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Equivariant Maps between Representation Spheres of a Torus

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

By

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§ 0. Introduction

The Borsuk-Ulam theorem [1] states that if $f: S^m \to S^n$ is an odd map between spheres, i.e., $f(-x) = -f(x)$ for all $x \in S^m$, then $m \leq n$. This theorem can be extended to a class of G-maps $SU \to SW$ between the unit spheres of linear representations U and W of a compact Lie group G. If G is a torus or a p-torus, i.e., if G is a product of circle groups, or of cyclic groups of order p with p prime, then the existence of a G-map $f: SU \to SW$ with the fixed point set $W^G = \{0\}$ implies $\dim U \leq \dim W$ (see [3] and the references there).

In this paper we will see that if we make an additional assumption on U, W or f then U must be a subrepresentation of W.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle group of complex numbers with absolute value 1. For any integer a let S^1 act on $V_a = \mathbb{C}$ via $(z, v) \mapsto z^a v$ for $z \in S^1$, $v \in V_a$. For a sequence $(a_1, \ldots, a_k)$ of integers,denote by $V(a_1, \ldots, a_k)$ the tensor product $V_{a_1} \otimes \cdots \otimes V_{a_k}$, which can be considered as a representation of the k-dimensional torus $T^k = S^1 \times \cdots \times S^1$. The set of such $V(a_1, \ldots, a_k)$ gives a complete set of irreducible unitary representations of T^k, and so any finite dimensional unitary representation U of T^k decomposes into a direct sum

$$U = \bigoplus V(a_1, \ldots, a_k)^{u(a_1, \ldots, a_k)}$$

where $u(a_1, \ldots, a_k)$ is a nonnegative integer and $V(a_1, \ldots, a_k)^{u(a_1, \ldots, a_k)}$ denotes the direct sum of $u(a_1, \ldots, a_k)$ copies of $V(a_1, \ldots, a_k)$.

Let $\mathbb{Z}[x_1, \ldots, x_k]_L$ denote the ring of Laurent polynomials in $x_1, \ldots, x_k$,

$$f(x_1, \ldots, x_k) = \sum_{i_1, \ldots, i_k} a(i_1, \ldots, i_k) x_1^{i_1} \cdots x_k^{i_k},$$

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where $i_1, \ldots, i_k$ run over the integers $\mathbb{Z}$, and $a(i_1, \ldots, i_k)$'s are integers and only finitely many of them are nonzero. $f(x_1, \ldots, x_k)$ is *irreducible* if it is not a unit and if whenever

$$f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k) \cdot h(x_1, \ldots, x_k)$$

then one of $g(x_1, \ldots, x_k)$ and $h(x_1, \ldots, x_k)$ is a unit.

Using the equivariant K-theory in the previous paper [2], we obtained a necessary condition for the existence of a G-map $SU \to SW$ in terms of the Euler classes of U and W. Along the line of this we will do a further study for the case of $G = T^k$, and obtain the following results :

Theorem 0.1. *Let*

$$U = \bigoplus V(a_1, \ldots, a_k)^{u(a_1, \ldots, a_k)}, \text{ and } W = \bigoplus V(a_1, \ldots, a_k)^{w(a_1, \ldots, a_k)}$$

be two unitary representations of T^k with $W^{T^k} = \{0\}$. Assume that whenever $w(a_1, \ldots, a_k)$ is nonzero then $1 - x_1^{a_1} \cdots x_k^{a_k}$ is irreducible in $\mathbb{Z}[x_1, \ldots, x_k]_L$. Then there exists a T^k-map $SU \to SW$ if and only if U is a subrepresentation of W as a real representation.

We see that $1 - x_1^{a_1} \cdots x_k^{a_k}$ is irreducible if $a_i = \pm 1$ for some $i\,(1 \leq i \leq k)$.

If U is a unitary representation, S^1 acts on SU via scalar multiplication. Then we obtain

Theorem 0.2. *Let U and W be two unitary representations of T^k decomposed into direct sum as in Theorem 0.1. Then there exists a T^k-map $f\colon SU \to SW$ such that $f(zu) = z^m f(u)$ for any $z \in S^1$ and $u \in SU$ where m is a fixed nonzero integer, if and only if $u(a_1, \ldots, a_k) \leq w(ma_1, \ldots, ma_k)$ for any $(a_1, \ldots, a_k)$ with $u(a_1, \ldots, a_k) \neq 0$.*

In this Theorem, if $m = 1$ then U must be a subrepresentation of W as a complex representation.

