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Dequantization Techniques for Weyl Quantization

By

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Abstract

By quantization we mean the linear bijection from $\mathcal{S}'(\Pi)$ to $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ due to Weyl, and by dequantization we mean its inverse. We propose two new, but related, dequantization schemes. The first is adapted to knowledge of the matrix elements (with respect to the Hermite–Gaussian functions) of the operator to be dequantized, while the second is adapted to its integral kernel. Our dequantization schemes are completely general. We apply these methods to the case where the operators in question are Toeplitz operators related to functions of angle on phase space. This enables us to compare the symbols of these Toeplitz operators with the functions of angle themselves.

§ 1. Introduction

When correctly constructed in the context of the rigged Hilbert triple $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$, Weyl quantization provides a linear bijection between the space $\mathcal{S}'(\Pi)$ of tempered distributions on phase space $\Pi \equiv \mathbb{R}^2$ and the space $\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ of continuous linear maps from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. However, effective formulae for dequantization are thin on the ground. Some symbolic formulae exist, but these are only rigorous when dequantizing observables which are particularly well-behaved. For example, a simple formula exists for the phase-space function whose Weyl quantization is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$. In cases where such formulae cannot be justified, it is not clear to what extent they can still be used to obtain information concerning the dequantization of observables.

In this paper we exhibit how explicit formulae for the dequantizations of general observables can be obtained. Two main approaches are discussed. Both of these approaches are essentially the same, but they have differing areas of

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utility, in that one is of use when considering observables for which an integral kernel is known, while the other is of use when considering observables for which the matrix coefficients with respect to the Hermite Gaussian functions are known. It is frequently the case that an observable is known in one or other of these forms — while either form gives, in principle, complete information about the observable, in practice it is often extremely difficult to analyze the properties of these observables. For example, it can be hard to derive the matrix coefficients of an observable from its integral kernel representation, or *vice versa*.

In particular we apply these results to considering the dequantizations of observables which are functions of the quantum phase angle alone. More explicitly, we consider that class of Toeplitz operators which are frequently cited as being quantum mechanical observables providing information about the quantum phase angle. We consider their dequantizations (with respect to Weyl quantization) and compare the result with the corresponding function of the fundamental phase space angle. We are therefore obtaining knowledge of the relationship between the Weyl quantizations of these functions of phase space angle and the matching Toeplitz operators. We show how the Toeplitz operators are, in some sense, deformations of the Weyl quantized observables, and provide some asymptotic information concerning the nature of these deformations.

Let us write $\Delta:\mathcal{S}'(\Pi)\to\mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ for Weyl quantization, so that, if $T\in\mathcal{S}'(\Pi)$, then $\Delta[T]$ is a continuous linear map from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. In many cases $\Delta[T]$ is more regular than that, for example $\Delta[T]$ may be a bounded operator on $L^2(\mathbb{R})$, but the rigged triple framework provides the necessary freedom to include essentially all the operators of interest in quantum mechanics.

Now, in practice, determining a closed form expression for $\Delta[T]$ in particular cases is not easy, but determining T explicitly from $\Delta[T]$ is even more difficult. Moreover, if a closed form for T is not known, the currently known approximation schemes are rather crude. For example, there is the formal expression

$$T(p, q)=\mathrm{Tr}(\Delta(p, q)\Delta[T]), \tag{1.1}$$

where

$$[\Delta(p, q)f](x)=2e^{2ip(x-q)}f(2q-x), \quad f\in L^2(\mathbb{R}). \tag{1.2}$$

Aside from the fact that $\Delta[T]$ will not usually be trace class, and $\Delta(p,q)$ never is, equation (1.1) is not easily evaluated. Using it as an approximation scheme, there is no assurance that an expression like

$$T^{(N)}(p, q)=\sum_{k=1}^{N}\langle e_k, \Delta(p,q)\Delta[T]e_k\rangle \tag{1.3}$$

will be close to T for any integer N or any choice of orthonormal basis $\{e_k:k\geq 1\}$ for $L^2(\mathbb{R})$.

In recent work [6, 7], we pointed out that in the modified Dicke model of a laser [2], the thermodynamic limit which is used in that model has the effect of dequantization, in a sense which we shall clarify below.

Setting aside the physics of the model, this provides us with a new technique for dequantization. As the function classes involved can be chosen explicitly, and the limits to be calculated are referred to appropriate topologies, the technique also supplies us with a rigorous approximation scheme.

Before introducing the new procedure, we briefly recall the basic theory of quantization and dequantization adapted to the rigged triple

$$\mathcal{S}(\mathbb{R})\subseteq L^2(\mathbb{R})\subseteq\mathcal{S}'(\mathbb{R}), \tag{1.4}$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space of C^∞-functions of rapid decrease at infinity, and $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. Quantization begins with the continuous homomorphism $\mathcal{G}$ of $\mathcal{S}(\mathbb{R}^2)$ into $\mathcal{S}(\Pi)$ given by the formula

$$[\mathcal{G}F](p,\ q)=\frac{1}{2\pi}\int_{\mathbb{R}}F\Big(q+\frac{1}{2}x,\ q-\frac{1}{2}x\Big)e^{ipx}dx \qquad F\in\mathcal{S}(\Pi). \tag{1.5}$$

Here and later, Π is the space $\mathbb{R}^2$ interpreted as phase space, and the Cartesian coordinates $(p,\ q)$ have the meaning of classical momentum and position (but not of anything in particular, in this paper) . Given $T\in\mathcal{S}'(\Pi)$, its Weyl quantization $\Delta[T]\in\mathcal{L}(\mathcal{S}(\mathbb{R}),\ \mathcal{S}'(\mathbb{R}))$ is defined by the formula (the square brackets indicate the canonical duality pairings)

$$[\Delta[T]g,\ f]=[T,\ \mathcal{G}(f\otimes g)] \qquad f,\ g\in\mathcal{S}(\mathbb{R}). \tag{1.6}$$

In order to determine the inverse of Δ, we observe that if $S\in\mathcal{L}(\mathcal{S}(\mathbb{R}),\ \mathcal{S}'(\mathbb{R}))$, then it defines a separately continuous bilinear map $\widehat{S}:\mathcal{S}(\mathbb{R})\times\mathcal{S}(\mathbb{R})\to\mathbb{C}$ given by

$$\widehat{S}(f,\ g)=[Sg,\ f] \qquad f,\ g\in\mathcal{S}(\mathbb{R}). \tag{1.7}$$

By the Banach-Steinhaus Theorem, $\widehat{S}$ is in fact jointly continuous, and hence defines a continuous linear functional $\widetilde{S}$ on $\mathcal{S}(\mathbb{R}^2)\equiv\mathcal{S}(\mathbb{R})\,\widehat{\otimes}_\pi\mathcal{S}(\mathbb{R})$, such that

$$[\widetilde{S},\ f\otimes g]=\widehat{S}(f,\ g)=[Sg,\ f] \qquad f,\ g\in\mathcal{S}(\mathbb{R}). \tag{1.8}$$

Thus we can define the map $D:\mathcal{L}(\mathcal{S}(\mathbb{R}),\ \mathcal{S}'(\mathbb{R}))\to\mathcal{S}'(\Pi)$ by

$$D(S)=(\mathcal{G}^{-1})'\widetilde{S}=\widetilde{S}\circ\mathcal{G}^{-1}, \qquad S\in\mathcal{L}(\mathcal{S}(\mathbb{R}),\ \mathcal{S}'(\mathbb{R})). \tag{1.9}$$

Regarding the properties of the map D, we have the following.

Lemma 1.1. *We have that* $\Delta[D(S)] = S$ *for any* $S \in \mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$, *while* $D(\Delta[T]) = T$ *for any* $T \in \mathscr{S}'(\Pi)$.

Proof. We see that

$$[\Delta[D(S)]g, f] = [D(S), \mathscr{G}(f \otimes g)] = [\widetilde{S}, f \otimes g] = [Sg, f]$$

for any $S \in \mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ and $f, g \in \mathscr{S}(\mathbb{R})$, while

$$[D(\Delta[T]), F] = [\tilde{\Delta}[T], \mathscr{G}^{-1}F] = [T, f]$$

for any $T \in \mathscr{S}'(\Pi)$ and $F \in \mathscr{S}(\Pi)$, so the result follows. ∎

Thus we have shown that Weyl quantization is bijective, and have obtained a formula (1.9) for the inverse $D = \Delta^{-1}$. However, equation (1.9) is of little or no practical use in finding the distribution $D(S)$ explicitly. In this paper we shall present two different solutions to this problem. In both cases we have a fully rigorous procedure for calculating the dequantization $D(S)$ for a given element of $S \in \mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$. The importance of the two methods lies in their ranges of utility. One method would be of use in dequantizing an element of $\mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ whose integral kernel is known explicitly, while the second is useful when dequantizing elements of $\mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ when what is known explicitly are the matrix coefficients with respect to the Schauder basis $\{h_n : n \geq 0\}$ for $\mathscr{S}(\mathbb{R})$ consisting of the Hermite-Gaussian functions. In the latter case, we also investigate the problem of dequantizing elements of $\mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ which are Toeplitz operators on $L^2(\mathbb{R})$ — many such operators are considered in the literature as being the correct quantizations of functions of the phase angle. From the point of view that we have set out in previous work [4, 5, 6, 11, 12, 14], it is important to compare these operators with Weyl quantizations of functions of the phase angle alone.

§ 2. The Convolution Method of Dequantization

The method of quantization that we shall now give was obtained from a consideration of a model of laser light (see above). For the sake of comparison, the parameter N in $\mathscr{E}_N$ refers to the number of gas atoms in the laser cavity and (c, d) in $\tau_{c,d}$ are parameters which fix the laser intensity.

