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Higher-Codimensional Boundary Value Problems and F-Mild Microfunctions —Local and Microlocal Uniqueness—

By

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Abstract

For the study of local and microlocal boundary value problems with a boundary of codimension greater than one, sheaves of F-mild hyperfunctions and F-mild microfunctions are introduced. They are refinements of the notions of hyperfunctions and microfunctions with real analytic parameters and have natural boundary values. F-mild solutions of a general $\mathscr{D}$-Module $\mathscr{M}$ (that is, a system of linear partial differential equations with analytic coefficients) are considered. In particular, local and microlocal uniqueness in the boundary value problem (the Holmgren type theorem) is proved if the boundary is non-characteristic for $\mathscr{M}$, or else if $\mathscr{M}$ is Fuchsian along the boundary.

Introduction

The purpose of this paper is to study the higher-codimensional boundary value problem for a general system of linear partial differential equations with analytic coefficients. In general, we must impose some regularity condition on the solutions in order to define their boundary values. We introduce the notion of F-mild hyperfunctions as this regularity condition, which is a refinement of that of hyperfunctions with real analytic parameters. We also define the notion of F-mild microfunctions as a microlocalization of that of F-mildness. Our main result is the local and microlocal uniqueness of F-mild hyperfunction (or microfunction) solutions of a system of linear partial differential equations which is Fuchsian along Y in the sense of Y. Laurent and T. Monteiro Fernandes [L-MF] or in the sense of N. S. Madi [M] and S. Yamazaki [Y].

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Let M be a real analytic manifold and N a closed real analytic submanifold of M of codimension $d \geq 2$. Then the sheaf $\mathscr{B}_{N|M}^F$ of F-mild hyperfunctions is defined on the normal bundle $T_N M$ of N (strictly speaking, the sheaf $\mathscr{B}_{N|M}^F$ depends on a partial complexification L of M). Let us take a local coordinate system $(t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n)$ of M such that N is defined by $t=0$. Assume that f is a section of $\mathscr{B}_{N|M}^F$ (that is, an F-mild hyperfunction) defined on a neighborhood of $0 + \partial/\partial t_1 \in T_N M$. Then f is actually regarded as a hyperfunction defined on a wedge domain

$$\{(t, x) \in \mathbb{R}^d \times \mathbb{R}^n;\ |t| < \varepsilon,\ |x| < \varepsilon,\ |t_j| < \varepsilon t_1\ (2 \leqslant j \leqslant d)\}$$

with edge N for some $\varepsilon > 0$. In addition, for any non-negative integers $\alpha_1, \ldots, \alpha_d$, $\partial_{t_1}^{\alpha_1} \cdots \partial_{t_d}^{\alpha_d} f(t, x)$ has a natural boundary value as t tends to zero as a hyperfunction of x.

The restriction of $\mathscr{B}_{N|M}^F$ to the zero-section of $T_N M$ coincides with the sheaf of hyperfunctions defined on a neighborhood of N which have t as real analytic parameters. Moreover, a section of $\mathscr{B}_{N|M}^F$ which is defined on $T_N M$ with the zero-section removed has also t as real analytic parameters on a neighborhood of N. Hence we may regard $\mathscr{B}_{N|M}^F$ as a tangential decomposition of the sheaf of hyperfunctions which have t as real analytic parameters.

We take complexifications X and Y of M and N respectively such that Y is a closed submanifold of X. We denote by $\mathscr{D}_X$ the sheaf on X of rings of linear partial differential operators (of finite order) with holomorphic coefficients.

Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-Module; that is, a system of linear partial differential equations with holomorphic coefficients (in this paper, we shall write Module with a capital letter, instead of sheaf of modules).

First, let us assume that Y is non-characteristic for $\mathscr{M}$. Then we prove that any hyperfunction solution to $\mathscr{M}$ which is defined on a wedge domain with edge N is F-mild, thus having boundary values with no further assumption. This case was studied by P. Schapira ([Sc 1], [Sc 2]) by using the theory of microlocalization of sheaves. The local uniqueness in this boundary value problem was proved in T. Oaku [O 4]. K. Takeuchi [Tk] formulated microlocal boundary value problem by using the theory of second microlocalization and proved the microlocal uniqueness in the non-characteristic case. Here we give another proof to the microlocal uniqueness by a natural extension of the method used in Oaku [O 4].

Next, suppose that $\mathscr{M}$ is Fuchsian along Y in the sense of Laurent and Monteiro Fernandes [L-MF]. In this case, not all the hyperfunction solutions to $\mathscr{M}$ are necessarily F-mild, but we can obtain the local and microlocal uniqueness for F-mild solutions. More precisely, we obtain a monomorphism (an injective sheaf homomorphism)

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}^F_{N|M}) \rightarrowtail \tau_N^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N),$$

where τ_N: $T_N M \to N$ is the canonical projection, $\mathscr{B}_N$ is the sheaf of hyperfunctions on N, and $\mathscr{M}_Y$ is the induced system (that is, the $\mathscr{D}$-Module theoretic restriction) of $\mathscr{M}$ to Y, which is a coherent $\mathscr{D}_Y$-Module. We can also obtain the microlocalization of this morphism, which is also injective.

Finally assume that $\mathscr{M}$ is a Fuchs-Goursat system in the sense of Yamazaki [Y], which is a generalization of a Fuchs-Goursat operator due to Madi [M]. In this case, since $\mathscr{M}_Y$ is not coherent over $\mathscr{D}_Y$ in general, we consider a kind of Goursat problem: Set $M_i = \{(t, x) \in \mathbb{R}^d \times \mathbb{R}^n;\ t_i = 0\}$ by using a local coordinate system as mentioned above. For an F-mild hyperfunction, we can define its restriction to M_i for $1 \leqslant i \leqslant d$, which can be regarded as Goursat data. Thus we prove the local and microlocal uniqueness of the F-mild solution to $\mathscr{M}$ whose Goursat data are zero. Note that Yamazaki [Y] proved the (micro-)local solvability of this Goursat problem for microfunctions with real analytic parameters under a kind of (micro-)hyperbolicity condition.

We should remark the following: The higher-codimensional boundary value problem for hyperfunctions was initiated by M. Kashiwara and T. Kawai [K-K] for elliptic systems of differential equations from the microlocal point of view. After that, M. Kashiwara and T. Oshima ([K-Os], [Os]) defined the boundary values of an arbitrary hyperfunction solution of $\mathscr{M}$ which is defined in $\{(t, x) \in \mathbb{R}^d \times \mathbb{R}^n;\ t_i > 0\ (1 \leqslant i \leqslant d)\}$ under a condition stronger than that of Fuchsian system in the sense of Laurent-Monteiro Fernandes [L-MF].

In Section 1, we assume the existence of a partial complexification L of M and introduce several sheaves attached to the boundary, which are higher-codimensionl analogues of those defined in Oaku [O 3].

Section 2 is devoted to concrete expressions of these sheaves.

In Section 3, also assuming the existence of L, we introduce the sheaf of F-mild hyperfunctions. One of the main results in this section is the edge of the wedge theorem, which gives a criterion for an F-mild hyperfunction to become zero in terms of its expression as a sum of boundary values of holomorphic functions. Note that the results of Section 3 were essentially stated in Oaku [O 4]. The main difference is that we use the notion of normal deformation (cf. Kashiwara-Schapira [K-S 2]) here instead of the real monoidal transform adopted in Oaku [O 4].

In Section 4, we microlocalize the notion of F-mildness. In particular, we prove that the sheaf of microfunctions with real analytic parameters can be embedded to the sheaf of F-mild microfunctions. We also give a concrete characterization of the singularity spectrum of an F-mild hyperfunction by using holomorphic functions which define it.

Section 5 is concerned with the non-characteristic boundary value problem. We prove that all the hyperfunction solutions defined on a wedge domain with edge N are F-mild.

In Section 6, we consider the higher-codimensional boundary value problem for a system which is Fuchsian along Y in the sense of Laurent-Monteiro Fernandes [L-MF], and the Goursat problem for a Fuchs-Goursat system in the sense of Madi [M] and Yamazaki [Y]. Main results are the local and microlocal uniqueness of F-mild solutions of both problems.

§1. Several Sheaves Attached to the Boundary

In this section, we introduce several sheaves attached to the higher-codimensional boundary as a natural extension of the one-codimensional case in Oaku [O 3].

We denote the sets of integers, real numbers and complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Further, we set $\mathbb{N}:=\{n\in\mathbb{Z};\ n\geqslant 1\}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

Let M be a $(d+n)$-dimensional real analytic manifold and N a n-dimensional closed real analytic submanifold of M. In this paper, we always assume that $d\geqslant 2$. There exist complexifications X and Y of M and N respectively such that Y is a closed submanifold of X. We assume that there exists a $(d+2n)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to the triplet $(\{0\}\times\mathbb{R}^n, \mathbb{R}^{d+n}, \mathbb{R}^d\times\mathbb{C}^n)$ by a local coordinate system (τ, z) of X around each point of N. We say such a local coordinate system *admissible*. We use the notation $\tau=t+\sqrt{-1}\,s\,(t, s\in\mathbb{R}^d)$, $z=x+\sqrt{-1}\,y\,(x, y\in\mathbb{R}^n)$, $|z|=\max\{|z_k|;\ 1\leqslant k\leqslant n\}$ and so on for an admissible local coordinate system (τ, z). Hence by an admissible local coordinate system the following inclusion relations are obtained:

$$\begin{tikzcd}
N=\{0\}\times\mathbb{R}^n_x \arrow[r, hook] \arrow[d, hook] & M=\mathbb{R}^d_t\times\mathbb{R}^n_x \arrow[d, hook, "i"'] \arrow[dr, hook, "i_M"] & \\
Y=\{0\}\times\mathbb{C}^n_z \arrow[r, hook, "i_Y"] & L=\mathbb{R}^d_t\times\mathbb{C}^n_z \arrow[r, hook, "i_L"] & X=\mathbb{C}^d_\tau\times\mathbb{C}^n_z.
\end{tikzcd}$$