After discussing some prerequisites in § 1 and § 2, we will prove Theorems 0.1 and 0.2 in § 3. Finally in § 4 we will correct the incorrect part of the previous paper [2].

§ 1. G-maps between Representation Spheres

In this section we will recall some prerequisites from [2].

Let $R(G)$ denote the complex representation ring of a compact Lie group G. The Euler class $\lambda_{-1}U$ of a unitary representation U of G is defined by

$$\lambda_{-1}U = \sum_i (-1)^i \Lambda^i U \in R(G),$$

where $\Lambda^i U$ is the i-th exterior power of U. The equivariant K-ring $K_G(SU)$ of the unit sphere SU of U is isomorphic to $R(G)$ divided by the ideal generated by $\lambda_{-1}U$:

$$K_G(SU)\cong R(G)/(\lambda_{-1}U).$$

For a second unitary representation W of G, let f: $SU\to SW$ be a G-map. We have a commutative diagram:

$$\begin{array}{ccc} R(G) & \xrightarrow{\text{identity}} & R(G) \\ \pi_2\downarrow & & \downarrow\pi_1 \\ R(G)/(\lambda_{-1}W)\cong K_G(SW) & \xrightarrow{f^*} & K_G(SU)\cong R(G)/(\lambda_{-1}U) \end{array}$$

where π_1 and π_2 are the canonical projections. Then we obtain

Proposition 1.1 ([2; Proposition 2.4]). *If there exists a G-map $SU\to SW$, then $\lambda_{-1}W\in(\lambda_{-1}U)$ in $R(G)$.*

Now we restrict our attension to the k-dimensional torus T^k. Then

$$R(T^k)\cong\mathbb{Z}[x_1,\ \dots,\ x_k]_L$$

(see [2; Proposition 3.1]). Under this isomorphism the representation $V(a_1,\ \dots,\ a_k)$ corresponds to the monomial $x_1^{a_1}\cdots x_k^{a_k}$

Let

$$U=\bigoplus V(a_1,\ \dots,\ a_k)^{u(a_1,\dots,a_k)}$$

be a unitary representation of T^k decomposed into a direct sum as in § 0. We have in $R(T^k)$ or hence in $\mathbb{Z}[x_1,\ \dots,\ x_k]_L$,

$$\begin{aligned}\lambda_{-1}U&=\prod\lambda_{-1}(V(a_1,\ \dots,\ a_k))^{u(a_1,\dots,a_k)}\\&=\prod(1-x_1^{a_1}\cdots x_k^{a_k})^{u(a_1,\dots,a_k)},\end{aligned}$$

where the product $\prod$ is taken over the sequences $(a_1,\ \dots,\ a_k)$.

Proposition 1.1 implies

Proposition 1.2. *Let*

$$U=\bigoplus V(a_1,\ \dots,\ a_k)^{u(a_1,\dots,a_k)},\ \text{and}\ W=\bigoplus V(a_1,\ \dots,\ a_k)^{w(a_1,\dots,a_k)}$$

be two unitary representations of T^k. If there exists a T^k-map $SU\to SW$, then in $\mathbb{Z}[x_1,\ \dots,\ x_k]_L$

$$\prod(1-x_1^{a_1}\cdots x_k^{a_k})^{w(a_1,\dots,a_k)}=\alpha(x_1,\ \dots,\ x_k)\prod(1-x_1^{a_1}\cdots x_k^{a_k})^{u(a_1,\dots,a_k)} \tag{1.3}$$

for some $\alpha(x_1,\ \dots,\ x_k)\in\mathbb{Z}[x_1,\ \dots,\ x_k]_L$.