Suppose that we have chosen a fixed element $F \in \mathscr{S}(\Pi)$ such that

$$\iint_{\Pi} F(p, q)\, dp dq = 1.$$

For any $a, b \in \mathbb{R}$ and $N > 0$ we consider the function $F_{N,a,b} \in \mathscr{S}(\Pi)$ defined by

$$F_{N,a,b} = \mathscr{G}^{-1} \tau_{-a,-b} \mathscr{E}_N F, \tag{2.1}$$

where $\mathscr{E}_M, \tau_{c,d} \in \mathscr{L}(\mathscr{S}(\Pi))$ are scalings and translations, respectively defined by the formulae

$$\begin{aligned}[\mathscr{E}_M F](p, q) &= MF(p\sqrt{M}, q\sqrt{M}) \\ [\tau_{c,d} F](p, q) &= F(p+c, q+d)\end{aligned} \qquad F \in \mathscr{S}(\Pi). \tag{2.2}$$

If $S \in \mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ we can define the functions $D_{F;N}(S)$ by

$$[D_{F;N}(S)](a,b) = [\widetilde{S}, F_{N,a,b}]. \tag{2.3}$$

The family $\{D_{F;N} : N \in \mathbb{N}\}$ gives an approximation scheme for D for each function F, as the following result shows.

Proposition 2.1. *Each function $D_{F;N}(S)$ belongs to $\mathscr{S}'(\Pi)$, and*

$$\lim_{N \to \infty} D_{F;N}(S) = D(S),$$

where the limit is taken with respect to the weak topology on $\mathscr{S}'(\Pi)$.

Proof. Direct calculation shows us that

$$[D_{F;N}(S)](a,b) = [D(S), \mathscr{G}(F_{N,a,b})] = [D(S), \tau_{-a,-b}\mathscr{E}_N F].$$

For any $G \in \mathscr{S}(\Pi)$, therefore, we have

$$[D_{F;N}(S), G] = \iint_\Pi [D(S), \tau_{-a,-b}\mathscr{E}_N F] G(a,b)\, da db = [D(S), \mathscr{E}_N F * G],$$

and so $D_{F;N}(S) \in \mathscr{S}'(\Pi)$. It is a standard result that

$$\lim_{N \to \infty} \mathscr{E}_N F * G = G$$

for any $G \in \mathscr{S}(\Pi)$, the convergence being with respect to the Fréchet topology on $\mathscr{S}(\Pi)$. Hence the result follows. ∎

We emphasize again that we therefore obtain a convergence scheme for each $F \in \mathscr{S}(\Pi)$.

If the operator S has a known integral kernel, Σ say, then its dequantization symbol is given by

$$D(S) = 2\pi \mathscr{G}(\Sigma) \tag{2.4}$$

when Σ is in the domain of $\mathscr{G}$, namely when $\Sigma \in \mathscr{S}(\Pi)$. Our method certainly gives this result. For in this case, using equations (1.8) and (2.3),

$$\begin{aligned}[D_{F;N}(S)](a,b) &= \iint_\Pi \Sigma(p, q) F_{N,a,b}(p, q)\, dp dq \\ &= 2\pi \iint_\Pi [\mathscr{G}\Sigma](p, q) [\tau_{-a,-b}\mathscr{E}_N F](p,q)\, dp dq\end{aligned}$$

$$=2\pi[\mathcal{G}\Sigma * \mathcal{E}_N F_-](a, b) \tag{2.5}$$

where $F_-(p, q)=F(-p, -q)$, and hence we deduce in this case that $D(S)\in \mathcal{S}(\Pi)$ and

$$D(S)=2\pi\mathcal{G}(\Sigma).$$

But, unlike equation (2.4), our method is not restricted to the cases where $S\in \mathcal{S}(\Pi)$. For example, if

$$Sf=f(0)\delta \qquad f\in\mathcal{S}(\mathbb{R}),$$

then S has the integral kernel $\Sigma=\delta\otimes\delta$. Direct calculation then shows us that

$$[D_{F;N}(S)](a,b)=\sqrt{N}F_1(\sqrt{N}b),$$

where

$$F_1(y)=\int_{\mathbb{R}} F(x, -y)\,dx.$$

Standard analysis then shows us that

$$D(S)=1\otimes\delta,$$

a result which can be readily checked.

So far, the choice of function F has played no role. This has been because our examples have been so simple. We are now going to consider a more difficult example, for which the choice

$$F(p, q)=\frac{1}{\pi}e^{-p^2-q^2} \tag{2.6}$$

will be useful. This example considers the quantization $\Delta[\varphi]$ of the angle function in phase space

$$\varphi(r\cos\beta, r\sin\beta)=\begin{cases}\beta, & r>0,\ -\pi<\beta\leq\pi,\\ 0, & r=0.\end{cases} \tag{2.7}$$

In previous work we have shown that

$$\langle f, \Delta[\varphi]g\rangle=\frac{1}{2}\pi\int_{\mathbb{R}}\mathrm{sgn}(q)\,\overline{f(q)}\,g(q)\,dq$$
$$-\frac{1}{2}i\lim_{L\to\infty}\iint_{\mathbb{R}^2}\mathrm{sgn}(y)\,e^{-|xy|}g_{I(L)}(y)\overline{f\left(y+\frac{1}{2}x\right)}g\left(y-\frac{1}{2}x\right)dxdy \tag{2.8a}$$

where

$$g_{I(L)}(x)=\begin{cases}x^{-1}, & L^{-1}<|x|<L,\\ 0, & \text{otherwise}.\end{cases} \tag{2.8b}$$

Then it is clear that

$$\Delta[\varphi]=\frac{1}{2}\pi\operatorname{sgn}(Q)-\frac{1}{2}iS, \tag{2.9}$$

where $S\in\mathscr{L}(\mathscr{S}(\mathbb{R}),\mathscr{S}'(\mathbb{R}))$ is such that

$$[\widetilde{S},G]=\lim_{L\to\infty}\iint_{\mathbb{R}^2}\operatorname{sgn}(y)e^{-|xy|}g_{I(L)}(x)G\Big(y+\frac{1}{2}x,\ y-\frac{1}{2}x\Big)dxdy \tag{2.10}$$

for any $G\in\mathscr{S}(\Pi)$. Clearly, then,

$$\varphi=\frac{1}{2}\pi 1\otimes\operatorname{sgn}-\frac{1}{2}iD(S). \tag{2.11}$$

We shall see how the above dequantization technique provides an alternative confirmation of the above identity. Choosing $F\in\mathscr{S}(\Pi)$ to be the Gaussian, equation (2.6), evidently

$$F_{N,a,b}(p,q)=\sqrt{\frac{N}{\pi}}\exp\Big[-\frac{1}{4}N(p+q-2b)^2-\frac{1}{4N}(p-q)^2-ia(p-q)\Big]. \tag{2.12}$$

Then

$$\begin{aligned}&[D_{F;N}(S)](p,q)\\&=\lim_{L\to\infty}\sqrt{\frac{N}{\pi}}\iint_{\mathbb{R}^2}\operatorname{sgn}(y)e^{-|xy|}g_{I(L)}(x)\exp\Big[-N(y-q)^2-\frac{1}{4N}x^2-ipx\Big]\,dxdy.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}&[D_{F;N}(S)](p,q)\\&=-2i\sqrt{\frac{N}{\pi}}\int_0^\infty\frac{\sin px}{x}\ \Big\{e^{-qx}\int_{-q+x/2N}^\infty e^{-Ny^2}dy-e^{qx}\int_{q+x/2N}^\infty e^{-Ny^2}dy\Big\}dx.\end{aligned}$$

As $D_{F;N}(S)$ is a smooth function, it is legitimate to differentiate it with respect to p under the integral sign. Doing so, we obtain

$$\Big[\frac{\partial D_N(S)}{\partial p}\Big](p,q)=-\frac{2iq}{p^2+q^2}+\frac{4iqe^{-Nq^2}}{p^2+q^2}\sqrt{\frac{N}{\pi}}\int_0^\infty e^{-Ny^2}\cos(2Npy)\,dy,$$

and we recognize the first term as a partial derivative of $2i\varphi(p,q)$, so it follows that

$$\Big|\Big[\frac{\partial}{\partial p}(D_{F;N}(S)-2i\varphi)\Big](p,q)\Big|\le\frac{4|q|e^{-Nq^2}}{p^2+q^2}.$$

Since $[D_{F;N}(S)](0,q)=0$ and $\varphi(0,q)=\frac{\pi}{2}\operatorname{sgn}(q)$ we deduce that

$$\Big|[D_{F;N}(S)](p,q)+i\frac{\pi}{2}\operatorname{sgn}(q)-2i\varphi(p,q)\Big|\le 4|q|e^{-Nq^2}\int_0^{|p|}\frac{dt}{t^2+q^2}\le 2\pi e^{-Nq^2}.$$

and hence

$$\lim_{N\to\infty} [D_{F;N}(S)](p, q) = 2i\varphi(p, q) - i\pi \operatorname{sgn}(q)$$

whenever $q \neq 0$. We note in passing that the above identity also holds (trivially) when $q=0$ and $p>0$, but not when $q=0$ and $p<0$, but the required identity then holds distributionally in $\mathscr{S}'(\Pi)$, as required.

§ 3. Special Hermite Functions

Suppose, on the other hand, that the operator $S \in \mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ is known explicitly in terms of its matrix coefficients $[Sh_m, h_n]$, where $m, n \geq 0$, with respect to the Schauder basis for $S(\mathbb{R})$ consisting of the Hermite-Gaussian functions. Then the above procedure for dequantization is not helpful in calculational terms, since we would have to express each function $F_{N,a,b}$ in terms of the Schauder basis $\{h_m \otimes h_n: m,n \geq 0\}$ for $\mathscr{S}(\Pi)$, and it is unlikely that we could do so simply enough for it to be used in calculations.

To deal with this case, we shall adopt an alternative approach to dequantization, using a different Schauder basis for $\mathscr{S}(\Pi)$ than that obtained from the Hermite-Gaussian functions. The Weyl quantization procedure can then be seen as simply changing the basis used for $\mathscr{S}'(\Pi)$, and hence the dequantization procedure is immediate, involving the opposite change of basis.

