

Singular Solutions with Asymptotic Expansion of Linear Partial Differential Equations in the Complex Domain

By

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Abstract

We consider a linear partial differential equation with holomorphic coefficients in a neighbourhood of $z=0$ in \mathbf{C}^{d+1} ,

$$P(z, \partial) u(z) = f(z),$$

where $u(z)$ and $f(z)$ admit singularities on the surface $K = \{z_0=0\}$. Our main result is the following:

For the operator P we define an exponent γ^* called the minimal irregularity of K and show that if $u(z)$ grows at most exponentially with exponent γ^* as z_0 tends to 0 and if $f(z)$ has a Gevrey type expansion of exponent γ^* with respect to z_0 , then $u(z)$ also has the same one.

§1. Introduction

Let $P(z, \partial)$ be a linear partial differential operator with holomorphic coefficients in a neighbourhood of $z=0$ in \mathbf{C}^{d+1} and $K = \{z_0=0\}$. Let us consider

$$P(z, \partial) u(z) = f(z).$$

Suppose that $f(z)$ is holomorphic except on K . Then one of the important problems is the existence of solutions with singularities on K . We refer results concerning it to Hamada, Leray and Wagschal [2], Kashiwara and Schapira [3], Ōuchi [7], Persson [11] and other papers cited in those papers.

Another problem is to study behaviours of singular solutions near K . The asymptotic behaviours of some singular solutions were considered in Ōuchi [4] and [5] and we showed that they grow really exponentially as z tends to K in some region and behave mildly as z tends to K in another region, which is similar to Stokes phenomenon in the theory of ordinary differential equations.

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We discussed in Ōuchi [8] and [9] solutions that grow at most exponential order as z tends to K in some region. It is the main result in [9] that for some class of operators if $f(z)$ behaves asymptotically $f(z) \sim \sum_{n=0}^{+\infty} f_n(z') z_0^n$ as $z_0 \rightarrow 0$ in a sector, where $|f_n(z')| \leq AB^n \Gamma(n/\gamma^* + 1)$ and γ^* is determined by $P(z, \partial)$ and if $u(z)$ grows at most some exponential order near $z_0 = 0$, that is, for any $\varepsilon > 0$ $|u(z)| \leq C_\varepsilon \exp(\varepsilon|z_0|^{-r^*})$ near $z_0 = 0$, then $u(z)$ has also the asymptotic expansion like $f(z)$ as z_0 tends to 0.

In order to show the results in [4], [5], [8] and [9] we used an integral representation of solutions with singularities on K .

This paper follows [9], where we imposed more strict conditions on $P(z, \partial)$ than Condition 1 in this paper. Our purposes are to improve the results in [9], that is, to weaken conditions on $P(z, \partial)$ as possible as we can and to show them by another method, which is simpler than that in [9].

Now in order to state results let us give notations and definitions. The coordinates of \mathbb{C}^{d+1} are denoted by $z = (z_0, z_1, \dots, z_d) = (z_0, z') \in \mathbb{C} \times \mathbb{C}^d$. $|z| = \max\{|z_i|; 0 \leq i \leq d\}$ and $|z'| = \max\{|z_i|; 1 \leq i \leq d\}$. Its dual variables are $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_d)$. $\mathbb{N} = \{0, 1, 2, \dots\}$. The differentiation is denoted by $\partial_i = \partial/\partial z_i$, and $\partial = (\partial_0, \partial_1, \dots, \partial_d) = (\partial_0, \partial')$. For a linear partial differential operator $A(z, \partial)$ we denote its principal symbol by P.S. $A(z, \xi)$. Let $\Omega = \Omega_0 \times \Omega'$ be a polydisk with $\Omega_0 = \{z_0 \in \mathbb{C}^1; |z_0| < R\}$ and $\Omega' = \{z' \in \mathbb{C}^d; |z'| < R\}$ for some positive constant R . Put $\Omega_0(\theta) = \{z_0 \in \Omega_0 - \{0\}; |\arg z_0| < \theta\}$ and $\Omega(\theta) = \Omega_0(\theta) \times \Omega'$.

Let K be a nonsingular complex hypersurface through the origin $z = 0$. We choose a coordinate system so that $K = \{z_0 = 0\}$. So the coordinate z_0 plays a distinguished role. $\mathcal{O}(\Omega)$ ($\mathcal{O}(\Omega')$, $\mathcal{O}(\Omega(\theta))$) is the set of all holomorphic functions on Ω (resp. Ω' , $\Omega(\theta)$). $\mathcal{O}(\Omega(\theta))$ contains multi-valued functions, if $\theta > \pi$.

In the following of this paper we consider an m -th order linear partial differential operator $P(z, \partial)$ with coefficients in $\mathcal{O}(\Omega)$ in the form:

$$(1.1) \quad P(z, \partial) := a_{k^*,0}(z) \partial_0^{k^*} + \sum_{\alpha \neq (k^*,0)} a_\alpha(z) \partial^\alpha,$$

where $a_\alpha(z)$'s are holomorphic on Ω and $a_{k^*,0}(0, z') \neq 0$. Let $j_\alpha \in \mathbb{N}$ be the valuation of $a_\alpha(z)$ with respect to z_0 , that is,

$$(1.2) \quad a_\alpha(z) = z_0^{j_\alpha} b_\alpha(z), \quad b_\alpha(0, z') \neq 0.$$

We put $j_\alpha = \infty$, if $a_\alpha(z) \equiv 0$.

We suppose that $P(z, \partial)$ satisfies the following condition:

Condition 1. For all α with $k^* < |\alpha| \leq m$ the sum of the valuation j_α of $a_\alpha(z)$ and k^* is greater than the order α_0 of differentiation with respect to z_0 :

$$(1.3) \quad j_\alpha + k^* > \alpha_0.$$

We understand that Condition 1 is satisfied when $k^* = m$.

Note that if Condition 1 is satisfied, inequality (1.3) holds for every $\alpha \neq (k^*, 0, 0, \dots, 0)$, because (1.3) holds trivially for any α with $|\alpha| \leq k^*$. Condition 1 is obviously satisfied, if $P(z, \partial)$ is normal with respect to ∂_0 , that is, $a_\alpha(z)$ vanishes identically for every α with $\alpha_0 \geq k^*$ with $\alpha \neq (k^*, 0, 0, \dots, 0)$. Examples are given after Theorem 1.4 and Corollary 1.5. Now we define an exponent γ^* , which plays an important role in our results.

Definition 1.1. *The minimal irregularity γ^* of the surface $K = \{z_0 = 0\}$ is defined by*

$$(1.4) \quad \begin{cases} \gamma^* = \min_{|\alpha| > k^*} \left\{ \frac{j_\alpha - \alpha_0 + k^*}{|\alpha| - k^*} \right\} & \text{if } 0 \leq k^* < m, \\ \gamma^* = \infty & \text{if } k^* = m. \end{cases}$$

If $k^* < m$, Condition 1 means that K is an irregular characteristic surface defined in Ōuchi [6]. If $k^* = m$, K is noncharacteristic. γ^* is named the minimal irregularity of K in Ōuchi [10].

Let us define some spaces of functions in order to give the main result.

Definition 1.2. *We say that $u(z) \in \mathcal{O}(\Omega(\theta))$ grows at most exponentially with exponent $\kappa > 0$ if for any $\varepsilon > 0$ and $0 < \theta' < \theta$ there exists a constant $C = C(\varepsilon, \theta')$ such that*

$$(1.5) \quad |u(z)| \leq C \exp(\varepsilon|z_0|^{-\kappa}) \text{ for } z \in \Omega(\theta').$$

We denote by $\mathcal{O}_{(\kappa)}(\Omega(\theta))$ the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ which grow at most exponentially with exponent κ . If $\kappa = \infty$, we put $\mathcal{O}_{(\infty)}(\Omega(\theta)) = \mathcal{O}(\Omega(\theta))$.

We note that if $u(z) \in \mathcal{O}_{(\kappa)}(\Omega(\theta))$, then $\partial^\alpha u(z) \in \mathcal{O}_{(\kappa)}(W(\theta))$ for any polydisk $W \subseteq \Omega$ with center $z = 0$.

Definition 1.3. *We say that $u(z) \in \mathcal{O}(\Omega(\theta))$ has a Gevrey type asymptotic expansion $\sum_{k=0}^\infty u_k(z') z_0^k$ with exponent κ , $0 < \kappa \leq \infty$ where $u_k(z') \in \mathcal{O}(\Omega')$, if for any $0 < \theta' < \theta$ there exist constants $A = A(\theta')$ and $B = B(\theta')$ such that*

$$(1.6) \quad \left| u(z) - \left(\sum_{k=0}^{n-1} u_k(z') z_0^k \right) \right| \leq AB^n \Gamma(n/\kappa + 1) |z_0|^n \text{ for } z \in \Omega(\theta')$$

holds for every $n = 0, 1, 2, \dots$. In this case we write $u(z) \sim \sum_{k=0}^\infty u_k(z') z_0^k$. We denote by $\text{Asy}_{(\kappa)}(\Omega(\theta))$ the set of all $u(z) \in \mathcal{O}(\Omega(\theta))$ which have Gevrey type asymptotic

expansion exponent κ .