§ 2. The Ring of Laurent Polynomials

Any unit in $\mathbb{Z}[x_1,\ \dots,\ x_k]_L$ is of the form $\pm x_1^{a_1}\cdots x_k^{a_k}$ for some integers $a_1,\ \dots,\ a_k$. Note that $1-x_1^{a_1}\cdots x_k^{a_k}$ and $1-x_1^{-a_1}\cdots x_k^{-a_k}$ differ by a unit factor. In fact

$$1-x_1^{a_1}\cdots x_k^{a_k}=-x_1^{a_1}\cdots x_k^{a_k}(1-x_1^{-a_1}\cdots x_k^{-a_k}).$$

$\mathbb{Z}[x_1, ..., x_k]$ denotes the (ordinary) polynomial ring over, $\mathbb{Z}$, which is contained in $\mathbb{Z}[x_1, ..., x_k]_L$ as a subring. Given $f[x_1, ..., x_k]\in\mathbb{Z}[x_1, ..., x_k]_L$, then $x_1^{a_1}\cdots x_k^{a_k}f(x_1, ..., x_k)$ is in $\mathbb{Z}[x_1, ..., x_k]$ for sufficiently large $a_i\geq 0$ $(1\leq i\leq k)$. Since $\mathbb{Z}[x_1, ..., x_k]$ is a unique factorization domain $x_1^{a_1}\cdots x_k^{a_k}f(x_1, ..., x_k)$ is uniquely expressible as a product of irreducible elements up to units $(=\pm 1)$ and the order of factors, i.e.,

$$(2.1)\qquad x_1^{a_1}\cdots x_k^{a_k}f(x_1, ..., x_k)=f_1(x_1, ..., x_k)\cdots f_m(x_1, ..., x_k),$$

where $f_i(x_1, ..., x_k)$ $(1\leq i\leq m)$ are irreducible polynomials in $\mathbb{Z}[x_1, ..., x_k]$ and are uniquely determined up to sign. The equation (2.1) gives

$$f(x_1, ..., x_k)=x_1^{-a_1}\cdots x_k^{-a_k}f_1(x_1, ..., x_k)\cdots f_m(x_1, ..., x_k)$$

in $\mathbb{Z}[x_1, ..., x_k]_L$. $f_i(x_1, ..., x_k)$ $(1\leq i\leq m)$ are also irreducible in $\mathbb{Z}[x_1, ..., x_k]_L$. This gives

Lemma 2.2. *$\mathbb{Z}[x_1, ..., x_k]_L$ is a unique factorization domain.*

Lemma 2.3. (i) *$1-x_1^{a_1}\cdots x_k^{a_k}$ divides $1-x_1^{b_1}\cdots x_k^{b_k}$ in $\mathbb{Z}[x_1, ..., x_k]_L$ if and only if $l(a_1, ..., a_k)=(b_1, ..., b_k)$ for some $l\in\mathbb{Z}$.*

(ii) *If $(b_1, ..., b_k)\neq(0, ..., 0)$ then any factorization of $1-x_1^{b_1}\cdots x_k^{b_k}$ has at most one factor of the form $1-x_1^{c_1}\cdots x_k^{c_k}$ $(c_i\in\mathbb{Z})$.*

Proof. First we prove the necessity of (i). This is clear if $(b_1, ..., b_k)=(0, ..., 0)$. So we assume $b_k\neq 0$. Then we see $a_k\neq 0$. We assume further that $a_k>0$ and $b_k>0$. (Noting that $1-x_1^{c_1}\cdots x_k^{c_k}$ is different from $1-x_1^{-c_1}\cdots x_k^{-c_k}$ only by a unit factor , the case of $a_k<0$ or $b_k<0$ can be deduced from the case of $a_k>0$ and $b_k>0$.) Letting $m=a_k>0$, $n=b_k>0$ and $x=x_k$, then $\mathbb{Z}[x_1, ..., x_k]_L$ can be considered as the ring of Laurent polynomials in x over $\mathbb{Z}[x_1, ..., x_{k-1}]_L$, i.e.,