The basis for $\mathscr{S}(\Pi)$ that we shall now consider involves the special Hermite functions. These functions are considered in [16]. The functions we shall use are not exactly the same as those used in that text, since we adopt slightly different scaling and normalization conventions.

For any $m, n \geq 0$, define the function $\Phi_{m,n} \in \mathscr{S}(\Pi)$ by the formula

$$\Phi_{m,n} = 2\pi \mathscr{G}(h_m \otimes h_n), \tag{3.1}$$

noting that $\Phi_{m,n} = \overline{\Phi_{n,m}}$ for all $m,n \geq 0$.

Proposition 3.1 *The set $\{\Phi_{m,n}: m,n \geq 0\}$ is a Schauder basis for $\mathscr{S}(\Pi)$, with $\left\{\frac{1}{2\pi}\Phi_{n,m}: m,n \geq 0\right\}$ being the matching dual Schauder basis for $\mathscr{S}'(\Pi)$.*

Proof. Since $\mathscr{G}$ is a continuous automorphism of $\mathscr{S}(\Pi)$, and since $\{h_m \otimes h_n: m,n \geq 0\}$ is a Schauder basis for $\mathscr{S}(\Pi)$, it is clear that $\{\Phi_{m,n}: m,n \geq 0\}$ is also a Schauder basis for $\mathscr{S}(\Pi)$. The result concerning the matching dual Schauder basis for $\mathscr{S}'(\Pi)$ follows from the orthogonality of the functions involved, since

$$[\Phi_{k,j}, \Phi_{m,n}] = \langle \Phi_{j,k}, \Phi_{m,n} \rangle = 2\pi \delta_{jm} \delta_{kn}, \qquad j, k, m, n \geq 0.$$

∎

It is also clear from the above that the infinite series

$$\sum_{m,n\geq 0} \xi_{m,n}\Phi_{m,n}$$

belongs to $\mathscr{S}(\Pi)$ if and only if the double series $(\xi_{m,n})$ belongs to $s^{(2)}$, so that

$$\sup_{m,n\geq 0} (m+1)^r(n+1)^s|\xi_{m,n}|<\infty$$

for all integers r, $s\geq 0$.

We introduce the generating function $\mathscr{P}_{s,t}$ for the functions $\Phi_{m,n}$ by defining

$$\mathscr{P}_{s,t}(p, q)=\sum_{m,n\geq 0} \frac{s^m t^n}{\sqrt{2^{m+n}m!n!}}\Phi_{m,n}(p, q)\ , \qquad s, t\in\mathbb{R}. \tag{3.2}$$

Proposition 3.2. *We have*

$$\mathscr{P}_{s,t}(p,q)=2\exp\left[-p^2-q^2+is(p-iq)-it(p+iq)-\frac{1}{2}st\right] \tag{3.3}$$

and hence we deduce that

$$\Phi_{m,n}(p, q) = (-1)^{\min(m,n)}i^{m-n}2^{1+\frac{1}{2}|m-n|}\sqrt{\frac{\min(m, n)!}{\max(m, n)!}}\,e^{-r^2}r^{|m-n|}e^{i(n-m)\beta}L_{\min(m,n)}^{(|m-n|)}(2r^2) \tag{3.4}$$

for all m, $n\geq 0$, *where we write* $p+iq=re^{i\beta}$.

Proof. The first result follows by observing that $\mathscr{P}_{s,t}=2\pi\mathscr{G}(G_s\otimes G_t)$, and the second result follows by equating coefficients of $s^m t^n$ for all m, $n\geq 0$. ∎

Special Hermite functions are associated with pairs of raising and lowering operators. Because our conventions differ from those of [16], we must modify his definitions slightly. To this end, consider the four vector fields

$$\begin{aligned} W&=\frac{1}{2}\left(\frac{\partial}{\partial p}+i\frac{\partial}{\partial q}\right)+(p+iq), \\ \overline{W}&=\frac{1}{2}\left(\frac{\partial}{\partial p}-i\frac{\partial}{\partial q}\right)-(p-iq), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} Z&=\frac{1}{2}\left(\frac{\partial}{\partial p}-i\frac{\partial}{\partial q}\right)+(p-iq), \\ \overline{Z}&=\frac{1}{2}\left(\frac{\partial}{\partial p}+i\frac{\partial}{\partial q}\right)-(p+iq). \end{aligned} \tag{3.6}$$

Together with the identity they generate an algebra on phase space which is isomorphic to the 3-dimensional Heisenberg algebra [7]. Their actions as raising and lowering operators are as follows:

Proposition 3.3. *For any m, $n\geq 0$ we have that*

$$W\Phi_{m,n}=i\sqrt{2m}\,\Phi_{m-1,n},\quad \overline{W}\Phi_{m,n}=i\sqrt{2m+2}\,\Phi_{m+1,n}, \tag{3.7}$$

and

$$Z\Phi_{m,n}=-i\sqrt{2n}\,\Phi_{m,n-1},\quad \overline{Z}\,\Phi_{m,n}=-i\sqrt{2n+2}\,\Phi_{m,n+1}. \tag{3.8}$$

∎

Combining these differential operators, we can define the two elliptic differential operators

$$L=-\frac{1}{2}(W\overline{W}+\overline{W}W),\qquad R=-\frac{1}{2}(Z\overline{Z}+\overline{Z}Z), \tag{3.9}$$

which can be written as

$$L=-\frac{1}{4}\left(\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right)+p^2+q^2+iN,\quad R=-\frac{1}{4}\left(\frac{\partial^2}{\partial p^2}+\frac{\partial^2}{\partial q^2}\right)+p^2+q^2-iN, \tag{3.10}$$

where the differential operator N is given by the formula

$$N=p\frac{\partial}{\partial q}-q\frac{\partial}{\partial p}. \tag{3.11a}$$

It is worth noting that, in polar coordinates, we have the identity

$$N=\frac{\partial}{\partial\beta}, \tag{3.11b}$$

which is of some interest, since then iN is the third component of angular momentum if Π is embedded in $\mathbb{R}^3\equiv\Pi\times\mathbb{R}$. The special Hermite functions are eigenvectors of these operators:

Proposition 3.4. *For any m, $n\geq 0$ we have that*

$$L\Phi_{m,n}=(2m+1)\,\Phi_{m,n},\quad R\Phi_{m,n}=(2n+1)\,\Phi_{m,n}. \tag{3.12}$$

∎

We shall now show the utility of this new Schauder basis for $\mathcal{S}(\Pi)$.

Proposition 3.5. *For any $M, N \geq 0$, the element $\Delta[\Phi_{M,N}] \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ is in fact the bounded operator*

$$\Delta[\Phi_{M,N}] = |h_N\rangle\langle h_M| \tag{3.13}$$

acting on $L^2(\mathbb{R})$.

Proof. This result follows from the identity

$$\begin{aligned}[\Delta[\Phi_{M,N}]h_m, h_n] &= [\Phi_{M,N}, \mathcal{G}(h_n \otimes h_m)] \\ &= \frac{1}{2\pi}[\Phi_{M,N}, \Phi_{n,m}] \\ &= \delta_{Mm}\delta_{Nn}\end{aligned}$$

for all $M, N, m, n \geq 0$. ∎

Another way of putting this result would be to say that $\Phi_{M,N}$ is the symbol, or dequantization, of the operator $|h_N\rangle\langle h_M|$.

We note, as also observed in [16], that this result implies that very easy formulae exist for the Moyal products of the special Hermite functions $\Phi_{M,N}$, since

$$\Phi_{J,K} * \Phi_{M,N} = \delta_{JN}\Phi_{M,K}, \qquad J, K, M, N \geq 0. \tag{3.14}$$

In view of these results, we now have a precise expression for the dequantization $D(S) \in \mathcal{S}'(\Pi)$ of any element $S \in L(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ in terms of this new basis for $\mathcal{S}'(\Pi)$. The following is now immediate:

Theorem 3.1. *If $S \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$, then its dequantization $D(S) \in \mathcal{S}'(\Pi)$ can be written as the infinite series*

$$D(S) = \sum_{M,N \geq 0} [Sh_M, h_N]\Phi_{M,N}, \tag{3.15}$$

which converges in $\mathcal{S}'(\Pi)$. ∎

Thus, for a given $S \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$, the problem of determining $D(S)$ is now simply a matter of investigating the properties of the above infinite series.

Before moving on to investigate specific cases of this dequantization technique, we note the following useful identity concerning the special Hermite functions, which will be of use in the theory of quantization in polar coordinates.

Proposition 3.6. *For any M, $N \geq 0$ we have that*

$$\int_0^\infty \Phi_{M,N}(r\cos\beta,\ r\sin\beta)\, r dr = i^{M-N} e^{i(N-M)\beta} g_{M,N}, \tag{3.16a}$$

where

$$g_{M,N} = \left[\frac{\max(M,N)!\, 2^{\min(M,N)}}{\min(M,N)!\, 2^{\max(M,N)}}\right]^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\min(M,N) + s(M,N)\right)}{\Gamma\left(\frac{1}{2}\max(M,N) + s(M,N)\right)}, \tag{3.16b}$$

with the quantity $s(M,N)$ being given by the formula

$$s(M,N) = \begin{cases} \frac{1}{2}, & \min(M,N)\ \textit{even}, \\ 1, & \min(M,N)\ \textit{odd}. \end{cases} \tag{3.16c}$$

∎

Proof. While this result can be proved directly from the definition for the functions $\Phi_{M,N}$, we choose another route. Note that

$$\int_0^\infty \mathscr{P}_{s,t}(r\cos\beta,\ r\sin\beta)\, r dr = 2\pi \int_0^\infty [\mathscr{G}(G_s \otimes G_t)](r\cos\beta,\ r\sin\beta)\, r dr$$

$$= \sum_{m,n\geq 0} \frac{i^{m-n} s^m t^n e^{i(n-m)\beta}}{2^{\max(m,n)} \min(m,n)!} \frac{\Gamma\left(\frac{1}{2}\min(m,n) + s(m,n)\right)}{\Gamma\left(\frac{1}{2}\max(m,n) + s(m,n)\right)}$$

for all $s,t \in \mathbb{R}$. Equating coefficients, the result is immediate.