It is obvious that $u(z) \in \text{Asy}_{l(\infty)}(\Omega(\theta))$ means that it is holomorphically extensible to $z=0$. Let $u(z) \in \text{Asy}_{l(\kappa)}(\Omega(\theta))$. Then from (1.6)

$$(1.7) \quad |u_n(z')| \leq AB^n \Gamma\left(\frac{n}{\kappa} + 1\right) \text{ for } z' \in \Omega'$$

and $\partial^\alpha u(z) \in \text{Asy}_{l(\kappa)}(W(\theta))$ and

$$(1.8) \quad |\partial_0^n u(z)| \leq AB^n \Gamma\left(\left(1 + \frac{1}{\kappa}\right)n + 1\right) \text{ for } z \in W(\theta')$$

for any polydisk $W \subset \subset \Omega$ with center $z=0$.

Now let us consider

$$(Eq) \quad P(z, \partial) u(z) = f(z),$$

where $u(z), f(z) \in \mathcal{O}(\Omega(\theta))$. We give our main results:

Theorem 1.4. *Suppose that Condition 1 holds. Let $u(z) \in \mathcal{O}(\Omega(\theta))$ be a solution of (Eq). If $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ and $f(z) \in \text{Asy}_{l(\gamma^*)}(\Omega(\theta))$, then $u(z)$ has a Gevrey type asymptotic expansion with exponent γ^* in $W(\theta)$, where $W \subset \subset \Omega$ is some polydisk with center $z=0$ in C^{d+1} .*

From Theorem 1.4 we obtain that if $f(z)$ is holomorphic near the origin and θ is sufficiently large, then $u(z)$ is also holomorphic at the origin.

Corollary 1.5. *Suppose that Condition 1 holds and $\theta > (\pi/2\gamma^*) + \pi$. Let $u(z) \in \mathcal{O}(\Omega(\theta))$ be a solution of (Eq). If $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ and $f(z)$ is holomorphic in Ω . Then $u(z)$ is also holomorphic in a neighbourhood of $z=0$.*

For another subspace $\mathcal{F}(\Omega(\theta))$ of $\mathcal{O}(\Omega(\theta))$ we can also show the following result, which is similar to Theorem 1.4 and Corollary 1.5:

$$\begin{cases} u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta)) \\ P(z, \partial) u(z) = f(z) \in \mathcal{F}(\Omega(\theta)) \end{cases} \Rightarrow u(z) \in \mathcal{F}(\Omega(\theta)).$$

This topic will be discussed more generally in the forthcoming paper.

Let us give some simple examples satisfying Condition 1.

Examples.

$$(Ex.1) \quad \partial_0^{k^*} + z_0^l A_m(z_0, z', \partial'),$$

where $A_m(z_0, z', \partial')$ is an operator with order $m(k^* < m)$ and $P.S.A_m(0, z', \xi') \neq 0$. We have $\gamma^* = (j + k^*) / (m - k^*)$.

$$(Ex.2) \quad \partial_0^2 + z_0^l A_{m_1}(z_0, z', \partial') \partial_0 + z_0^p A_{m_0}(z_0, z', \partial'),$$

where $A_{m_i}(z_0, z', \partial')$'s are operators with order $m_i, m_0 > 2, m_1 > 1$, and $P.S.A_{m_i}(0, z', \xi') \neq 0$. We have $k^* = 2$ and $\gamma^* = \min\{(j_1 + 1)/(m_1 - 1), (j_0 + 2)/(m_0 - 2)\}$.

$$(Ex.3) \quad I_d + z_0^2 \partial_0 + z_0 \partial_1^2,$$

which is not normal with respect to ∂_0 and $k^* = 0$ and $\gamma^* = 1/2$.

Remark. We give a comment concerning the condition that the solution $u(z)$ belongs to $\mathcal{O}_{(\gamma^*)}(\Omega(\theta))$. Let us show the necessity of this condition by a counter example. Put $\Omega_K = \Omega - K$ and denote by $\tilde{\Omega}_K$ its universal covering space. Let us return to (Ex.1). Put $P(z, \partial) = \partial_0^{k^*} + z_0^j A_m(z_0, z', \partial')$, where we assume $P.S.A_m(0, 0, \xi') \neq 0$. Then $\gamma^* = (j + k^*) / (m - k^*)$ and let $0 < \theta < \pi/2\gamma^*$. We can show that there exists $u(z) \in \mathcal{O}(\tilde{\Omega}_K)$ in a polydisk Ω such that

$$(1.9) \quad P(z, \partial) u(z) = 0,$$

$$(1.10) \quad |u(z)| \leq A_0 \exp(c_0 |z_0|^{-r^*}),$$

$$(1.11) \quad |u(z_0, 0)| \geq A_1 \exp(c_1 |z_0|^{-r^*}) \text{ for } z \in \Omega(\theta) \cap \{z' = 0\},$$

where A_i and c_i are positive constants. It follows from (1.10) and (1.11) that $u(z) \notin \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ and do not have an asymptotic expansion in $\Omega(\theta)$.

The results similar to Theorem 1.4 and Corollary 1.5 were obtained in Ōuchi [8] and [9] under more strict conditions than Condition 1. We showed them by detailed analysis of an integral representation of solutions with singularities on K . In order to do so, several conditions were imposed on the operator $P(z, \partial)$, and the proofs were long and not easy. Operators such as (Ex.1) for $j > 0$, (Ex.2) for $j_0, j_1 > 0$ and (Ex.3) do not satisfy the conditions in those papers.

On the other hand we assume in this paper the only one condition, Condition 1. We can improve the results in the former papers and state the main results very simply. The proof is different from those in [8] and [9], that is, we do not use integral representations to show Theorem 1.4, but estimate the derivatives of $u(z)$ and apply the following Theorem 1.6, which is itself interesting. You will find the proof less complicated.

Theorem 1.6. *Let κ be a positive rational number. Suppose that $v(z) \in \mathcal{O}(\Omega(\theta))$ satisfies the following estimate. For any $\varepsilon > 0$ and $0 < \theta' < \theta$ there exist constants $C = C(\varepsilon, \theta')$ and $B = B(\theta')$ such that*

$$(1.12) \quad \left| \left(\frac{\partial}{\partial z_0} \right)^n v(z) \right| \leq C \exp(\varepsilon |z_0|^{-\kappa}) B^n \Gamma((1 + \kappa^{-1})n + 1) \text{ for } z \in \Omega(\theta')$$

holds for all $n = 0, 1, 2, \dots$. Then $v(z) \in \text{Asy}_{(\kappa)}(W(\theta))$ for some polydisk W with center $z = 0$ in \mathbb{C}^{d+1} .

In §2 first we estimate the derivatives $\partial_0^n u(z)$ for all $n \in \mathbb{N}$. Secondly, by using the estimates and Theorem 1.6, we show Theorem 1.4. In §3 we give the proof of Theorem 1.6. In §4 we show a proposition, which is assumed in the proof of Theorem 1.6. The author thanks the referee who read the manuscript carefully and gave him advice.

§2. Estimates of Derivatives of $u(z)$ and Proof of Theorem 1.4

In this section first we obtain estimates of derivatives of a solution $u(z)$ of (Eq) and secondly show Theorem 1.4 by assuming Theorem 1.6, which we prove in the next section. For this purpose the method of majorant functions is available. Let $A(x) = \sum_{\alpha} A_{\alpha} x^{\alpha}$ and $B(x) = \sum_{\alpha} B_{\alpha} x^{\alpha}$ be formal power series of N variables $x = (x_1, x_2, \dots, x_N)$ and $\alpha \in \mathbb{N}^N$. $A(x) \gg 0$ and $A(x) \ll B(x)$ mean $A_{\alpha} \geq 0$ and $|A_{\alpha}| \leq B_{\alpha}$ for all α , respectively.

Lemma 2.1. *Let $\theta(t)$ be a formal power series of one variable t such that $\theta(t) \gg 0$ and $(R - t)\theta(t) \gg 0$ for some $R > 0$. Then for derivatives $\theta^{(j)}(t) = (d/dt)^j \theta(t)$, $j=0, 1, \dots$, we have*

$$(2.1) \quad (R - t)\theta^{(j)}(t) \gg 0 \quad R'\theta^{(j+1)}(t) \gg \theta^{(j)}(t)$$

and for R_0 with $R_0 > R'$

$$(2.2) \quad (R_0 - t)^{-1}\theta^{(j)}(t) \ll (R_0 - R')^{-1}\theta^{(j)}(t).$$

For the proof of Lemma 2.1 we refer to Wagshal [13].

