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On a Solution of a q-Difference Analogue of Lauricella's D-Type Hypergeometric Equation with $|q|=1$

By

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Abstract

By using the double sine function, an integral solution of a q-difference analogue of Lauricella's D-type hypergeometric equation is constructed in the case when $|q|=1$. Furthermore, its contiguity operators are considered

§ 1. Introduction

q-Analysis with $|q|=1$ is important for the studies on the XXZ model in the gapless regime [8], on the massive field theory [10], [18] and on a representation theory of the quantum group $SL_q(2, \mathbf{R})$ [12]. However, it is not satisfactorily explored yet from the viewpoint of the theory of q-special functions.

In [15], the author and K. Ueno presented a method of q-analysis with $|q|=1$. By using Kurokawa's double sine function in place of the q-shifted factorial, they proved that the Euler type integral makes sense and that it gives a solution of a q-difference analogue of Gauss' hypergeometric equation under some conditions on parameters.

In this paper, we apply this method to a q-difference analogue of Lauricella's D-type hypergeometric equation (We call it "*Lauricella's q-HGE*" in this paper.), which is a system of q-difference equations of complex n-variables $z:=(z_1, z_2, \cdots z_n)$:

$$
(1)\quad \begin{aligned}
&\{(1-cq^{-1}T_q)(1-T_{q,z_j})-z_j(1-aT_q)(1-b_jT_{q,z_j})\}f(z)=0, \quad (j=1, 2, \cdots n)\\
&\{z_j(1-b_jT_{q,z_j})(1-T_{q,z_k})-z_k(1-T_{q,z_j})(1-b_kT_{q,z_k})\}f(z)=0, \quad (1\leq j<k\leq n),
\end{aligned}
$$

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where a, b_j ($j=1, 2, \cdots n$) and c are complex parameters. T_{q,z_k} is a q-shift operator acting on z_k and $T_q := T_{q,z_1}T_{q,z_2}\cdots T_{q,z_n}$. This equation was discussed by Noumi [16] in his studies on the quantum Grassmanian.

In the case when $|q|<1$, the equation (1) has a solution $\phi_D(z)=\phi_D(z_1, z_2, \cdots z_n)$ represented by Jackson integral.

$$\phi_D(z)=\phi_D\begin{pmatrix} a;\ b_1\ b_2\ \dots\ b_n \\ c \end{pmatrix};z;q\Big) \tag{2}$$

$$=\frac{1}{B(a, c-a; q)}\int_0^1 t^a \frac{(tq)_\infty}{(tq^{c-a})_\infty}\prod_{k=1}^{n}\frac{(tz_k q^{b_k})_\infty}{(tz_k)_\infty}\frac{d_q t}{t},$$

where $B(x, y; q)$ is the q-beta function (see [6].) and $(x)_\infty := \prod_{j=1}^{\infty}(1-xq^j)$.

We investigate a solution of (1) with $|q|=1$. In the similar way to the case of a q-analogue of Gauss' hypergeometric equation [15], we construct an integral solution of the equation in which multiplicative variables of (1) are transformed to the additive variables.

This paper is organized as follows: In Section 2, we give the definition of a certain function, which plays the role of the q-shifted factorials, and recall its basic properties. We remark that similar function has been defined by Faddeev [3] and Ruijsenaars [17]. In Section 3, We define a function $\Psi_D(x)$ by using integration of the Euler type and we show that $\Psi_D(x)$ gives a solution of Lauricella's q-HGE with $|q|=1$. In Section 4, we consider contiguity operators acting on $\Psi_D(x)$. In the case when $|q|<1$, contiguous relations are studied on for various hypergeometric q-difference systems [2], [7], [13], [14], [16]. In the case when $|q|=1$, we can also obtain contiguity operators in the same way as Noumi's result [16]. However, the relation between them and the representations of $U_q(\mathfrak{gl}(m))$ is not fully understood because we have to impose some conditions on parameters in order that this integral representation makes sense.

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§ 2. Kurokawa's Double Sine Function and a "q-Shifted Factorial with $|q|=1$"

In this section, we give definitions of the functions $\langle z\rangle$, $\tilde{\Gamma}(z)$, and $\tilde{B}(z)$, which are important in the following argument. They play the roles of the q-shifted factorials, of the q-gamma function and of the q-beta function respectively in the case when $|q|=1$. For this end, we introduce Kurokawa's double sine function $S_2(z|\omega_1, \omega_2)$ [9].