We shall mainly follow the notation of Kashiwara-Schapira [K-S 2]; we denote the normal deformations of N and Y in M and L by $\widetilde{M}_N$ and $\widetilde{L}_Y$ respectively. For example, by an admissible coordinate system, we see that $\widetilde{M}_N=\{(r, t, x);\ r\in\mathbb{R}, (r, t, x)\in M\}$, $\Omega_M=\widetilde{M}_N\cap\{(r, t, x);\ r>0\}$, $T_NM\simeq\widetilde{M}_N\cap\{(r, t, x);\ r=0\}$ and p_M: $\widetilde{M}_N\ni(r, t, x)\mapsto(r, t, x)\in M$. Then, we can regard $\widetilde{M}_N$ as a submanifold of $\widetilde{L}_Y$. Therefore we have the following commutative diagram:

$$\begin{tikzcd}
T_NM \arrow[rr, hook, "s_M"] \arrow[dr, "\tau_N"] \arrow[dd, hook, "i'"'] & & \widetilde{M}_N \arrow[dr, "p_M"] \arrow[dd, "\tilde{i}'"] & & \Omega_M \arrow[ll, hook', "j_M"'] \arrow[dl, "\tilde{p}_M"'] \arrow[dd, "\tilde{i}"] & \\
& N \arrow[rr] \arrow[dd, hook, "i|_N"] & & M \arrow[rr, "i_M"] \arrow[dd, hook, "i"] & & X \arrow[dd, equal] \\
T_YL \arrow[rr, "s_L"] \arrow[dr, "\tau_Y"] & & \widetilde{L}_Y \arrow[dr, "p_L"] & & \Omega_L \arrow[ll, "j_L"'] \arrow[dl, "\tilde{p}_L"'] & \\
& Y \arrow[rr, hook, "i_Y"] & & L \arrow[rr, hook, "i_L"] & & X.
\end{tikzcd}$$

Moreover, we have the following:

$$\begin{tikzcd}
T_NM \arrow[r, hook, "s_M"] \arrow[d, hook, "i'"] \arrow[dr, phantom, "\square"] & \widetilde{M}_N \arrow[d, hook, "\tilde{i}'"] \arrow[dr, phantom, "\square"] & \Omega_M \arrow[l, hook', "j_M"'] \arrow[r, "\tilde{p}_M"] \arrow[d, hook, "\tilde{i}"] \arrow[dr, phantom, "\square"] & M \arrow[d, hook, "i"] \\
T_YL \arrow[r, hook, "s_L"] & \widetilde{L}_Y & \Omega_L \arrow[l, hook', "j_L"'] \arrow[r, "\tilde{p}_L"] & L,
\end{tikzcd}$$

where we mean by $\square$ the square is *Cartesian*. We denote by $\mathscr{C}_M$ and $\mathscr{B}_M$ the sheaf of *microfunctions* on T_M^*X and that of *hyperfunctions* on M respectively as usual. Further we denote by $\mathscr{C}\mathscr{O}_L$ and $\mathscr{B}\mathscr{O}_L$ the sheaf of *microfunctions with holomorphic parameters* on T_L^*X and the sheaf of *hyperfunctions with holomorphic parameters* on L respectively. In particular

$$\mathscr{B}\mathscr{O}_L := \mathscr{H}_L^d(\mathscr{O}_X) \otimes or_{L/X} \simeq \boldsymbol{R}\Gamma_L(\mathscr{O}_X) \otimes or_{L/X}[d],$$

where $\mathscr{O}_X$ is *the sheaf of holomorphic functions* on X and $or_{L/X}$ is *the relative orientation sheaf* with respect to $i_L\colon L \to X$.

1.1 Lemma. *For any non-zero integer k, the following equalities hold:*

$$R^k(j_L)_* \tilde{p}_L^{-1} \mathscr{B}\mathscr{O}_L = 0,$$
$$R^k(j_M)_* \tilde{p}_M^{-1} \mathscr{B}_M = 0.$$

Proof. We may assume that $X = \mathbb{C}_\tau^d \times \mathbb{C}_z^n$, $L = \mathbb{R}_t^d \times \mathbb{C}_z^n$ and $Y = \{0\} \times \mathbb{C}_z^n$. By an argument similar to the proof of Theorem 4.2.3 of Kashiwara-Schapira [K-S 2], for any $z^* \in \widetilde{L}_Y$ we have

$$(R^k(j_L)_* \tilde{p}_L^{-1} \mathscr{B}\mathscr{O}_L)_{z^*} \simeq \varinjlim_W H^k(\tilde{p}_L(W \cap \Omega_L); \mathscr{B}\mathscr{O}_L),$$

where W ranges through a fundamental neighborhood system of z^* in $\widetilde{L}_Y$. We may assume that $\tilde{p}_L(W \cap \Omega_L)$ is the product of an open set of $\mathbb{R}^d$ and a Stein open set of $\mathbb{C}^n$. Thus we can see that

$$H^k(\widetilde{p}_L(W\cap\Omega_L);\mathscr{B}\mathscr{O}_L)=0$$

by the well-known property of $\mathscr{B}\mathscr{O}_L$. Hence the first equality is proved. The proof of the second equality is similar. □

1.2 Definition. We set:

$$\mathscr{B}\mathscr{O}_{Y|L}:=(j_L)_*\widetilde{p}_L^{-1}\mathscr{B}\mathscr{O}_L\simeq\mathbb{R}(j_L)_*\widetilde{p}_L^{-1}\mathscr{B}\mathscr{O}_L,$$
$$\mathscr{B}\mathscr{O}_{Y,L}:=s_L^{-1}\mathscr{B}\mathscr{O}_{Y,L}\simeq\nu_Y(\mathscr{B}\mathscr{O}_L),$$

where ν_Y denotes *the specialization functor* along Y. Thus $\mathscr{B}\mathscr{O}_{Y,L}$ and $\mathscr{B}\mathscr{O}_{Y|L}$ are sheaves on $\widetilde{L}_Y$ and T_YL respectively.

Note that by the definition, $\mathscr{B}\mathscr{O}_{Y,L}$ and $\mathscr{B}\mathscr{O}_{Y|L}$ are conic sheaves.

1.3 Proposition. *Suppose that $X=\mathbb{C}^{d+n}$, $L=\mathbb{R}^d\times\mathbb{C}^n$, $M=\mathbb{R}^{d+n}$ and $Y=\{0\}\times\mathbb{C}^n$. Identify $\widetilde{L}_Y$ with $\mathbb{R}_r\times\mathbb{R}^d_t\times\mathbb{C}^n_z$. Then:*

(1) *Suppose that open sets V of U of $\mathbb{R}$ and $\mathbb{R}^d$ satisfy the following: The mapping $V_{>0}\times U\ni(r,t)\mapsto rt\in\mathbb{R}^d$ has connected fibers, where $V_{>0}:=\{r\in V;\ r>0\}$.*
Then for any Stein open set Ω of $\mathbb{C}^n$ and $k\neq0$, it follows that

$$H^k(V\times U\times\Omega;\mathscr{B}\mathscr{O}_{Y,L})=0.$$

(2) *Set $\widetilde{\gamma}$: $\widetilde{L}_Y\to\widetilde{L}_Y/\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}:=\{c\in\mathbb{R};\ c>0\}$. Then the flabby dimension of $\widetilde{\gamma}_*\mathscr{B}\mathscr{O}_{Y,L}$ is equal to n.*

(3) *For any proper convex closed cone G of $\mathbb{R}^n$ and $k\neq n$,*

$$\mathscr{H}^k_{\mathbb{R}^{d+1}\times(\mathbb{R}^n+\sqrt{-1}G)}(\mathscr{B}\mathscr{O}_{Y,L})|_{\mathbb{R}^{n+d+1}}=0.$$

1.4 Corollary. *The complex $\mu_{\widetilde{M}_N}(\mathscr{B}\mathscr{O}_{Y,L})$ is concentrated in degree n, where $\mu_{\widetilde{M}_N}$ denotes the microlocalization functor along $\widetilde{M}_N$.*

Proof of Proposition 1.3. (1) Set $V_{>0}U:=\{rt\in\mathbb{R}^d;\ r\in V_{>0},\ t\in U\}$. Then by the definition, we have

$$H^k(V\times U\times\Omega;\mathscr{B}\mathscr{O}_{Y,L})\simeq H^k(V_{>0}U\times\Omega;\mathscr{B}\mathscr{O}_L).$$

Thus the proof of (1) is reduced to the corresponding property of $\mathscr{B}\mathscr{O}_L$.

(2) follows from the following fact: (i) the flabby dimension of $\mathscr{B}\mathscr{O}_L$ is equal to n, (ii) if F is a flabby sheaf on L, then $(j_L)_*\widetilde{p}_L^{-1}F$ is a conically flabby sheaf on $\widetilde{L}_Y$.

(3) By (2) we have for $k>n$

$$\mathscr{H}^k_{\mathbb{R}^{d+1}\times(\mathbb{R}^n+\sqrt{-1}G)}(\mathscr{B}\mathscr{O}_{Y,L})|_{\mathbb{R}^{n+d+1}}=0.$$

In order to prove (3) for the case where $k<n$, we use the following abstract

edge of the wedge theorem due to Kashiwara-Laurent [K-L]:

1.5 Lemma (Théorème 1.4.1 of [K-L]). *Let T be a topological space. Suppose that there exists a contravarient functor which associates with each complex manifold W a sheaf $\mathscr{F}_W$ on $T\times W$ such that:*

(H 0) *$\mathscr{F}_W$ is a $p^{-1}\mathscr{O}_W$-Module, where p: $T\times W\to W$ denotes the first projection.*

(H 1) *If V is an open subset of T and $U'\subset U$ are open subsets of W such that U is connected and U' is not empty, then it follows that*

$$\Gamma_{V\times(U\setminus U')}(V\times U;\mathscr{F}_W)=0.$$

(H 2) *Let f: $W\to\mathbb{C}$ be a holomorphic function with $df\neq 0$ on W and set $Z:=f^{-1}(0)$. Let i: $Z\to W$ be the canonical imbedding. Then there exists an exact sequence:*

$$0\longrightarrow\mathscr{F}_W\xrightarrow{f\cdot}\mathscr{F}_W\xrightarrow{i^*}i_*\mathscr{F}_Z\longrightarrow 0.$$

(H 3) *Let W and Z be complex manifolds with Z compact. Let f be the canonical projection of $T\times W\times Z$ to $T\times W$. Then for any integer k the following holds:*

$$R^kf_*\mathscr{F}_{W\times Z}\simeq\mathscr{F}_W\underset{\mathbb{C}}{\otimes}H^k(Z;\mathscr{O}_Z).$$

In addition to these conditions, suppose that G is a closed convex set of $\mathbb{C}^n$ containing z, and that there does not exist $\mathbb{C}$-linear subvariety Γ of $\mathbb{C}^n$ of dimension $n-q+1$ containing z, such that $\Gamma\cap G$ is a neighborhood of z in Γ. Then for any $t\in T$ and $k<q$ the following holds:

$$\mathscr{H}^k_{T\times G}(\mathscr{F}_{\mathbb{C}^n})_{(t,z)}=0.$$

End of proof of Proposition 1.3. Let us identify the normal deformation of W in $\mathbb{R}^d\times W$ with $\mathbb{R}^{d+1}\times W$ for any complex manifold W. Let j_W be a natural inclusion $\mathbb{R}_{>0}\times\mathbb{R}^d\times W\to\mathbb{R}^{d+1}\times W$ and $\tilde{p}_W$ a mapping $\mathbb{R}_{>0}\times\mathbb{R}^d\times W\ni(r,t,w)\mapsto(rt,w)\in\mathbb{R}^d\times W$. Let us set

$$\mathscr{F}_W:=(j_W)_*\tilde{p}_W^{-1}\mathscr{B}\mathscr{O}_{\mathbb{R}^d\times W}.$$