Define for $0 < R < 1$ and $r \geq 0$

$$(2.3) \quad \Theta_R^{(r)}(t) = \frac{\Gamma(r+1)}{(R-t)^{r+1}}.$$

We see that

$$(2.4) \quad \Theta_R^{(n-r)}(t) \ll \frac{R}{n-r+1} \Theta_R^{(n-r+1)} \ll \dots \ll \frac{R^r}{(n-r+1) \dots n} \Theta_R^{(n)}(t).$$

We have from (2.4)

Lemma 2.2. *Let $0 < c < 1$. Then there is a constant $C = C(c)$ such that*

$$(2.5) \quad \sum_{r=0}^n \frac{n!}{(n-r)!} c^r \Theta_R^{(j+n-r)}(t) \ll C \Theta_R^{(j+n)}(t).$$

Now let $v(z) \in \mathcal{O}(\Omega(\theta))$. For a fixed $z_0 \in \Omega_0(\theta)$ let M_{z_0} be $\sup\{|v(z)|; z' \in \Omega'\}$. Then $v(z) \underset{z'}{\ll} M_{z_0} \Theta_R^{(0)}(z_1 + z_2 + \dots + z_d)$, where $\underset{z'}{\ll}$ means the inequality holds

as functions of z' . In the sequel for simplicity we put $t = z_1 + z_2 + \dots + z_d$.

Lemma 2.3. *Let $v(z) \in \mathcal{O}(\Omega(\theta))$. Suppose that $v(z) \ll_{z'} M(|z_0|) \Theta_R^{(k)}(t)$ for all $z_0 \in \Omega_0(\theta)$, where $M(s)$ is a positive nonincreasing function on $(0, R]$. Then for any θ' with $0 < \theta' < \theta$ there are positive constants C and $0 < \delta < 1/2$ such that for $z_0 \in \Omega_0(\theta')$*

$$(2.6) \quad z'_0 \partial_0^l v(z) \ll_{z'} M((1-\delta)|z_0|) C' \Theta_R^{(l+k)}(t)$$

holds for all $l \in \mathbb{N}$.

Proof. Let $v(z) = \sum_{\alpha'} v_{\alpha'}(z_0) (z')^{\alpha'}$ and $\Theta^{(k)}(t) = \sum_{\alpha'} c_{\alpha'} (z')^{\alpha'}$. Then $|v_{\alpha'}(z_0)| \leq M(|z_0|) c_{\alpha'}$. Let us estimate $\partial_0^l v_{\alpha'}(z_0)$

$$\partial_0^l v_{\alpha'}(z_0) = \frac{l!}{2\pi i} \int_L \frac{v_{\alpha'}(\zeta)}{(\zeta - z_0)^{l+1}} d\zeta.$$

Let $0 < 2\delta < \min\{\theta - \theta', 1\}$. Choose the path $L = \{\zeta = z_0 + |z_0|(\sin\delta) e^{i\rho}; 0 \leq \rho \leq 2\pi\}$. Then we have

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{v_{\alpha'}(\zeta)}{(\zeta - z_0)^{l+1}} |z_0| (\sin\delta) d\rho \right| \\ & \leq \frac{1}{2\pi |z_0| \sin\delta|^l} \int_0^{2\pi} |v_{\alpha'}(z_0 + e^{i\rho}|z_0| \sin\delta)| d\rho \leq \frac{c_{\alpha'}}{|z_0| \sin\delta|^l} M((1-\delta)|z_0|). \end{aligned}$$

Hence

$$|z'_0 \partial_0^l v_{\alpha'}(z_0)| \leq \frac{M((1-\delta)|z_0|) c_{\alpha'} l!}{|\sin\delta|^l}.$$

Therefore for a fixed z_0 , taking $C = 1/|\sin\delta|$, we have

$$z'_0 \partial_0^l v(z) \ll_{z'} M((1-\delta)|z_0|) l! C' \Theta_R^{(k)}(t) \ll_{z'} M((1-\delta)|z_0|) C' \Theta_R^{(l+k)}(t).$$

Now we return to our equation

$$(Eq) \quad P(z, \partial) u(z) = f(z).$$

We need the following inequalities that Condition 1 implies to prove Proposition 2.5.

Lemma 2.4. *Suppose that Condition 1 holds and put $s = 1 + 1/\gamma^*$. Then we have for $\alpha \neq (k^*, 0)$, $j_\alpha - \alpha_0 + k^* \geq 1$ and $s(j_\alpha - \alpha_0 + k^*) \geq (|\alpha'| + j_\alpha)$.*

Proof. The first statement is obvious. It follows from the definition of γ^* that $j_\alpha - \alpha_0 + k^* \geq \gamma^* (|\alpha| - k^*)$. The second inequality follows from this inequality.

Let us estimate the derivatives of $u(z)$ in (Eq).

Proposition 2.5. *Suppose $m > k^*$ and $a_{k^*,0}(z) \neq 0$ in Ω . Let $u(z) \in \mathcal{O}_{(\gamma^*)}(\Omega(\theta))$ be a solution of (Eq) and assume $f(z) \in \text{Asy}_{(\gamma^*)}(\Omega(\theta))$. Let W be a polydisk with center 0 such that $W \subseteq \Omega$. Then for any $\varepsilon > 0$ and $0 < \theta' < \theta$ there exist constants $A = A(\theta', W)$, $B = B(\theta', W)$ and $C = C(\varepsilon, \theta', W)$ such that*

$$(2.7) \quad z'_0 \partial_0^{l+n} u(z) \ll C \exp(\varepsilon |z_0|^{-r^*}) A^{l+sn} B^n \Theta_R^{l+sn}(t) \text{ for } z \in W(\theta')$$

holds for all $l, n \in \mathbb{N}$, where $s = 1 + 1/\gamma^*$.

Proof. Since $m > k^*$, $\gamma^* \neq \infty$ and $s > 1$. Put $N_0 = [(m - k^*) / (s - 1)] + 1$ and $\theta' < \theta_1 < \theta$, where $[a]$ means the integral part of $a \in \mathbb{R}$. For $0 \leq n < \max\{N_0, k^*\}$ we may assume

$$\partial_0^n u(z) \ll C \exp(\varepsilon |z_0|^{-r^*}) B^n \Theta_R^{(sn)}(t) \text{ for } z \in W(\theta_1).$$

By Lemma 2.3, the inequality (2.7) holds for $0 \leq n < \max\{N_0, k^*\}$ and for all $l \in \mathbb{N}$. Now we assume (2.7) holds for $0 \leq n \leq N - 1$ and all $l \in \mathbb{N}$, where $N \geq \max\{N_0, k^*\}$. We have, by differentiating (Eq) n times, $n = l + N - k^*$, with respect to z_0

$$\begin{aligned} (2.8) \quad z'_0 \partial_0^{l+N} u(z) &= z'_0 \partial_0^{k^*+n} u(z) \\ &= z'_0 \partial_0^n \left(- \sum_{\alpha \neq (k^*, 0)} z_0^\alpha b_\alpha(z) \partial^\alpha u(z) + f(z) \right) \\ &= - \sum_{\alpha \neq (k^*, 0)} \sum_{n'=0}^n \binom{n}{n'} j_\alpha (j_\alpha - 1) \cdots (j_\alpha - n' + 1) \\ &\quad \times \underbrace{\left(\sum_{r=0}^{n-n'} \binom{n-n'}{r} \right)}_{(I)} \underbrace{z_0^{j_\alpha - n' + l} \partial_0^r b_\alpha(z) \partial^{\alpha'} \partial_0^{n-n'-r+\alpha_0} u(z)}_{(E2)} \\ &\quad + z'_0 \partial_0^n f(z). \end{aligned}$$

To estimate (I) we divide it into two cases: $(E1) \leq (E2)$ and $(E1) \geq (E2)$. If $(E1) \leq (E2)$, we have

$$(E2) - (E1) = n + \alpha_0 - j_\alpha - r - l = N - k^* + \alpha_0 - j_\alpha - r \leq N - 1.$$

Therefore we can apply the inductive hypothesis to (I) and obtain

$$(I) \ll C \exp(\varepsilon |z_0|^{-r^*}) \overbrace{B_1^{r+1} r! A^{j_\alpha - n' + l + s(N - k^* + \alpha_0 - j_\alpha - r)}}^{(E3)}$$

$$\times \overbrace{B^{N-k^*+\alpha_0-j\alpha-r}}^{(E4)} \overbrace{\Theta_R^{(j\alpha-n'+l+|\alpha|+s(N-k^*+\alpha_0-j\alpha-r))}}^{(E5)}(t),$$

where B_1 is a constant depending only on $b_\alpha(z)$. From Lemma 2.4 we see that $(E3) \leq (E5) \leq l+s(N-r)-n' \leq l+sN-r-n'$ and $(E4) \leq N-1-r$. Hence

$$(2.9) \quad (I) \ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_1^{r+1} r! A^{l+sN-r-n'} B^{N-1-r} \Theta_R^{(l+sN-r-n')}(t).$$

