\mathfrak{a}$ we obtain an $\mathbf{F}$-linear isomorphism

$$F(\lambda) \cong \mathfrak{a}^-.$$

Since $\mathfrak{a}^-$ is abelian we may regard it as the algebra of polynomials in the variables $y_1, y_2, \cdots$. Then we see that z acts on $\mathfrak{a}^- \cong \mathbb{F}[y_1, y_2, \cdots]$ by the multiplication of λ, x_p acts by $\lambda \frac{\partial}{\partial y_p}$. By this realization we immediately have the following lemma.

Lemma 3.1.1. *$F(\lambda)$ is an irreducible $\mathfrak{a}$-module.*

Fix an skew-symmetric $(n+1) \times (n+1)$-matrix with integral coefficients $M = (m_{ij})_{0 \le i,j \le n}$. We say that $\kappa \in \mathbb{Q}(q)^*$ is generic with respect to M if for any $k \in \mathbb{Z}_{>0}$ the matrix $([k\langle h_i, \alpha_j \rangle] \kappa^{-km_{ij}})$ is invertible.

Note that if $n = 1$ any κ is not generic with respect to any M. Since the matrix $([k\langle h_i, \alpha_j \rangle])_{0 \le i,j \le n}$ is invertible for $n > 1$, there exists a generic κ for $n > 1$.

Lemma 3.1.2. (1) *For a fixed M, we assume that $\kappa \in \mathbb{Q}(q)^*$ is generic with respect to M (in particular $n > 1$). Then $\mathcal{F}_n$ is an irreducible S_n-module.*
(2) *$\mathcal{F}_1$ is not irreducible.*

Proof. (1) Set $G(k) = (g(k)_{ij}) ([k\langle h_i, \alpha_j \rangle] \kappa^{-km_{ij}})$. Since κ is generic with respect to M there exists its inverse $G(k)^{-1} = (g(k)^{ij})$ for any k. Note that by the definition $\sum_{0 \le s \le n} g(k)^{is} g(k)_{sj} = \delta_{ij}$.

We set

$$\tilde{H}_{ik} = \begin{cases} \sum_{0 \le s \le n} \frac{k}{[k]} g(k)^{is} H_{s,k}, & \text{for } k > 0, \\ H_{i,k}, & \text{for } k < 0. \end{cases}$$

Then we have

$$[\tilde{H}_{i,k}, \tilde{H}_{j,l}] = [\tilde{H}_{i,-k}, \tilde{H}_{j,-l}] = 0,$$

$$[\tilde{H}_{i,k}, \tilde{H}_{j,-l}] = \delta_{k,l} \delta_{ij},$$

for $k, l > 0$. Since all $G(k)$ are regular, $\tilde{H}_{i,k} (0 \le i \le n, k > 0)$ generate S_n^+.

We shall use Lemma 3.1.1. Put $\mathbb{F} = \mathbb{Q}(q)$, $\mathfrak{a} = S_n$, $\mathfrak{a}^\pm = S_n^\pm$, $\lambda = 1$, $x_p = \tilde{H}_{i,k}$ $(k > 0)$, $y_r = \tilde{H}_{j,l} (l < 0)$ where $p = (k-1)(n+1) + i + 1$ and $r = (-l-1)(n+1) + j + 1$.

Then it is clear that $F(1) = \mathcal{F}_n$. By Lemma 3.1.1 we conclude that $\mathcal{F}_n$ is an irreducible S_n-module.

(2) It is easy to see that $\kappa^{-km_{01}} H_{0,-k} + H_{1,-k}$ is a central element of S_1 for each $k \in \mathbb{Z}_{>0}$. Therefore $\mathcal{F}_1$ has infinitely many singular vectors. □

3.2. Construction of level (1,0) modules. In this subsection we assume $\kappa = 1$. Note that $1 \in \mathbb{Q}(q)$ is generic with respect to any M with $n > 1$.

Note that one can rewrite $\bar{P}=\oplus_{i=2}^{n}\mathbf{Z}\bar{\alpha}_i\oplus\mathbf{Z}\bar{\Lambda}_n$. We introduce a twisted version of the group algebra $\mathbf{Q}(q)[\bar{P}]$ by $\mathbf{Z}/2\mathbf{Z}$. We denote it by $\mathbf{Q}(q)\{\bar{P}\}$. This is the $\mathbf{Q}(q)$-algebra generated by symbols $e^{\bar{\alpha}_2}, e^{\bar{\alpha}_3}, \ldots, e^{\bar{\alpha}_n}, e^{\bar{\Lambda}_n}$ which satisfy the following relations:

$$e^{\bar{\alpha}_i}e^{\bar{\alpha}_j}=(-1)^{\langle\bar{h}_i,\bar{\alpha}_j\rangle}e^{\bar{\alpha}_j}e^{\bar{\alpha}_i} \tag{3.2.1}$$

$$e^{\bar{\alpha}_i}e^{\bar{\Lambda}_n}=(-1)^{\delta_{in}}e^{\bar{\Lambda}_n}e^{\bar{\alpha}_i}. \tag{3.2.2}$$

For $\bar{\alpha}=\sum_{i=2}^{n}m_i\bar{\alpha}_i+m_{n+1}\bar{\Lambda}_n$ we denote $e^{\bar{\alpha}}=(e^{\bar{\alpha}_2})^{m_2}(e^{\bar{\alpha}_3})^{m_3}\cdots(e^{\bar{\alpha}_n})^{m_n}(e^{\bar{\Lambda}_n})^{m_{n+1}}$. For example $e^{\bar{\alpha}_1}=e^{-2\bar{\alpha}_2}e^{-3\bar{\alpha}_3}\cdots e^{-n\bar{\alpha}_n}e^{(n+1)\bar{\Lambda}_n}$, $e^{\bar{\Lambda}_i}=e^{-\bar{\alpha}_{i+1}}e^{-2\bar{\alpha}_{i+2}}\cdots e^{-(n-i)\bar{\alpha}_n}e^{(n-i+1)\bar{\Lambda}_n}$ where $\bar{\Lambda}_i$ is the i-th fundamental weight. We denote $\bar{\alpha}_0=-\sum_{i=1}^{n}\bar{\alpha}_i$ and $\bar{h}_0=-\sum_{i=1}^{n}\bar{h}_i$.

Note that $\langle h_i,\alpha_j\rangle=\langle\bar{h}_i,\bar{\alpha}_j\rangle$ for $0\leq i,j\leq n$.

We denote by $\mathbf{Q}(q)\{\bar{Q}\}$ the subalgebra of $\mathbf{Q}(q)\{\bar{P}\}$ generated by $e^{\bar{\alpha}_i}$ $(1\leq i\leq n)$.

Set

$$W(p)_n=\mathcal{F}_n\otimes\mathbf{Q}(q)\{\bar{Q}\}e^{\bar{\Lambda}_p}\qquad\text{for } 1\leq p\leq n$$

and

$$W(0)_n=\mathcal{F}_n\otimes\mathbf{Q}(q)\{\bar{Q}\}.$$

We define the operators $H_{i,l}, e^{\bar{\alpha}}(\bar{\alpha}\in\bar{Q}), \partial_{\bar{\alpha}_i}$ on $W(p)_n$ for $i=0,1,\ldots,n$ as follows: for $v\otimes e^{\bar{\beta}}=H_{i1,-k1}\cdots H_{iN,-kN}v_0\otimes e^{\bar{\beta}}\in W(p)_n$,

$$H_{i,l}(v\otimes e^{\bar{\beta}})=(H_{i,l}v)\otimes e^{\bar{\beta}},$$

$$e^{\bar{\alpha}}(v\otimes e^{\bar{\beta}})=v\otimes e^{\bar{\alpha}}e^{\bar{\beta}},$$

$$\partial_{\bar{\alpha}_i}(v\otimes e^{\bar{\beta}})=\langle\bar{h}_i,\bar{\beta}\rangle v\otimes e^{\bar{\beta}},$$

$$d(v\otimes e^{\bar{\beta}})=\Big(-\sum_{s=1}^{N}k_s-\frac{(\bar{\beta}|\bar{\beta})}{2}+\frac{(\bar{\Lambda}_p|\bar{\Lambda}_p)}{2}\Big)v\otimes e^{\bar{\beta}}.$$

We have the following lemma.