§ 4. Dequantization of Toeplitz Operators

Of particular interest in the theory of Weyl quantization of functions of the phase angle are the so-called Toeplitz operators. Such operators are widely used in the literature to model phase angle phenomena. To be specific, for any function $f \in L^\infty(\mathbb{T})$ we can define a bounded operator $\widehat{T}(f) \in \mathscr{L}(L^2(\mathbb{R}))$ such that

$$\langle h_m,\ \widehat{T}(f) h_n \rangle = i^{m-n} \widehat{f}_{m-n} \qquad m,\ n \geq 0. \tag{4.1}$$

This map is obtained in the following manner. Composing the embedding map from Hardy space $H^2(\mathbb{T})$ to $L^2(\mathbb{T})$ with the map from $L^2(\mathbb{T})$ to itself consisting of multiplication by f, and finally the Szegö projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$ yields a bounded linear operator on $H^2(\mathbb{T})$. The map $\widehat{T}(f)$ is the image of this map under the unitary transformation from $H^2(\mathbb{T})$ to $L^2(\mathbb{R})$ obtained by mapping the basis element $e^{in\beta}$ of $H^2(\mathbb{T})$ with the Hermite-Gaussian function $i^n \cdot h^n$ for all $n \geq 0$.

By the "standard" Toeplitz operator we mean $X = \widehat{T}(\Theta)$, where $\Theta(e^{i\beta}) = \beta$ for

$-\pi<\beta\leq\pi$, which has been proposed as a candidate for a quantum phase operator [9, 8, 10, 13]. Also of interest is the associated operator $U=\widehat{T}(e^{i\theta})$, which is a right-shift operator on $L^2(\mathbb{R})$, since $Uh_n=ih_{n+1}$ for all $n\geq 0$. Another Toeplitz operator is its adjoint map $U^*=\widehat{T}(e^{-i\theta})$, which is $-i$ times the left-shift operator on $L^2(\mathbb{R})$ [15].

It has been the burden of much of our previous work on quantum phase that such operators are not the primary ones to consider in connection with the physical phenomena. We believe, in this regard, that attention should be paid to the operators which are the Weyl quantizations of phase space functions of the phase angle alone. To re-establish our notation for such operators, for any $f\in L^\infty(\mathbb{T})$ we can consider the function $f_{\text{ang}}\in\mathcal{S}'(\Pi)$ given by the formula

$$f_{\text{ang}}(r\cos\beta,\ r\sin\beta)=f(e^{i\beta}). \tag{4.2}$$

Our proposal, then, is that the Weyl quantization $\Delta[f_{\text{ang}}]$ of f_{ang} is the operator which should be considered rather than $\widehat{T}(f)$. However, it is clear that, in some sense, the operator $\Delta[f_{\text{ang}}]$ is a deformation of the operator $\widehat{T}(f)$, and it would be interesting to understand this deformation in detail. To do so we must consider the difference between the distributions f_{ang} and $\mathfrak{D}(f)$ in $\mathcal{S}'(\Pi)$, where $\mathfrak{D}(f)=\mathfrak{D}(\widehat{T}(f))$ is the dequantization of the Toeplitz operator $\widehat{T}(f)$. We can deduce the following result immediately.

Proposition 4.1. *For any $f\in L^\infty(\mathbb{T})$ we have that*

$$f_{\text{ang}}=\sum_{m,n\geq 0} i^{n-m}g_{m,n}\widehat{f}_{n-m}\Phi_{m,n} \tag{4.3}$$

$$\mathfrak{D}(f)=\sum_{m,n\geq 0} i^{n-m}\widehat{f}_{n-m}\Phi_{m,n}. \tag{4.4}$$

■

We note in passing that the above formula for f_{ang} yields the following general distributional identities:

$$\sum_{m\geq 0} g_{m,m+k}\Phi_{m,m+k}(r\cos\beta,\ r\sin\beta)=i^{-k}e^{ik\beta},\qquad k\geq 0, \tag{4.5}$$

and their complex conjugates. We also remark that these results show clearly the importance of the coefficients $g_{m,n}$ for angular quantization.
The identities f_{ang} and $\mathfrak{D}(f)$ may be simplified further, if for any $k\geq 0$ we introduce the distribution F_k defined by the formula

$$F_k(r)=2^{1+\frac{1}{2}k}r^k e^{-r^2}\sum_{M\geq 0}(-1)^M\sqrt{\frac{M!}{(M+k)!}}L_M^{(k)}(2r^2). \tag{4.6}$$

Then

Proposition 4.2. *For any $f \in L^\infty(\mathbb{T})$ we have the distributional identities*

$$[f_{\mathrm{ang}}](r\cos\beta,\ r\sin\beta) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ik\beta}, \tag{4.7}$$

$$[\mathfrak{D}(f)](r\cos\beta,\ r\sin\beta) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ik\beta} F_{|k|}(r)\ . \tag{4.8}$$

∎

Thus we see that the radial dependence of the distributions $\mathfrak{D}(f)$ is completely contained in the distributions F_k. We proceed by investigating the properties of these distributions F_k. We need to make the following auxiliary definitions.
For any $k \geq 0$ and $r \geq 0$ we define the function $Z_{k,r} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by setting

$$Z_{k,r}(V) = \begin{cases} \exp\left[-r^2 \tanh\left(\frac{1}{2}V\right)\right] \dfrac{e^{\frac{1}{2}kV}}{\cosh^{k+1}\left(\frac{1}{2}V\right)}, & V>0, \\ 0, & V \leq 0, \end{cases} \tag{4.9}$$

and for any $\alpha > 0$ we define $\xi_\alpha \in L^1(\mathbb{R})$ by the formula

$$\xi_\alpha(V) = \begin{cases} \dfrac{1}{2\sqrt{V}} e^{-\frac{1}{2}\alpha V}, & V>0, \\ 0, & V \leq 0. \end{cases} \tag{4.10}$$

We now have the following result.

Proposition 4.3. *We have that $F_0(r) = 1$. Moreover, for any $k \in \mathbb{N}, F_k(r)$ is a smooth polynomially bounded function given by the formula*

$$F_k(r) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}k} r^k \int_{\mathbb{R}} Z_{k,r}(V)\,[\xi_1 * \xi_3 * \ldots * \xi_{2k-1}](V)\,dV, \qquad r \geq 0. \tag{4.11}$$

Moreover we have the inequality

$$|F_k(r)| \leq \frac{2^k r^k}{\sqrt{(2k-3)!!}}, \qquad r \geq 0,\ k \geq 1. \tag{4.12}$$

Proof. The result concerning F_0 is a standard distributional result for the Laguerre polynomials. For general k we start with equation (4.6) , and substitute the expression

$$\sqrt{\frac{M!}{(M+k)!}} = \pi^{-\frac{1}{2}k} \int_{\mathbb{R}^k} e^{-M\|x\|^2} \exp\left[-\sum_{j=1}^{k} j x_j^2\right] dx_1 \ldots dx_k\,.$$

The order of the sum and integral may be interchanged, and the resulting sum over M inside the integral sign may be recognized as a generating function for the Laguerre polynomials (the absolute and uniform convergence of this series justifies interchanging the order of the sum and the integral). Changing to polar coordinates in $\mathbb{R}^k$, the angular integrals can be performed immediately, as the integrand only depends on the radius R. It is then convenient to change variables to $V=R^2$, and so we can write

$$F_k(r)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}k} r^k \int_0^\infty \frac{\exp\left[-r^2\tanh\left(\frac{1}{2}V\right)\right]}{\cosh^{k+1}\left(\frac{1}{2}V\right)\Big]} e^{\frac{1}{2}kV}\mathscr{G}_k'(V)\,dV$$

for any $k\in\mathbb{N}$, where

$$\mathscr{G}_k(V)=\int_{\substack{x_1,\dots,x_k\geq 0\\ x_1^2+\cdots+x_k^2\leq V}} \exp\left[-\frac{1}{2}\sum_{j=1}^{k}(2j-1)x_j^2\right] dx_1\dots dx_k$$

$$=2^{-k}\int_{\substack{y_1,\dots,y_k\geq 0\\ y_1+\cdots+y_k\leq V}} \exp\left[-\frac{1}{2}\sum_{j=1}^{k}(2j-1)y_j\right] \frac{dy_1\dots dy_k}{\sqrt{y_1\dots y_k}}$$

for any $V>0$. Thus we deduce that

$$\mathscr{G}_k'(V)=[\xi_1*\xi_3*\dots*\xi_{2k-1}](V)$$

for any $V>0$ and $k\geq 1$, so that

$$F_k(r)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}k} r^k\int_{\mathbb{R}} Z_{k,r}(V)\,[\xi_1*\xi_3*\dots*\xi_{2k-1}](V)\,dV$$

for all $k\geq 1$, as required. The fact that $F_k(r)$ is a smooth function of r is now immediate. Now

$$\int_{\mathbb{R}}\xi_\alpha(V)e^{-ixV}dV=\frac{\sqrt{\pi}}{\sqrt{2(\alpha+2ix)}} \qquad x\in\mathbb{R}$$

(where we take the square root with positive real part) for any $\alpha>0$, and hence $\xi_1*\xi_3*\dots*\xi_{2k-1}$ belongs to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ for all $k\geq 2$, with Fourier transform

$$[\mathscr{F}(\xi_1*\xi_3*\dots*\xi_{2k-1})](x)=\frac{\pi^{\frac{1}{2}(k-1)}}{2^{\frac{1}{2}(k+1)}\prod_{j=1}^{k}\sqrt{2j-1+2ix}} \qquad x\in\mathbb{R}.$$