Suppose that $(E1) > (E2)$. Then

$$(I) \ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) \overbrace{B_1^{r+1} r! A^{n-n'-r+\alpha_0}}^{(E6)} \times \overbrace{\Theta_R^{(n-n'+|\alpha|-r)}}^{(E7)}(t).$$

We have $(E6) \leq (E7) = l+N-k^*+|\alpha|-n'-r$. It follows from the assumption that $|\alpha|-k^* \leq (s-1)N_0 \leq (s-1)N$ and $(E7) \leq l+sN-r-n'$. So

$$(2.10) \quad (I) \ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_1^{r+1} r! A^{l+sN-r-n'} \Theta_R^{(l+sN-r-n')}(t).$$

Thus in both cases

$$(2.11) \quad (I) \ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_1^{r+1} r! A^{l+sN-n'} B^{N-1-r} \Theta_R^{(l+sN-r-n')}(t).$$

We proceed to estimate (II) . We have

$$(II) \ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) \sum_{r=0}^{n-n'} \binom{n-n'}{r} B_1^{r+1} r! A^{l+sN-n'} B^{N-1-r} \Theta_R^{(l+sN-r-n')}(t)$$

by choosing B with $B_1/B < 1/2$ and by Lemma 2.2,

$$\begin{aligned} &\ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_1 A^{l+sN-n'} B^{N-1} \left(\sum_{r=0}^{n-n'} \frac{(n-n')!}{(n-n'-r)!} \left(\frac{1}{2}\right)^r \Theta_R^{(l+sN-n'-r)}(t) \right) \\ &\ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_2 A^{l+sN-n'} B^{N-1} \Theta_R^{(l+sN-n')}(t). \end{aligned}$$

Therefore, we have for $A > 2$ and large B

$$\begin{aligned} z_0^l \partial_0^{l+N} u(z) &\ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_2 A^{l+sN} B^{N-1} \\ &\quad \sum_{\alpha \neq (k^*, 0)} \left\{ \sum_{n'=0}^n \binom{n}{n'} (j_\alpha(j_\alpha-1) \cdots (j_\alpha-n'+1)) A^{-n'} \Theta_R^{(l+sN-n')}(t) \right\} \\ &\ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) B_3 A^{l+sN} B^{N-1} \left(\sum_{n'=0}^n \left(\frac{n!}{(n-n')!} \left(\frac{1}{2}\right)^{n'} \Theta_R^{(l+sN-n')}(t) \right) \right) \\ &\ll_{z'} C \exp(\varepsilon|z_0|^{-r^*}) A^{l+sN} B^N \Theta_R^{(l+sN)}(t). \end{aligned}$$

It follows from Lemma 2.3 that $z'_0 \partial_0^{l+N-k^*} f(z) \ll_{z'} CA^l B^{N-1} \Theta_R^{(l+sN)}(t)$ and we obtain the desired estimate for $z'_0 \partial_0^{l+N} u(z)$.

We have immediately from Proposition 2.5

Corollary 2.6. *Suppose that the same conditions as in Proposition 2.5 hold. Then there exists a polydisk $U = \{z, |z| < r\}$ such that for any $\varepsilon > 0$ and any $0 < \theta' < \theta$*

$$(2.12) \quad |\partial_0^n u(z)| \leq C \exp(\varepsilon |z_0|^{-\gamma}) B^n \Gamma((1 + 1/\gamma^*)n + 1)$$

holds for $z \in U(\theta')$ and all $n \in \mathbb{N}$, where $B = B(\theta')$ and $C = C(\varepsilon, \theta')$.

Now let us proceed to the proofs of Theorem 1.4 and Corollary 1.5, assuming Theorem 1.6.

Proof of Theorem 1.4. Suppose $a_{k^*,0}(z) \neq 0$ in Ω . If $m = k^*$, then $\gamma^* = \infty$ and Theorem 1.4 follows from Cauchy Kowalevskja's Theorem with precise estimates of the radius of convergence (see Zerner [14]). If $m > k^*$, by combining Corollary 2.6 and Theorem 1.6, we have Theorem 1.4. Otherwise let $a_{k^*,0}(0, z') \neq 0$. Then there are positive constants r_i ($1 \leq i \leq d$) such that $a_{k^*,0}(0, z') \neq 0$ on $M = \{z'; |z_i| = r_i, 1 \leq i \leq d\}$. Suppose that $m > k^*$. Then for each point $\tilde{z}' \in M$ there exists a neighbourhood $V_{\tilde{z}'} = \{z, |z_0| < R\tilde{z}, |z' - \tilde{z}'| < R\tilde{z}\}$ of $(0, \tilde{z}')$ such that (2.12) holds on $V_{\tilde{z}'}(\theta')$. By maximal principle of holomorphic functions estimate (2.12) holds on $V(\theta')$, where $V = \{z, |z_i| < r_i, 0 \leq i \leq d\}$ for some r_0 . Hence it follows from Theorem 1.6 that $u(z) \in \text{Asy}_{\gamma^*}(W(\theta))$ for some polydisk W . If $m = k^*$, $u(z) \in \mathcal{O}(\Omega(\theta))$ is also holomorphic in a neighbourhood of $\{z \in \mathbb{C}^{d+1}; z_0 = 0, z' \in M\}$ by Cauchy Kowalevskaja's Theorem. So it follows from Hartogs extension Theorem that $u(z)$ is holomorphic at $z = 0$.

Proof of Corollary 1.5. $u(z) \in \text{Asy}_{\gamma^*}(W(\theta))$ by Theorem 1.4. Put $\theta_0 = \theta - \pi$. Then by the assumption we have $\theta_0 > \pi/2\gamma^*$. Define $w(z) = u(z_0 e^{\pi i}, z') - u(z_0 e^{-\pi i}, z')$ for $\{z_0; |\arg z_0| < \theta_0\}$. Then we have $w(z) \in \text{Asy}_{\gamma^*}(W(\theta_0))$ and $w(z) \sim 0$, that is, there are some constants B and C such that for $z \in W(\theta_1)$ with $\pi/2\gamma^* < \theta_1 < \theta_0$

$$(2.13) \quad |w(z)| \leq CB^n |z_0|^n \Gamma(n/\gamma^* + 1) \text{ for } z \in W(\theta_1)$$

for all $n \in \mathbb{N}$. This implies that there is $c > 0$ such that $|w(z)| \leq C \exp(-c|z_0|^{-r^*})$ (see [9]). Since $\theta_1 > \pi/2\gamma^*$, we have $w(z) \equiv 0$, namely $u(z_0 e^{\pi i}, z') \equiv u(z_0 e^{-\pi i}, z')$. Therefore $u(z)$ is single valued and holomorphic at $\{z_0 = 0\}$.

§3. Proof of Theorem 1.6

In this section we prove Theorem 1.6. Only the variable z_0 is essential in Theorem 1.6. So by replacing z_0 with t , we write $v(t)$ instead of $v(z_0, z')$ and

regard it as a function of one variable t and other variables $z' = (z_1, z_2, \dots, z_n)$ may be considered as parameters. To clarify the situation we restate the assumptions on $v(t)$. $v(t) \in \mathcal{O}(\Omega_0(\theta))$ with $\Omega_0(\theta) = \{t; 0 < |t| < R, |\arg t| < \theta\}$ and for any $\varepsilon > 0$ and any $0 < \theta' < \theta$ there exist constants $C = C(\varepsilon, \theta')$ and $B = B(\theta')$ such that

$$(3.1) \quad \left| \left(\frac{d}{dt} \right)^n v(t) \right| \leq C \exp(\varepsilon |t|^{-\kappa}) B^n \Gamma((1 + \kappa^{-1})n + 1)$$

holds for $t \in \Omega_0(\theta')$ and $n = 0, 1, 2, \dots$.

Now put $\kappa = q/p$, where $p, q \in \mathbb{N}$ and they are relatively prime. Define

$$(3.2) \quad V(t, \zeta) = \sum_{n=0}^{\infty} \frac{\zeta^{n(p+q)} (d/dt)^{nq} v(t)}{\Gamma(n(p+q) + 1)}.$$

We can easily see the following properties on $V(t, \zeta)$.

Lemma 3.1. (i). *There is a constant $r_{\theta'} > 0$ such that $V(t, \zeta)$ is holomorphic in $\{(t, \zeta); t \in \Omega_0(\theta'), |\zeta| \leq r_{\theta'}\}$.*

(ii). *$V(t, \zeta)$ satisfies*

$$(3.3) \quad \left(\frac{\partial}{\partial t} \right)^q V(t, \zeta) - \left(\frac{\partial}{\partial \zeta} \right)^{p+q} V(t, \zeta) = 0.$$

We omit the proof.

Define a modified Laplace transform of $V(t, \zeta)$ by

$$(3.4) \quad \widehat{V}(\xi, \zeta) = \int_0^R \exp(-\xi t^{-\kappa}) V(t, \zeta) dt$$

and put $\mathcal{C}(\phi) = \{\xi \neq 0; |\arg \xi| < \phi\}$. Let us investigate the properties of $\widehat{V}(\xi, \zeta)$, in particular, its behaviour near $\xi = 0$.