Lemma 3.2.1. *As operators on* $W(p)_n$,

$$e^{\bar{\alpha}_i}e^{\bar{\alpha}_j}=(-1)^{\langle\bar{h}_i,\bar{\alpha}_j\rangle}e^{\bar{\alpha}_j}e^{\bar{\alpha}_i},$$

$$e^{\bar{\alpha}_i}q^{\partial_{\bar{\alpha}_j}}=q^{\langle\bar{h}_i,\bar{\alpha}_j\rangle}q^{\partial_{\bar{\alpha}_j}}e^{\bar{\alpha}_i},$$

$$e^{\bar{\alpha}_i}z^{\partial_{\bar{\alpha}_j}}=z^{\langle\bar{h}_i,\bar{\alpha}_j\rangle}z^{\partial_{\bar{\alpha}_j}}e^{\bar{\alpha}_i}$$

for $0\leq i,j\leq n$.

We introduce the following generating functions:

$$E_i(z) = \sum_{k\in\mathbf{Z}} E_{i,k} z^{-k},$$

$$F_i(z) = \sum_{k\in\mathbf{Z}} F_{i,k} z^{-k}.$$

Proposition 3.2.2. *Let $c=1$ and $\kappa=1$. Then for each p and n, the following action gives a $U_q^{\phi'}(\mathfrak{g}_{tor})$-module structure on $W(p)_n$:*

$$q^{\frac{1}{2}c} \mapsto q^{\frac{1}{2}},$$

$$q^{d_1} \mapsto q^{d},$$

$$E_i(z) \mapsto \exp\left(\sum_{k\geq 1} \frac{H_{i,-k}}{[k]}(q^{-1/2}z)^k\right) \exp\left(\sum_{k\geq 1} -\frac{H_{i,k}}{[k]}(q^{1/2}z)^{-k}\right) e^{\overline{\alpha}_i} z^{\partial_{\overline{\alpha}_i}+1},$$

$$F_i(z) \mapsto \exp\left(\sum_{k\geq 1} -\frac{H_{i,-k}}{[k]}(q^{1/2}z)^k\right) \exp\left(\sum_{k\geq 1} \frac{H_{i,k}}{[k]}(q^{-1/2}z)^{-k}\right) e^{-\overline{\alpha}_i} z^{-\overline{\alpha}_i+1},$$

$$K_i^+(z) \mapsto \exp\left((q-q^{-1})\sum_{k\geq 1} H_{i,k} z^{-k}\right) q^{\partial_{\overline{\alpha}}},$$

$$K_i^-(z) \mapsto \exp\left(-(q-q^{-1})\sum_{k\geq 1} H_{i,-k} z^{k}\right) q^{-\partial_{\overline{\alpha}}}$$

for $0\leq i\leq n$.

The proof will be given in Appendix.
We have immediately the following lemma.

Lemma 3.2.3. *The $U_q^{\phi'}(\mathfrak{g}_{tor})$-module $W(p)_n$ is cyclic:*

$$W(p)_n = U_q^{\phi'}(\mathfrak{g}_{tor})(v_0 \otimes e^{\Lambda_p}).$$

Theorem 3.2.4. *If $n>1$ then $W(p)_n$ is irreducible for any p.*

Proof. Since $\mathscr{F}_n$ is irreducible with respect to the action of S_n. it is enough to show that for any non-zero $v = v_0 \otimes \sum_{\overline{\alpha}\in\overline{Q}} a_{\overline{\alpha}} e^{\overline{\alpha}} e^{\overline{\Lambda}_i}$ ($a_{\overline{\alpha}} \in \mathbb{Q}(q)$) there exists $X \in U_q^{\phi'}(\mathfrak{g}_{tor})$ such that $Xv = v_0 \otimes e^{\overline{\Lambda}_p}$. Let $\overline{S_n}$ be the subalgebra of S_n generated by $H_{i,l}$ ($1\leq i\leq n, l\in\mathbb{Z}\setminus\{0\}$) and $q^{\frac{1}{2}c}$. Let $\overline{\mathscr{F}}_n$ be the $\overline{S_n}$-submodule of $\mathscr{F}_n$ generated by $v_0 \otimes e^{\overline{\Lambda}_i}$, and let $\overline{W(p)_n} = \overline{\mathscr{F}}_n \otimes \mathbb{Q}(q)\{\overline{Q}\} e^{\overline{\Lambda}_p}$. As already known $\overline{W(p)_n}$ is an irreducible $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module. It is obvious that $v \in \overline{W(p)_n}$. Therefore there exists $X \in U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1}) \subset U_q^{\phi'}(\mathfrak{g}_{tor})$ such that $Xv = v_0 \otimes e^{\overline{\Lambda}_p}$. □

Remark 3.2.5. Since $\mathscr{F}_1$ is not irreducible as an S_1-module $W(p)_1$ is not

irreducible.

Remark 3.2.6. Since $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ and $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$ are subalgebras of $U_q^{\phi'}(\mathfrak{g}_{tor})$ we can regard $W(p)_n$ as a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module or as a $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$-module. As a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module, $W(p)_n$ is a level 1 module. On the other hand it is a level 0 module as a $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$-module.

3.3. On the structure of level (1,0) modules. In this subsection we will study the level 1 $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module structure of $W(p)_n$.

Let M be a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module. We can regard M as a $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module via a surjective homomorphism $U_q(\widehat{\mathfrak{sl}}_{n+1}) \to U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$. As a $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module, we denote the character of M by ch_M.

Let $L(\Lambda_p)$ be the irreducible highest weight $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module with highest weight Λ_p. Note that the following identity holds:

$$\mathrm{ch}_{L(\Lambda_p)} = \frac{e^{\Lambda_p}\sum_{\alpha\in\overline{Q}} e^{\alpha-(\frac{1}{2}(\alpha|\alpha)-(\alpha|\overline{\Lambda_p}))\delta}}{\varphi(e^{-\delta})^n}. \tag{3.3.1}$$

Here $\varphi(x)=\prod_{k>0}(1-x^k)$.

We denote δ by the null root of $U_q(\widehat{\mathfrak{sl}}_{n+1})$.

By the definition of $W(p)_n$ and (3.2.1) it is immediate to see the following proposition.

Proposition 3.3.1. *As a $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module, we have*

$$ch_{W(p)_n} = \frac{e^{\Lambda_p}\sum_{\alpha\in\overline{Q}} e^{\alpha-(\frac{1}{2}(\alpha|\alpha)-(\alpha|\overline{\Lambda_p}))\delta}}{\varphi(e^{-\delta})^{n+1}}$$

$$= \frac{ch_{L(\Lambda_p)}}{\varphi(e^{-\delta})}.$$

Lemma 3.3.2. *For each $l\in\mathbb{Z}\setminus\{0\}$ there exist $\widehat{H}_l=\sum_{i=0}^n a_{i,l}H_{i,l}$ $(a_{i,l}\in\mathbf{Q}(q))$ such that*

$$[\widehat{H}_l, H_{j,k}]=0$$

for any $1\le j\le n$ and $k\in\mathbb{Z}\setminus\{0\}$. Moreover such $\widehat{H}_l$ is unique up to scalar.

Proof. Note that $\kappa=1$. The rank of $n\times(n+1)$-matrix $([l\langle h_i,\alpha_j\rangle])_{1\le i\le n,0\le j\le n}$ is equal to n. The lemma follows form this fact immediately. □

By the definition of $\widehat{H}_l$ we have

$$[\widehat{H}_k,\widehat{H}_l]=\delta_{k+l,0}\gamma_k. \tag{3.3.2}$$

where $\gamma_k \in \mathbb{Q}(q)$. We fix a normalization of $\widehat{H}_l$ by putting $\gamma_k = 1$ for all k.

Let $\widehat{S}_n$ be the subalgebra of S_n generated by $\widehat{H}_l$. By the definition, $\widehat{S}_n$ acts on $W(p)_n$.

The following two lemmas are easy to see.

Lemma 3.3.3. *For $l>0$, $\widehat{H}_l(v_0 \otimes e^{\beta}) = 0$.*

Lemma 3.3.4. *The action of $U_q'(\widehat{\mathfrak{sl}}_{n+1})$ on $W(p)_n$ commutes with the action of $\widehat{S}_n$.*

Let $\widehat{S}_n^-$ be the subalgebra of $\widehat{S}_n$ generated by $\widehat{H}_l (l<0)$.