Thus

$$|[\mathscr{F}(\xi_1*\xi_3*\dots*\xi_{2k-1})](x)|^2\leq\frac{\pi^{k-1}}{2^{k+1}(2k-5)!![(2k-3)^2+4x^2]},$$

and hence

$$\|\mathscr{F}(\xi_1*\xi_3*\dots*\xi_{2k-1})\|^2\leq\frac{\pi^k}{2^{k+2}(2k-3)!!},$$

so that

$$|F_k(r)|^2 \leq \left(\frac{2}{\pi}\right)^k r^{2k} \|Z_{k,r}\|^2 \|\xi_1 * \xi_3 * \ldots * \xi_{2k-1}\|^2$$

$$= \left(\frac{2}{\pi}\right)^k r^{2k} \|Z_{k,r}\|^2 \|\mathscr{F}(\xi_1 * \xi_3 * \ldots * \xi_{2k-1})\|^2$$

$$\leq \frac{r^{2k}}{4(2k-3)!!} \int_0^\infty \frac{\exp\left[-2r^2 \tanh\left(\frac{1}{2}V\right)\right]}{\cosh^{2(k+1)}\left(\frac{1}{2}V\right)} e^{kV} dV$$

$$\leq \frac{2^{2k-2} r^{2k}}{(2n-3)!!} \int_0^\infty \frac{dV}{\cosh^2\left(\frac{1}{2}V\right)}$$

$$= \frac{2^{2k-1} r^{2k}}{(2k-3)!!} .$$

for all $k \geq 2$. Moreover, we see that

$$F_1(r) = \sqrt{\frac{2}{\pi}} r \int_0^\infty \frac{\exp\left[-r^2 \tanh\left(\frac{1}{2}V\right)\right]}{\cosh^2\left(\frac{1}{2}V\right)} \frac{dV}{2\sqrt{V}}$$

$$\leq \frac{r}{\sqrt{2\pi}} \int_0^\infty \frac{dV}{\sqrt{V} \cosh^2\left(\frac{1}{2}V\right)}$$

$$\leq \frac{r}{\sqrt{2\pi}} \left[\int_0^1 \frac{dV}{\sqrt{V}} + \int_1^\infty \frac{dV}{\cosh^2\left(\frac{1}{2}V\right)}\right]$$

$$\leq \frac{4r}{\sqrt{2\pi}}$$

so we deduce that

$$|F_k(r)| \leq \frac{2^k r^k}{\sqrt{(2k-3)!!}} \qquad k \geq 1,$$

as required. ∎

This result has the following immediate and important corollary.

Corollary 4.1. *For any $f \in L^\infty(\mathbb{T})$, the distribution $\mathscr{D}(f)$ is a smooth polynomially bounded function on* Π.

Proof. Since $|\widehat{f}_k| \leq \|f\|_\infty$ for all $k \in \mathbb{Z}$, the above inequalities for $F_k(r)$ show that the series

$$[\mathscr{D}(f)](r\cos\beta, r\sin\beta) = \sum_{k \in \mathbb{Z}} \widehat{f}_k e^{ik\beta} F_{|k|}(r)$$

converges absolutely on Π, and moreover converges uniformly on $\{(a,b)\in\Pi : a^2+b^2\leq R^2\}$ for any $R>0$. Thus $\mathscr{D}(f)$ is certainly a function. It is a straightforward, but lengthy, exercise to show that similar results can be deduced about the corresponding series of partial derivatives, and hence we deduce that the function $\mathscr{D}(f)$ is smooth. Although the inequalities we have obtained above are not sufficient to show this, the fact that $\mathscr{D}(f)\in\mathscr{S}'(\Pi)$ guarantees that $\mathscr{D}(f)$ is polynomially bounded. ∎

It is intriguing to notice the connections of the above formulae with the Exp transform considered by Bayen et al. [3]. Recall that the Exp transform is the Moyal product equivalent of standard exponentiation for phase space functions, so that

$$\mathrm{Exp}_*(K)=\sum_{n\geq 0}\frac{1}{n!}(K)^n_* \tag{4.13}$$

for suitable distributions K, where $(K)^n_*$ denotes the n-fold Moyal product of K with itself, so that

$$\Delta[\mathrm{Exp}_*(K)]=\exp(\Delta[K]) \tag{4.14}$$

whenever these formulae make sense. We recall that [3]

$$\mathrm{Exp}_*\left(-\frac{1}{2}iV(p^2+q^2)\right)=\frac{1}{\cosh\left(\frac{1}{2}V\right)}\exp\left[-(p^2+q^2)\tanh\left(\frac{1}{2}V\right)\right] \tag{4.15}$$

for any $V>0$, which is equal to $Z_{0,r}(V)$ for $r^2=p^2+q^2$.

§ 5. Further Properties of the Functions F_k

The functions F_k are a measure of the difference between f_{ang} and $\mathscr{D}(f)$, and so we investigate these further, shedding more light on the relation between Toeplitz and Weyl symbols.

We have shown above that the functions F_k are polynomially bounded. In fact they satisfy further, more useful, conditions. We shall determine some of these.

Proposition 5.1. *The functions F_k obey the lower*

$$F_k(r)\geq\frac{r^k}{(r^2+2k)^{\frac{1}{2}k}},\qquad k\in\mathbb{N}, \tag{5.1}$$

and upper bounds

$$F_k(r) \leq \begin{cases} 1, & k=1, \\ \dfrac{(1+\varepsilon)^{\frac{1}{2}k}}{(1-\varepsilon)^{\frac{1}{2}k-1}} + 2^{\frac{1}{2}k+1} r^k e^{-\varepsilon r^2}, & k \geq 2, \end{cases} \tag{5.2}$$

for any $r \geq 0$ and $0 < \varepsilon < 1$. Thus we deduce that each function F_k is bounded on $[0, \infty)$ and that

$$\lim_{r\to\infty} F_k(r) = 1. \tag{5.3}$$

Proof. Since

$$\int_{\substack{y_1,\ldots,y_k \geq 0 \\ y_1+\cdots+y_k=V}} \frac{dy_1 \ldots dy_{k-1}}{\sqrt{y_1 \ldots y_k}} = \frac{\pi^{\frac{1}{2}k} V^{\frac{1}{2}k-1}}{\Gamma\left(\frac{1}{2}k\right)}$$

for all $k \geq 2$, we deduce that,

$$\frac{\pi^{\frac{1}{2}k} V^{\frac{1}{2}k-1}}{2^k \Gamma\left(\frac{1}{2}k\right)} e^{-\frac{1}{2}(2k-1)V} \leq \mathscr{G}_k'(V) \leq \frac{\pi^{\frac{1}{2}k} V^{\frac{1}{2}k-1}}{2^k \Gamma\left(\frac{1}{2}k\right)} e^{-\frac{1}{2}V}$$

for all $k \in \mathbb{N}$ and $V > 0$. We can show from this that

$$\begin{aligned} F_k(r) &\geq \frac{r^k}{2^{\frac{1}{2}k}\Gamma\left(\frac{1}{2}k\right)} \int_0^\infty \frac{\exp\left[-r^2 \tanh\left(\frac{1}{2}V\right)\right]}{\cosh^{k+1}\left(\frac{1}{2}V\right)} e^{\frac{1}{2}kV} V^{\frac{1}{2}k-1} e^{-\frac{1}{2}(2k-1)V} dV \\ &\geq \frac{r^k}{2^{\frac{1}{2}k}\Gamma\left(\frac{1}{2}k\right)} \int_0^\infty e^{-\frac{1}{2}(r^2+2k)V} V^{\frac{1}{2}k-1} dV, \qquad (k \geq 2), \end{aligned}$$

from which equation (5.1) follows, while for $k=1$ we have

$$\begin{aligned} F_1(r) &= \frac{r}{\sqrt{2\pi}} \int_0^\infty \frac{\exp\left[-r^2 \tanh\left(\frac{1}{2}V\right)\right]}{\cosh^2\left(\frac{1}{2}V\right)} \frac{dV}{\sqrt{V}} \\ &\leq \frac{r}{\sqrt{\pi}} \int_0^1 e^{-r^2 u} \frac{du}{\sqrt{u}} \leq 1. \end{aligned}$$

To prove (5.2) we use the bound

$$\begin{aligned} F_k(r) &\leq \frac{r^k}{2^{\frac{1}{2}k}\Gamma\left(\frac{1}{2}k\right)} \int_0^\infty \frac{\exp\left[-r^2 \tanh\left(\frac{1}{2}V\right)\right]}{\cosh^{k+1}\left(\frac{1}{2}V\right)} e^{\frac{1}{2}(k-1)V} V^{\frac{1}{2}k-1} dV \\ &= \frac{2r^k}{2^{\frac{1}{2}k}\Gamma\left(\frac{1}{2}k\right)} \int_0^1 e^{-r^2 U} (1+U)^{k-1} \left[\log\left(\frac{1+U}{1-U}\right)\right]^{\frac{1}{2}k-1} dU \end{aligned}$$

for any $k\geq 2$, where we have changed variable to $U=\tanh\left(\frac{V}{2}\right)$. Now

$$2U<\log\left(\frac{1+U}{1-U}\right)<\frac{2}{1-\varepsilon^2}U \qquad 0<U<\varepsilon,$$

and so for $0<U<\varepsilon$ we have

$$\int_0^\varepsilon e^{-r^2U}(1+U)^{k-1}\left[\log\left(\frac{1+U}{1-U}\right)\right]^{\frac{1}{2}k-1}dU<\frac{2^{\frac{1}{2}k-1}(1+\varepsilon)^{\frac{1}{2}k}\Gamma\left(\frac{1}{2}k\right)}{r^k(1-\varepsilon)^{\frac{1}{2}k-1}},$$

while for $\varepsilon<U<1$ we have

$$\int_\varepsilon^1 e^{-r^2U}(1+U)^{k-1}\left[\log\left(\frac{1+U}{1-U}\right)\right]^{\frac{1}{2}k-1}dU\leq 2^k e^{-\varepsilon r^2}\Gamma\left(\frac{1}{2}k\right).$$

Putting these two results together, we obtain the desired inequality. The facts that the function F_k is bounded and that $\lim_{r\to\infty}F_k(r)=1$ are now evident. ∎

We now know that F_k is bounded on $[0,\infty)$, we have an upper and a lower bound for it, and we know its limit as $r\to\infty$. To proceed further in our study of F_k, we need to know the convolution of $\mathscr{G}'_k$ with itself.