Then it is easy to see that $W\mapsto\mathscr{F}_W$ defines a contravariant functor. Hence it suffices to verify (H 0)-(H 3) of Lemma 1.5 for this $\mathscr{F}_W$ with $T=\mathbb{R}^{d+1}$. (H 0) is trivial. (H 1) follows from the unique continuation property of sections of $\mathscr{B}\mathscr{O}$ with respect to holomorphic parameters. Let Z be as in (H 2). Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^{d+1}\times Z \arrow[d, hook, "\tilde{i}'"] \arrow[r, phantom, "\square", shift right=20] & \mathbb{R}_{>0}\times\mathbb{R}^d\times Z \arrow[l, hook', "j_Z"'] \arrow[r, "\tilde{p}_Z"] \arrow[d, hook, "\tilde{i}"] & \mathbb{R}^d\times Z \arrow[r, hook, "i_Z"] \arrow[d, hook, "i"] & \mathbb{C}^d\times Z \arrow[d, hook, "i"] \\
\mathbb{R}^{d+1}\times W & \mathbb{R}_{>0}\times\mathbb{R}^d\times W \arrow[l, hook', "j_W"'] \arrow[r, "\tilde{p}_W"] & \mathbb{R}^d\times W \arrow[r, hook, "i_W"] & \mathbb{C}^d\times W.
\end{tikzcd}$$

Hence by applying the functor $\mathbb{R}(j_W)_* \tilde{p}_{\bar{W}}^{-1} i_W^!(*)[d]$ to the distinguished triangle

$$\mathcal{O}_{\mathbb{C}^d\times W} \xrightarrow{f\cdot} \mathcal{O}_{\mathbb{C}^d\times W} \xrightarrow{i^*} \mathbb{R}i_*\mathcal{O}_{\mathbb{C}^d\times Z} \xrightarrow{+1},$$

we have

$$\mathcal{F}_W \xrightarrow{f\cdot} \mathcal{F}_W \xrightarrow{i^*} \mathbb{R}(j_W)_* \tilde{p}_{\bar{W}}^{-1} i_W^! \mathbb{R}i_* \mathcal{O}_{\mathbb{C}^d\times Z}[d] \xrightarrow{+1}.$$

Then, it follows that

$$\begin{aligned}
&\mathbb{R}(j_W)_* \tilde{p}_{\bar{W}}^{-1} i_W^! \mathbb{R}i_* \mathcal{O}_{\mathbb{C}^d\times Z}[d] \\
&\quad\simeq \mathbb{R}(j_W)_* \tilde{p}_{\bar{W}}^{-1} i_W^! \mathbb{R}i_* i_Z^! \mathcal{O}_{\mathbb{C}^d\times Z}[d] \simeq \mathbb{R}(j_W)_* \tilde{p}_{\bar{W}}^{-1} \mathbb{R}i_* \mathcal{B}\mathcal{O}_{\mathbb{R}^d\times Z}[-1] \\
&\quad\simeq \mathbb{R}(j_W)_* \mathbb{R}\tilde{i}_* \tilde{p}_Z^! \mathcal{B}\mathcal{O}_{\mathbb{R}^d\times Z}[-1] \simeq \mathbb{R}\tilde{i}'_* \mathbb{R}(j_Z)_* \tilde{p}_{\bar{Z}}^{-1} \mathcal{B}\mathcal{O}_{\mathbb{R}^d\times Z} \\
&\quad\simeq \mathbb{R}\tilde{i}'_* \mathcal{F}_Z.
\end{aligned}$$

Hence we have

$$0 \longrightarrow \mathcal{F}_W \xrightarrow{f\cdot} \mathcal{F}_W \xrightarrow{i^*} \tilde{i}'_* \mathcal{F}_Z \longrightarrow 0.$$

Therefore (H 2) is verified. Lastly let us verify (H 3). Let W and Z be as in (H 3). Then, it is well-known that

$$\mathbb{R}f_* \mathcal{O}_{\mathbb{C}^d\times W\times Z} \simeq \mathcal{O}_{\mathbb{C}^d\times W} \underset{\mathbb{C}}{\otimes} \mathbb{R}\Gamma(Z; \mathcal{O}_Z).$$

Thus by the same argument as in (H 2) we have

$$\mathbb{R}f_* \mathcal{F}_{W\times Z} \simeq \mathcal{F}_W \underset{\mathbb{C}}{\otimes} \mathbb{R}\Gamma(Z; \mathcal{O}_z).$$

Thus (H 3) is verified. Since $\mathbb{R}^n + \sqrt{-1}\, G$ does not contain non-zero $\mathbb{C}$-linear subspace, (2) follows from Lemma 1.5. The proof is complete. □

By Corollary 1.4, we can define several sheaves as follows:

1.6 Definition. We set:

$\mathcal{C}_{N,M} := \mathcal{H}^n(\mu_{\tilde{M}_N}(\mathcal{B}\mathcal{O}_{Y,L}) \otimes or_{M/L}) \simeq \mu_{\tilde{M}_N}(\mathcal{B}\mathcal{O}_{Y,L}) \otimes or_{M/L}[n]$,

$\mathcal{B}_{N,M} := \mathcal{C}_{N,M}|_{\tilde{M}_N} = \mathcal{H}^n_{\tilde{M}_N}(\mathcal{B}\mathcal{O}_{Y,L}) \otimes or_{M/L} \simeq \mathbb{R}\Gamma_{\tilde{M}_N}(\mathcal{B}\mathcal{O}_{Y,L}) \otimes or_{M/L}[n]$,

$\tilde{\mathcal{A}}_{N,M} := \mathcal{H}^0(\nu_{\tilde{M}_N}(\mathcal{B}\mathcal{O}_{Y,L}))$.

Thus $\mathcal{C}_{N,M}$, $\mathcal{B}_{N,M}$ and $\tilde{\mathcal{A}}_{N,M}$ are sheaves on $T^*_{\tilde{M}_N}\tilde{L}_Y$, $\tilde{M}_N$ and $T_{\tilde{M}_N}\tilde{L}_Y$ respectively. Note that by Proposition 1.3 (2), $\mathcal{B}_{N,M}$ is conically flabby.

1.7 Proposition. *Suppose that* $X = \mathbb{C}^{d+n}$, $L = \mathbb{R}^d \times \mathbb{C}^n$, $M = \mathbb{R}^{d+n}$ *and* $Y = \{0\} \times \mathbb{C}^n$. *Identity* $T_Y L$ *with* $\mathbb{R}^d \times \mathbb{C}^n$. *Then:*

(1) *For any proper convex open cone* U *of* $\mathbb{R}^d$, *Stein open set* Ω *of* $\mathbb{C}^n$ *and* $k \neq 0$,

$$H^k(U\times\Omega;\ \mathcal{B}\mathcal{O}_{Y|L})=0.$$

(2) *Set* $\gamma\colon T_YL\to T_YL/\mathbb{R}_{>0}=Y\sqcup S_YL$. *Then the flabby dimension of* $\gamma_*\mathcal{B}\mathcal{O}_{Y|L}$ *is equal to* n.

(3) *For any proper convex closed cone* G *of* $\mathbb{R}^n$ *and* $k\neq n$,

$$\mathcal{H}^k_{\mathbb{R}^d\times(\mathbb{R}^n+\sqrt{-1}G)}(\mathcal{B}\mathcal{O}_{Y|L})|_{\mathbb{R}^{n+d}}=0.$$

This proposition can be proved by the same method as in Proposition 1.3. Note that in the proof of (1), we use Siu's theorem (cf. [O 3]).

1.8 Corollary. *The complex* $\mu_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})$ *is concentrated in degree* n.

Therefore, by the same manner as in Definition 1.6, we define several sheaves as follows:

1.9 Definition. We set:

$$\begin{aligned}
&\tilde{\mathcal{C}}_{N|M}:=(\mu_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})\otimes or_{N/Y})\simeq\mu_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})\otimes or_{N/Y}[n],\\
&\mathcal{B}_{N|M}:=\tilde{\mathcal{C}}_{N|M}|_{T_NM}=\mathcal{H}^n_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})\otimes or_{N/Y}\simeq\boldsymbol{R}\Gamma_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})\otimes or_{N/Y}[n],\\
&\tilde{\mathcal{A}}_{N|M}:=\mathcal{H}^0(\nu_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})).
\end{aligned}$$

Thus $\tilde{\mathcal{C}}_{N,M}$, $\tilde{\mathcal{B}}_{N|M}$ and $\tilde{\mathcal{A}}_{N|M}$ are sheaves on $T^*_{T_NM}T_YL$, T_NM and $T_{T_NM}T_YL$ respectively.

Note that by Proposition 1.7 (2), $\tilde{\mathcal{B}}_{N|M}$ is conically flabby.

Now, let us consider the following canonical mappings (cf. [K-S 2]):

$$T^*_{\tilde{M}_N}\tilde{L}_Y\xleftarrow[s_{L\pi}]{}T_NM\underset{\tilde{M}_N}{\times}T^*_{\tilde{M}_N}\tilde{L}_Y\xrightarrow[{}^ts'_L]{\sim}T^*_{T_NM}T_YL.$$

1.10 Definition. We set:

$$\begin{aligned}
&\mathcal{C}_{N|M}:=({}^ts'_L)_!\,s_{L\pi}^{-1}\mathcal{C}_{N,M}\simeq\boldsymbol{R}({}^ts'_L)_!\,s_{L\pi}^{-1}\mathcal{C}_{N,M},\\
&\mathcal{B}_{N|M}:=\mathcal{C}_{N|M}|_{T_NM}\simeq s_M^{-1}\mathcal{B}_{N,M}.
\end{aligned}$$

Thus $\mathcal{C}_{N|M}$ and $\mathcal{B}_{N|M}$ are sheaves on $T^*_{T_NM}T_YL$ and T_NM respectively.