Definition 2.1. *For* $\boldsymbol{\omega} := (\omega_1, \omega_2) \in \boldsymbol{C}^2$ *we define* $\zeta_2(s, z|\boldsymbol{\omega})$, $\Gamma_2(z|\boldsymbol{\omega})$ *and* $S_2(z|\boldsymbol{\omega})$ *by*

$$\zeta_2(s, z|\boldsymbol{\omega}) := \sum_{m_1, m_2 \in \boldsymbol{Z}_{\geq 0}} (z+m_1\omega_1+m_2\omega_2)^{-s},$$

$$\Gamma_2(z|\boldsymbol{\omega}) := \exp\left(\frac{\partial}{\partial s}\zeta_2(s, z|\boldsymbol{\omega})|_{s=0}\right),$$

$$S_2(z|\boldsymbol{\omega}) := \Gamma_2(z|\boldsymbol{\omega})^{-1}\Gamma_2(\omega_1+\omega_2-z|\boldsymbol{\omega}).$$

It is known that the double sine function satisfies the functional relation

$$\frac{S_2(z+\omega_1|\boldsymbol{\omega})}{S_2(z|\boldsymbol{\omega})} = \frac{1}{2\sin\frac{\pi z}{\omega_2}}. \tag{3}$$

By using this function, we can construct a "q-shifted factorial with $|q|=1$". We suppose that $q=e^{2\pi i\omega}$ $(0<\omega<1,\ \omega \notin \mathbf{Q})$, *i.e.* $|q|=1$ and q is not a root of unity. From now on, we take such a branch of logarithm that $\log q = 2\pi i\omega$.

Definition 2.2. *We define* $\langle z\rangle$, $\tilde{\Gamma}(z)$ *and* $\tilde{B}(z)$ *by*

$$\langle z\rangle = \langle z; q\rangle := i^{1-z}q^{-\frac{z(z-1)}{4}}S_2\left(z\left|\left(1, \frac{1}{\omega}\right)\right.\right),$$

$$\tilde{\Gamma}(z) = \tilde{\Gamma}(z; q) := \sqrt{\omega}^{-1}(q-1)^{1-z}\langle z\rangle^{-1},$$

$$\tilde{B}(a, b) = \tilde{B}(a, b; q) := \frac{\tilde{\Gamma}(a)\tilde{\Gamma}(b)}{\tilde{\Gamma}(a+b)}.$$

These functions have the following properties:

Lemma 2.3. (1) $\langle z\rangle$ *and* $\tilde{\Gamma}(z)$ *satisfy functional equations*

$$\langle z\rangle = (1-q^z)\langle z+1\rangle,$$

$$\tilde{\Gamma}(z+1) = \frac{1-q^z}{1-q}\tilde{\Gamma}(z), \quad \tilde{\Gamma}(1) = 1.$$

(2) $\langle z\rangle$ *has simple poles*

$$z = n_1 + \frac{n_2}{\omega} \quad (n_1, n_2 \in \mathbf{Z}_{>0}),$$

and simple zeros

$$z=n_1+\frac{n_2}{\omega} \quad (n_1, n_2\in\mathbb{Z}_{\leq 0}).$$

(3) *As* $|z|\to\infty$ *within a sector not containing the real axis,* $\langle z\rangle$ *has an asymptotic behavior*

$$\langle z\rangle=\begin{cases}\exp[-\pi iz+O(1)] & \Im z>0,\\ \exp[-\pi i\omega(z^2-z)+O(1)] & \Im z<0.\end{cases}$$

This lemma follows from results of the papers [8], [19].

§ 3. Construction of a Solution

In this section, we construct an integral solution of Lauricella's q-HGE with $|q|=1$ by using the functions defined in the previous section.

First, we define an integrand $\phi_D(s, x)$ of the integral.