Lemma 3.2. (i). *$\widehat{V}(\xi, \zeta) \in \mathcal{O}(\mathcal{C}(\kappa\theta' + \pi/2) \times \{|\zeta| \leq r_{\theta'}\})$.*

(ii). *$\widehat{V}(\xi, \zeta)$ satisfies an equation of Fuchsian type*

$$(3.5) \quad \{(-\partial_{\xi})^p \prod_{h=1}^q (\kappa \xi \partial_{\xi} - h)\} \widehat{V}(\xi, \zeta) - (\partial_{\zeta})^{p+q} \widehat{V}(\xi, \zeta) = F(\xi, \zeta),$$

where $F(\xi, \zeta) \in \mathcal{O}(\mathcal{C} \times \{|\zeta| \leq r_{\theta'}\})$.

Proof. Since $V(t, \zeta)$ is holomorphic in $\{(t, \zeta); t \in \Omega_0(\theta'), |\zeta| \leq r_{\theta'}\}$, by deforming the integration path, we have (i). We proceed to show (ii). It follows from integration by parts that

$$(\kappa \xi \partial_{\xi} - 1) \widehat{V}(\xi, \zeta) = \int_0^R \exp(-\xi t^{-\kappa}) t \partial_t V(t, \zeta) dt + \exp(-R^{-\kappa} \xi) I_0(\xi, \zeta),$$

where $I_0(\xi, \zeta) = -RV(R, \zeta)$. By repeating integration by parts, we have

$$\prod_{h=1}^q (\kappa \xi \partial_\xi - h) \widehat{V}(\xi, \zeta) = \int_0^R \exp(-\xi t^{-\kappa}) t^q \partial_t^q V(t, \zeta) dt + \exp(-R^{-\kappa} \xi) I_{q-1}(\xi, \zeta),$$

where $I_{q-1}(\xi, \zeta)$ is a polynomial in ξ whose coefficients are determined by $\{\partial_\xi^n \widehat{V}(\xi, \zeta)|_{\xi=R}; 0 \leq n \leq q-1\}$. Hence we have by Lemma 3.1

$$\begin{aligned} (-\partial_\xi)^p \prod_{h=1}^q (\kappa \xi \partial_\xi - h) \widehat{V}(\xi, \zeta) &= \int_0^R \exp(-\xi t^{-\kappa}) t^{q-p\kappa} \partial_t^q V(t, \zeta) dt + F(\xi, \zeta) \\ &= \int_0^R \exp(-\xi t^{-\kappa}) \partial_t^q V(t, \zeta) dt + F(\xi, \zeta) \\ &= \int_0^R \exp(-\xi t^{-\kappa}) \partial_\xi^{p+q} V(t, \zeta) dt + F(\xi, \zeta) \\ &= \partial_\xi^{p+q} \widehat{V}(\xi, \zeta) + F(\xi, \zeta), \end{aligned}$$

where $F(\xi, \zeta)$ is holomorphic in $\mathcal{C} \times \{\zeta; |\zeta| \leq r_{\theta^*}\}$.

Since $\widehat{V}(\xi, \zeta)$ satisfies an equation of Fuchsian type, we have a representation of $\widehat{V}(\xi, \zeta)$ near $\xi = 0$ from Proposition 4.1 given in the next section.

Lemma 3.3. $\widehat{V}(\xi, \zeta)$ is represented around $\xi = 0$ in the following form:

$$(3.6) \quad \widehat{V}(\xi, \zeta) = \Psi_0(\xi, \zeta) + \sum_{h=1}^{q-1} \xi^{h/\kappa} \Psi_h(\xi, \zeta) + \xi^p \log \xi \Psi_q(\xi, \zeta),$$

where $\Psi_h(\xi, \zeta)$ ($h = 0, 1, 2, \dots, q$) are holomorphic in $\{(\xi, \zeta); |\xi| \leq r_0, |\zeta| \leq r_1\}$ with some positive constants r_0 and r_1 and in particular for $1 \leq h \leq q$ $\Psi_h(\xi, \zeta)$ is a power series in $\xi^{p/\kappa}$:

$$(3.7) \quad \Psi_h(\xi, \zeta) = \sum_{s=0}^{\infty} \xi^{ps} \phi_{h,s}(\zeta).$$

Next proposition concerns functions defined by integral of Laplace type and their asymptotic expansion.

Proposition 3.4. Let $\kappa = q/p$ be a positive rational number, where $p, q \in \mathbb{N}$ and are relatively prime. Let $\psi(\xi) = \sum_{s=0}^{\infty} a_s \xi^{ps}$ be a power series in ξ^p which satisfies $|a_s| \leq AR_1^{-ps}$ for all $s \in \mathbb{N}$ with some $R_1 > 0$. Let $0 < r_0 < R_1$ and $h \in \mathbb{N}$ with $1 \leq h \leq q$ and

$$(3.8) \quad v_h(t) = t^{1-\kappa} \int_0^{r_0} \exp(-rt^{-\kappa}) r^{h/\kappa} \psi(r) dr.$$

Then $v_h(t) \in \text{Asy}_{(\kappa)}(\mathbf{C}(\pi/2\kappa))$.

Proof. Define

$$(3.9) \quad \phi_0^N(\xi) = \sum_{s=0}^N a_s \xi^{ps}, \quad \phi_1^N(\xi) = \sum_{s=N+1}^{\infty} a_s \xi^{ps}.$$

We have for some constants A and C independent of N

$$(3.10) \quad \begin{cases} |\phi_0^N(\xi)| \leq AC^{-Np} |\xi|^{pN} & \text{for } |\xi| \geq r_0, \\ |\phi_1^N(\xi)| \leq AC^{-(N+1)p} |\xi|^{p(N+1)} & \text{for } |\xi| \leq r_0. \end{cases}$$

Put

$$(3.11) \quad \begin{cases} I_N(t) = t^{-1-\kappa} \int_0^{+\infty} \exp(-rt^{-\kappa}) r^{h/\kappa} \phi_0^N(r) dr, \\ J_{1,N}(t) = t^{-1-\kappa} \int_{r_0}^{+\infty} \exp(-rt^{-\kappa}) r^{h/\kappa} \phi_0^N(r) dr, \\ J_{2,N}(t) = t^{-1-\kappa} \int_0^{r_0} \exp(-rt^{-\kappa}) r^{h/\kappa} \phi_1^N(r) dr. \end{cases}$$

Then

$$(3.12) \quad v_h(t) = I_N(t) - J_{1,N}(t) + J_{2,N}(t).$$

We have

$$(3.13) \quad \begin{aligned} I_N(t) &= \sum_{s=0}^N t^{h-1+qs} a_s \int_0^{+\infty} \exp(-r) r^{ps+h/\kappa} dr \\ &= \sum_{s=0}^N t^{h-1+ps} a_s \Gamma(h/\kappa + ps + 1). \end{aligned}$$

Let us proceed to estimating $J_{1,N}(t)$ and $J_{2,N}(t)$. Suppose $|\arg t| < \theta < \pi/2\kappa$. Then we have from (3.10)

$$\begin{aligned} |J_{1,N}(t)| &\leq AC^{-pN} |t|^{-1-\kappa} \int_{r_0}^{+\infty} \exp(-r(\Re t^{-\kappa})) r^{pN+h/\kappa} dr \\ &\leq AC^{-pN} |t|^{-1-\kappa} \exp(-r_0(\Re t^{-\kappa})/2) \int_0^{+\infty} \exp(-r(\Re t^{-\kappa})/2) r^{pN+h/\kappa} dr \\ &\leq AC_1^{-pN} |t|^{h-1+qN} \exp(-r_0(\Re t^{-\kappa})/2) \Gamma(h/\kappa + pN + 1) \end{aligned}$$

and

$$\begin{aligned} |J_{2,N}(t)| &\leq AC^{-p(N+1)} |t|^{-1-\kappa} \int_0^{r_0} \exp(-r(\Re t^{-\kappa})) r^{h/\kappa+p(N+1)} dr \\ &\leq AC_2^{-p(N+1)} |t|^{h-1+q(N+1)} \Gamma(h/\kappa + p(N+1) + 1). \end{aligned}$$

Hence

$$|v_h(t) - I_N(t)| \leq |J_{1,N}(t)| + |J_{2,N}(t)| \leq AB^{h-1+q(N+1)} |t|^{h-1+q(N+1)} \Gamma((h-1+q(N+1))/\kappa+1)$$

for some positive constant $B=B(\theta')$, which implies $v_h(t) \in \text{Asy}_{\{\kappa\}}(\mathbb{C}(\pi/2\kappa))$.

Now we have prepared to prove Theorem 1.6.