1.11 Proposition. *There exist sheaf isomorphisms*:

$$\begin{aligned}
(j_M)_*\tilde{p}_M^{-1}\mathcal{B}_M&\simeq\mathcal{B}_{N,M},\\
\nu_N(\mathcal{B}_M)&\simeq\mathcal{B}_{N|M}.
\end{aligned}$$

Proof. By Lemma 1.1 and Corollary 1.4 we have

$$\begin{aligned}
(j_M)_*\tilde{p}_M^{-1}\mathcal{B}_M&\simeq\boldsymbol{R}(j_M)_*\tilde{p}_M^{-1}\mathcal{B}_M\simeq\boldsymbol{R}(j_M)_*\tilde{p}_M^{-1}i_M^!\mathcal{O}_X\otimes or_{M/X}[n+d]\\
&\simeq\boldsymbol{R}(j_M)_*\tilde{p}_M^!\,i^!\,i_L^!\mathcal{O}_X\otimes or_{M/X}[n+d-1]
\end{aligned}$$

$$
\begin{aligned}
&\simeq \mathbb{R}(j_M)_* \tilde{i}^! \tilde{p}_L^! i_L^! \mathcal{O}_X \otimes or_{L/X}[d] \otimes or_{L/X}^{\otimes -1} \otimes or_{M/X}[n-1] \\
&\simeq (\tilde{i}')^! \mathbb{R}(j_L)_* \tilde{i}^! \tilde{p}_L^{-1} i_L^! \mathcal{O}_X \otimes or_{L/X}[d] \otimes or_{M/L}[n] \\
&\simeq (\tilde{i}')^! \mathbb{R}(j_L)_* \tilde{p}_L^{-1} \mathcal{B}\mathcal{O}_L \otimes or_{M/L}[n] \simeq (\tilde{i}')^! \mathcal{B}\mathcal{O}_{Y,L} \otimes or_{M/L}[n] \\
&\simeq \mathcal{B}_{N,M}.
\end{aligned}
$$

Thus we have the first isomorphism. Applying the functor s_M^{-1} to this isomorphism we obtain the second one. Thus the proposition is proved. □

§2. Concrete Expressions

In this section, we give the concrete expressions of the sheaves defined in Section 1.

We denote the canonical projections by:

$$
\tau_{N,M}\colon T_{\tilde{M}_N} \tilde{L}_Y \to \tilde{M}_N, \quad \tau_{N|M}\colon T_{T_N M} T_Y L \to T_N M,
$$
$$
\pi_{N,M}\colon T^*_{\tilde{M}_N} \tilde{L}_Y \to \tilde{M}_N, \quad \pi_{N|M}\colon T^*_{T_N M} T_Y L \to T_N M.
$$

Moreover, $\dot{\pi}_{N,M}$ denotes the restriction of $\pi_{N,M}$ to $\dot{T}^*_{\tilde{M}_N} \tilde{L}_Y := T^*_{\tilde{M}_N} \tilde{L}_Y \setminus T^*_{\tilde{M}_N} \tilde{M}_N$. Similarly $\dot{\pi}_{N|M}$ denotes the restriction of $\pi_{N|M}$ to $\dot{T}^*_{T_N M} T_Y L := T^*_{T_N M} T_Y L \setminus T^*_{T_N M} T_N M$. Hence in the notation above, by the same arguments as in the theory of usual microfunctions (see for example Sato-Kawai-Kashiwara [S-K-K]) we can show that there exist monomorphisms (*boundary value morphisms*):

$$
\mathrm{b}_{N,M}\colon \tilde{\mathcal{A}}_{N,M} \rightarrowtail \tau_{N,M}^{-1} \mathcal{B}_{N,M},
$$
$$
\tilde{\mathrm{b}}_{N|M}\colon \tilde{\mathcal{A}}_{N|M} \rightarrowtail \tau_{N|M}^{-1} \tilde{\mathcal{B}}_{N|M},
$$

and epimorphism (*spectral morphisms*):

$$
\mathrm{sp}_{N,M}\colon \pi_{N,M}^{-1} \mathcal{B}_{N,M} \twoheadrightarrow \mathcal{C}_{N,M},
$$
$$
\tilde{\mathrm{sp}}_{N|M}\colon \pi_{N|M}^{-1} \tilde{\mathcal{B}}_{N|M} \twoheadrightarrow \tilde{\mathcal{C}}_{N|M}.
$$

Note that boundary value morphisms are induced by the canonical morphisms $\mathrm{id} \to \tau_{N,M}^! \mathbb{R}(\tau_{N,M})_!$ and $\mathrm{id} \to \tau_{N|M}^! \mathbb{R}(\tau_{N|M})_!$ respectively. Similarly the canonical morphisms $\pi_{N,M}^{-1} \mathbb{R}(\pi_{N,M})_* \to \mathrm{id}$ and $\pi_{N|M}^{-1} \mathbb{R}(\pi_{N|M})_* \to \mathrm{id}$ induce spectral morphisms. Moreover, let us consider the following commutative diagram:

$$
\begin{tikzcd}
T_N M \underset{\tilde{M}_N}{\times} T^*_{\tilde{M}_N} \tilde{L}_Y \arrow[r, "\sim", "{}^t s'_L"'] \arrow[dr, "s_{L\pi}"'] & T^*_{T_N M} T_Y L \arrow[r, "\pi_{N|M}"] & T_N M \arrow[r, hook, "i'"'] \arrow[d, hook, "s_M"'] \arrow[dr, phantom, "\square"] & T_Y L \arrow[d, hook, "s_L"] \\
& T^*_{\tilde{M}_N} \tilde{L}_Y \arrow[r, "\pi_{N,M}"'] & \tilde{M}_N \arrow[r, hook, "\tilde{i}'"'] & \tilde{L}_Y.
\end{tikzcd}
$$

Then we easily see that $({}^t s'_L)_! \, s_{L\pi}^{-1} \pi_{N,M}^{-1} \simeq \pi_{N|M}^{-1} s_M^{-1}$. Hence $\mathrm{sp}_{N,M}$ induces a spectral morphism

$$\mathrm{sp}_{N|M}\colon \pi_{N|M}^{-1}\mathscr{B}_{N|M} \twoheadrightarrow \mathscr{C}_{N|M}.$$

2.1 Proposition. (1) *In $\widetilde{M}_N$ there exists an exact sequence*

$$0 \to \mathscr{B}\mathscr{O}_{Y,L}|_{\widetilde{M}_N} \to \mathscr{B}_{N,M} \to (\dot{\pi}_{N,M})_* \mathscr{C}_{N,M} \to 0.$$

(2) *In $T_N M$ there exist exact sequences*

$$0 \to \mathscr{B}\mathscr{O}_{Y|L}|_{T_N M} \to \mathscr{B}_{N|M} \to (\dot{\pi}_{N|M})_* \mathscr{C}_{N|M} \to 0,$$
$$0 \to \mathscr{B}\mathscr{O}_{Y|L}|_{T_N M} \to \widetilde{\mathscr{B}}_{N|M} \to (\dot{\pi}_{N|M})_* \mathscr{C}_{N|M} \to 0.$$

Proof. By Sato's fundamental distinguished triangle, we have (1). Further applying the functor s_M^{-1}, we have an exact sequence

$$0 \to \mathscr{B}\mathscr{O}_{Y|L}|_{T_N M} \to \mathscr{B}_{N|M} \to s_M^{-1}(\dot{\pi}_{N,M})_* \mathscr{C}_{N,M} \to 0,$$

and a canonical morphism

$$s_M^{-1}(\dot{\pi}_{N,M})_* \mathscr{C}_{N,M} \to (\dot{\pi}_{N|M})_* \mathscr{C}_{N|M}.$$

It is easy to see that this morphism is an isomorphism. Hence we have the first exact sequence of (2). The second one is also obtained by Sato's fundamental distinguished triangle. □

In general, let $\tau\colon E \to Z$ be a vector bundle and $\pi\colon E^* \to Z$ its dual vector bundle. If A is a subset of E, *the polar set* A° is defined by

$$A^\circ := \{\xi \in E^*;\ \pi(\xi) \in \tau(A),\ \langle \eta, \xi \rangle \geqslant 0 \quad \text{for any } \eta \in \tau^{-1}(\pi(\xi)) \cap A\}.$$

Further, we set $A^{\circ a} := \{\xi \in E^*;\ -\xi \in A^\circ\}$.

2.2 Proposition. *Let U be an open convex subset of $T_{\widetilde{M}_N}\widetilde{L}_Y$ with connected fiber, V the convex hull of U.*

(1) *If φ is a section of $\widetilde{\mathscr{A}}_{N,M}$ on U, then* $\mathrm{supp}\,(\mathrm{sp}_{N,M}(\mathrm{b}_{N,M}(\varphi))) \subset U^{\circ a}$. *Conversely, if a section f of $\mathscr{B}_{N,M}$ on $\tau_{N,M}(U)$ satisfies* $\mathrm{supp}\,(\mathrm{sp}_{N,M}(f)) \subset U^{\circ a}$, *then there exists a unique section φ of $\widetilde{\mathscr{A}}_{N,M}$ on U such that* $\mathrm{b}_{N,M}(\varphi) = f$.

(2) *The natural restriction $\Gamma(V; \widetilde{\mathscr{A}}_{N,M}) \to \Gamma(U; \widetilde{\mathscr{A}}_{N,M})$ is an isomorphism.*

Proof. Set

$$P^+ := \{(\sqrt{-1}\eta, \sqrt{-1}\xi) \in T_{\widetilde{M}_N}\widetilde{L}_Y \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N}\widetilde{L}_Y;\ \langle \sqrt{-1}\eta, \sqrt{-1}\xi \rangle = -\langle \eta, \xi \rangle > 0\}$$

and denote by p_k^+ the k-th projection on P^+ $(k=1, 2)$. Then, by Corollary A. 2 of M. Uchida [U] (cf. [S-K-K]), we have an exact sequence:

$$0 \longrightarrow \widetilde{\mathscr{A}}_{N,M} \xrightarrow{\mathrm{b}_{N,M}} \tau_{N,M}^{-1}\mathscr{B}_{N,M} \longrightarrow (p_1^+)_*(p_2^+)^{-1}\mathscr{C}_{N,M}.$$

We remark that $U^{\circ a} = V^{\circ a}$. Since $p_2^+|_{(p_1^+)^{-1}(U)}$ is a continuous open mapping with connected fiber, it follows that

$$\begin{aligned}\Gamma(U; (p_1^+)_*(p_2^+)^{-1}\mathscr{C}_{N,M}) &\simeq \Gamma((p_1^+)^{-1}(U); (p_2^+)^{-1}\mathscr{C}_{N,M}) \simeq \Gamma(p_2^+((p_1^+)^{-1}(U)); \mathscr{C}_{N,M}) \\ &\simeq \Gamma(T^*_{\widetilde{M}_N}\widetilde{L}_Y \backslash U^{\circ a}; \mathscr{C}_{N,M}) \simeq \Gamma(T^*_{\widetilde{M}_N}\widetilde{L}_Y \backslash V^{\circ a}; \mathscr{C}_{N,M}) \\ &\simeq \Gamma(V; (p_1^+)_*(p_2^+)^{-1}\mathscr{C}_{N,M}).\end{aligned}$$