Definition 3.1. *For* $x=(x_1, x_2, \cdots x_n)\in\mathbb{C}^n$ *and* $a, b_j\,(j=1, 2, \cdots n), c\in\mathbb{C}$, *let us put*

(4)
$$\phi_D(s, x)=\phi_D\left(\begin{matrix}a;\ b_1\ b_2, \cdots\ b_n\\ c\end{matrix}\ ; s, x\ ; q\right)$$

$$:=q^{as}\frac{\langle s+1\rangle}{\langle s+c-a\rangle}\prod_{k=1}^{n}\frac{\langle s+x_k+b_k\rangle}{\langle s+x_k\rangle}.$$

By means of Lemma 2.3, we see that $\phi_D(s, x)$ has simple poles at

(5)
$$s=-1+m_1+\frac{m_2}{\omega},\quad s=-x_k-b_k+m_1+\frac{m_2}{\omega},$$
$$s=a-c+n_1+\frac{n_2}{\omega},\quad s=-x_k+n_1+\frac{n_2}{\omega},$$
$$(m_1, m_2\in\mathbb{Z}_{>0},\quad n_1, n_2\in\mathbb{Z}_{\leq 0},\quad k=1, 2, \cdots n)$$

and that

(6)
$$\phi_D(s, x)=\begin{cases}\exp[-2\pi|s|\omega\Re a+O(1)] & \text{as}\quad s\to+i\infty,\\ \exp\left[-2\pi|s|\omega\Re\left(c-\sum_{k=1}^{n}b_k-1\right)+O(1)\right] & \text{as}\quad s\to-i\infty,\end{cases}$$

for x in a bounded domain on $\mathbb{C}^n$.

We, now, integrate $\phi_D(s, x)$ along a contour on the s-plane. In order that the integral makes sense, we impose the following conditions on the parameters a, $b_j (j=1, 2, \cdots n)$ and c:

Conditions on parameters: *We assume that*

$$a-c \notin \mathbf{Z}_{>0} \tag{7}$$

$$b_j \notin \mathbf{Z}_{<0} \quad \text{for} \quad j=1, 2, \cdots n \tag{8}$$

$$\Re a>0, \quad \Re\left(c-\sum_{j=1}^{n} b_j-2\right)>0. \tag{9}$$

Under these conditions, we can define the Euler integral $\Psi_D(x)$. Let us denote by K any bounded domain in the region $\{x \in \mathbf{C}^n | \forall x_j \notin \mathbf{Z}_{<0}\}$.

Definition 3.2. *Suppose that a, b_j and c satisfy (7) ~ (9) then, for $x \in K$, we define $\Psi_D(x)$ by*

$$\Psi_D(x)=\Psi_D\left(\begin{matrix} a; b_1\ b_2 \cdots b_n \\ c \end{matrix} ; x ; q\right) \tag{10}$$

$$:= \frac{1}{\widetilde{B}(a, c-a)} \int_{-i\infty}^{+i\infty} \phi_D(s, x)\, ds$$

where the contour (cf. Fig.1) *lies on the right of the poles*

$$s=-x_k+n_1+\frac{n_2}{\omega}, \quad s=a-c+n_1+\frac{n_2}{\omega}, \quad (n_1, n_2 \in \mathbf{Z}_{\leq 0}),$$

and on the left of the poles

$$s=-x_k-b_k+m_1+\frac{m_2}{\omega}, \quad s=-1+m_1+\frac{m_2}{\omega}, \quad (m_1, m_2 \in \mathbf{Z}_{>0}),$$

where $k=1, 2, \cdots n$.

Thanks to the condition (7), (8) and (9), we can see that the integral (10) converges uniformly and defines an analytic function of x.

Next, we prove that $\Psi_D(x)$ is a solution of the system of difference equations which is obtained by transforming the multiplicative variables of Lauricella's q-HGE (1) to the additive variables.

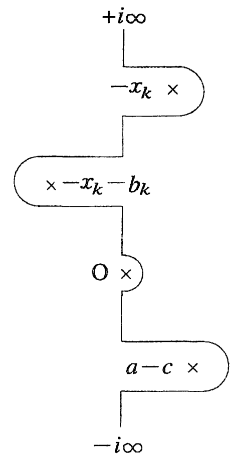

Figure 1: Contour of the integral $\Psi_D(x)$

Let us introduce some notations on difference operators. We define T_{x_j} by a usual difference operator acting on x_j i.e.

$$(T_{x_j}f)(x_1, x_2, \cdots x_n) := f(x_1, \cdots x_{j-1}, x_j+1, x_{j+1}, \cdots x_n),$$

and define T and $T_{>k}$ by

$$T := T_{x_1}T_{x_2}\cdots T_{x_n}, \quad T_{>k} := T_{x_{k+1}}\cdots T_{x_n}. \tag{11}$$

Then, the following theorem holds.