Proof of Theorem 1.6. We have the inversion formula

$$(3.14) \quad V(t, \zeta) = \frac{\kappa t^{-1-\kappa}}{2\pi i} \int_{a-i\infty}^{a+i\infty} \exp(\xi t^{-\kappa}) \widehat{V}(\xi, \zeta) d\xi \quad a > 0.$$

Let us deform the integration path. Take a constant ϕ_0 so that $\pi/2 < \phi_0 < \min\{\pi/2 + \kappa\theta', \pi\}$ and define contours $A_\nu, \nu = \pm 0, \pm 1, \pm 2$ by

$$\begin{cases} A_{-2} = \{\xi = r e^{-i\phi_0}, r_0 \leq r < +\infty\} & A_{+2} = \{\xi = r e^{+i\phi_0}, r_0 \leq r < +\infty\} \\ A_{-1} = \{\xi = r_0 e^{i\phi}, -\pi \leq \phi \leq -\phi_0\} & A_{+1} = \{\xi = r_0 e^{i\phi}, \phi_0 \leq \phi \leq +\pi\} \\ A_{-0} = \{\xi = r e^{-i\pi}, 0 \leq r \leq r_0\} & A_{+0} = \{\xi = r e^{+i\pi}, 0 \leq r \leq r_0\}. \end{cases}$$

Put

$$(3.15) \quad V_\nu(t, \zeta) = \frac{\kappa t^{-1-\kappa}}{2\pi i} \int_{A_\nu} \exp(\xi t^{-\kappa}) \widehat{V}(\xi, \zeta) d\xi,$$

where $\nu = \pm 2, \pm 1, \pm 0$, and

$$(3.16) \quad \begin{cases} V^*(t, \zeta) = - \sum_{i=-2, \pm 1} V_i(t, \zeta) + V_{+2}(t, \zeta) \\ V^0(t, \zeta) = - V_{-0}(t, \zeta) + V_{+0}(t, \zeta). \end{cases}$$

It follows from Lemma 3.3 that $\widehat{V}(\xi, \zeta)$ is integrable at $\xi = 0$ and holomorphically extensible around $\xi = 0$. Therefore we can deform the integration path in (3.14) to $-A_{-2} - A_{-1} - A_{-0} + A_{+0} - A_{+1} + A_{+2}$ and we have

$$(3.17) \quad V(t, \zeta) = V^*(t, \zeta) + V^0(t, \zeta).$$

$V^*(t, \zeta)$ decays exponentially as t tends to 0. More precisely let $0 < \theta'' < (\phi_0 - \pi/2)\kappa^{-1}$ and $|\arg t| < \theta''$. Then there are positive constants $A=A(\theta'')$ and $d=d(\theta'')$ such that $|V^*(t, \zeta)| \leq A \exp(-d|t|^{-\kappa})$ for $t \in \Omega_0(\theta'')$. As for $V^0(t, \zeta)$ we have by Lemma 3.3

$$(3.18) \quad \begin{aligned} V^0(t, \zeta) &= \frac{\kappa t^{-1-\kappa}}{2\pi i} \int_{-A_{-0}+A_{+0}} \exp(\xi t^{-\kappa}) \widehat{V}(\xi, \zeta) d\xi \\ &= \frac{\kappa t^{-1-\kappa}}{2\pi i} \int_{-A_{-0}+A_{+0}} \exp(\xi t^{-\kappa}) (\Psi_0(\xi, \zeta) + \sum_{h=1}^{q-1} \xi^{h/\kappa} \Psi_h(\xi, \zeta) + \xi^\nu \log \xi \Psi_q(\xi, \zeta)) d\xi. \end{aligned}$$

We have

$$(3.19) \quad \int_{-A_{-0}+A_{+0}} \exp(\xi t^{-\kappa}) \Psi_0(\xi, \zeta) d\xi = 0.$$

On the other hand it holds that for $h=1, 2, \dots, q-1$

$$(3.20) \quad \frac{1}{2\pi i} \int_{-A_{-0}+A_{+0}} \exp(\xi t^{-\kappa}) \xi^{h/\kappa} \Psi_h(\xi, \zeta) d\xi = \frac{-\sin h\pi/\kappa}{\pi} \int_0^{r_0} \exp(-rt^{-\kappa}) r^{h/\kappa} \Psi_h(-r, \zeta) dr$$

and

$$(3.21) \quad \frac{1}{2\pi i} \int_{-A_{-0}+A_{+0}} \exp(\xi t^{-\kappa}) \xi^p (\log \xi) \Psi_q(\xi, \zeta) d\xi = - \int_0^{r_0} \exp(-rt^{-\kappa}) (-r)^p \Psi_q(-r, \zeta) dr.$$

It follows from Proposition 3.4 that the integrals (3.20) and (3.21) have Gevrey type expansion. Hence we see that $V^0(t, \zeta)$ has the desired asymptotic expansion as t tends to 0 in $\{t; |\arg t| < \pi/2\kappa\}$ and $v(t) = V^0(t, 0) + V^*(t, 0) \in \text{Asy}_{(\kappa)}(\mathcal{O}_0(\theta'))$. By applying the same method to $v_\varphi(t) = v(te^{i\varphi})$, we have $v(t) \in \text{Asy}_{(\kappa)}(\mathcal{O}_0(\theta))$.

Remark. The author conjectured that the conclusion of Theorem 1.6 would hold for all real $\kappa > 0$. Prof. Honda (Hokkaido Univ.) indicated him that the conjecture was valid by using regularizers in Gevrey class.

§4. Solutions of Some Fuchsian Equations

In this section we consider a special Fuchsian equation which appeared in Lemma 3.2,

$$(4.1) \quad ((-\partial_\xi)^p \prod_{h=1}^q (\kappa \xi \partial_\xi - h)) \Psi(\xi, \zeta) - (\partial_\zeta)^{p+q} \Psi(\xi, \zeta) = F(\xi, \zeta),$$

where $F(\xi, \zeta) \in \mathcal{O}(\mathcal{C} \times \{|\zeta| \leq r\})$ and $\kappa = q/p$ with relatively prime integers p, q . The structure of solution of Fuchsian equations was investigated in Tahara [12], where he imposed a condition on zeros of the indicial polynomial. The operator (4.1) is simple but does not satisfy his condition. So we give the structure of solutions of (4.1) in a neighbourhood of $\xi=0$. We can apply our method employed to show Proposition 4.1 to general Fuchsian operators.

Proposition 4.1. *Let $\Psi(\xi, \zeta) \in \mathcal{O}(\{0 < |\xi| < r; |\arg \xi| < \theta\} \times \{|\zeta| < r\})$ be a solution of (4.1). Then $\Psi(\xi, \zeta)$ is represented around $\xi=0$ in the following form:*

$$(4.2) \quad \Psi(\xi, \zeta) = \Psi_0(\xi, \zeta) + \sum_{h=1}^{q-1} \xi^{h/\kappa} \Psi_h(\xi, \zeta) + \xi^p (\log \xi) \Psi_q(\xi, \zeta),$$

where $\kappa = q/p$. $\Psi_h(\xi, \zeta)$'s ($h = 0, 1, 2, \dots, q$) are holomorphic in $\{(\xi, \zeta); |\xi| < r_0, |\zeta| < r'_0\}$ for some positive constants r_0 and r'_0 . In particular for $1 \leq h \leq q$

$$(4.3) \quad \Psi_h(\xi, \zeta) = \sum_{s=0}^{\infty} \xi^{ps} \phi_{h,s}(\zeta),$$

where $\phi_{h,s}(\zeta)$'s are holomorphic in $\{|\zeta| < r'_0\}$.

The operator $((-\partial_\xi)^p \prod_{h=1}^q (\kappa \xi \partial_\xi - h)) - (\partial_\zeta)^{p+q}$ is Fuchsian in the sense of Baouendi-Goulaouic [1]. Its indicial polynomial is $H(\lambda) = (-1)^p \lambda(\lambda-1) \cdots (\lambda-p+1) \prod_{h=1}^q (\kappa \lambda - h)$ and its indicial exponents are $\{0, 1, 2, \dots, p-1, p/q, 2p/q, \dots, (q-1)p/q, p\}$.

Lemma 4.2. Let $H_n(\lambda) = \lambda \prod_{i=1}^n H(\lambda + ip)$ for $n = 1, 2, \dots$.

(i) All the zeros of $H_n(\lambda)$ ($n \geq 1$) are in the interval $I_n = [-pn, 0]$ on the real axis.

(ii) Let \mathcal{Z} be the set of zeros of $H_n(\lambda - np)$. Then $\mathcal{Z} = \cup_{h=0}^q \mathcal{Z}_h$, where $\mathcal{Z}_0 = \{0, 1, \dots, p-1, p, \dots, np-1, np\}$ and $\mathcal{Z}_h = \{h/\kappa, p+h/\kappa, \dots, (n-1)p+h/\kappa\}$ for $1 \leq h \leq q$. We have $\mathcal{Z}_q = \{p, 2p, \dots, np\} \subset \mathcal{Z}_0$ and the zeros in \mathcal{Z}_q are double and other zeros are simple.