Lemma 5.1. *For any $k\in\mathbb{N}$,*

$$(\mathscr{G}'_k*\mathscr{G}'_k)(V)=\frac{\pi^k}{2^{k+1}(k-1)!}e^{-\frac{1}{2}kV}\sinh^{k-1}\left(\frac{1}{2}V\right), \qquad V>0. \qquad (5.4)$$

Proof. We note that

$$(\xi_\alpha*\xi_\alpha)(V)=\frac{1}{4}e^{-\frac{1}{2}\alpha V}\int_0^V\frac{dU}{\sqrt{U(V-U)}}=\frac{1}{4}\pi e^{-\frac{1}{2}\alpha V} \qquad V>0$$

for any $\alpha>0$, and so for $k=1$,

$$(\mathscr{G}'_1*\mathscr{G}'_1)(V)=(\xi_1*\xi_1)(V)=\frac{1}{4}\pi e^{-\frac{1}{2}V}, \qquad V>0,$$

as required. We now proceed by induction. If $k\in\mathbb{N}$ and we suppose that the above identity for $\mathscr{G}'_k*\mathscr{G}'_k$ holds, then

$$\begin{aligned}(\mathscr{G}'_{k+1}*\mathscr{G}'_{k+1})(V)&=[(\mathscr{G}'_k*\mathscr{G}'_k)*(\xi_{2k+1}*\xi_{2k+1})](V)\\&=\frac{\pi^{k+1}}{2^{k+3}(k-1)!}\int_0^V e^{-\frac{1}{2}kU}\sinh^{k-1}\left(\frac{1}{2}U\right)e^{-\frac{1}{2}(2k+1)(V-U)}dU\\&=\frac{\pi^{k+1}e^{-\frac{1}{2}(2k+1)V}}{2^{k+3}(k-1)!}\int_0^V e^U\left(\frac{e^U-1}{2}\right)^{k-1}dU\end{aligned}$$

$$=\frac{\pi^{k+1}}{2^{k+2}k!}e^{-\frac{1}{2}(k+1)V}\sinh^k\left(\frac{1}{2}V\right)$$

for all $V>0$, as required. ∎

We are going to use this lemma in an analysis of the functions F_k'. This requires us to make use of the sequence of functions

$$X_{k,\beta}(r)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}k}r^{k+\beta}\int_0^\infty\frac{\exp\left[-r^2\tanh\left(\frac{1}{2}V\right)\right]}{\cosh^{k+1}\left(\frac{1}{2}V\right)}\tanh^\beta\left(\frac{1}{2}V\right)e^{\frac{1}{2}kV}\mathscr{G}_k'(V)\,dV,$$

$$r\geq 0,\ k\in\mathbb{N},\ \beta\geq 0, \tag{5.5}$$

since $X_{k,0}=F_k$ and

$$F_k'(r)=\frac{k}{r}F_k(r)-2X_{k,1}(r)=\frac{k}{r}X_{k,0}(r)-2X_{k,1}(r). \tag{5.6}$$

We are first going to show that $F_k'\in L^2[0,\infty)$, and to that end we must consider the square-integrability of $r^{-\alpha}X_{k,\beta}(r)$ for certain values of α and β. Note that $X_{k,\beta}$ is certainly a continuous function on $[0,\infty)$.

Lemma 5.2. *If $k\in\mathbb{N}$, $\alpha\in\mathbb{R}$ and $\beta\geq 0$ are such that*

$$\frac{1}{2}-\beta<\alpha<k+\beta+\frac{1}{2}$$

then the function

$$r\mapsto r^{-\alpha}X_{k,\beta}(r)$$

belongs to $L^2[0,\infty)$, with

$$\int_0^\infty r^{-2\alpha}X_{k,\beta}(r)^2dr\leq\frac{2^{\alpha+\beta+\frac{1}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)}{(2\alpha+2\beta-1)(k-1)!}. \tag{5.7}$$

Proof. Consider the function $Y_{k,\alpha,\beta}:[0,\infty)^3\to[0,\infty)$ given by the formula

$$Y_{k,\alpha,\beta}(r,V,W)$$
$$=\left(\frac{2}{\pi}\right)^k r^{2(k+\beta-\alpha)}\frac{\exp\left[-r^2\left(\tanh\left(\frac{1}{2}V\right)+\tanh\left(\frac{1}{2}W\right)\right)\right]}{\cosh^{k+1}\left(\frac{1}{2}V\right)\cosh^{k+1}\left(\frac{1}{2}W\right)}$$
$$\tanh^\beta\left(\frac{1}{2}V\right)\tanh^\beta\left(\frac{1}{2}W\right)e^{\frac{1}{2}k(V+W)}\mathscr{G}_k'(V)\mathscr{G}_k'(W).$$

Then $Y_{k,\alpha,\beta}$ is measurable, and

$$\int_0^\infty Y_{k,\alpha,\beta}(r,V,W)\,dr$$

$$=\frac{1}{2}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^k \frac{\tanh^\beta\left(\frac{1}{2}V\right)\tanh^\beta\left(\frac{1}{2}W\right)e^{\frac{1}{2}k(V+W)}\mathscr{G}_k'(V)\mathscr{G}_k'(W)}{\left[\tanh\left(\frac{1}{2}V\right)+\tanh\left(\frac{1}{2}W\right)\right]^{k+\beta-\alpha+\frac{1}{2}}\cosh^{k+1}\left(\frac{1}{2}V\right)\cosh^{k+1}\left(\frac{1}{2}W\right)}$$

$$=\frac{1}{2}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^k \frac{\sinh^\beta\left(\frac{1}{2}V\right)\sinh^\beta\left(\frac{1}{2}W\right)e^{\frac{1}{2}k(V+W)}\mathscr{G}_k'(V)\mathscr{G}_k'(W)}{\sinh^{k+\beta-\alpha+\frac{1}{2}}\left(\frac{1}{2}(V+W)\right)\cosh^{\alpha+\frac{1}{2}}\left(\frac{1}{2}V\right)\cosh^{\alpha+\frac{1}{2}}\left(\frac{1}{2}W\right)}$$

$$\leq\frac{1}{2}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^k \frac{e^{\frac{1}{2}k(V+W)}\mathscr{G}_k'(V)\mathscr{G}_k'(W)}{\sinh^{k-\beta-\alpha+\frac{1}{2}}\left(\frac{1}{2}(V+W)\right)\cosh^{\alpha+\beta+\frac{1}{2}}\left(\frac{1}{2}V\right)\cosh^{\alpha+\beta+\frac{1}{2}}\left(\frac{1}{2}W\right)}$$

$$\leq 2^{\alpha+\beta-\frac{1}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)\frac{e^{\frac{1}{2}k(V+W)}\mathscr{G}_k'(V)\mathscr{G}_k'(W)}{\sinh^{k-\beta-\alpha+\frac{1}{2}}\left(\frac{1}{2}(V+W)\right)\cosh^{\alpha+\beta+\frac{1}{2}}\left(\frac{1}{2}(V+W)\right)}\ .$$

Thus we deduce that

$$\int_0^V\int_0^\infty Y_{k,\alpha,\beta}(r,V-W,W)\,drdW$$

$$\leq 2^{\alpha+\beta-\frac{1}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)\left(\frac{2}{\pi}\right)^k\frac{e^{\frac{1}{2}kV}(\mathscr{G}_k' * \mathscr{G}_k')(V)}{\sinh^{k-\beta-\alpha+\frac{1}{2}}\left(\frac{1}{2}V\right)\cosh^{\alpha+\beta+\frac{1}{2}}\left(\frac{1}{2}V\right)}$$

$$=\frac{2^{\alpha+\beta-\frac{3}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)}{(k-1)!}\frac{\sinh^{\alpha+\beta-\frac{3}{2}}\left(\frac{1}{2}V\right)}{\cosh^{\alpha+\beta+\frac{1}{2}}\left(\frac{1}{2}V\right)}$$

$$=\frac{2^{\alpha+\beta-\frac{3}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)}{(k-1)!}\tanh^{\alpha+\beta-\frac{3}{2}}\left(\frac{1}{2}V\right)\operatorname{sech}^2\left(\frac{1}{2}V\right)$$

for all $V>0$. Using Tonelli's Theorem, we deduce that $Y_{k,\alpha,\beta}\in L^1([0,\infty)^3)$, and that

$$\int_0^\infty\int_0^\infty\int_0^\infty Y_{k,\alpha,\beta}(r,V,W)\,drdVdW$$

$$\leq\frac{2^{\alpha+\beta-\frac{3}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)}{(k-1)!}\int_0^\infty\tanh^{\alpha+\beta-\frac{3}{2}}\left(\frac{1}{2}V\right)\operatorname{sech}^2\left(\frac{1}{2}V\right)dV$$

$$=\frac{2^{\alpha+\beta+\frac{1}{2}}\Gamma\left(k+\beta-\alpha+\frac{1}{2}\right)}{(2\alpha+2\beta-1)(k-1)!}.$$

The result now follows, since it is clear that

$$r^{-2\alpha}X_{k,\beta}(r)^2=\int_0^\infty\int_0^\infty Y_{k,\alpha,\beta}(r,V,W)\,dVdW$$

for all $r\geq 0$. ∎

We are now able to establish the square-integrability of the functions F_k'.