Theorem 3.3. *$\Psi_D(x)$ satisfies the following system of difference equations:*

$$\begin{gathered}\{(1-q^{c-1}T)(1-T_{x_j})-q^{x_j}(1-q^aT)(1-q^{b_j}T_{x_j})\}\Psi_D(x)=0,\\ (j=1, 2, \cdots n),\\ \{q^{x_j}(1-q^{b_j}T_{x_j})(1-T_{x_k})-q^{x_k}(1-T_{x_j})(1-q^{b_k}T_{x_k})\}\Psi_D(x)=0,\\ (1\leq j<k\leq n).\end{gathered} \tag{12}$$

Proof. First, we prove the formula

$$\begin{aligned}&\int_{-i\infty}^{+i\infty} q^{a(s+1)}\frac{\langle s+1\rangle}{\langle s+c-a\rangle}\frac{\langle s+x_j+b_j+2\rangle}{\langle s+x_j+1\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l+1\rangle}{\langle s+x_l+1\rangle}ds\\ &=\int_{-i\infty}^{+i\infty} q^{as}\frac{\langle s\rangle}{\langle s+c-a-1\rangle}\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds.\end{aligned} \tag{13}$$

It can be shown as follows: We note that the both side of (13) converges because of Condition (9). By Chauchy's theorem, we can see that

$$\int_{C_R} q^{a(s+1)} \frac{\langle s+1\rangle}{\langle s+c-a\rangle} \frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle} \prod_{l\neq j} \frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle} ds=0$$

where C_R is a cycle defined by Fig. 2.

From (6), it follows that the integral from $+iR$ to $+iR+1$ and the integral from $-iR$ to $-iR+1$ vanish as $R\to\infty$. Thus, we can see that

$$\int_{-i\infty}^{+i\infty} q^{a(s+1)} \frac{\langle s+1\rangle}{\langle s+c-a\rangle} \frac{\langle s+x_j+b_j+2\rangle}{\langle s+x_j+1\rangle} \prod_{l\neq j} \frac{\langle s+x_l+b_l+1\rangle}{\langle s+x_l+1\rangle} ds$$

$$=\int_{-i\infty+1}^{+i\infty+1} q^{a(s+1)} \frac{\langle s+1\rangle}{\langle s+c-a\rangle} \frac{\langle s+x_j+b_j+2\rangle}{\langle s+x_j+1\rangle} \prod_{l\neq j} \frac{\langle s+x_l+b_l+1\rangle}{\langle s+x_l+1\rangle} ds$$

$$=\int_{-i\infty}^{+i\infty} q^{as} \frac{\langle s\rangle}{\langle s+c-a-1\rangle} \frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle} \prod_{l\neq j} \frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle} ds.$$

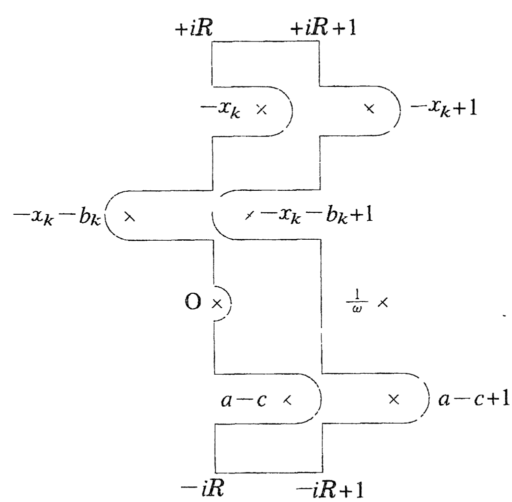

Figure 2: Cycle C_R

So, we have proved (13). By using this formula, we have

$$(1-q^{c-1}T)(1-T_{x_j})\Psi_D(x)$$

$$=\frac{1-q^{b_j}}{\widetilde{B}(a,c-a)}$$

$$\times\left\{q^{x_j}\int_{-i\infty}^{+i\infty}q^{as}\frac{\langle s+1\rangle}{\langle s+c-a\rangle}\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds\right.$$