For the same reason, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(V; \widetilde{\mathscr{A}}_{N,M}) & \xrightarrow{\mathrm{b}_{N,M}} & \Gamma(\tau_{N,M}(V); \mathscr{B}_{N,M}) & \longrightarrow & \Gamma(T^*_{\widetilde{M}_N}\widetilde{L}_Y \backslash V^{\circ a}; \mathscr{C}_{N,M}) \\ & & \downarrow & & \| & & \| \\ 0 & \longrightarrow & \Gamma(U; \widetilde{\mathscr{A}}_{N,M}) & \xrightarrow{\mathrm{b}_{N,M}} & \Gamma(\tau_{N,M}(U); \mathscr{B}_{N,M}) & \longrightarrow & \Gamma(T^*_{\widetilde{M}_N}\widetilde{L}_Y \backslash U^{\circ a}; \mathscr{C}_{N,M}), \end{array}$$

here all the rows are exact. Hence we can easily prove the proposition. □

2.3 Proposition. *Let U be an open convex subset of $T_{T_N M} T_Y L$ with connected fiber, V the convex hull of U.*

(1) *If φ is a section of $\widetilde{\mathscr{A}}_{N|M}$ on U, then* $\operatorname{supp}(\widetilde{\mathrm{sp}}_{N|M}(\mathrm{b}_{N|M}(\varphi))) \subset U^{\circ a}$. *Conversely, if a section f of $\widetilde{\mathscr{B}}_{N|M}$ on $\tau_{N|M}(U)$ satisfies* $\operatorname{supp}(\widetilde{\mathrm{sp}}_{N|M}(f)) \subset U^{\circ a}$, *then there exists a unique section φ of $\widetilde{\mathscr{A}}_{N|M}$ on U such that $\mathrm{b}_{N|M}(\varphi) = f$.*

(2) *The natural restriction $\Gamma(V; \widetilde{\mathscr{A}}_{N|M}) \to \Gamma(U; \widetilde{\mathscr{A}}_{N|M})$ is an isomorphism.*

The proof is same to that of Proposition 2.2.

Now suppose that $X = \mathbb{C}^d_\tau \times \mathbb{C}^n_z$, $L = \mathbb{R}^d_t \times \mathbb{C}^n_z$, $Y = \{0\} \times \mathbb{C}^n_z$, $M = \mathbb{R}^d_t \times \mathbb{R}^n_x$ and $N = \{0\} \times \mathbb{R}^n_x$. Let us identify $\widetilde{M}_N$ and $\widetilde{L}_Y$ with $\mathbb{R}_r \times \mathbb{R}^d_t \times \mathbb{R}^n_x$ and $\mathbb{R}_r \times \mathbb{R}^d_t \times \mathbb{C}^n_z$ respectively. Then we identify the normal deformation of $\widetilde{M}_N$ in $\widetilde{L}_Y$ with $\mathbb{R}^{d+2} \times \mathbb{R}^n \times \sqrt{-1}\mathbb{R}^n$. Let U be an open subset of $\widetilde{M}_N$. Therefore by the same arguments as in the theory of hyperfunctions we can represent any section $f(t, x)$ of $\mathscr{B}_{N,M}$ on U as

$$f(t, x) = \sum_{j=1}^{J} F_j(t, x + \sqrt{-1}\Gamma_j 0)$$

for some natural number J. Here Γ_j are open cones of $\mathbb{R}^n$ and $F_j(t, z)$ are sections of $\widetilde{\mathscr{A}}_{N,M}$ on $U + \sqrt{-1}\Gamma_j$. Moreover, the following holds:

2.4 Lemma. *Let f be a germ of $\mathscr{B}_{N,M}$ at $x^* \in \widetilde{M}_N$. Then $(x^*; \sqrt{-1}\xi^*) \in$*

$T^*_{\tilde{M}_N}\tilde{L}_Y$ is not contained in $\mathrm{supp}\,(\mathrm{sp}_{N,M}(f))$ *if and only if there exist sections $F_j(t, z)$ of $\tilde{\mathscr{A}}_{N,M}$ on $U+\sqrt{-1}\Gamma_j$ such that U is a neighborhood of x^* and Γ_j is an open cone of $\mathbb{R}^n$ with $\xi^*\notin\Gamma_j^\circ$ and that*

$$f(t, x)=\sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0).$$

Let U be an open set of $\tilde{M}_N$ such that all the fibers of the mapping $\tilde{p}_M$: $U\cap\Omega_M\rightarrow M$ are connected and that $\tilde{p}_M(U\cup\Omega_M)$ is bounded. Then we see that for any open convex cone $\Gamma\subset\mathbb{R}^n$

$$\Gamma(U+\sqrt{-1}\Gamma;\ \tilde{\mathscr{A}}_{N,M})\simeq\varinjlim_{W}\Gamma(W;\ \mathscr{B}\mathscr{O}_L),$$

where W ranges through the family of open subsets of $\tilde{L}_Y$ such that $C_{\tilde{M}_N}(\tilde{L}_N\setminus W)\cap(U+\sqrt{-1}\Gamma)=\emptyset$. Here for any subset S of $\tilde{L}_N$, $C_{\tilde{M}_N}(S)$ denotes the normal cone of S along $\tilde{M}_N$. Thus we can assume that such a W satisfies the following: all the fibers of the mapping $\tilde{p}_L$: $W\cap\Omega_L\rightarrow L$ are connected and $\tilde{p}_L(W\cap\Omega_L)$ is bounded. Then it follows that for any open cone Γ of $\mathbb{R}^n$

$$\Gamma(U+\sqrt{-1}\Gamma;\ \tilde{\mathscr{A}}_{N,M})\simeq\varinjlim_{W}\Gamma(\tilde{p}_L(W\cap\Omega_L);\ \mathscr{B}\mathscr{O}_L).$$

Therefore any section of $\tilde{\mathscr{A}}_{N,M}$ on $U+\sqrt{-1}\Gamma$ can be represented by a section of $\mathscr{B}\mathscr{O}_L$ on $\tilde{p}_L(W\cap\Omega_L)$ for some W. Hence we can obtain the following proposition in the same way as Proposition 1.10 of Oaku [O 3]:

2.5 Proposition. *Under the preceding notation, the following holds:*

(1) *Assume that $f(t, x)$ is a section of $\mathscr{B}_{N,M}$ on U and that* $\mathrm{supp}\,(\mathrm{sp}_{N,M}(f))$ *is contained in the interior of $U+\sqrt{-1}\,(\bigcup_{j=1}^{J}\Gamma_j^\circ)$. Then there exist sections $F_j(t, z)$ of $\tilde{\mathscr{A}}_{N,M}$ on $U+\sqrt{-1}\Gamma_j$ such that*

$$f(t, x)=\sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0).$$

(2) *Assume that sections $F_j(t, z)$ of $\tilde{\mathscr{A}}_{N,M}$ on $U+\sqrt{-1}\Gamma_j\,(1\leqslant j\leqslant J)$ satisfy*

$$\sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0)=0$$

as a section of $\mathscr{B}_{N,M}$ on U. Then for any subcone $\Gamma_j'\Subset\Gamma_j$, there exist sections F_{jk} of $\tilde{\mathscr{A}}_{N,M}$ on $U+\sqrt{-1}\,(\Gamma_j'+\Gamma_k')$ such that

$$F_j=\sum_{k=1}^{J} F_{jk}, \quad F_{jk}+F_{kj}=0 \qquad (1\leqslant j,\ k\leqslant J).$$

Similarly we can identify the normal deformation of T_NM in T_YL with $\mathbb{R}^{d+1}\times\mathbb{R}^n\times\sqrt{-1}\,\mathbb{R}^n$. Let U be an open subset of T_NM. Then, we can represent any section $f(t,x)$ of $\widetilde{\mathscr{B}}_{N|M}$ on U as

$$f(t,x)=\sum_{j=1}^{J} F_j(t,\ x+\sqrt{-1}\Gamma_j 0)$$

for some natural number J. Here Γ_j are open cones of $\mathbb{R}^n$ and $F_j(t,z)$ are sections of $\widetilde{\mathscr{A}}_{N|M}$ on $U+\sqrt{-1}\Gamma_j$. Moreover, the following holds:

2.6 Lemma. *Let f be a germ of $\widetilde{\mathscr{B}}_{N|M}$ at $x^*\in T_NM$. Then $(x^*;\sqrt{-1}\,\xi^*)\in T^*_{T_NM}T_YL$ is not contained in* supp $(\widetilde{\mathrm{sp}}_{N|M}(f))$ *if and only if there exists sections $F_j(t,z)$ of $\widetilde{\mathscr{A}}_{N|M}$ on $U+\sqrt{-1}\Gamma_j$ such that U is a neighborhood of x^* and Γ_j is an open cone of $\mathbb{R}^n$ with $\xi^*\notin\Gamma_j^\circ$ and that*

$$f(t,x)=\sum_{j=1}^{J} F_j(t,\ x+\sqrt{-1}\Gamma_j 0).$$

2.7 Proposition. *Let U be an open convex set of T_NM such that $\tau_N(U)$ is bounded. Then the following holds:*

(1) *Assume that a section $f(t,x)$ of $\widetilde{\mathscr{B}}_{N|M}$ on U satisfies that* supp $(\widetilde{\mathrm{sp}}_{N|M}(f))$ *is contained in the interior of $U+\sqrt{-1}\,(\bigcup_{j=1}^{J}\Gamma_j^\circ)$. Then there exists sections $F_j(t,z)$ on $\widetilde{\mathscr{A}}_{N|M}$ on $U+\sqrt{-1}\Gamma_j$ such that*

$$f(t,x)=\sum_{j=1}^{J} F_j(t,\ x+\sqrt{-1}\Gamma_j 0).$$

(2) *Assume that sections $F_j(t,z)$ of $\widetilde{\mathscr{A}}_{N|M}$ on $U+\sqrt{-1}\Gamma_j\,(1\leqslant j\leqslant J)$ satisfy*

$$\sum_{j=1}^{j} F_j(t,\ x+\sqrt{-1}\Gamma_j 0)=0$$

as a section of $\widetilde{\mathscr{B}}_{N|M}$ on U. Then for any subcone $\Gamma'_j\Subset\Gamma_j$, there exists sections $F_{jk}(t,z)$ of $\widetilde{\mathscr{A}}_{N|M}$ on $U+\sqrt{-1}\,(\Gamma'_j+\Gamma'_k)$ such that

$$F_j=\sum_{k=1}^{J} F_{jk}, \qquad F_{jk}+F_{kj}=0 \qquad (1\leqslant j,\ k\leqslant J).$$