Proposition 3.3.5. *As $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module*

$$W(p)_n \cong L(\Lambda_p)^{\oplus \infty}.$$

Proof. Set $\deg(\widehat{H}_k) = k$. Let $M_k = M_k(\widehat{H}_{-1}, \widehat{H}_{-2}, \ldots)$ be a monomial of degree k in variables $\widehat{H}_{-1}, \widehat{H}_{-2}, \ldots$. Then by the above two lemmas, $M_k v_0 \otimes e^{\Lambda_p}$ is a singular vector of $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module $W(p)_n$. Let W_{M_k} be the $U_q(\widehat{\mathfrak{sl}}_{n+1})$-submodule which is generated by $M_k v_0 \otimes e^{\Lambda_p}$. Then by the definition of the action of $U_q(\widehat{\mathfrak{sl}}_{n+1})$ on $W(p)_n$, we have

$$W_{M_k} \cong L(\Lambda_p - k\delta) \cong L(\Lambda_p).$$

The vectors $\{M_k v_0 \otimes e^{\Lambda_p}\}$ are linearly independent. The number of the monomials of degree k is equal to the k-th partition number $p(k)$. Therefore there is a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-submodule W of $W(p)_n$ which is isomorphic to $\oplus_{k \geq 0} L(\Lambda_p - k\delta)^{\oplus p(k)}$. By Proposition 3.3.1 it coincides with $W(p)_n$. This completes proof. □

By Lemma 3.3.4 and the proof of Proposition 3.3.5, the following corollary follows immediately.

Corollary 3.3.6. *As $U_q'(\widehat{\mathfrak{sl}}_{n+1}) \otimes \widehat{S}_n^-$-module $W(p)_n$ is isomorphic to $L(\Lambda_p) \otimes \widehat{S}_n^-$.*

3.4. Construction of level (1,1) modules. We introduce a twisted version of the group algebra $\mathbb{Q}(q)[Q]$ by $\mathbb{Z} \backslash 2\mathbb{Z}$. We denoted it by $\mathbb{Q}(q)\{Q\}$. This is the $\mathbb{Q}(q)$-algebra generated by symbols $e^{\alpha_0}, e^{\alpha_1}, \ldots, e^{\alpha_n}$ which satisfy the following relations:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i}. \tag{3.4.1}$$

Similarly to §3.2, we denote $e^{\alpha} = (e^{\alpha_0})^{m_0} (e^{\alpha_1})^{m_1} \ldots (e^{\alpha_n})^{m_n}$ for $\alpha = \sum_{i=0}^{n} m_i \alpha_i \in Q$.

Let

$$V(p)_n = \mathcal{F}_n \otimes \mathbf{Q}(q)\{Q\} e^{\Lambda_p}.$$

Here we regard e^{Λ_p} only a symbol indexed by p.

We define the operators $H_{i,l}$ $(0 \leq i \leq n,\ l \neq 0)$, e^{α} $(\alpha \in Q)$, ∂_{α_i} and $z^{H_{i,0}}$ $(0 \leq i \leq n)$ on $V(p)_n$ as follows:

for $v \otimes e^{\beta} e^{\Lambda_p} = H_{i_1,-k_1} \cdots H_{i_N,-k_N} v_0 \otimes e^{\beta} e^{\Lambda_p} \in V(p)_n$ $(\beta = \sum_{k=0}^{n} m_k \alpha_k \in Q)$,

$$H_{i,l}(v \otimes e^{\beta} e^{\Lambda_p}) = (H_{i,l} v) \otimes e^{\beta} e^{\Lambda_p},$$

$$e^{\alpha}(v \otimes e^{\beta} e^{\Lambda_p}) = v \otimes (e^{\alpha} e^{\beta}) e^{\Lambda_p},$$

$$\partial_{\alpha_i}(v \otimes e^{\beta} e^{\Lambda_p}) = \langle h_i, \beta + \Lambda_p \rangle v \otimes e^{\beta} e^{\Lambda_p},$$

$$z^{H_{i,0}}(v \otimes e^{\beta} e^{\Lambda_p}) = z^{\langle h_i, \beta + \Lambda_p \rangle} \kappa^{\frac{1}{2} \sum_{k=0}^{n} \langle h_i, m_k \alpha_k \rangle m_{ik}} v \otimes e^{\beta} e^{\Lambda_p},$$

$$d_1(v \otimes e^{\beta} e^{\Lambda_p}) = \left(-\sum_{s=1}^{N} k_s - \frac{(\beta|\beta)}{2} - (\beta|\Lambda_p) \right) v \otimes e^{\beta} e^{\Lambda_p},$$

$$d_2(v \otimes e^{\beta} e^{\Lambda_p}) = m_0 (v \otimes e^{\beta} e^{\Lambda_p}).$$

The following lemma is easy.

Lemma 3.4.1. *As operators on* $V(p)_n$,

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i}$$

$$q^{\partial_{\alpha_i}} e^{\alpha_j} = q^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} q^{\partial_{\alpha_i}}$$

$$z^{H_{i,0}} e^{\alpha_j} = z^{\langle h_i, \alpha_j \rangle} \kappa^{\frac{1}{2} \langle h_i, \alpha_j \rangle m_{ij}} e^{\alpha_j} z^{H_{i,0}}.$$

Proposition 3.4.2. *Assume* $c = 1$ *then for each* p *and* n, *the following action gives a* $U_q(\mathfrak{g}_{tor})$*-module structure on* $V(p)_n$:

$$q^{\frac{1}{2}c} \mapsto q^{\frac{1}{2}},$$

$$q^{d_1} \mapsto q^{d_1},$$

$$q^{d_2} \mapsto q^{d_2},$$

$$E_i(z) \mapsto \exp\left(\sum_{k \geq 1} \frac{H_{i,-k}}{[k]} (q^{-1/2} z)^k \right) \exp\left(\sum_{k \geq 1} -\frac{H_{i,k}}{[k]} (q^{1/2} z)^{-k} \right) e^{\alpha_i} z^{H_{i,0}+1},$$

$$F_i(z) \mapsto \exp\left(\sum_{k \geq 1} -\frac{H_{i,-k}}{[k]} (q^{1/2} z)^k \right) \exp\left(\sum_{k \geq 1} \frac{H_{i,k}}{[k]} (q^{-1/2} z)^{-k} \right) e^{-\alpha_i} z^{-H_{i,0}+1},$$

$$K_i^{+}(z) \mapsto \exp\left((q - q^{-1}) \sum_{k \geq 1} H_{i,k} z^{-k} \right) q^{\partial_{\alpha_i}}.$$

$$K_i^-(z) \mapsto \exp\left(-(q-q^{-1})\sum_{k\geq 1} H_{i,-k}z^k\right)q^{-\partial_{\alpha_i}}$$

for $0\leq i\leq n$.

The proof will be given in Appendix.

It is easy to see the following lemma.

Lemma 3.4.3. *$V(p)_n$ is a cyclic $U_q(\mathfrak{g}_{tor})$-modulc: $V(p)_n = U_q(\mathfrak{g}_{tor})(v_0 \otimes e^{\Lambda_p})$.*

Lemma 3.4.4. *$V(p)_n$ has level 1 as a $U_q^{(1)}(\hat{\mathfrak{sl}}_{n+1})$-module and as a $U_q^{(2)}(\hat{\mathfrak{sl}}_{n+1})$-module.*

Proof. It is clear that $V(p)_n$ is a level 1 $U_q^{(1)}(\hat{\mathfrak{sl}}_{n+1})$-module. The center of $U_q^{(2)}(\hat{\mathfrak{sl}}_{n+1})$ is $\prod_{k=0}^{n} K_i$. By the definition it acts as the scalar q on $V(p)_n$. □

§4. On $U_q(\mathfrak{sl}_{2,tor})$

4.1. In this section we assume that $\mathfrak{g}=\mathfrak{sl}_2$. We shall try to find finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$.