(iii) Define for complex numbers $c \neq 0$ and $w_{0,j}, j = 0, 1, \dots, p+q-1$,

$$(4.4) \quad w(\xi, c) = \frac{1}{2\pi i} \int_{L_1} \frac{\xi^{\lambda+p} c^{-\lambda} (\sum_{j=0}^{p+q-1} \lambda^{p+q-j} w_{0,j})}{\lambda H(\lambda+p)} d\lambda,$$

where the path L_1 is a Jordan curve surrounding the interval I_1 . Then for $0 \leq h \leq p+q-1$

$$(4.5) \quad (\xi \partial_\xi)^h w(\xi, c)|_{\xi=c} = \frac{(-1)^p c^p w_{0,h}}{\kappa^q} + \text{linear combinations of } \{w_{0,j}; 0 \leq j < h\}.$$

(iv) Let $w(\lambda) = g(\lambda)/H_n(\lambda)$ be a rational function in λ , where $g(\lambda)$ is a polynomial in λ with degree $p+q$ and $|g(\lambda)| \leq M_g(1+|\lambda|)^{p+q}$. Then

$$(4.6) \quad w(\lambda - np) = \sum_{l=0}^{np} \frac{w_{0,l}}{(\lambda-l)} + \sum_{h=1}^{q-1} \sum_{l=0}^{n-1} \frac{w_{h,l}}{\lambda - lp - h/\kappa} + \sum_{l=0}^{n-1} \frac{w_{q,l}}{(\lambda - lp - p)^2}$$

and there are constants A and B such that

$$(4.7) \quad |w_{h,l}| \leq \frac{M_g A B^n}{((n-1)(p+q))!}.$$

Proof. It is easy to show (i). So we omit the proof. Since

(4.8)

$$\begin{aligned}
 H_n(\lambda - np) &= (\lambda - np) \prod_{l=0}^{n-1} H(\lambda - lp) \\
 &= (-1)^{np} \kappa^{np} \prod_{l=0}^{np} (\lambda - l) \prod_{l=0}^{n-1} (\lambda - lp - p) \prod_{h=1}^{q-1} \prod_{l=0}^{n-1} (\lambda - lp - h/\kappa),
 \end{aligned}$$

we have (ii). Let us show (iii). We have by calculating residue at $\lambda = \infty$

$$\begin{aligned}
 (4.9) \quad (\xi \partial_\xi)^h w(\xi, c)|_{\xi=c} &= \frac{c^p}{2\pi i} \int_{C_1} \frac{(\lambda + p)^h (\sum_{j=0}^{p+q-1} \lambda^{p+q-j} w_{0,j})}{\lambda H(\lambda + p)} d\lambda \\
 &= \frac{c^p}{2\pi i} \int_{C_1} \frac{(\lambda + p)^h (\sum_{j=0}^h \lambda^{p+q-j} w_{0,j})}{\lambda H(\lambda + p)} d\lambda \\
 &= \frac{(-1)^p c^p w_h}{\kappa^q} + \text{linear combinations of } \{w_{0,j}; 0 \leq j < h\}.
 \end{aligned}$$

Finally we show (iv). The identity (4.6) is the representation of $w_n(\lambda - np)$ by the partial fractions. We only show (4.7) for w_{0,l_0p} . As for other cases we can show more easily. Put

$$G(\lambda) = (-1)^{np} \kappa^{nq} \prod_{\substack{0 \leq l \leq np \\ l \neq l_0p}} (\lambda - l) \prod_{\substack{1 \leq l \leq n \\ l \neq l_0}} (\lambda - lp) \prod_{h=1}^{q-1} \prod_{l=0}^{n-1} (\lambda - lp - h/\kappa).$$

Then

$$w_{0,l_0p} = \lim_{\lambda \rightarrow l_0p} \frac{d}{d\lambda} \frac{(\lambda - l_0p)^2 g(\lambda - np)}{H_n(\lambda - np)} = \frac{g'(l_0p - np)G(l_0p) - g(l_0p - np)G'(l_0p)}{G(l_0p)^2}$$

and

$$|w_{0,l_0p}| \leq \frac{|g'(l_0p - np)G(l_0p)| + |g(l_0p - np)G'(l_0p)|}{|G(l_0p)|^2}.$$

Since

$$\begin{aligned}
 |g(l_0p - np)| &\leq M_g A (1+n)^{p+q}, \quad |g'(l_0p - np)| \leq M_g A (1+n)^{p+q-1}, \\
 |G'(l_0p)| &\leq AB^{n-1} (n(p+q) - 2)!, \\
 A'B^{-(n-1)} (n(p+q) - 1)! &\leq |G(l_0p)| \leq AB^{n-1} (n(p+q) - 1)!
 \end{aligned}$$

for some constants A, A' and B , we have (4.7) for w_{0,l_0p} .

Proof of Proposition 4.1. Let $F_0(\zeta) = F(0, \zeta)$ and $F_*(\xi, \zeta) = F(\xi, \zeta) - F_0(\zeta)$. Since $H(\lambda) \neq 0$ for $\lambda = p+1, p+2, \dots$ and $F_*(0, \zeta) \equiv 0$, it follows from the theory of partial differential equations of Fuchsian type that there exists $W_*(\xi, \zeta)$ which is holomorphic in a neighbourhood of $\xi = 0$ such that

$$(4.10) \quad ((-\partial_\xi)^p \prod_{h=1}^q (\kappa \xi \partial_\xi - h)) W_*(\xi, \zeta) - (\partial_\zeta)^{p+q} W_*(\xi, \zeta) = F_*(\xi, \zeta)$$

(see [1] or [12]). Therefore we may assume that $\Psi(\xi, \zeta)$ satisfies

$$(4.11) \quad ((-\partial_\xi)^p \prod_{h=1}^q (\kappa \xi \partial_\xi - h)) \Psi(\xi, \zeta) - (\partial_\zeta)^{p+q} \Psi(\xi, \zeta) = F_0(\zeta).$$

Now let $c > 0$ be a small constant and $w_{0,j}(\zeta, c)$'s ($0 \leq j \leq p + q - 1$) be functions determined by Lemma 4.2 so that

$$w_*(\xi, \zeta, c) = \frac{1}{2\pi i} \int_{L_1} \frac{\xi^{\lambda+p} c^{-\lambda} (\sum_{j=0}^{p+q-1} \lambda^{p+q-j} w_{0,j}(\zeta, c))}{\lambda H(\lambda+p)} d\lambda$$

satisfies for all $h=0, 1, \dots, p+q-1$

$$(\xi \partial_\xi)^h w_*(\xi, \zeta, c)|_{\xi=c} = (\xi \partial_\xi)^h \Psi(\xi, \zeta)|_{\xi=c}.$$

Define $w_0(\zeta, \lambda, c) = \sum_{j=0}^{p+q-1} \lambda^{p+q-j} w_{0,j}(\zeta, c)$ and

$$(4.12) \quad \begin{cases} w_1(\zeta, \lambda, c) = \frac{w_0(\zeta, \lambda, c) + F_0(\zeta)}{\lambda H(\lambda+p)} \\ w_n(\zeta, \lambda, c) = \frac{\partial_\zeta^{p+q} w_{n-1}(\zeta, \lambda, c)}{H(\lambda+np)} \text{ for } n \geq 2. \end{cases}$$

Then we have for $n \geq 1$

$$(4.13) \quad w_n(\zeta, \lambda, c) = \frac{\partial_\zeta^{(n-1)(p+q)} (w_0(\zeta, \lambda, c) + F_0(\zeta))}{H_n(\lambda)},$$

which is a rational function in λ whose denominator is $H_n(\lambda)$. Let L_n be a Jordan curve surrounding the interval $I_n = [-pn, 0]$. Define for $n \geq 1$

$$(4.14) \quad \begin{aligned} W_n(\xi, \zeta, c) &= \frac{1}{2\pi i} \int_{L_n} \xi^{\lambda+pn} c^{-\lambda} w_n(\zeta, \lambda, c) d\lambda \\ &= \frac{c^{pn}}{2\pi i} \int_{L_n+pn} \xi^\lambda c^{-\lambda} w_n(\zeta, \lambda-np, c) d\lambda \\ &= \frac{c^{pn}}{2\pi i} \int_{L_n+pn} \xi^\lambda c^{-\lambda} \left\{ \frac{\partial_\zeta^{(n-1)(p+q)} (\sum_{j=0}^{p+q-1} (\lambda-np)^{p+q-j} w_{0,j}(\zeta, c) + F_0(\zeta))}{H_n(\lambda-np)} \right\} d\lambda \end{aligned}$$

and put

$$(4.15) \quad W(\xi, \zeta, c) = \sum_{n=1}^{\infty} W_n(\xi, \zeta, c).$$

Let us calculate $W_n(\xi, \zeta, c)$ and show the convergence of the series $W(\xi, \zeta, c)$.