$$\left.-q^{c-1+x_j}\int_{-i\infty}^{+i\infty}q^{as}\frac{\langle s+1\rangle}{\langle s+c-a\rangle}\frac{\langle s+x_j+b_j+2\rangle}{\langle s+x_j+1\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l+1\rangle}{\langle s+x_l+1\rangle}ds\right\}$$

$$=\frac{1-q^{b_j}}{\widetilde{B}(c,c-a)}$$

$$\times\left\{q^{x_j}\int_{-i\infty}^{+i\infty}q^{as}\frac{\langle s+1\rangle}{\langle s+c-a\rangle}\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds\right.$$

$$\left.-q^{c-a-1+x_j}\int_{-i\infty}^{+i\infty}q^{as}\frac{\langle s\rangle}{\langle s+c-a-1\rangle}\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds\right\}.$$

$$=\frac{q^{x_j}(1-q^{b_j})(1-q^{c-a-1})}{\widetilde{B}(a,c-a)}$$

$$\times\int_{-i\infty}^{+i\infty}q^{as}\frac{\langle s+1\rangle}{\langle s+c-a-1\rangle}\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds.$$

Similarly, we can see that

$$q^{x_j}(1-q^aT)(1-q^{b_j}T_{x_j})\Psi_D$$

$$=\frac{q^{x_j}(1-q^{b_j})(1-q^{c-a-1})}{\widetilde{B}(a,c-a)}$$

$$\times\int_{-i\infty}^{+i\infty}q^{as}\frac{\langle s+1\rangle}{\langle s+c-a-1\rangle}\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\prod_{l\neq j}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds.$$

Thus, the first formula is proved.

The second formula follows from straightforward calculation. We can see that

$$q^{x_j}(1-q^{b_j}T_{x_j})(1-T_{x_k})\Psi_D(x)$$

$$=q^{x_k}(1-T_{x_j})(1-q^{b_k}T_{x_k})\Psi_D(x)$$

$$=\frac{q^{x_j+x_k}(1-q^{b_j})(1-q^{b_k})}{\widetilde{B}(a,c-a)}\left[\int_{-i\infty}^{+i\infty}q^{(a+1)s}\frac{\langle s+1\rangle}{\langle s+c-a\rangle}\right.$$

$$\left.\times\frac{\langle s+x_j+b_j+1\rangle}{\langle s+x_j\rangle}\frac{\langle s+x_k+b_k+1\rangle}{\langle s+x_k\rangle}\prod_{l\neq j,k}\frac{\langle s+x_l+b_l\rangle}{\langle s+x_l\rangle}ds\right].$$

∎

§ 4. Contiguity Operators Acting on $\Psi_D(x)$

In this section, we give contiguity operators acting on $\Psi_D(x)$. They are obtained by replacing q-shift operators in Noumi's operator [16] by addititive shift operators. However, in order to make the integral (12) converge, we have to impose the condition (16).

We use the same notation as [16]. Let us put

$$F(\lambda, x) := \Psi_D\left(\begin{matrix} \lambda_2+1 \,;\, -\lambda_4 \cdots -\lambda m \\ \lambda_2+\lambda_3+2, \end{matrix} \;;\; q;\, x_4 \cdots x_m\right)$$

where $x := (x_4, x_5, \cdots x_m)$ is complex $m-3$ variables and $\lambda := (\lambda_1, \lambda_2, \cdots \lambda_m) \in \mathbf{C}^m$ is a set of complex parameters satisfying

$$\sum_{j=1}^{m} \lambda_j = -2 \tag{14}$$

$$\Re\lambda_1 > -2, \qquad \Re\lambda_2 > 0. \tag{15}$$

We define contiguity oprators by

$$E_{1,2} := \frac{q^{\frac{\lambda_1-\lambda_2}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \left\{ (1-q^{\lambda_2+\lambda_3+1}T_{>3}) - q^{\lambda_2} \sum_{k=4}^{m} q^{x_k} (1-q^{-\lambda_k}T_{x_k}) T_{>k} \right\},$$

$$E_{2,3} := -\frac{q^{\frac{-\lambda_1-\lambda_2-\lambda_3-1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} (1-q^{\lambda_2+1}T_{>3}),$$