Let

$$E_i = E_{i,0},\ F_i = F_{i,0},\ q^{\pm h_i} = K_i^{\pm},\ \text{for } i=0,1,$$

$$E_{-1} = F_{0,1}K_0^-,\ F_{-1} = K_0^+ E_{0,-1},\ q^{\pm h_{-1}} = q^{\pm c}K_0^{\mp}.$$

Proposition 4.1.1. *$U_q(\mathfrak{sl}_{2,tor})$ is generated by E_i, F_i, $q^{\pm h_i}$ $(i=-1,0,1)$, $q^{\pm\frac{1}{2}c}$, $q^{\pm d_1}$, $q^{\pm d_2}$.*

Proof. Let $\mathcal{A}$ be the subalgebra of $U_q(\mathfrak{sl}_{2,tor})$ generated by E_i, F_i, $q^{\pm h_i}$ $(i=-1,0,1)$, $q^{\pm\frac{1}{2}c}$, $q^{\pm d_1}$, $q^{\pm d_2}$. By the definition we have $E_{0,-1} = q^{-h_0}F_{-1}$ and $F_{0,1} = E_{-1}q^{h_0}$. Since

$$[E_{0,0}, F_{0,1}] = \frac{1}{q-q^{-1}} q^{-\frac{1}{2}c}K_{0,1}^+$$

$$= q^{-\frac{1}{2}c}q^{h_0}H_{0,1}$$

and

$$[E_{0,-1}, F_{0,0}] = -\frac{1}{q-q^{-1}} q^{\frac{1}{2}c}K_{0,-1}^-$$

$$= q^{\frac{1}{2}c}q^{-h_0}H_{0,-1}.$$

We deduce $H_{0,1}$, and $H_{0,-1}\in\mathcal{A}$. We recall (2.2.7)

$$[H_{i,k},E_{j,l}]=\frac{1}{k}[k\langle h_i,\alpha_j\rangle]q^{-\frac{1}{2}|k|c}\kappa^{-km_U}E_{j,k+l},$$

$$[H_{i,k},F_{j,l}]=-\frac{1}{k}[k\langle h_i,\alpha_j\rangle]q^{\frac{1}{2}|k|c}\kappa^{-km_U}F_{j,k+l}.$$

By these formulas we have $E_{i,k}$, $F_{i,k}\in\mathscr{A}$ for $i=0,1$, $k\in\mathbf{Z}$ in inductively.

On the other hand we know

$$[E_{0,1},F_{0,1}]=\frac{1}{q-q^{-1}}q^{-\frac{1}{2}c}K^{+}_{0,2}$$

$$=\frac{1}{2}(q-q^{-1})H^2_{0,1}+H_{0,2}.$$

Therefore we get $H_{0,2}\in\mathscr{A}$. Similary we have $H_{i,l}\in\mathscr{A}$ for any i,l.

This completes the proof. □

Lemma 4.1.2. *The following relations hold in* $U_q(\mathfrak{sl}_{2,tor})$:

(4.1.1) $$[q^{\pm h_i},q^{\pm h_j}]=0,$$

(4.1.2) $$[q^{\pm d_i},q^{\pm h_j}]=0,$$

(4.1.3) $$q^{d_1}E_jq^{-d_1}=q^{\delta_{j,-1}}E_j,$$

$$q^{d_1}F_jq^{-d_1}=q^{-\delta_{j,-1}}F_j,$$

(4.1.4) $$q^{d_2}E_jq^{-d_2}=q^{-1+\delta_{j,1}}E_j,$$

$$q^{d_2}F_jq^{-d_2}=q^{1-\delta_{j,1}}F_j,$$

(4.1.5) $$q^{h_i}E_jq^{-h_i}=q^{a_{ij}}E_j,$$

$$q^{h_i}F_jq^{-h_i}=q^{-a_{ij}}F_j,$$

where

$$(a_{ij})_{-1\le i,j\le 1}=\begin{pmatrix}2&-2&2\\-2&2&-2\\2&-2&2\end{pmatrix},$$

(4.1.6) $$[E_i,F_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}},\quad \textit{for } |i-j|\le 1,$$

(4.1.7) $$E^3_{-1}F_1-q^{-2}[3]E^2_{-1}F_1E_{-1}+q^{-4}[3]E_{-1}F_1E^2_{-1}-q^{-6}F_1E^3_{-1}=0,$$

(4.1.8) $$E^3_1F_{-1}-q^2[3]E^2_1F_{-1}E_1+q^4[3]E_1F_{-1}E^2_1-q^6F_{-1}E^3_1=0,$$

(4.1.9) $$F^3_{-1}E_1-q^{-2}[3]F^2_{-1}E_1F_{-1}+q^{-4}[3]F_{-1}E_1F^2_{-1}-q^{-6}E_1F^3_{-1}=0,$$

$$(4.1.10)\qquad F_1^3E_{-1}-q^2[3]F_1^2E_{-1}F_1+q^4[3]F_1E_{-1}F_1^2-q^6E_{-1}F_1^3=0,$$

$$(4.1.11)\qquad E_i^3E_j-[3]E_i^2E_jE_i+[3]E_iE_jE_i^2-E_jE_i^3=0,\qquad \text{for } |i-j|=1,$$

$$(4.1.12)\qquad F_i^3F_j-[3]F_i^2F_jF_i+[3]F_iF_jF_i^2-F_jF_i^3=0,\qquad \text{for } |i-j|=1,$$

$$(4.1.13)\qquad E_{-1}E_1-q^2E_1E_{-1}=0,$$

$$(4.1.14)\qquad F_{-1}F_1-q^2F_1F_{-1}=0.$$

Proof. By the definition of $U_q(\mathfrak{g}_{tor})$ and E_i, F_i and q^{h_i} it is easy to check these relations. □

Let $\mathcal{U}$ be an associative algebra over $\mathbb{Q}(q)$ generated by E_i, F_i, $q^{\pm h_i}$ $(i=-1,0,1)$, $q^{\pm\frac{1}{2}c}$, $q^{\pm d_1}$, $q^{\pm d_2}$ with relations (4.1.1) – (4.1.14). Then we have,

Corollary 4.1.3. *There is a canonical surjective algebra homomorphism* $\Psi:\mathcal{U}\to U_q(\mathfrak{sl}_{2,tor})$.

Remark 4.1.4. Ψ has a highly nontrivial kernel. It is important to determine it. For example the following formulas holds in $U_q(\mathfrak{sl}_{2,tor})$:

$$\kappa^{m_{01}}E_{0,0}E_{1,-1}-q^{-2}\kappa^{m_{01}}E_{1,-1}E_{0,0}=q^{-2}E_{0,-1}E_{1,0}-E_{1,0}E_{0,-1},$$

$$E_{0,-1}=q^{-h_0}F_{-1}$$

and

$$E_{1,-1}=\frac{\kappa^{-m_{01}}}{[-2]}[F_{-1}F_0-q^{-2}F_0F_{-1},E_1].$$

Therefore we have

$$\begin{aligned}X=&\frac{1}{[-2]}E_0[F_{-1}F_0-q^{-2}F_0F_{-1},E_1]-\frac{q^{-2}}{[-2]}[F_{-1}F_0-q^{-2}F_0F_{-1},E_1]E_0\\&-q^{-2}q^{-h_0}F_{-1}E_1-E_1q^{-h_0}F_{-1}\\=&0\end{aligned}$$

in $U_q(\mathfrak{sl}_{2,tor})$. Thus $X\in\operatorname{Ker}\Psi$. But , as an element of $\mathcal{U}$, X is not equal to 0.

4.2. Let

$$E_{0*}=F_{1,1}K_1^-,\qquad F_{0*}=K_1^+E_{1,-1},\qquad q^{\pm h_{0*}}=q^{\pm c}K_1^{\mp}.$$

Proposition 4.2.1. *The subalgebra generated by* E_i, F_i, and q^{h_i} *for* $i=0,1$, E_{0*}, F_{0*}, $q^{h_{0'}}$, $q^{\pm c}$, $q^{\pm d_1}$, $q^{\pm d_2}$, *is equal to* $U_q(\mathfrak{sl}_{2,tor})$. *That is, they are generators of*

$U_q(\mathfrak{sl}_{2,tor})$. *Moreover these generators satisfy relations similar to the ones in Lemma* 4.1.2.

Proof. This proposition is proved in the same way as Proposition 4.1.1 and Lemma 4.1.2. □

We have immediately the following lemma.