Proposition 5.2. *For any $k\in\mathbb{N}$ the function F_k' belongs to $L^2[0,\infty)$, and there exists a constant $A>0$ such that*

$$\int_0^\infty F_k'(r)^2dr\leq A^2k^{\frac{3}{2}} \qquad k\in\mathbb{N}. \tag{5.8}$$

Proof. Using (5.6), we have that

$$\int_0^\infty r^{-2}X_{k,0}(r)^2dr\leq\frac{2^{\frac{3}{2}}\Gamma\left(k-\frac{1}{2}\right)}{(k-1)!}, \qquad \int_0^\infty X_{k,1}(r)^2dr\leq\frac{2^{\frac{3}{2}}\Gamma\left(k+\frac{3}{2}\right)}{(k-1)!}.$$

Thus we can use Stirling's formula to deduce the existence of constants $\gamma,\delta>0$ such that

$$\int_0^\infty r^{-2}X_{k,0}(r)^2dr\leq\gamma^2k^{-\frac{1}{2}}, \qquad \int_0^\infty X_{k,1}(r)^2dr\leq\delta^2k^{\frac{3}{2}},$$

for all $k\in\mathbb{N}$. Thus we deduce that $F_k'\in L^2[0,\infty)$, with

$$\left[\int_0^\infty F_k'(r)^2dr\right]^{\frac{1}{2}}\leq k\left[\int_0^\infty r^{-2}X_{k,0}^2dr\right]^{\frac{1}{2}}+2\left[\int_0^\infty X_{k,1}(r)^2dr\right]^{\frac{1}{2}}\leq(\gamma+2\delta)k^{\frac{3}{4}},$$

so that result follows by putting $A=\gamma+2\delta$. ∎

We can sharpen this bound, including a factor of r^{-1} in the integral.

Proposition 5.3. *For any $k\geq 2$ the function $r^{-\frac{1}{2}}F_k'(r)$ belongs to $L^2[0,\infty)$, and there exists a constant $B>0$ such that*

$$\int_0^\infty r^{-1}F_k'(r)^2dr\leq B^2k \qquad k\geq 2. \tag{5.9}$$

Proof. The proof is similar to the above one, since

$$\int_0^\infty r^{-3}X_{k,0}(r)^2dr\leq\frac{2}{k-1},\qquad \int_0^\infty r^{-1}X_{k,1}(r)^2dr\leq 2k,$$

for all $k\geq 2$, and so that result follows with $B=2+2\sqrt{2}$. ∎

Thus we have seen that the functions F_k' are fairly well–behaved. Unfortunately, it is not the case that the functions F_k themselves belong to $L^2[0,\infty)$, because of their behaviour at ∞. However, it will suffice to consider the difference between the functions F_k and the constant function 1. These final results concerning the F_k will provide us with information concerning the rapidity with which they converge to 1 at infinity.

Lemma 5.3. *For any $k\geq 0$ we define functions*

$$\mathcal{M}_k(r\cos\beta,\ r\sin\beta)=e^{ik\beta}[1-F_{|k|}(r)]=i^k\sum_{M\geq 0}[g_{M,\,M+k}-1]\Phi_{M,M+k}(r\cos\beta,\ r\sin\beta).\quad(5.10)$$

and we define $\mathcal{M}_{-k}=\overline{\mathcal{M}_k}$.

For any $k\in\mathbb{Z}$ the function $\mathcal{M}_k$ belongs to $L^2(\Pi)$. Moreover, for any $k\geq 0$ the function $r^{\frac{1}{2}}(1-F_k(r))$ belongs to $L^2(0,\infty)$, and there exists $D>0$ such that

$$\frac{1}{2\pi}\|\mathcal{M}_k\|^2=\int_0^\infty r(1-F_{|k|}(r))^2dr\leq D^2|k|^{\frac{5}{4}},\qquad k\in\mathbb{Z}.\quad(5.11)$$

Proof. The coefficients

$$Y_{m,n}=\left(\frac{\min(m,n)+1}{\max(m,n)+1}\right)^{s(m,n)-\frac{3}{4}}\qquad m,n\geq 0,$$

were used in [11] to estimate the coefficients $g_{m,n}$. For our purposes here, we observe that there exists a constant $C>0$ such that

$$|g_{m,n}-Y_{m,n}|$$
$$\leq\frac{C}{\min(m,n)+1}Y_{m,n}\leq\frac{C}{\min(m,n)+1}\left(\frac{\max(m,n)+1}{\min(m,n)+1}\right)^{\frac{1}{4}}\leq\frac{C}{\min(m,n)+1}(|m-n|+1)^{\frac{1}{4}}$$

for all $m,n\geq 0$, while

$$|Y_{m,n}-1|\leq\left(\frac{\max(m,n)+1}{\min(m,n)+1}\right)^{\frac{1}{4}}-1$$
$$\leq\frac{|m-n|}{4(\min(m,n)+1)^{\frac{5}{8}}(\max(m,n)+1)^{\frac{3}{8}}}\leq\frac{1}{4}\left(\frac{|m-n|+1}{\min(m,n)+1}\right)^{\frac{5}{8}}.$$

Thus we can find $\widehat{C}>0$ such that

$$|g_{m,n}-1|\leq\widehat{C}\left(\frac{|m-n|+1}{\min(m,n)+1}\right)^{\frac{5}{8}}$$

for all $m,n\geq 0$. Thus we deduce that $\mathcal{M}_k\in L^2(\Pi)$ for all $k\in\mathbb{Z}$, with

$$\begin{aligned}\|\mathcal{M}_k\|^2&=2\pi\int_0^\infty[1-F_{|k|}(r)]^2rdr\\&=2\pi\sum_{M\geq 0}(g_{M,M+|k|}-1)^2\\&\leq 2\pi\widehat{C}^2(|k|+1)^{\frac{5}{4}}\sum_{M\geq 0}\frac{1}{(M+1)^{\frac{5}{4}}}\\&\leq 10\pi\widehat{C}^2(|k|+1)^{\frac{5}{4}}\end{aligned}$$

for all $k\in\mathbb{Z}$. Since $\mathcal{M}_0=0$, finding the constant D is now simple. ∎

We can now put these results to obtain the following further properties of the functions F_k.

Proposition 5.4. *For any $k\geq 2$, the function $1-F_k$ is bounded, with*

$$|1-F_k(r)|\leq\sqrt{2BD}k^{\frac{9}{16}},\qquad k\geq 2,\tag{5.12}$$

and

$$|1-F_k(r)|\leq\frac{\sqrt{2AD}k^{\frac{11}{16}}}{r^{\frac{1}{4}}},\qquad r>0,\ k\geq 2.\tag{5.13}$$

Proof. We observe that $(1-F_k)F_k'$ belongs to $L^1[0,\infty)$ for all $k\geq 2$, with

$$\int_0^\infty|1-F_k(r)||F_k'(r)|dr\leq\left[\int_0^\infty r[1-F_k(r)]^2dr\right]^{\frac{1}{2}}\left[\int_0^\infty r^{-1}F_k'(r)^2dr\right]^{\frac{1}{2}}\leq BDk^{\frac{9}{8}}.$$

Since

$$\frac{d}{dr}[(1-F_k(r))^2]=-2(1-F_k(r))F_k'(r)$$
$$\lim_{r\to\infty}F_k(r)=1,$$

we deduce that

$$(1-F_k(r))^2=2\int_r^\infty(1-F_k(r))F_k'(r)dr\leq 2\int_0^\infty|1-F_k(r)||F_k'(r)|dr\leq 2BDk^{\frac{9}{8}},$$

as required. Moreover we see that

$$\begin{aligned}(1-F_k(r))^2&=2\int_r^\infty(1-F_k(r))F_k'(r)dr\\&\leq 2\left[\int_r^\infty t(1-F_k(t))^2dt\right]^{\frac{1}{2}}\left[\int_t^\infty t^{-1}F_k'(t)^2dt\right]^{\frac{1}{2}}\end{aligned}$$

$$\le \frac{2}{r^{\frac{1}{2}}}\left[\int_0^\infty t(1-F_k(t))^2 dt\right]^{\frac{1}{2}}\left[\int_0^\infty F_k'(t)\,dt\right]^{\frac{1}{2}}$$

$$\le \frac{2ADk^{\frac{11}{8}}}{r^{\frac{1}{2}}}$$

for all $r>0$, as required. ■

Our above analysis has only been valid for the functions F_k with $k\geq 2$. However we can use the fact that $F_0=1$ and the inequality

$$\frac{r}{\sqrt{r^2+2}}\le F_1(r)\le 1$$

to yield the following result, which summarizes the behaviour of $1-F_k$ at large r for all k.

Corollary 5.1. *We can find constants $E, F>0$ such that*

$$|1-F_k(r)|\le Ek^{\frac{9}{16}}, \tag{5.14}$$

and

$$|1-F_k(r)|\le \frac{Fk^{\frac{11}{16}}}{r^{\frac{1}{4}}}, \tag{5.15}$$

for all $k\geq 0$ and $r>0$. ■

We note that we have shown that $1-F_k(r)=O(r^{-\frac{1}{4}})$ as $r\to\infty$. To do this we have used the fact that F_k' belongs to $L^2[0,\infty)$. In fact, for each integer k, the asymptotic behaviour of F_k as $r\to\infty$ is much better than this, as can be obtained by using Laplace's method for integrals on the original definition for F_k. However, using Laplace's method does not gives us results concerning the asymptotic behaviour of these functions expressed uniformly in the integer k, and so are not sufficiently precise for us to be able to use then in what follows.

§ 6. Dequantization of Toeplitz Operators (Continued)

Now that we have established several properties concerning the functions F_k, we are in a position to prove certain results concerning the distribution $\mathscr{D}(f)\in \mathscr{S}(\Pi)$, which is the dequantization of the Toeplitz operator $\widehat{T}(f)$ for some $f\in L^\infty(\mathbb{T})$. Recall that we have already shown that $\mathscr{D}(f)$ is a polynomially bounded function on Π. Our first result is not as strong as we would wish, but is certainly useful in some cases.