$$E_{3,4} := \frac{q^{\frac{\lambda_4}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \{ (1-q^{-\lambda_4}T_{x_4}) - q^{-x_4+\lambda_3+1} (1-T_{x_4}) \},$$

$$E_{j,j+1} := \frac{q^{\frac{\lambda_{j+1}}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \{ (1-q^{-\lambda_{j+1}}T_{x_{j+1}}) - q^{x_j-x_{j+1}-2} (1-T_{x_{j+1}}) \} T_{x_j}^{-1} \quad (4 \leq j \leq m-1),$$

$$E_{2,1} := -\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \left\{ (1-q^{\lambda_2+1}T_{>3}) - q^{-\lambda_1} \sum_{k=4}^{m} q^{-x_k-\sum_{l=k+1}^{m}\lambda_l} (1-T_{x_k}) T_{>k} \right\},$$

$$E_{3,2} := -\frac{q^{\frac{\lambda_1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \left\{ (1-q^{\lambda_3+1}T_{>3}) - q^{-\lambda_1} \sum_{k=4}^{m} q^{x_k} (1-q^{-\lambda_k}T_{x_k}) T_{>k} \right\} T_{>3}^{-1},$$

$$E_{4,3} := -\frac{q^{\frac{-\lambda_3}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} \{ (1-q^{\lambda_2+\lambda_3+1}T_{>3}) - q^{x_4-\lambda_4-1} (1-q^{\lambda_2}T_{>3}) \},$$

$$E_{j+1,j} := \frac{q^{\frac{\lambda_j}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\{(1-q^{\lambda_j}T_{x_j})-q^{x_{j+1}-x_j-\lambda_{j+1}-1}(1-T_{x_j})\}. \quad (4\leq j\leq m-1).$$

$T_{>k}$ are defined in (11). Formally, these operators generate a representation of $U_q(\mathfrak{gl}(m))$. We can see that the following proposition holds:

Proposition 4.1. *Operetors $E_{j,j+1}$ and $E_{j+1,j}$ act on $F(\lambda, x)$ as follows:*

$$E_{1,2}F(\lambda, x) = -q^{\frac{1}{2}(\lambda_1+\lambda_3+1)}[\lambda_2+\lambda_3+1]F(\lambda+\alpha_1, x),$$

$$E_{2,3}F(\lambda, x) = q^{-\frac{1}{2}(\lambda_1+\lambda_3)}[\lambda_2+1]F(\lambda+\alpha_2, x),$$

$$E_{3,4}F(\lambda, x) = q^{-\frac{1}{2}(\lambda_2+1)}\frac{[\lambda_3+1][\lambda_4]}{[\lambda_2+\lambda_3+2]}F(\lambda+\alpha_3, x),$$

$$E_{j,j+1}F(\lambda, x) = [\lambda_{j+1}]F(\lambda+\alpha_j, x) \quad (4\leq j\leq m-1),$$

$$E_{2,1}F(\lambda, x) = q^{\frac{1}{2}(\lambda_1+\lambda_3)}\frac{[\lambda_1][\lambda_2+1]}{[\lambda_2+\lambda_3+2]}F(\lambda-\alpha_1, x),$$

$$E_{3,2}F(\lambda, x) = q^{\frac{1}{2}(\lambda_2+\lambda_3+1)}[\lambda_3+1]F(\lambda-\alpha_2, x),$$

$$E_{4,3}F(\lambda, x) = q^{\frac{1}{2}(\lambda_2+1)}[\lambda_2+\lambda_3+1]F(\lambda-\alpha_3, x),$$

$$E_{j+1,j}F(\lambda, x) = [\lambda_{j+1}]F(\lambda-\alpha_j, x) \quad (4\leq j\leq m-1),$$

where,

$$[a] := \frac{q^{\frac{a}{2}}-q^{-\frac{a}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},$$

and

$$\alpha_j := (0, 0, \cdots \overset{\overset{j}{\cup}}{1}, \overset{\overset{j+1}{\cup}}{-1}, 0, \cdots, 0).$$

Proof. These formulas can be proved through straightforward calculation. For example, we prove the first formula and the fifth formula. By the same argument as (13), we can see that