Lemma 4.2.2. *Let* $U_q^{(1)}$ *be the subalgebras generated by* E_i, F_i, q^{h_i} *for* $i=1,0^*$ *and* $q^{\pm(d_1+d_2)}$, $U_q^{(2)}$ *the subalgebras generated by* E_i, F_i, q^{h_i} *for* $i=0$, 1 *and* $q^{\pm(d_1+d_2)}$, $U_q^{(3)}$ *the subalgebras generated by* E_i, F_i, $q^{\pm h_i}$ *for* $i=0^*,-1$ *and* $q^{\pm(d_1+d_2)}$, *and* $U_q^{(4)}$ *the subalgebras generated by* E_i, F_i, $q^{\pm h_i}$ *for* $i=0,-1$ *and* $q^{\pm(d_1+d_2)}$. *Then* $U_q^{(i)}$ $(i=1,2,3,4)$ *are isomorphic to* $U_q(\widehat{\mathfrak{sl}}_2)$.

Those four algebras are schematically visualized by Fig. 1.

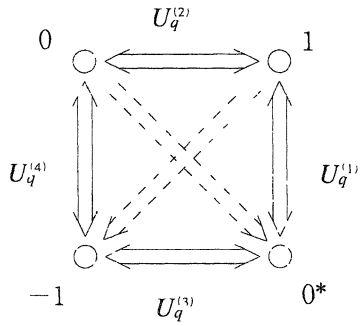

Fig. 1.

Let $U_q(\mathfrak{sl}_2)_{(i)}$ $(i=-1,0,1,0^*)$ be the subalgebra of $U_q(\mathfrak{sl}_{2,tor})$ generated by E_i, F_i, $q^{\pm h_i}$. All $U_q(\mathfrak{sl}_{2,(i)})$ $(i\in\{-1,0,1,0^*\})$ are isomorphic to $U_q(\mathfrak{sl}_2)$. The upper left circle in Fig. 1 means $U_q(\mathfrak{sl}_2)_{(0)}$, the upper right one means $U_q(\mathfrak{sl}_2)_{(1)}$, the lower left one means $U_q(\mathfrak{sl}_2)_{(-1)}$ and the lower right one means $U_q(\mathfrak{sl}_2)_{(0^*)}$. The diagram

$$O^i \Longleftrightarrow O^j \qquad (i,j\in\{-1,0,1,0^*\})$$

means the algebra generated by $U_q(\mathfrak{sl}_2)_{(i)}$ and $U_q(\mathfrak{sl}_2)_{(j)}$ is isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$. For example $O^0 \Longleftrightarrow O^1$ means the algebra generated by $U_q(\mathfrak{sl}_2)_{(0)}$ and $U_q(\mathfrak{sl}_2)_{(-1)}$ which we call $U_q^{(2)}$ is isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$. The meaning of the diagram

$$O^i ======\Rightarrow O^j$$

is as follows: In the algebra generated by $U_q(\mathfrak{sl}_2)_{(i)}$ and $U_q(\mathfrak{sl}_2)_{(j)}$, the following relations hold

$$q^{h_i}E_jq^{-h_i}=q^2E_j, \qquad q^{h_j}E_iq^{-h_j}=q^2E_i, \tag{4.2.1}$$

(4.2.2) $$q^{h_i}F_jq^{-h_i}=q^{-2}F_j, \qquad q^{h_j}F_iq^{-h_j}=q^{-2}F_i.$$

(4.2.3) $$E_i^3F_j-q^{-2}[3]E_i^2F_jE_i+q^{-4}[3]E_iF_jE_i^2-q^{-6}F_jE_i^3=0,$$

$$E_j^3F_i-q^2[3]E_j^2F_iE_j+q^4[3]E_jF_iE_j^2-q^6F_iE_j^3=0,$$

(4.2.4) $$F_i^3E_j-q^{-2}[3]F_i^2E_jF_i+q^{-4}[3]F_iE_jF_i^2-q^{-6}E_jF_i^3=0,$$

$$F_j^3E_i-q^2[3]F_j^2E_iF_j+q^4[3]F_jE_iF_j^2-q^6E_iF_j^3=0,$$

(4.2.5) $$E_iE_j-q^2E_jE_i=0,$$

(4.2.6) $$F_iF_j-q^2F_jF_i=0.$$

Appendix A.

A.1. Proof of Proposition 3.2.2 and 3.4.2. For the proof we rewrite the defining relation $U_q(\mathfrak{g}_{tor})$ generating function level.

(A.1.1) $$q^{\pm\frac{1}{2}c} \quad \text{are central.}$$

(A.1.2) $$K_i^+K_i^-=K_i^-K_i^+=1,$$

(A.1.3) $$K_i^\pm(z)K_j^\pm(w)=K_j^\pm(w)K_i^\pm(z)$$

(A.1.4)
$$\theta_{-\langle h_i,\alpha_j\rangle}\left(q^{-c}\kappa^{-m_v}\frac{z}{w}\right)K_i^-(z)K_j^+(w)=\theta_{-\langle h_i,\alpha_j\rangle}\left(q^{c}\kappa^{-m_v}\frac{z}{w}\right)K_j^+(w)K_i^-(z)$$

(A.1.5) $$q^{d_1}K_j^\pm(z)q^{-d_1}=K_j^\pm(q^{-1}z),$$

(A.1.6) $$[q^{d_2}, K_j^\pm(z)]=0,$$

(A.1.7) $$q^{d_1}E_j(z)q^{-d_1}=E_j(q^{-1}z),$$

$$q^{d_1}F_j(z)q^{-d_1}=F_j(q^{-1}z),$$

(A.1.8) $$q^{d_2}E_j(z)q^{-d_2}=q^{\delta_{j0}}E_j(z),$$

$$q^{d_2}F_j(z)q^{-d_2}=q^{-\delta_{j0}}F_j(z),$$

(A.1.9) $$K_i^+(z)E_j(w)=\theta_{-\langle h_i,\alpha_j\rangle}\left(q^{-\frac{1}{2}c}\kappa^{-m_v}\frac{w}{z}\right)E_j(w)K_i^+(z)$$

$$K_i^-(z)E_j(w)=\theta_{\langle h_i,\alpha_j\rangle}\left(q^{-\frac{1}{2}c}\kappa^{m_v}\frac{z}{w}\right)E_j(w)K_i^-(z)$$

$$K_i^+(z)F_j(w)=\theta_{\langle h_i,\alpha_j\rangle}\left(q^{\frac{1}{2}c}\kappa^{-m_v}\frac{w}{z}\right)F_j(w)K_i^+(z)$$

$$K_i^-(z)F_j(w)=\theta_{-\langle h_i,\alpha_j\rangle}\left(q^{\frac{1}{2}c}\kappa^{m_{ij}}\frac{z}{w}\right)F_j(w)K_i^-(z)$$

(A.1.10)

$$[E_i(z),F_j(w)]=\delta_{i,j}\frac{1}{q-q^{-1}}\left\{\delta\left(q^c\frac{w}{z}\right)K_i^+(q^{\frac{1}{2}c}w)-\delta\left(q^c\frac{z}{w}\right)K_i^-(q^{\frac{1}{2}c}z)\right\}$$

(A.1.11) $$(\kappa^{m_{ij}}z-q^{\langle h_i,\alpha_j\rangle}w)E_i(z)E_j(w)=(q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z-w)E_j(w)E_i(z)$$

$$(\kappa^{m_{ij}}z-q^{-\langle h_i,\alpha_j\rangle}w)F_i(z)F_j(w)=(q^{-\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z-w)F_j(w)F_i(z)$$

(A.1.12)

$$\sum_{\sigma\in\mathfrak{S}_m}\sum_{r=0}^{m}(-1)^r\begin{bmatrix}m\\r\end{bmatrix}_q E_i(z_{\sigma(1)})\cdots E_i(z_{\sigma(r)})E_j(w)E_i(z_{\sigma(r+1)})\cdots E_i(z_{\sigma(m)})=0$$

$$\sum_{\sigma\in\mathfrak{S}_m}\sum_{r=0}^{m}(-1)^r\begin{bmatrix}m\\r\end{bmatrix}_q F_i(z_{\sigma(1)})\cdots F_i(z_{\sigma(r)})F_j(w)F_i(z_{\sigma(r+1)})\cdots F_i(z_{\sigma(m)})$$

where $i\neq j$ and $m=1-\langle h_i,\alpha_j\rangle$.