By the following theorem, we regard $\mathscr{C}_{N|M}$ as a subsheaf of $\widetilde{\mathscr{C}}_{N|M}$:

2.8 Theorem. *There exist natural monomorphisms*:

$$\tilde{\alpha}_{N|M}\colon \mathscr{C}_{N|M}\rightarrowtail\widetilde{\mathscr{C}}_{N|M},$$
$$\tilde{\alpha}_{N|M}\colon \mathscr{B}_{N|M}\rightarrowtail\widetilde{\mathscr{B}}_{N|M},$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
\pi_{N|M}^{-1}\mathscr{B}_{N|M} & \xrightarrow{\mathrm{sp}_{N|M}} & \mathscr{C}_{N|M} \\
\downarrow \tilde{\alpha}_{N|M} & & \downarrow \tilde{\alpha}_{N|M} \\
\pi_{N|M}^{-1}\widetilde{\mathscr{B}}_{N|M} & \xrightarrow{\widetilde{\mathrm{sp}}_{N|M}} & \widetilde{\mathscr{C}}_{N|M}.
\end{array}$$

Proof. We have

$$\begin{aligned}
\mathscr{C}_{N|M} &\simeq \boldsymbol{R}({}^t s_L')_* s_{L\pi}^{-1}(\mu_{\tilde{M}_N}(\mathscr{B}\mathscr{O}_{Y,L})\otimes or_{M/L}[n]) \\
&\to \mu_{T_NM}(s_L^{-1}\mathscr{B}\mathscr{O}_{Y,L})\otimes or_{N/Y}\overset{L}{\otimes}\omega_{M/L}\overset{L}{\otimes}\omega_{T_NM/\tilde{M}_N}^{\otimes -1}[n] \\
&\simeq \mu_{T_NM}(\mathscr{B}\mathscr{O}_{Y|L})\otimes or_{N/Y}[n] \\
&\simeq \widetilde{\mathscr{C}}_{N|M},
\end{aligned}$$

where $\omega_{M/N}$ denotes *the relative dualiting complex*. Hence we have the first morphism. Applying the functor $(\pi_{N|M})_*$ we have the second morphism. The proof of the commutativity is straightforward. The proof of the injectivity is similar to that of Theorems 1.1 and 1.2 of Oaku [O 3]. □

§3. F-Mild Hyperfunctions

We shall define the sheaf of F-mild hyperfunctions for the higher-codimensional boundary case. Note that the results of this section are essentially contained in Oaku [O 4]. However in [O 4] we worked not on the normal bundle but on the sphere bundle. Thus for the convenience of the reader, we give the detailed proofs in this section.

We inherit the notation of the preceding section.

Let us set

$$D(V,\varepsilon,\Gamma):=\{(t,z)\in\mathbb{R}^d\times\mathbb{C}^n;(t,\operatorname{Re} z)\in V,\ |\operatorname{Im} z|<\varepsilon,\ \operatorname{Im} z\in\Gamma\}$$

for a subset V of M, a constant $\varepsilon>0$ and a cone Γ of $\mathbb{R}^n$.

In general, we mean by Cl and Int *the closure* and *the interior* of a set respectively.

3.1 Definition. Let x^* be a point of $T_N M$ and (τ, z) an admissible local coordinate system of X around $\mathring{x}:=\tau_N(x^*)$ such that $z(\mathring{x})=0$. Then a germ $u(t, x)$ of $\mathscr{B}_{N|M}$ at x^* is said to be *F-mild* (with respect to a partial complexification L) at x^* if there exist a natural number J and holomorphic functions $F_j(\tau, x)$ $(1 \leqslant j \leqslant J)$ defined on a neighborhood of $D(p_M(U \cap \mathrm{Cl}\,\Omega_M), \varepsilon, \Gamma_j)$ in X such that

$$u(t, x)=\sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0)$$

as a hyperfunction on $\tilde{p}_M(U \cap \Omega_M)$. Here U is an open neighborhood of x^* in $\tilde{M}_N$ such that the all the fibers of the mapping $\tilde{p}_M$: $U \cap \Omega_M \to M$ are connected, ε is a positive constant and $\Gamma_1, \ldots, \Gamma_J$ are open convex cones in $\mathbb{R}^n$. We denote by $\mathscr{B}^F_{N|M}$ the sheaf of sections of $\mathscr{B}_{N|M}$ which are F-mild at each point of their defining domains. Sections of $\mathscr{B}^F_{N|M}$ are called *F-mild hyperfunctions.*

3.2 Example. Let $u(s, x)$ be an F-mild hyperfunction at the origin of $\mathbb{R}_s \times \mathbb{R}^n_x$ from the positive side $\{(s, x) \in \mathbb{R}^{1+n};\ s>0\}$ (see [O 1] for an example of F-mild hyperfunction which is not mild in the sense of ([Kt 1]). Then, $u(t_1^2-\sum_{j=2}^{d} t_j^2, x)$ is an F-mild hyperfunction on a neighborhood of

$$\left\{0+\langle t, \frac{\partial}{\partial t}\rangle \in T_N M;\ t_1^2-\sum_{j=2}^{d} t_j^2>0\right\}$$

in $T_N M$ with $M=\mathbb{R}^d_t \times \mathbb{R}^n_x$ and $N=\{0\} \times \mathbb{R}^n_x$.

We denote the natural inclusion $\mathscr{B}^F_{N|M} \rightarrowtail \mathscr{B}_{N|M}$ by $\beta^F_{N|M}$.

Let us denote by $\mathscr{B}^A_M$ *the sheaf of hyperfunctions which have t as real analytic parameters on M.* Moreover, set

$$\mathscr{B}^A_{N|M}:=\mathscr{B}^A_M|_N.$$

3.3 Lemma. (1) *The following equality holds*:

$$\mathscr{B}^F_{N|M}|_N=\mathscr{B}^A_{N|M}.$$

(2) *There exists a natural monomorphism*

$$\alpha_{N|M}\colon \tau_N^{-1}\mathscr{B}^A_{N|M} \rightarrowtail \mathscr{B}^F_{N|M}.$$

In particular, any germ of $\mathscr{B}^A_{N|M}$ *is F-mild on a whole fiber of* τ_N.

Proof. The proof of (1) is obvious by Definition 3.1. Similarly we have a natural morphism $\alpha_{N|M}$: $\tau_N^{-1}\mathcal{B}^A_{N|M}\to\mathcal{B}^F_{N|M}$. The injectivity of $\alpha_{N|M}$ follows from the Holmgren uniqueness theorem for hyperfunctions. □

3.4 Proposition. *Let x^* be a point of T_NM. Then F-mildness (with respect to L) at x^* of a section of $\mathcal{B}_{N|M}$ does not depend on an admissible local coordinate system (τ, z) of X taken in Definition* 3.1.

Proof. Let (τ, z) and (τ', z') be two admissible local coordinate systems of X around x^* such that $x^*=0+\partial/\partial\tau_1=0+\partial/\partial\tau'_1$. Since Im $z=$Im $z'=0$ on L, as a mapping of (τ', z'), τ depends only on τ'. This proves the proposition. □

On the other hand, since T_NM is a one-codimensional boundary of $\mathrm{Cl}\Omega_M$, there exists *the sheaf $\mathcal{B}^F_{T_NM|\mathrm{Cl}\Omega_M}$ of F-mild hyperfunctions for a one-codimensional boundary* defined by Oaku [O 1]. For simplicity we denote $\mathcal{B}^F_{T_NM|\mathrm{Cl}\Omega_M}$ by $\mathcal{B}^F_1$. Then by the definition $\mathcal{B}^F_1$ is a subsheaf of $\mathcal{B}_{T_NM|\mathrm{Cl}\Omega_M}$.

Let $\widetilde{X}^{\mathbb{C}}_Y$ be the complex normal deformation of Y in X (cf. the proof of Proposition 10.3.19 of [K-S 2]). If (τ, z) is an admissible local coordinate system, then we see that $\widetilde{X}^{\mathbb{C}}_Y=\{(\rho, \tau, z);\ \rho\in\mathbb{C},\ (\rho\tau, z)\in X\}$. Note that $\widetilde{X}^{\mathbb{C}}_Y$ is a complexification of $\widetilde{M}_N$ and $\widetilde{L}_Y$ is regarded as a submanifold of $\widetilde{X}^{\mathbb{C}}_Y$.

3.5 Lemma. *There exists a natural monomorphism:*

$$\tilde{p}^*_M\colon \mathcal{B}_{N|M}\rightarrowtail\mathcal{B}_{T_NM|\mathrm{Cl}\Omega_M}.$$

Proof. Since $\tilde{p}_M$ is a smooth mapping, we have a substitution morphism

$$\tilde{p}^*_M\colon \tilde{p}_M^{-1}\mathcal{B}_M\longrightarrow\mathcal{B}_{\Omega_M},$$

such that

$$\mathrm{supp}(\tilde{p}^*_M u)=\tilde{p}_M^{-1}(\mathrm{supp}(u))=\{(r, t, x)\in\Omega_M; (rt, x)\in\mathrm{supp}(u)\}.$$

Thus, we see that $\tilde{p}^*_M$ is injective. Therefore applying the functor $s_M^{-1}(j_M)_*$ we have

$$\tilde{p}^*_M\colon \mathcal{B}_{N|M}\longrightarrow s_M^{-1}(j_M)_*\mathcal{B}_{\Omega_M}\simeq s_M^{-1}(j_M)_*\, j_M^{-1}\mathcal{B}_{\widetilde{M}_N}=\mathcal{B}_{T_NM|\mathrm{Cl}\Omega_M}.$$

We prove this morphism is injective. For any $x^*\in T_NM$, we have

$$\begin{aligned}(\mathcal{B}_{N|M})_{x^*}&\simeq\varinjlim_W\Gamma(W\cap\Omega_M;\ \tilde{p}_M^{-1}\mathcal{B}_M)\xrightarrow{\tilde{p}_M}\varinjlim_W\Gamma(W\cap\Omega_M;\ \mathcal{B}_{\widetilde{M}_N})\\&\simeq(\mathcal{B}_{T_NM|\mathrm{Cl}\Omega_M})_{x^*},\end{aligned}$$

where W ranges through a neighborhood system of x^* in $\widetilde{M}_N$. Hence the injectivity is obvious. □

Let us set

$$\widetilde{D}(V, \delta, \widetilde{\Gamma}) := \{(r, \tau, z) \in \mathbb{R} \times \mathbb{C}^d \times \mathbb{C}^n; (r, \operatorname{Re} \tau, \operatorname{Re} z) \in V, |\operatorname{Im} z| < \delta, (\operatorname{Im} \tau, \operatorname{Im} z) \in \widetilde{\Gamma}\}$$

for a subset V of $\widetilde{M}_N$, a positive constant δ and cone $\widetilde{\Gamma}$ of $\mathbb{R}^{d+n}$, where we identify $\widetilde{M}_N$ with $\mathbb{R}_r \times \mathbb{R}^d_t \times \mathbb{R}^n_x$ by an admissible coordinate system.