$$\mathbb{Z}[x_1, ..., x_k]_L=\mathbb{Z}[x_1, ..., x_{k-1}]_L[x]_L.$$

Letting $\mathbb{a}=(a_1, ..., a_{k-1})$ and $\mathbb{b}=(b_1, ..., b_{k-1})$, we put $\alpha(\mathbb{a})=x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}$ and $\alpha(\mathbb{b})=x_1^{b_1}\cdots x_{k-1}^{b_{k-1}}$. By the assumption, $1-\alpha(\mathbb{a})x^m$ divides $1-\alpha(\mathbb{b})x^n$, i.e.,

$$(2.4)\qquad 1-\alpha(\mathbb{b})x^n=(1-\alpha(\mathbb{a})x^m)(\alpha_r x^r+\alpha_{r+1}x^{r+1}+\cdots+\alpha_{r+s}x^{r+s}),$$

where $r, s\in\mathbb{Z}$, $\alpha_r, \alpha_{r+1}, ..., \alpha_{r+s}\in\mathbb{Z}[x_1, ..., x_{k-1}]_L$, s is nonnegative, α_r and α_{r+s} is nonzero. It should be asserted here that $\alpha_r=1$ and $r=0$. Then (2.4) becomes

$$(2.5)\quad 1-\alpha(\mathbb{b})x^n=1+\alpha_1 x+\cdots+\alpha_s x^s-\alpha(\mathbb{a})x^m-\alpha(\mathbb{a})\alpha_1 x^{m+1}-\cdots-\alpha(\mathbb{a})\alpha_s x^{m+s}.$$

If $s=0$, we see $m=n$, $\alpha(\mathbb{a})=\alpha(\mathbb{b})$ and hence $(a_1, ..., a_k)=(b_1, ..., b_k)$. If $s>0$, then we divide into the two cases: $s<m$ and $m\leq s$. For the first case, comparing the coefficients of esch x^i on the both sides of (2.5), we see that this

case can not occur. For the second case, comparing the coefficients again, we see that $n=m+s$, and s is a multiple of m, say $s=(l-1)m$, then $\alpha(\boldsymbol{b})=\alpha(\boldsymbol{a})^l=\alpha(l\boldsymbol{a})$. This implies $l(a_1, \ldots, a_k)=(b_1, \ldots, b_k)$, and completes the proof of the necessity

The sufficiency is easy. In fact, assume $l(a_1, \ldots, a_k)=(b_1, \ldots, b_k)$ and let $X=x_1^{a_1}\cdots x_k^{a_k}$. Then

$$
\begin{aligned}
1-x_1^{b_1}\cdots x_k^{b_k}&=1-X^l\\
&=\begin{cases}(1+X+X^2+\cdots+X^{l-1})(1-X) & \text{if } l>0\\ -X^{-1}(X^{l+1}+\cdots+X^{-2}+X^{-1}+1)(1-X) & \text{if } l<0\\ 0 & \text{if } l=0\,.\end{cases}
\end{aligned}
$$

This shows the sufficiency of (i), and (ii). □

§ 3. Proof of Theorems 0.1, 0.2

Proof of Theorem 0.1. If U is a subrepresentation of W, then there is the inclusion map $SU \hookrightarrow SW$, which is a T^k-map.

If conversely there is a T^k-map $SU\to SW$, then we obtain the equation (1.3) from Proposition 1.2. From the assumption and Lemma 2.3 (ii) we see that $1-x_1^{a_1}\cdots x_k^{a_k}$ is irreducible if $u(a_1, \ldots, a_k)$ or $w(a_1, \ldots, a_k)$ is nonzero, and further that

$$u(a_1, \ldots, a_k)+u(-a_1, \ldots, -a_k)\leq w(a_1, \ldots, a_k)+w(-a_1, \ldots, -a_k)$$

since $\mathbb{Z}[x_1, \ldots, x_k]_L$ is a unique factorization domain. This means that U is a subrepresentation of W as a real representation, since $V(a_1, \ldots, a_k)$ and $V(-a_1, \ldots, -a_k)$ are isomorphic to each other as real representations. □

For unitary representations U, W of a compact Lie group G, and an integer m, let $U'=U\otimes V_1$ and $W'=W\otimes V_m$, where V_1, V_m are the representations of S^1 given in § 0. Then U' and W' become representations of $G\times S^1$, and we note that the following (3.1) and (3.2) are equivalent:

(3.1) *There is a G-map $f\colon SU\to SW$ such that $f(zu)=z^m f(u)$ for $z\in S^1$, $u\in SU$.*

(3.2) *There is a $G\times S^1$-map $SU'\to SW'$.*

$X*Y$ denotes the join of the topological spaces X and Y. If X and Y are G-spaces, then $X*Y$ admits the canonical G-action. Two G-maps $f\colon X\to X'$ and $g: Y\to Y'$ canonically induce the G-map $f*g: X*Y\to X'*Y'$. For two representations U_1 and U_2 of G, we see $SU_1*SU_2\approx S(U_1\oplus U_2)$. So G-maps $h: SU_1\to SW_1$ and $j: SU_2\to SW_2$ induce the G-map $h*j: S(U_1\oplus U_2)\to S(W_1\oplus W_2)$.

We will now prove Theorem 0.2.

Proof of Theorem 0.2. For unitary representations U and W of T^k decomposed into direct sum as in Theorem 0.1, representations $U'=U\otimes V_1$ and

$W' = W \otimes V_m$ of $T^k \times S^1$ are decomposed as follows:

$$U' = \bigoplus V(a_1, \ldots, a_k, 1)^{u(a_1,\ldots,a_k)}, \quad W' = \bigoplus V(a_1, \ldots, a_k, m)^{w(a_1,\ldots,a_k)},$$

where both the direct sums are taken over the sequences $(a_1, \ldots, a_k)$.

First we assume that $u(a_1, \ldots, a_k) \le w(ma_1, \ldots, ma_k)$ if $u(a_1, \ldots, a_k) \ne 0$. The map $p: S^1 \to S^1$ with $p(z) = z^m$ for $z \in S^1$ yields a $T^k \times S^1$-map from $S(V(a_1, \ldots, a_k, 1))$ to $S(V(ma_1, \ldots, ma_k, m))$. Taking the join of such $T^k \times S^1$-maps for all $(a_1, \ldots, a_k)$, we obtain a $T^k \times S^1$-map

$$SU' = S(\bigoplus V(a_1, \ldots, a_k, 1)^{u(a_1,\ldots,a_k)}) \to S(\bigoplus V(ma_1, \ldots, ma_k, m)^{u(a_1,\ldots,a_k)}).$$

This yields a $T^k \times S^1$-map $SU' \to SW'$, since

$$S(\bigoplus V(ma_1, \ldots, ma_k, m)^{u(a_1,\ldots,a_k)}) \subset S(\bigoplus V(a_1, \ldots, a_k, m)^{w(a_1,\ldots,a_k)}) = SW'$$

by the assumption. This shows the existence of a T^k-map $SU \to SW$ with the desired property.

If conversely there is a $T^k \times S^1$-map $SU' \to SW'$, then from Proposition 1.2 we obtain, in $\mathbb{Z}[x_1, \ldots, x_k, x]_L$,

$$\prod (1 - x_1^{a_1} \cdots x_k^{a_k} x^m)^{w(a_1,\ldots,a_k)} = \alpha(x_1, \ldots, x_k, x) \prod (1 - x_1^{a_1} \cdots x_k^{a_k} x)^{u(a_1,\ldots,a_k)}$$

for some $\alpha(x_1, \ldots, x_k, x) \in \mathbb{Z}[x_1, \ldots, x_k, x]_L$, where both the products $\prod$ are taken over the sequences $(a_1, \ldots, a_k)$. Since $1 - x_1^{a_1} \cdots x_k^{a_k} x$ is irreducible, Lemmas 2.2, 2.3 imply $u(a_1, \ldots, a_k) \le w(ma_1, \ldots, ma_k)$ if $u(a_1, \ldots, a_k) \ne 0$. □

§ 4. Correction to the Previous Paper

Finally we should correct the previous paper [2]. On page 729 of [2] it is asserted that $U \cong \bar{U}$, but this is incorrect. If we modify the definition of $|\gamma|$ as $|\gamma| := a_1 + \cdots + a_k + b_1 + \cdots + b_l$ for $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$, we can still prove Theorem 1.1 of [2] with this modification of $|\gamma|$. The new proof can be done along a similar line of the previous one in [2; § 4].

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