In these formulas we denote $\theta_m(z)=\frac{zq^m-1}{z-q^m}$ for $m\in\mathbb{Z}$, $\delta(z)=\sum_{k\in\mathbb{Z}}z^k$.

If Proposition 3.4.2 holds, then, from Lemma 3.2.1 and 3.4.1, we have Proposition 3.2.2 by putting $h_i\mapsto\overline{h_i}$, $\alpha_i\mapsto\overline{\alpha_i}$, $\kappa\mapsto 1$ and $z^{H_{i,0}}\mapsto z^{\overline{\partial_i}}$. Therefore it is enough to show Proposition 3.4.2.

The relations (A.1.1), (A.1.2) and (A.1.3) are trivial. (A.1.4) is just the commutation relations of Heisenberg algebra S_n. Therefore, by the definition of $V(p)_n$, it is clear that (A.1.4) holds. The relations (A.1.5), (A.1.6) immediately follows from the definition of d_1 and d_2.

Let us show (A.1.7) and (A.1.8). Take $v\otimes e^{\beta}e^{\Lambda_p}\in V(p)_n$ where $\beta=\sum_{k=0}^{n}m_k\alpha_k\in Q$. Then we have

$$\begin{aligned}
&q^{d_1}e^{\pm\alpha_j}z^{\pm H_{j,0}+1}q^{-d_1}(v\otimes e^{\beta}e^{\Lambda_p})\\
&=q^{\mp(\alpha_j|\beta+\Lambda_p)-1}z^{\pm\langle h_j,\beta+\Lambda_p\rangle+1}\kappa^{\frac{1}{2}\sum_{k=0}^{n}\langle h_i,m_i\alpha_k\rangle m_{jk}}v\otimes e^{\pm\alpha_j}e^{\beta}e^{\Lambda_p}\\
&=(q^{-1}z)^{\pm\langle h_j,\beta+\Lambda_p\rangle+1}\kappa^{\frac{1}{2}\sum_{k=0}^{n}\langle h_i,m_i\alpha_k\rangle m_{jk}}v\otimes e^{\pm\alpha_j}e^{\beta}e^{\Lambda_p}\\
&=e^{\pm\alpha_j}(q^{-1}z)^{\pm H_{j,0}+1}(v\otimes e^{\beta}e^{\Lambda_p}).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
q^{d_1}E_j(z)q^{-d_1}&=q^{d_1}\exp\left(\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{-\frac{1}{2}k}z^k\right)\exp\left(-\sum_{k\geq 1}\frac{H_{j,k}}{[k]}q^{-\frac{1}{2}k}z^{-k}\right)e^{\alpha_j}z^{H_{j,0}+1}q^{-d_1}\\
&=\exp\left(\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{-\frac{1}{2}k}(q^{-1}z)^k\right)\exp\left(-\sum_{k\geq 1}\frac{H_{j,k}}{[k]}q^{-\frac{1}{2}k}(q^{-1}z)^{-k}\right)e^{\alpha_j}(q^{-1}z)^{H_{j,0}+1}
\end{aligned}$$

$=E_j(q^{-1}z)$.

Similarly we have $q^{d_1}F_j(z)q^{-d_1}=F_j(q^{-1}z)$.
It is clear that

(A.1.13) $\quad q^{d_2}e^{\pm\alpha_j}z^{\pm H_{j,0}+1}q^{-d_2}(v\otimes e^{\beta}e^{\Lambda_p})=q^{\pm\delta_{j0}}e^{\pm\alpha_j}z^{\pm H_{j,0}+1}(v\otimes e^{\beta}e^{\Lambda_p})$.

From (A.1.13) and the fact that q^{d_2} commutes with $H_{j,k}$, we have (A.1.8).
We shall show (A.1.9). We denote

$$E_i^+(z)=\exp\Big(\sum_{k\geq 1}\frac{H_{i,-k}}{[k]}q^{-\frac{1}{2}k}z^k\Big),$$

$$E_i^-(z)=\exp\Big(-\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{-\frac{1}{2}k}z^{-k}\Big).$$

$$F_i^+(z)=\exp\Big(-\sum_{k\geq 1}\frac{H_{i,-k}}{[k]}q^{\frac{1}{2}k}z^k\Big),$$

$$F_i^-(z)=\exp\Big(\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{\frac{1}{2}k}w^{-k}\Big).$$

Let us proof

$$K_i^+(z)E_j(w)=\theta_{-\langle h_i,\alpha_j\rangle}\Big(q^{-\frac{1}{2}}\kappa^{m_{ij}}\frac{w}{z}\Big)E_j(w)K_i^+(z).$$

We have

$$\Big[(q-q^{-1})\sum_{k\geq 1}H_{i,k}z^{-k},\sum_{l\geq 1}\frac{H_{j,-l}}{[l]}q^{-\frac{1}{2}l}w^l\Big]=\sum_{k,l}(q-q^{-1})\frac{1}{[l]}[H_{i,k},H_{j,-l}]z^{-k}q^{-\frac{1}{2}l}w^l$$

$$=\sum_k(q-q^{-1})\frac{[k\langle h_i,\alpha_j\rangle]}{k}\kappa^{-km_{ij}}q^{-\frac{1}{2}k}\Big(\frac{w}{z}\Big)^k$$

$$=\sum_k\frac{1}{k}(q^{k\left(\langle h_i,\alpha_j\rangle-\frac{1}{2}\right)}-q^{k\left(-\langle h_i,\alpha_j\rangle-\frac{1}{2}\right)})\kappa^{-km_{ij}}\Big(\frac{w}{z}\Big)^k$$

$$=\log\frac{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}{1-q^{\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}$$

and

$$\Big[(q-q^{-1})\sum_{k\geq 1}H_{i,k}z^{-k},-\sum_{l\geq 1}\frac{H_{j,l}}{[l]}q^{-\frac{1}{2}l}w^{-l}\Big]=0.$$

We recall Campbell-Hausdorff formula: let A and B be noncommutative operators and $C=[A,B]$. If $[C,A]=[C,B]=0$ then we have $e^Ae^B=e^Ce^Be^A$.

By Campbell-Hausdorff formula we get

$$\exp\Big((q-q^{-1})\sum_{k\geq 1}H_{i,k}z^{-k}\Big)E_j^+(w)E_j^-(w)$$

$$=\frac{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}\kappa^{-m_{ij}}\frac{w}{z}}}{1-q^{\langle h_i,\alpha_j\rangle-\frac{1}{2}\kappa^{-m_{ij}}\frac{w}{z}}}E_j^+(w)E_j^-(w)\exp\Big((q-q^{-1})\sum_{k\geq 1}H_{i,k}z^{-k}\Big).$$

On the other hand, by Lemma 3.4.1 we have

$$q^{\partial_{\alpha_i}}e^{\alpha_i}w^{H_{j,0}}=q^{\langle h_i,\alpha_j\rangle}e^{\alpha_i}w^{H_{j,0}}q^{\partial_{\alpha_i}}.$$

Thus we get

$$K_i^+(z)E_j(w)$$

$$=\exp\Big((q-q^{-1})\sum_{k\geq 1}H_{i,k}z^{-k}\Big)E_j^+(w)E_j^-(w)q^{\partial_{\alpha_i}}e^{\alpha_i}w^{H_{j,0}+1}$$

$$=\frac{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}\kappa^{-m_{ij}}\frac{w}{z}}}{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}\kappa^{-m_{ij}}\frac{w}{z}}}E_j(w)K_i^+(z)$$

$$=\theta_{-\langle h_i,\alpha_j\rangle}\Big(q^{-\frac{1}{2}}\kappa^{m_{ij}}\frac{w}{z}\Big)E_j(w)K_i^+(z).$$