3.6 Lemma. *The monomorphism $\widetilde{p}^*_M : \mathscr{B}_{N|M} \rightarrowtail \mathscr{B}_{T_N M|\mathrm{Cl}\Omega_M}$ induces a monomorphism*

$$\mathscr{B}^F_{N|M} \rightarrowtail \mathscr{B}^F_1 .$$

Proof. We have only to prove that $\widetilde{p}^*_M \mathscr{B}^F_{N|M}$ is contained in $\mathscr{B}^F_1$. Let x^* be a point of $T_N M$ and (τ, z) an admissible local coordinate system around $\tau_N(x^*)$ such that $\tau_N(x^*) = 0$ in this coordinate system. Since the proof is similar, we consider only the case where x^* is a point of $\dot{T}_N M$. We may assume that $x^* = 0 + \partial/\partial t_1$. Let $F(t, x + \sqrt{-1}\Gamma 0)$ be a germ of $\mathscr{B}^F_{N|M}$ at x^* with an open cone Γ of $\mathbb{R}^n$. Then we may assume that $F(\tau, z)$ is a holomorphic function defined on a neighborhood of $D(p_M(U \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma)$ with a neighborhood U of x^* in $\widetilde{M}_N$ and a positive constant ε. Let us set

$$G(\rho, \tau, z) := F(\rho\tau, z).$$

Then the boundary value of G represents $\widetilde{p}^*_M F(t, x + \sqrt{-1}\Gamma 0)$. Set $\tau' := (\tau_2, \dots, \tau_d)$ and so on. Then $G(\rho, \tau, z)$ is holomorphic on a neighborhood of

$$\{(r, t, z) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^n; 0 \leqslant r < \delta, |t'| < \delta |t_1|, |z| < \delta, \operatorname{Im} z \in \Gamma\}$$

with a constant $\delta > 0$. Hence by applying the local version of Bochner's tube theorem to $G(r^2, \tau, z)$, we can see that $G(\rho, \tau, z)$ is holomorphic on a neighborhood of $\widetilde{D}(V \cap \mathrm{Cl}\Omega_M, \delta_1, \widetilde{\Gamma})$ with an open neighborhood V of $(0, x^*)$ in $\widetilde{M}_N$, a constant $\delta_1 > 0$ and an open cone $\widetilde{\Gamma}$ of $\mathbb{R}^{d+n}$ containing $\{0\} \times \Gamma$ with $0 \in \mathbb{R}^d$. Therefore it follows that the boundary value of G represents a germ of $\mathscr{B}^F_1$ at x^*. □

By this lemma, we regard $\mathscr{B}^F_{N|M}$ as a subsheaf of $\mathscr{B}^F_1$.

Let us denote by $\mathbb{C}^\times$ the multiplicative group $\mathbb{C} \setminus \{0\}$. Then we recall that $\mathbb{C}^\times$ acts on $\widetilde{X}^{\mathbb{C}}_Y$. By using an admissible coordinate system, we can describe this action by

$$\mathbb{C}^\times \times \widetilde{X}^{\mathbb{C}}_Y \ni (c, (\rho, \tau, z)) \mapsto \left(\frac{\rho}{c}, c\tau, z\right) \in \widetilde{X}^{\mathbb{C}}_Y .$$

Let us denote by ϑ the infinitesimal generator of this action. Thus by an admissible coordinate system, we have

$$\vartheta = \langle \tau, \frac{\partial}{\partial \tau} \rangle - \rho \frac{\partial}{\partial \rho}.$$

3.7 Lemma. *Let W be a connected open set of $\tilde{L}_Y$ such that $W \cap \mathrm{Cl}\,\Omega_L \neq \emptyset$ and all the fibers of the mapping p_L: $W \to L$ are connected. Let $G(\rho, \tau, z)$ be a holomorphic function defined on a neighborhood of $W \cap \mathrm{Cl}\,\Omega_L$ in $\tilde{X}_Y^{\mathbb{C}}$.*

(1) *Suppose G satisfies*

$$\vartheta G(\rho, \tau, z) = 0.$$

Then G is extended to a holomorphic function on a neighborhood of $\tilde{W} := \tilde{p}_L^{-1}(p_L(W \cap \Omega_L))$.

(2) *Assume moreover that G can be extended analytically to a neighborhood of $T_Y Y \cap \mathrm{Cl}\,\tilde{W}$. Then there exists a holomorphic a function $F(\tau, z)$ defined on a neighborhood of $p_L(W \cap \mathrm{Cl}\,\Omega_L)$ in $\mathbb{C}^{n+d}$ such that $G(\rho, \tau, z) = F(\rho\tau, z)$.*

Proof. Since $G(r, t, z)$ is constant along each fiber of $\tilde{p}_L$: $W \cap \Omega_L \to L$, it is easy to see that there exists a holomorphic function $F(\tau, z)$ on a neighborhood of $p_L(\tilde{W} \cap \Omega_L) = p_L(W \cap \Omega_L)$ such that $G(\rho, \tau, z) = F(\rho\tau, z)$ holds. Hence (1) is proved. To prove (2), we have only to show that $F(\tau, z)$ can be continued analytically to a neighborhood of each point of $p_L(W) \cap Y$. Let z^* be an arbitrary point of $W \cap T_Y L$. We may assume $z^* = z_0 + \partial/\partial t_1$ with $z_0 \in \mathbb{C}^n$. Set $\tau' := (\tau_2, \ldots, \tau_n)$. Then G can be developed into a power series of the form

$$G(\rho, \tau, z) = \sum_{\nu=0}^{\infty} \sum_{\alpha' \in \mathbb{N}_0^{d-1}} a_{\nu\alpha'}(\tau_1, z) \rho^{\nu} (\tau')^{\alpha'}$$

on the set

$$\{(\rho, \tau, z) \in \mathbb{C} \times \mathbb{C}^d \times \mathbb{C}^n;\ |\rho| < \delta,\ |\tau_1 - 1| < \delta,\ |\tau'| < \delta,\ |z - z_0| < \delta\}$$

with some $\delta > 0$. Since $\vartheta G = 0$, $a_{\nu\alpha'}(\tau_1, z)$ satisfies

$$a_{\nu\alpha'}(\tau_1, z) = \tau_1^{\nu - |\alpha'|} a_{\nu\alpha'}(1, z).$$

Hence we have

$$\begin{aligned} G(\rho, \tau, z) &= \sum_{\nu=0}^{\infty} \sum_{\alpha' \in \mathbb{N}_0^{d-1}} a_{\nu\alpha'}(1, z) \rho^{\nu} \tau_1^{\nu - |\alpha'|} (\tau')^{\alpha'} \\ &= \sum_{\nu=0}^{\infty} \sum_{\alpha' \in \mathbb{N}_0^{d-1}} a_{\nu\alpha'}(1, z)\, (\rho\tau_1)^{\nu} \left(\frac{\tau'}{\tau_1}\right)^{\alpha'}, \end{aligned}$$

which is holomorphic on

$$\{(\rho, \tau, z) \in \mathbb{C}\times\mathbb{C}^d\times\mathbb{C}^n;\ |\rho\tau_1|<\delta,\ |\tau'|<\delta|\tau_1|,\ |z-z_0|<\delta\}.$$

By the assumption, G is holomorphic on a neighborhood of $(\rho, \tau, z)=(0, 0, z_0)$. Hence we have $a_{\nu\alpha'}(1, z)=0$ for $\nu<|\alpha'|$. Hence we conclude that

$$F(\tau, z):=G\left(\tau_1, 1, \frac{\tau'}{\tau_1}, z\right)=\sum_{\alpha'\in\mathbb{N}_0^{d-1}}\sum_{\nu\geqslant|\alpha'|}a_{\nu\alpha'}(1, z)\,\tau_1^{\nu-|\alpha'|}(\tau')^{\alpha'}$$

is holomorphic on a neighborhood of $(\tau, z)=(0, z_0)$. This completes the proof. □

3.8 Theorem (the Edge of the Wedge Theorem for $\mathcal{F}$-Mild Hyperfunctions). *Let x^* be a point of T_NM and (τ, z) an admissible local coordinate system around $\tau_N(x^*)$ such that $\tau_N(x^*)=0$ in this system. Let ε be a positive constant, U an open neighborhood x^* in $\widetilde{M}_N$, and Γ_j $(1\leqslant j\leqslant J)$ open convex cones of $\mathbb{R}^n$. Let $F_j(\tau, z)$ be holomorphic functions defined on a neighborhood of $D(p_M(U\cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_j)$ such that*

$$\sum_{j=1}^{J}F_j(t, x+\sqrt{-1}\Gamma_j 0)=0$$

holds as a hyperfunction on $\tilde{p}_M(U\cap\Omega_M)$. Then for any open convex cones Γ_j' such that $\Gamma_j'\Subset\Gamma_j$, there exist a positive constant δ, an open neighborhood V of x^* in $\widetilde{M}_N$, and holomorphic functions $F_{jk}(\tau, z)$ defined on a neighborhood of $D(p_M(V\cap\mathrm{Cl}\Omega_M), \delta, \Gamma_j'+\Gamma_k')$ such that

$$F_j(\tau, z)=\sum_{k=1}^{J}F_{jk}(\tau, z), \qquad F_{jk}(\tau, z)+F_{kj}(\tau, z)=0 \qquad (1\leqslant j, k\leqslant J).$$