Proposition 6.1. *If $f \in L^\infty(\mathbb{T})$ is such that $(|k|^{\frac{9}{16}}\widehat{f}_k)_{k\in Z}$ belongs to $\ell^1(\mathbb{Z})$, then the distribution $\mathfrak{D}(f)$ is a bounded function on Π. If, moreover, the sequence $(|k|^{\frac{11}{16}}\widehat{f}_k)_{k\in Z}$ belongs to $\ell^1(\mathbb{Z})$, then the bounded function $\mathfrak{D}(f)$ is such that*

$$[\mathfrak{D}(f)](r\cos\beta,\ r\sin\beta) \to f(e^{i\beta}) \qquad \text{as} \qquad r \to \infty, \tag{6.1}$$

uniformly in β.

Proof. Under the first condition we observe that the distributional identity

$$[\mathfrak{D}(f)](r\cos\beta,\ r\sin\beta) = \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{ik\beta} F_{|k|}(r)$$

is an absolutely convergent series, and the boundedness of $\mathfrak{D}(f)$ is then evident. In the second case, we note that the identity

$$f(e^{i\beta}) - [\mathfrak{D}(f)](r\cos\beta,\ r\sin\beta) = \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{ik\beta}[1 - F_{|k|}(r)] = \sum_{k\in\mathbb{Z}} \widehat{f}_k \mathcal{M}_k$$

is also an absolutely convergent series, with

$$|f(e^{i\beta}) - [\mathfrak{D}(f)](r\cos\beta,\ r\sin\beta)| \leq \frac{F}{r^{\frac{1}{4}}} \sum_{k\in\mathbb{Z}} |\widehat{f}_k||k|^{\frac{11}{16}} < \infty$$

for all $r>0$, proving the second result. ∎

We note that the distributional identity for the Fourier series,

$$f(e^{i\beta}) = \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{ik\beta},$$

is absolutely convergent whenever $(\widehat{f}_k)_{k\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$ (which is true in all of the above cases), in which case f is a continuous function on $\mathbb{T}$. The condition $(\widehat{f}_k)_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ is rather strong, and we should like conditions which are somewhat weaker, but which still give us control on $\Delta[f_{\rm ang}] - \widehat{T}(f)$. The next result provides such a condition.

Proposition 6.2. *If $f \in L^\infty(\mathbb{T})$ is such that $(|k|^{\frac{5}{8}}\widehat{f}_k)_{k\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$, then the function $f_{\rm ang} - \mathfrak{D}(f)$ belongs to $L^2(\Pi)$, and hence the operator $\Delta[f_{\rm ang}] - \widehat{T}(f)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{Z})$.*

Proof. This result is an immediate consequence of the identity

$$f_{\rm ang}-\mathscr{D}(f)=\sum_{k\in\mathbb{Z}}\widehat{f}_k\mathscr{M}_k$$

and the fact that the functions $\{\mathscr{M}_k:k\in\mathbb{Z}\}$ form an orthogonal sequence in $L^2(\Pi)$, with the already-established upper bounds on their norms. The fact that the Weyl quantization of an element of $L^2(\Pi)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$ is standard. ■

The condition in this last Proposition is considerably weaker than the one used previously, but is still sufficiently strong to ensure that the function f is continuous on $\mathbb{T}$, and that the Fourier series for f converges absolutely. Although this result implies that $\mathscr{D}(f)$ tends towards $f_{\rm ang}$ in some sense as $r\to\infty$, we cannot automatically deduce pointwise convergence.

If we wish to investigate further the properties of $\mathscr{D}(f)$ as $r\to\infty$, it is sensible to introduce the following definition. For any $f\in L^\infty(\mathbb{T})$ and any $r>0$, define the function $\mathscr{D}^{(r)}(f)\in\mathscr{C}(\mathbb{T})$ by setting

$$[\mathscr{D}^{(r)}(f)](e^{i\beta})=[\mathscr{D}(f)](r\cos\beta,\ r\sin\beta)\qquad e^{i\beta}\in\mathbb{T},\tag{6.2}$$

thereby extracting the angular dependence. We would like to find out results concerning the convergence of the functions $\mathscr{D}^{(r)}(f)$ as $r\to\infty$.

Proposition 6.3. *If $f\in L^\infty(\mathbb{T})$ is such that the sequence $(|k|^{\frac{11}{16}}\widehat{f}_k)_{k\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$, then*

$$\lim_{r\to\infty}\|f-\mathscr{D}^{(r)}(f)\|=0\tag{6.3}$$

Proof. This follows since

$$\|f-\mathscr{D}^{(r)}(f)\|^2=\sum_{k\in\mathbb{Z}}|\widehat{f}_k|^2(F_{|k|}(r)-1)^2\leq\frac{F^2}{r^{\frac{1}{2}}}\sum_{k\in\mathbb{Z}}|k|^{\frac{11}{8}}|\widehat{f}_k|^2$$

for all $r>0$. ■

Thus, if we are satisfied with convergence with respect to the $L^2(\mathbb{T})$ norm, this result gives us what we would like. It would, however, be interesting to obtain details concerning the pointwise convergence of these functions, but as yet we have not been able to establish suitable results.
However, it is possible to obtain results of a more distributional nature which do not require any special conditions, and hence hold for all functions $f\in L^\infty(\mathbb{T})$. Thus we shall see that there is always a sense in which $\mathscr{D}^{(r)}(f)$ converges to f as $r\to\infty$.

Proposition 6.4. *For any $f \in L^\infty(\mathbb{T})$ we have that*

$$\lim_{r\to\infty} \mathfrak{D}^{(r)}(f)_{\mathrm{ang}} = f_{\mathrm{ang}} \tag{6.4}$$

where convergence is with respect to the weak topology in $\mathcal{S}'(\Pi)$.

Proof. For any $G \in \mathcal{S}(\Pi)$ we have

$$G = \sum_{M,N\geq 0} G_{M,N}\Phi_{M,N},$$

and hence

$$\int_0^\infty G(r\cos\beta,\ r\sin\beta)rdr = \sum_{M,N\geq 0} G_{M,N} i^{M-N} e^{i(N-M)\beta} g_{M,N} = \sum_{k\in\mathbb{Z}} \gamma_k e^{ik\beta},$$

where the sequence (γ_k) is such that

$$\sum_{k\in\mathbb{Z}} |k|^s |\gamma_k| < \infty$$

for all $s \geq 0$. Since we have that

$$[f_{\mathrm{ang}} - \mathfrak{D}^{(r)}(f)_{\mathrm{ang}}, G] = 2\pi \sum_{k\in\mathbb{Z}} \gamma_{-k} \hat{f}_k (F_{|k|}(r) - 1)$$

we deduce that

$$\left| [f_{\mathrm{ang}} - \mathfrak{D}^{(r)}(f)_{\mathrm{ang}}, G] \right| \leq \frac{2\pi F}{r^{\frac{1}{4}}} \sum_{k\in\mathbb{Z}} |k|^{\frac{11}{16}} |\gamma_{-k}| |\hat{f}_k|$$

for any $r > 0$, which implies the required result. ∎

§ 7. Comments

The results of Propositions 6.1, 6.2, 6.3 and 6.4 all, in their slightly different ways, indicate how each function $\mathfrak{D}(f)$ is a deformation of the phase space function f_{ang} obtained from a given function $f \in L^\infty(\mathbb{T})$. While Proposition 6.4 is sufficiently general to handle all possible types of function f, the result is the least concrete of the four. Unfortunately, only this last result is sufficiently weak to deal with the function $f = \Theta$, about which we would most like to have information, since this in turn would inform us concerning the relationship between the Weyl quantization $\Delta[\varphi]$ and the Toeplitz operator X.

However, there are a couple of points which are worth noting in particular. In the first case, if the function f is a trigonometric polynomial, so that the

Fourier series expansion

$$f(e^{i\beta}) = \sum_{k\in\mathbb{Z}} \widehat{f}_k e^{ik\beta} \tag{7.1}$$

is in fact a finite series, then it is clear that the function f satisfies the conditions of all four of Propositions 6.1, 6.2, 6.3 and 6.4, and so $\mathfrak{D}(f)$ is a bounded function on Π which converges to f as r tends to infinity uniformly in β. Moreover the difference function $\mathfrak{D}(f) - f_{\mathrm{ang}}$ belongs to $L^2(\Pi)$, and so on. It should be noted in particular that this result holds for angular functions such as

$$e^{ik\beta} \qquad (k\in\mathbb{Z}) \qquad \text{and} \qquad P_N(\cos\beta) \qquad (N\geq 0).$$

In another direction, suppose that the real-valued function f is such that the conditions of Proposition 6.2 are satisfied. Then we see that $\Delta[f_{\mathrm{ang}}] - \widehat{T}(f)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$, and so $\Delta[f_{\mathrm{ang}}]$ is a deformation of the Toeplitz operator $\widehat{T}(f)$ by a Hilbert-Schmidt, and therefore compact, operator. Moreover, the Toeplitz operator $\widehat{T}(f)$ is self-adjoint (since f is real-valued), and its spectral theory is well-understood. We can therefore apply standard theorems to obtain some information about the spectral properties of such operators $\Delta[f_{\mathrm{ang}}]$. The conditions of Proposition 6.2 are such that the sequence of Fourier coefficients $(\widehat{f}_k)_{k\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$, and so it follows that f is a continuous function on $\mathbb{T}$. Thus we deduce that the spectrum of $\widehat{T}(f)$ is equal to the range of the function f — if f is not constant almost everywhere, then $\widehat{T}(f)$ has no point spectrum, and so the spectrum, of $\widehat{T}(f)$ is equal to its continuous spectrum. Using a theorem of Weyl [1], we deduce that the continuous spectrum of $\Delta[f_{\mathrm{ang}}]$ is the same as the continuous spectrum of $\widehat{T}(f)$, and hence is contained in the range of f. Unfortunately, as also mentioned in [1], simply identifying $\Delta[f_{\mathrm{ang}}]$ as a compact deformation of $\widehat{T}(f)$ does not give us any information concerning the point spectrum of $\Delta[f_{\mathrm{ang}}]$.

In this paper we have concentrated on a few particular applications of our dequantization technique. It would seem clear that this technique is a powerful one, which could be used in a number of other contexts. We hope, in later work, to report on other applications.

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