The other formulas in (A.1.9) can be checked by similar arguments. Let us show (A.1.10). We have

$$\Big[-\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{-\frac{1}{2}k}z^{-k},-\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{\frac{1}{2}k}w^k\Big]$$

$$=\sum_{k\geq 1}\frac{1}{k[k]}[k\langle h_i,\alpha_j\rangle]\kappa^{-m_{ij}}\Big(\frac{w}{z}\Big)^k$$

$$=\begin{cases}\log\dfrac{1}{(1-q\frac{w}{z})(1-q^{-1}\frac{w}{z})}, & \langle h_i,\alpha_j\rangle=2\ \ (i=j),\\ \log(1-q\kappa^{-m_{ij}}\frac{w}{z})(1-q^{-1}\kappa^{-m_{ij}}\frac{w}{z}), & \langle h_i,\alpha_j\rangle=-2,\\ \log(1-\kappa^{-m_{ij}}\frac{w}{z}), & \langle h_i,\alpha_j\rangle=-1,\\ 0, & \langle h_i,\alpha_j\rangle=0.\end{cases}$$

For example we will show in the case of $\langle h_i,\alpha_j\rangle=-2$. By Campbell-Hausdorff formula we get

$$E_i^\pm(z)F_j^\pm(w)=F_j^\pm(w)E_i^\pm(z)$$

and

$$E_i^-(z)F_j^+(w)=\left(1-q\kappa^{-m_{ij}}\frac{w}{z}\right)\left(1-q^{-1}\kappa^{-m_{ij}}\frac{w}{z}\right)F_j^+(w)E_i^-(z).$$

On the other hand, by Lemma 3.4.1, we have

$$z^{H_{i,0}}e^{-\alpha_j}=z^2\kappa^{m_{ij}}e^{-\alpha_j}z^{H_{i,0}}.$$

Therefore we get

$$\begin{aligned}E_i(z)F_j(w)&=E_i^+(z)E_i^-(z)e^{\alpha_i}z^{H_{i,0}+1}F_j^+(w)F_j^-(w)e^{-\alpha_j}w^{-H_{j,0}+1}\\&=(z\kappa^{\frac{1}{2}m_{ij}}-qw\kappa^{-\frac{1}{2}m_{ij}})(z\kappa^{\frac{1}{2}m_{ij}}-q^{-1}w\kappa^{-\frac{1}{2}m_{ij}})E_i^+(z)F_j^+(w)E_i^-(z)F_j^-(w)\\&\quad\times e^{\alpha_j}e^{-\alpha_j}z^{H_{i,0}+1}w^{-H_{i,0}+1}.\end{aligned}$$

By a similar argument we have

$$\begin{aligned}F_j(w)E_i(z)&=(w\kappa^{-\frac{1}{2}m_{ij}}-qz\kappa^{\frac{1}{2}m_{ij}})(w\kappa^{-\frac{1}{2}m_{ij}}-q^{-1}z\kappa^{\frac{1}{2}m_{ij}})E_i^+(z)F_j^+(w)E_i^-(z)F_j^-(w)\\&\quad\times e^{\alpha_i}e^{-\alpha_i}z^{H_{i,0}+1}w^{-H_{i,0}+1}.\end{aligned}$$

Therefore we get

$$[E_i(z),F_j(w)]=0.$$

Similarly one can check the other formulas.
We will show (A.1.11). We have

$$\begin{aligned}&\left[-\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{-\frac{1}{2}k}z^{-k},\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{-\frac{1}{2}k}w^k\right]\\&=\sum_{k\geq 1}\frac{1}{k[k]}[k\langle h_i,\alpha_j\rangle]\kappa^{-m_{ij}}\left(\frac{w}{z}\right)^k\\&=\begin{cases}\log\left(1-\dfrac{w}{z}\right)\left(1-q^{-2}\dfrac{w}{z}\right), & i=j,\\ \log\dfrac{1}{\left(1-\kappa^{-m_{ij}}\dfrac{w}{z}\right)\left(1-q^{-2}\kappa^{-m_{ij}}\dfrac{w}{z}\right)}, & \langle h_i,\alpha_j\rangle=-2,\\ \log\dfrac{1}{1-q^{-1}\kappa^{-m_{ij}}\dfrac{w}{z}}, & \langle h_i,\alpha_j\rangle=-1,\\ 0, & \langle h_i,\alpha_j\rangle=0.\end{cases}\end{aligned}$$

For example let us show in the case of $\langle h_i,\alpha_j\rangle=-2$. If $\langle h_i,\alpha_j\rangle\neq -2$ one can show the formula by a similar argument. By Lemma 3.4.1 we get

$$(z\kappa^{m_{ij}}-q^{\langle h_i,\alpha_j\rangle}w)E_i(z)E_j(w)$$

$$= (z\kappa^{m_{iv}} - q^{-2}w) E_i^+(z) E_i^-(z) e^{\alpha_i} z^{H_{i,0}+1} E_j^+(w) E_j^-(w) e^{\alpha_j} w^{H_{j,0}+1}$$

$$= \frac{z\kappa^{m_{iv}} - q^{-2}w}{z^2\kappa^{m_{iv}}\left(1-\kappa^{-m_{iv}}\frac{w}{z}\right)\left(1-q^{-2}\kappa^{-m_{iv}}\frac{w}{z}\right)} E_i^+(z) E_i^+(w) E_i^-(z) E_j^-(w)$$

$$\times e^{\alpha_i} e^{\alpha_j} z^{H_{i,0}+1} w^{H_{j,0}+1}$$

$$= \frac{1}{(z-\kappa^{-m_{iv}}w)} E_i^+(z) E_i^+(w) E_j^-(z) E_j^-(w) e^{\alpha_i} e^{\alpha_i} z^{H_{i,0}+1} w^{H_{j,0}+1}.$$

On the other hand we have

$$(q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{iv}} z - w) E_j(w) E_i(z)$$

$$= \frac{q^{-2}\kappa^{m_{iv}} z - w}{w^2\kappa^{-m_{iv}}\left(1-\kappa^{m_{iv}}\frac{z}{w}\right)\left(1-q^{-2}\kappa^{m_{iv}}\frac{z}{w}\right)} E_j^+(w) E_i^+(z) E_j^-(w) E_i^-(z)$$

$$\times e^{\alpha_i} e^{\alpha_j} w^{H_{j,0}+1} z^{H_{i,0}+1}$$

$$= \frac{1}{(z-\kappa^{-m_{iv}}w)} E_i^+(z) E_j^+(w) E_i^-(z) E_j^-(w) e^{\alpha_i} e^{\alpha_j} z^{H_{i,0}+1} w^{H_{i,0}+1}.$$

Thus we have $(z\kappa^{m_{iv}} - q^{\langle h_i,\alpha_j\rangle} w) E_i(z) E_j(w) = (q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{iv}} z - w) E_j(w) E_i(z)$.

The formula $(z\kappa^{m_{iv}} - q^{-\langle h_i,\alpha_j\rangle} w) F_i(z) F_j(w) = (q^{-\langle h_i,\alpha_j\rangle}\kappa^{m_{iv}} z - w) F_j(w) F_i(z)$ is proved similarly.

Let us prove (A.1.12). Assume that $\langle h_i,\alpha_j\rangle = -2$. This is the most complicated case. The other cases can be proved similarly.

We have following formulas:

$$E_i(z_1) E_i(z_2) E_i(z_3) E_j(w)$$

$$= \frac{(z_1-z_2)(z_1-q^{-2}z_2)(z_1-z_3)(z_1-q^{-2}z_3)(z_2-z_3)(z_2-q^{-2}z_3)}{(\kappa^{\frac{1}{2}m_{iv}}z_1-\kappa^{-\frac{1}{2}m_{iv}}w)(\kappa^{\frac{1}{2}m_{iv}}z_1-q^{-2}\kappa^{-\frac{1}{2}m_{iv}}w)(\kappa^{\frac{1}{2}m_{iv}}z_2-\kappa^{-\frac{1}{2}m_{iv}}w)}$$

$$\times \frac{1}{(\kappa^{\frac{1}{2}m_{iv}}z_2-q^{-2}\kappa^{-\frac{1}{2}m_{iv}}w)(\kappa^{\frac{1}{2}m_{iv}}z_3-\kappa^{-\frac{1}{2}m_{iv}}w)(\kappa^{\frac{1}{2}m_{iv}}z_3-q^{-2}\kappa^{-\frac{1}{2}m_{iv}}w)}$$

$$\times E_i^+(z_1) E_i^+(z_2) E_i^+(z_3) E_j^+(w) E_i^-(z_1) E_i^-(z_2) E_i^-(z_3) E_j^-(w)$$