(16) $\quad (1-q^{(\lambda_2+\lambda_3+1)}T_{>3})F(\lambda, x)$

$$= \frac{1}{\tilde{B}(\lambda_2+1, \lambda_3+1)}\left\{\int_{-i\infty}^{+i\infty} q^{(\lambda_2+1)s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+z_k+1\rangle}{\langle s+x_k\rangle}ds\right.$$

$$\left. -\int_{-i\infty}^{+i\infty} q^{(\lambda_2+1)s}\frac{\langle s\rangle}{\langle s+\lambda_3\rangle}\prod_{k=4}^{m}\frac{\langle s+x_k-\lambda_k\rangle}{\langle s+x_k\rangle}ds\right\}$$

$$=\frac{1}{\tilde{B}(\lambda_2+1,\ \lambda_3+1)}\left\{\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+x_k-\lambda_k\rangle}{\langle s+x_k\rangle}ds\right.$$

$$\left.-\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s\rangle}{\langle s+\lambda_3\rangle}\prod_{k=4}^{m}\frac{\langle s+x_k-\lambda_k\rangle}{\langle s+x_k\rangle}ds\right\}.$$

On the other hand, noting that

$$q^{x_k}(1-q^{-\lambda_k}T_{>k})F(\lambda,\ x)$$

$$=\frac{1}{\tilde{B}(\lambda_2+1.\ \lambda_3)}\times$$

$$\left\{-\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{l=4}^{k-1}\frac{\langle s+x_k-\lambda_l\rangle}{\langle s+x_l\rangle}\frac{\langle s+x_k-\lambda_k+1\rangle}{\langle s+x_k\rangle}\prod_{l=k-1}^{m}\frac{\langle s+x_l-\lambda_l+1\rangle}{\langle s+x_l+1\rangle}ds\right.$$

$$\left.+\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{l=4}^{k}\frac{\langle s+x_l-\lambda_l\rangle}{\langle s+x_l\rangle}\frac{\langle s+x_{k+1}-\lambda_{k+1}+1\rangle}{\langle s+x_{k+1}\rangle}\prod_{l=k+2}^{m}\frac{\langle s+x_l-\lambda_l+1\rangle}{\langle s+x_l+1\rangle}ds\right\},$$

we have

$$\sum_{k=4}^{m}(1-q^{-\lambda_k}T_{x_k})T_{>k}F(\lambda,\ x) \tag{17}$$

$$=\frac{1}{\tilde{B}(\lambda_2+1,\ \lambda_3+1)}\left\{-\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+x_k-\lambda_k+1\rangle}{\langle s+x_k+1\rangle}ds\right.$$

$$\left.+\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+x_k-\lambda_k\rangle}{\langle s+x_k\rangle}ds\right\}.$$

Therefore, from (16) and (17), it follows that

$$E_{1,2}F(\lambda,\ x)$$

$$=\frac{q^{\frac{\lambda_1-\lambda_2}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\frac{1-q^{\lambda_2}}{\tilde{B}(\lambda_2+1,\ \lambda_3+1)}\int_{-i\infty}^{+i\infty} q^{\lambda_2 s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+x_k-\lambda_k\rangle}{\langle s+x_k\rangle}ds$$

$$=q^{\frac{1}{2}(\lambda_1+\lambda_3+1)}[\lambda_2+\lambda_3+1]F(\lambda+\alpha_1,\ x).$$

Next, we prove the fifth formula. It is easily seen that

$$(1-q^{\lambda_3}T_{>3})F(\lambda,\ z)$$

$$=\frac{1-q^{\lambda_3}}{\tilde{B}(\lambda_2+1,\ \lambda_3+1)}\int_{-i\infty}^{+i\infty} q^{(\lambda_2+2)s}\frac{\langle s+1\rangle}{\langle s+\lambda_3\rangle}\prod_{k=4}^{m}\frac{\langle s+z_k-\lambda_k\rangle}{\langle s+z_k\rangle}ds.$$