We have for some $A=A(c)$ and $0 < r < r'$

$$w_0(\zeta, \lambda, c) + F_0(\zeta) \ll_{\zeta} A(1+|\lambda|)^{p+q}(r-\zeta)^{-1}$$

and

$$\partial_{\zeta}^{(n-1)(p+q)}(w_0(\zeta, \lambda, c) + F_0(\zeta)) \ll_{\zeta} \frac{A((n-1)(p+q))!(1+|\lambda|)^{p+q}}{(r-\zeta)^{(n-1)(p+q)+1}}.$$

Let $|\zeta| \leq r/2$. Then there are constants $A=A(c)$ and B which is independent of c such that

$$(4.16) \quad |\partial_{\zeta}^{(n-1)(p+q)}(w_0(\zeta, \lambda, c) + F_0(\zeta))| \leq AB^n((n-1)(p+q))!(1+|\lambda|)^{p+q}.$$

We have by partial fractions of $w_n(\zeta, \lambda - np, c)$

$$(4.17) \quad w_n(\zeta, \lambda - np, c) = \sum_{l=0}^{np} \frac{w_{n,0,l}(\zeta, c)}{(\lambda - l)} + \sum_{h=1}^{q-1} \sum_{l=0}^{n-1} \frac{w_{n,h,l}(\zeta, c)}{\lambda - lp - h/\kappa} + \sum_{l=0}^{n-1} \frac{w_{n,q,l}(\zeta, c)}{(\lambda - lp - p)^2}.$$

where by Lemma 4.2 and (4.16) there are constants $A=A(c)$ and B such that $|w_{n,h,l}(\zeta, c)| \leq AB^n$. We have from (4.17) and calculation of the residues

$$(4.18) \quad W_n(\xi, \zeta, c) = c^{pn}(W_{n,0}(\xi, \zeta, c) + \sum_{h=1}^{q-1} (\xi/c)^{h/\kappa} W_{n,h}(\xi, \zeta, c) + (\xi/c)^p \log(\xi/c) W_{n,q}(\xi, \zeta, c)),$$

where

$$(4.19) \quad W_{n,0}(\xi, \zeta, c) = \sum_{l=0}^{pn} (\xi/c)^l w_{n,0,l}(\zeta, c),$$

$$W_{n,h}(\xi, \zeta, c) = \sum_{l=0}^{n-1} (\xi/c)^{lp} w_{n,h,l}(\zeta, c) \text{ for } 1 \leq h \leq q.$$

Note that $W_{n,h}(\xi, \zeta, c)$'s ($0 \leq h \leq q$) are polynomials of ξ . Let us show the convergence of $W(\xi, \zeta, c)$. Let $|\xi| \leq 2c$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} c^{pn} |W_{n,0}(\xi, \zeta, c)| &\leq \sum_{n=1}^{\infty} c^{pn} \left(\sum_{l=0}^{np} |\xi/c|^l |w_{n,0,l}(\zeta, c)| \right) \\ &\leq A \sum_{n=1}^{\infty} c^{pn} B^n \left(\sum_{l=0}^{np} |\xi/c|^l \right) \leq A \sum_{n=1}^{\infty} c^{pn} B^n 2^{np+1}. \end{aligned}$$

So if c is small, $\sum_{n=1}^{\infty} c^{pn} W_{n,0}(\xi, \zeta, c)$ converges on $\{\xi; |\xi| < 2c\}$. We can show by

the same method convergence of $\sum_{n=1}^{\infty} c^{np} W_{n,h}(\xi, \zeta, c)$ for $1 \leq h \leq q$. Hence $W(\xi, \zeta, c)$ converges and holomorphic in $\{\xi; 0 < |\xi| < 2c\}$ and from (4.12)

$$\begin{aligned} & (-\partial_{\xi})^p \prod_{h=1}^q (\kappa \xi \partial_{\xi} - h) W(\xi, \zeta, c) \\ &= ((-\partial_{\xi})^p \prod_{h=1}^q (\kappa \xi \partial_{\xi} - h)) \left(\sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{L_n} \xi^{\lambda+pn} c^{-\lambda} w_n(\zeta, \lambda, c) d\lambda \right) \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{L_{n+1}} \xi^{\lambda+pn} c^{-\lambda} H(\lambda+p(n+1)) w_{n+1}(\zeta, \lambda, c) d\lambda \\ &= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{L_{n+1}} \xi^{\lambda+pn} c^{-\lambda} \partial_{\xi}^{p+q} w_n(\zeta, \lambda, c) d\lambda + \frac{1}{2\pi i} \int_{L_1} \xi^{\lambda} c^{-\lambda} \frac{w_0(\zeta, \lambda, c) + F_0(\zeta)}{\lambda} d\lambda \\ &= \partial_{\xi}^{p+q} W(\xi, \zeta, c) + \frac{1}{2\pi i} \int_{L_1} \xi^{\lambda} c^{-\lambda} \frac{F_0(\zeta)}{\lambda} d\lambda = \partial_{\xi}^{p+q} W(\xi, \zeta, c) + F_0(\zeta). \end{aligned}$$

We also have

$$(\xi \partial_{\xi})^h W(\xi, \zeta, c) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{L_n} (\lambda+pn)^h \xi^{\lambda+pn} c^{-\lambda} w_n(\zeta, \lambda, c) d\lambda.$$

It follows from the definition of $w_{0,j}(\zeta, c)$ that for $0 \leq h \leq p+q-1$

$$\begin{aligned} & (\xi \partial_{\xi})^h W(\xi, \zeta, c)|_{\xi=c} = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{L_n} c^{pn} (\lambda+pn)^h w_n(\zeta, \lambda, c) d\lambda \\ &= \frac{c^p}{2\pi i} \int_{L_1} (\lambda+p)^h w_1(\zeta, \lambda, c) d\lambda = \frac{c^p}{2\pi i} \int_{L_1} (\lambda+p)^h \frac{w_0(\zeta, \lambda, c) + F_0(\zeta)}{\lambda H(\lambda+p)} d\lambda \\ &= \frac{c^p}{2\pi i} \int_{L_1} (\lambda+p)^h \frac{w_0(\zeta, \lambda, c)}{\lambda H(\lambda+p)} d\lambda = (\xi \partial_{\xi})^h w_*(\xi, \zeta, c)|_{\xi=c} = (\xi \partial_{\xi})^h \Psi(\xi, \zeta)|_{\xi=c}. \end{aligned}$$

Both $W(\xi, \zeta, c)$ and $\Psi(\xi, \zeta)$ satisfies (4.1) and the same initial conditions on $\xi = c$. Hence $W(\xi, \zeta, c) = \Psi(\xi, \zeta, c)$ and it follows from (4.19) that $\Psi(\xi, \zeta, c)$ is represented in the form (4.2) with (4.3).

References

- [1] Baouendi, M. S. and Goulaouic, C., Cauchy problems with characteristic initial hypersurface. *Comm. Pure Appl. Math.*, **26** (1973), 455-475.
- [2] Hamada, Y., Leray, J. et Wagschal, C., Système d'équation aux dérivées partielles à caractéristique multiples; problème de Cauchy ramifié; hyperbolicité partielle. *J. Math. Pures Appl.*, **55** (1976), 297-352.
- [3] Kashiwara, M. et Schpira, P., Problème de Cauchy pour les systèmes microdifférentiels dans le domaine complexe. *Inv. Math.*, **46** (1978), 17-38.

- [4] Ōuchi, S., Asymptotic behavior of singular solutions of linear partial differential equations in the complex domain, *J. Fac. Sci. Univ. Tokyo*, **27** (1980), 1-36.
- [5] _____, An integral representation of singular solutions of linear partial differential equations in the complex domain. *J. Fac. Sci. Univ. Tokyo*, **27** (1980), 37-85.
- [6] _____, Index, localization and classification of characteristic surfaces for linear partial differential operators, *Proc. Japan Acad.*, **60** (1984), 189-192.
- [7] _____, Existence of singular solutions and null solutions for linear partial differential operators, *J. Fac. Sci. Univ. Tokyo*, **32** (1985), 457-498.
- [8] _____, An integral representation of singular solutions and removable singularities to linear partial differential equations, *Publ. RIMS Kyoto Univ.*, **26** (1990), 735-783.
- [9] _____, The behaviour of solutions with singularities on a characteristic surface to linear partial differential equations in the complex domains, *Publ. RIMS Kyoto Univ.*, **29** (1993), 63-120.
- [10] _____, Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations. *J. Math. Sci. Univ. Tokyo*, **2** (1995), 375-417.
- [11] Persson, J., Singular holomorphic solutions of linear partial differential equations with holomorphic coefficients and nonanalytic solutions with analytic coefficients, *Asterisque* 89-90, *analytic solutions of partial differential equations (Trento 1981)*. Soc Math. France, 233-247.
- [12] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, *Japan. J. Math. (N. S.)*, **5** (1979), 245-347.
- [13] Wagschal, C., Problème de Cauchy analytique à données méromorphes, *J. Math. Pures Appl.*, **51** (1972), 375-397.
- [14] Zerner, M., Domaines d'holomorphic des fonctions vérifiant une équation aux dérivées partielles, *C. R. Acad. Sci. Paris Ser. I Math.*, **272** (1971), 1646-1648.