$$\times e^{3\alpha_i} e^{\alpha_j} z_1^{H_{i,0}+1} z_2^{H_{i,0}+1} z_3^{H_{i,0}+1} w^{H_{j,0}+1},$$

$$E_i(z_1) E_i(z_2) E_j(w) E_i(z_3)$$

$$= \frac{(z_1-z_2)(z_1-q^{-2}z_2)(z_1-z_3)(z_1-q^{-2}z_3)(z_2-z_3)(z_2-q^{-2}z_3)}{(\kappa^{\frac{1}{2}m_{iv}}z_1-\kappa^{-\frac{1}{2}m_{iv}}w)(\kappa^{\frac{1}{2}m_{iv}}z_1-q^{-2}\kappa^{-\frac{1}{2}m_{iv}}w)(\kappa^{\frac{1}{2}m_{iv}}z_2-\kappa^{-\frac{1}{2}m_{iv}}w)}$$

$$\times\frac{1}{(\kappa^{\frac{1}{2}m_v}z_2-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{-\frac{1}{2}m_v}w-\kappa^{\frac{1}{2}m_v}z_3)\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_3)}$$

$$\times E_i^+(z_1)E_i^+(z_2)E_i^+(z_3)E_j^+(w)E_i^-(z_1)E_i^-(z_2)E_i^-(z_3)E_j^-(w)$$

$$\times e^{3\alpha_t}e^{\alpha_j}z_1^{H_{i,0}+1}z_2^{H_{i,0}+1}z_3^{H_{i,0}+1}w^{H_{j,0}+1},$$

$$E_i(z_1)E_j(w)E_t(z_2)E_t(z_3)$$

$$=\frac{(z_1-z_2)\,(z_1-q^{-2}z_2)\,(z_1-z_3)\,(z_1-q^{-2}z_3)\,(z_2-z_3)\,(z_2-q^{-2}z_3)}{(\kappa^{\frac{1}{2}m_v}z_1-\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{\frac{1}{2}m_v}z_1-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{-\frac{1}{2}m_v}w-\kappa^{\frac{1}{2}m_v}z_2)}$$

$$\times\frac{1}{(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_2)\,(\kappa^{-\frac{1}{2}m_v}w-\kappa^{\frac{1}{2}m_v}z_3)\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_3)}$$

$$\times E_i^+(z_1)E_t^+(z_2)E_i^+(z_3)E_j^+(w)E_i^-(z_1)E_i^-(z_2)E_i^-(z_3)E_j^-(w)$$

$$\times e^{3\alpha_t}e^{\alpha_j}z_1^{H_{i,0}+1}z_2^{H_{i,0}+1}z_3^{H_{i,0}+1}w^{H_{j,0}+1},$$

$$E_j(w)E_t(z_1)E_t(z_2)E_t(z_3)$$

$$=\frac{(z_1-z_2)\,(z_1-q^{-2}z_2)\,(z_1-z_3)\,(z_1-q^{-2}z_3)\,(z_2-z_3)\,(z_2-q^{-2}z_3)}{(\kappa^{-\frac{1}{2}m_v}w-\kappa^{\frac{1}{2}m_v}z_1)\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_1)\,(\kappa^{-\frac{1}{2}m_v}w-\kappa^{\frac{1}{2}m_v}z_2)}$$

$$\times\frac{1}{(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_2)\,(\kappa^{-\frac{1}{2}m_v}w-\kappa^{\frac{1}{2}m_v}z_3)\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_3)}$$

$$\times E_i^+(z_1)E_t^+(z_2)E_t^+(z_3)E_j^+(w)E_i^-(z_1)E_t^-(z_2)E_i^-(z_3)E_j^-(w)$$

$$\times e^{3\alpha_t}e^{\alpha_j}z_1^{H_{i,0}+1}z_2^{H_{i,0}+1}z_3^{H_{i,0}+1}w^{H_{j,0}+1}.$$

Therefore it is enough to show that

(A.1.14)

$$\sum_{\sigma\in\mathfrak{S}_3}\frac{\prod_{i<j}(z_{\sigma(i)}-z_{\sigma(j)})\,(z_{\sigma(i)}-q^{-2}z_{\sigma(j)})}{\prod_{i=1}(\kappa^{\frac{1}{2}m_v}z_{\sigma(i)}-\kappa^{-\frac{1}{2}m_v}w)}$$

$$\times\left\{\frac{1}{(\kappa^{\frac{1}{2}m_v}z_{\sigma(1)}-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{\frac{1}{2}m_v}z_{\sigma(2)}-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{\frac{1}{2}m_v}z_{\sigma(3)}-q^{-2}\kappa^{-\frac{1}{2}m_v}w)}\right.$$

$$+\frac{q^2+1+q^{-2}}{(\kappa^{\frac{1}{2}m_v}z_{\sigma(1)}-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{\frac{1}{2}m_v}z_{\sigma(2)}-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_{\sigma(3)})}$$

$$\left.+\frac{q^2+1+q^{-2}}{(\kappa^{\frac{1}{2}m_v}z_{\sigma(1)}-q^{-2}\kappa^{-\frac{1}{2}m_v}w)\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_{\sigma(2)})\,(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_{\sigma(3)})}\right.$$

$$+\frac{1}{(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_{\sigma(1)})(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_{\sigma(2)})(\kappa^{-\frac{1}{2}m_v}w-q^{-2}\kappa^{\frac{1}{2}m_v}z_{\sigma(3)})}\Bigg\}$$

$=0.$

This identity is proved by a direct calculation.
Thus the proposition is proved.

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References

[Be] Beck, J., Braid group action and quantum affine algebras, *Comm. Math. Phys.*, **165** (1994), 555-568.

[B] Berman, S., On generators and relations for certain involutory subalgebras Kac-Moody Lie algebras, *Comm. Alg.*, **17** (12) (1989) 3165-3185.

[BM] Berman, S. and Moody, R., Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy, *Inv. Math.*, **108** (1992), 323-347.

[C] Cherednik, I., Double-affine Hecke algebras, Knizhink-Zamolodchikov equations and Macdonald's operators, *Intern. Math. Research Notices*, **9** (1992), 171-180.

[CP] Chari, V. and Pressly, A., Quantum affine algebras and affine Hecke algebras, *Pacific J. Math.*, **174** (1996), 295-326.

[D1] Drinfeld, V., Quantum groups, *Proc. ICM Berkeley*, **1** (1986), 789-820.

[D2] ———, A new realization of Yangians and quantized affine algebras, *Sov. Math. Dokl.*, **36** (1988), 212-216.

[EM] Eswara Rao, S. and Moody, R. V., Vertex representations for N-toroidal Lie algebras and a Generalization of the Virasoro algebra, *Comm. Math. Phys.*, **159** (1994), 239-264.

[FJ] Frenkel, I. and Jing, N., Vertex representations of quantum affine algebras, *Proc. Natl. Acad. Sci. USA*, **85** (1988), 9373-9377.

[FK] Frenkel, I. and Kac, V., Basic Representations of affine Lie algebras and dual Resonance models, *Inv. Math.*, **62** (1980), 23-66.

[GKV] Ginzburg, V., Kapranov, M. and Vasserot, E., Langlands reciprocity for algebraic surface, *Math. Res. Lett.*, **2** (1995), 147-160.

[K] Koyama, Y., Staggered polarization of vertex model with $U_q(\widehat{\mathfrak{sl}}(n))$-symmetry, *Comm. Math. Phys.*, **164** (1994), 277-291.

[MEY] Moody, R. V., Eswara Rao, S. and Yokomuma, T., Toroidal Lie algebras and vertex representations, *Geom. Dedicata*, **35** (1990), 283-307.

[S] Slodowy, S., Beyond Kac-Moody algebras, and inside, *Canadian Math. Soc. Conference Proceedings*, **5** (1986), 361-371.

[V] Vasserot, E., Private communication.

[VV] Varagnolo, M. and Vesserot, E., Schur duality in the toroidal setting, *Comm. Math. Phys.*, **182** (1996), 469-484.

[Y] Yamada, H., Extended affine Lie algebra and their vertex representations, *Publ. RIMS. Kyoto Univ.*, **25** (1989), 587-603.