Furthermore, by using the formula

$$\frac{\langle s+z_k-\lambda_k+1\rangle}{\langle s+z_k\rangle}=\frac{1}{1-q^{-\lambda_k}}\frac{\langle s+z_k-\lambda_k\rangle}{\langle s+z_k\rangle}-\frac{q^{-\lambda_k}}{1-q^{-\lambda_k}}\frac{\langle s+z_k-\lambda_k+1\rangle}{\langle s+z_k+1\rangle},$$

we obtain that

$$\begin{aligned}
&q^{-z_k}(1-T_k)T_{>k}F(\lambda;z)\\
&=\frac{1}{\widetilde{B}(\lambda_2+1,\lambda_3+1)}\\
&\times\left\{\int_{-i\infty}^{+i\infty}q^{(\lambda_2+2)s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{l=4}^{k-1}\frac{\langle s+z_l-\lambda_l\rangle}{\langle s+z_l\rangle}\prod_{l=k}^{m}\frac{\langle s+z_l-\lambda_l+1\rangle}{\langle s+z_l+1\rangle}ds\right.\\
&\left.-q^{-\lambda_k}\int_{-i\infty}^{+i\infty}q^{(\lambda_2+2)s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{l=4}^{k}\frac{\langle s+z_l-\lambda_k\rangle}{\langle s+z_l\rangle}\prod_{l=k+1}^{m}\frac{\langle s+z_l-\lambda_l+1\rangle}{\langle s+z_l+1\rangle}ds\right\}.
\end{aligned}$$

Noting that

$$-\sum_{k=4}^{n}\lambda_k=\lambda_1+\lambda_2+\lambda_3+2,$$

we have

$$\begin{aligned}
&\sum_{k=4}^{n}q^{-z_k-\sum_{l=k+1}^{m}\lambda_l}(1-T_k)T_{>k}F(\lambda;z)\\
&=\frac{1}{\widetilde{B}(\lambda_2+1,\lambda_3+1)}\left\{q^{\lambda_1+\lambda_3}\int_{-i\infty}^{+i\infty}q^{(\lambda_2+2)s}\frac{\langle s\rangle}{\langle s+\lambda_3\rangle}\prod_{k=4}^{m}\frac{\langle s+z_k-\lambda_k+1\rangle}{\langle s+z_k+1\rangle}ds\right.\\
&\left.-\int_{-i\infty}^{+i\infty}q^{(\lambda_2+2)s}\frac{\langle s+1\rangle}{\langle s+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+z_k-\lambda_k\rangle}{\langle s+zk\rangle}ds\right\}.
\end{aligned}$$

Thus, we can see that

$$\begin{aligned}
&E_{2,1}F(\lambda,x)\\
&=\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\frac{1}{\widetilde{B}(\lambda_2+1,\lambda_3+1)}\\
&\quad\times\left[\int_{-i\infty}^{+i\infty}\{(1-q^{\lambda_3})+q^{\lambda_3}(1-q^{s})\}q^{(\lambda_2+2)s}\frac{\langle s+1\rangle}{\langle s+\lambda_3\rangle}\prod_{k=4}^{m}\frac{\langle s+z_k-\lambda_k\rangle}{\langle s+z_k\rangle}ds\right.\\
&\quad\left.-q^{\lambda_1}\int_{-i\infty}^{+i\infty}q^{(\lambda_2+2)s}\frac{\langle s+1\rangle}{\langle z+\lambda_3+1\rangle}\prod_{k=4}^{m}\frac{\langle s+z_k-\lambda_k\rangle}{\langle s+z_k\rangle}ds\right]
\end{aligned}$$

$$=-q^{-\frac{1}{2}(\lambda_1+\lambda_3)}\frac{[\lambda_1][\lambda_2+1]}{[\lambda_2+\lambda_3+1]}F(\lambda-\alpha_1;z). \qquad \blacksquare$$

§ 5. Concluding Remarks

In this paper, we construct an integral solution of Lauricella's q-HGE with $|q|=1$. Formally, it looks similar to the solution with $|q|<1$. However, some analytic aspects of the case when $|q|=1$ are quite different from of the case when $|q|<1$ because of properties of $\langle z\rangle$.

For this reason, some important problems about hypergeometric q-difference systems with $|q|=1$ are left unclear. For example,

(1) The structure of the solution space are not known yet.

(2) The counterparts of the identities of basic hypergeometric series are not found at present.

These remain to be solved in the future.

More generally, from the functional equation (3), it is natural to construct 2-parameter difference analogue of hypergeometric systems by using ω_1-shift and ω_2-shift. Are they related with algebraic structure like elliptic quantum groups [4], [5] or like multiparameter quantum groups [11]? It would be a challanging problem to answer this question.

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