Proof. By virtue of Lemma 3.3 (1), we have only to prove this theorem in the case where $x^*=0+\partial/\partial t_1$. We prove by induction on J. First assume that $J=2$. Set

$$G_j(\rho, \tau, z):=F_j(\rho\tau, z).$$

Then by the proof of Lemma 3.6 each $G_j(\rho, \tau, z)$ is holomorphic on a neighborhood of $\widetilde{D}(V\cap\mathrm{Cl}\Omega_M, \varepsilon_1, \widetilde{\Gamma}_j)$ with an open neighborhood V of $(0, x^*)$ in $\widetilde{M}_N$, a constant $\varepsilon_1>0$, and an open cone $\widetilde{\Gamma}_j$ of $\mathbb{R}^{d+n}$ containing $\{0\}\times\Gamma_j$. In view of the edge of the wedge theorem for $\mathscr{B}_1^F$ (Theorem 1 of Oaku [O 1]), for any open cones $\Gamma_j'\Subset\Gamma_j$ $(j=1, 2)$ there exists a holomorphic function $G(\rho, \tau, z)$ defined on a neighborhood of $\widetilde{D}(W\cap\mathrm{Cl}\Omega_M, \varepsilon_2, \widetilde{\Gamma})$ with an open neighborhood W of $(0, x^*)$ in $\widetilde{M}_N$, a constant $\varepsilon_2>0$ and an open convex cone $\widetilde{\Gamma}$ of $\mathbb{R}^{d+n-1}$ containing $\{0\}\times(\Gamma_1'+\Gamma_2')$ such that $G=G_1$ on $\widetilde{D}(W\cap\mathrm{Cl}\Omega_M, \varepsilon_2, \widetilde{\Gamma}_1)$ and $G=-G_2$ on

$\widetilde{D}(W \cap \mathrm{Cl}\Omega_M, \varepsilon_2, \widetilde{\Gamma}_2)$. It is easy to see that G satisfies the assumptions of Lemma 3.7 (2). Thus we can find a holomorphic function $F(\tau, z)$ on a neighborhood of $D(p_M(W \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_1' + \Gamma_2')$ such that

$$F(\rho\tau, z) = G(\rho, \tau, z).$$

Since $F = F_1$ on $D(p_M(W \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_1)$ and $F = -F_2$ on $D(p_M(W \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_2)$, we obtain the theorem in the case where $J = 2$.

To prove Theorem 3.8 in the case where $J \geqslant 3$, we need several results.

Let us set

$$D_0(\delta, \Gamma) := \{(0, z) \in \mathbb{C}^{d+n};\ |z| < \delta,\ \mathrm{Im}\, z \in \Gamma\}$$

for a positive constant δ and an open cone Γ of $\mathbb{R}^n$.

3.9 Lemma. *Under the same assumption as in Theorem* 3.8, *for any open convex cones Γ_j' such that $\Gamma_j' \Subset \Gamma_j$ there exist a positive constant δ and holomorphic functions $F_{jk}(\tau, z)$ defined on a neighborhood of $D_0(\delta, \Gamma_j' + \Gamma_k')$ such that*

$$F_j(\tau, z) = \sum_{k=1}^{J} F_{jk}(\tau, z), \qquad F_{jk}(\tau, z) + F_{kj}(\tau, z) = 0 \qquad (1 \leqslant j, k \leqslant J)$$

Proof. We may assume that $x^* = 0 + \partial/\partial t_1$. Let us set

$$G_j(\rho, \tau, z) := F_j(\rho\tau, z).$$

Then by the proof of Lemma 3.6 each $G_j(\rho, \tau, z)$ is holomorphic on a neighborhood of $\widetilde{D}(V \cap \mathrm{Cl}\Omega_M, \varepsilon_1, \widetilde{\Gamma}_j)$ for an open neighborhood V of $(0, x^*)$ in $\widetilde{M}_N$, a positive constant ε_1 and open convex cones $\widetilde{\Gamma}_j$ of $\mathbb{R}^{d+n}$ containing $\{0\} \times \Gamma_j$. Therefore by the edge of the wedge theorem for $\mathscr{B}_1^F$, for any open cones $\Gamma_j' \Subset \Gamma_j$ there exist holomorphic functions $G_{jk}(\rho, \tau, z)$ defined on a neighborhood of $\widetilde{D}(W \cap \mathrm{Cl}\Omega_M, \varepsilon_2, \widetilde{\Gamma}_{jk})$ for an open neighborhood W of $(0, x^*)$ in $\widetilde{M}_N$, a positive constant ε_2 and open convex cones $\widetilde{\Gamma}_{jk}$ of $\mathbb{R}^{d+n}$ containing $\{0\} \times (\Gamma_j' + \Gamma_k')$ such that

$$G_j = \sum_{k=1}^{J} G_{jk}, \qquad G_{jk} + G_{kj} = 0 \qquad (1 \leqslant j, k \leqslant J).$$

Set $\tau' := (\tau_2, \ldots, \tau_n)$. Then as in the proof of Lemma 3.7, each G_{jk} can be developed into a power series of the form

$$G_{jk}(\rho, \tau, z) = \sum_{\nu=0}^{\infty} \sum_{\alpha' \in \mathbb{N}_0^{d-1}} a_{jk,\nu\alpha'}(\tau_1, z)\, \rho^{\nu} (\tau')^{\alpha'}.$$

Let us set

$$G_{jk}^{\circ}(\rho, \tau, z) := \sum_{\alpha' \in \mathbb{N}_0^{d-1}} \sum_{\nu \geqslant |\alpha'|} a_{jk,\nu\alpha'}(1, z)\rho^{\nu} \tau_1^{\nu-|\alpha|}(\tau')^{\alpha'}.$$

Further let us define G_j° for any G_j similarly. Note that since each G_j satisfies the assumptions of Lemma 3.7, we have $G_j^{\circ} = G_j$. Let us set

$$F_{jk}(\tau, z) = G_{jk}^{\circ}\Big(\tau_1, 1, \frac{\tau'}{\tau_1}, z\Big).$$

Then each F_{jk} is holomorphic on a neighborhood of $D_0(\delta, \Gamma_j' + \Gamma_k')$ for a positive constant δ. Since

$$G_j(\rho, \tau, z) = \sum_{k=1}^{J} G_{jk}(\rho, \tau, z), \qquad G_{jk}(\rho, \tau, z) + G_{kj}(\rho, \tau, z) = 0 \qquad (1 \leqslant j, k \leqslant J),$$

it follows that

$$G_j = G_j^{\circ} = \sum_{k=1}^{J} G_{jk}^{\circ}, \qquad G_{jk}^{\circ} + G_{kj}^{\circ} = 0 \qquad (1 \leqslant j, k \leqslant J).$$

In particular, we see that

$$F_j(\tau, z) = \sum_{k=1}^{J} F_{jk}(\tau, z), \qquad F_{jk}(\tau, z) + F_{kj}(\tau, z) = 0 \qquad (1 \leqslant j, k \leqslant J).$$

This completes the proof. □

For $(z, \zeta) \in \mathbb{C}^n \times (\mathbb{C}^n \setminus \{0\})$, let us set

$$W(z; \zeta) := \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \times \frac{\Big(1 - \sqrt{-1}\frac{\langle z, \zeta\rangle}{\sqrt{\langle\zeta, \zeta\rangle}}\Big)^{n-2}\Big(1 - \sqrt{-1}\frac{\langle z, \zeta\rangle}{\sqrt{\langle\zeta, \zeta\rangle}} - \Big(\langle z, z\rangle - \frac{\langle z, \zeta\rangle^2}{\langle\zeta, \zeta\rangle}\Big)\Big)}{\Big(\langle z, \zeta\rangle + \sqrt{-1}\Big(\langle z, z\rangle\sqrt{\langle\zeta, \zeta\rangle} - \frac{\langle z, \zeta\rangle^2}{\sqrt{\langle\zeta, \zeta\rangle}}\Big)\Big)^n},$$

where we set $\langle z, \zeta\rangle := \sum_{j=1}^{n} z_j \zeta_j$ and choose a branch as $\sqrt{1} = 1$.

3.10 Proposition. *Let x^* be a point of $T_N M$ and (τ, z) an admissible local coordinate system around $\tau_N(x^*)$ such that $\tau_N(x^*) = 0$ in this system. Let ε be a positive constant, U an open neighborhood of x^* in $\widetilde{M}_N$, and Γ_j $(1 \leqslant j \leqslant J)$ open convex cones of $\mathbb{R}^n$. Suppose that $p_M(U \cap \mathrm{Cl}\Omega_M) \supset \{(0, x) \in N; |x| < \varepsilon\}$. Let $F_j(\tau, z)$ be a holomorphic function defined on a neighborhood of $D(p_M(U \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_j)$. Set*

$$F(\tau,\ z;\ \zeta):=\sum_{j=1}^{J}\int_{C(\varepsilon,y_j)}F_j(\tau,\ w)\ W(z-w;\ \zeta)\ dw,$$

where

$$C(\varepsilon,\ y_j):=\left\{w\in\mathbb{C}^n;\ |\mathrm{Re}\ w|\leqslant\frac{\varepsilon}{2},\ \mathrm{Im}\ w=y_j\right\}$$

for a $y_j\in\Gamma_j$. *Then the following two conditions are equivalent for any* $\xi^*\in\mathbb{R}^n\setminus\{0\}$:

(1) *There exists a positive constant* δ *such that* $F(\tau,\ z;\ \zeta)$ *is holomorphic on a neighborhood of* $(0,\ \xi^*)\in\mathbb{C}^{d+n}\times\mathbb{R}^n$ *if* $y_j\in\Gamma_j$ *and* $|y_j|<\delta$.

(2) *There exist a natural number* K *and holomorphic functions* $G_k(\tau,\ z)$ $(1\leqslant k\leqslant K)$ *on a neighborhood of* $D(p_M(V\cap\mathrm{Cl}\Omega_M),\ \delta,\ \Xi_k)$ *for a neighborhood* V *of* x^* *in* $\widetilde{M}_N$, *a positive constant* δ *and open convex cones* Ξ_k *of* $\mathbb{R}^n$ *with* $\xi^*\notin\Xi_k{}^\circ$ *such that*

$$\sum_{j=1}^{J}F_j(t,\ x+\sqrt{-1}\Gamma_j\, 0)=\sum_{k=1}^{K}G_k(t,\ x+\sqrt{-1}\Xi_k\, 0)$$

holds on $\tilde{p}_M(V\cap\Omega_M)$.

Note that the defining domain of $F(\tau,\ z;\ \zeta)$ is conic with respect to ζ since $W(z;\ \zeta)$ is homogeneous of degree $-n$ with respect to ζ.

Proof. Assume (1). Let Δ_0 be a proper convex open cone of $\mathbb{R}^n$ such that $\xi^*\in\mathrm{Int}\Delta_0{}^\circ$ and that $F(\tau,\ z;\ \zeta)$ is holomorphic on a neighborhood of $\{0\}\times(\Delta_0{}^\circ\setminus\{0\})$. Thus we can choose a natural number K and proper convex open cones $\Delta_k(1\leqslant k\leqslant K)$ such that $\mathbb{R}^n=\bigcup_{k=0}^{K}\Delta_k{}^\circ$ and $\omega(\Delta_j{}^\circ\cap\Delta_k{}^\circ\cap S^{n-1})=0$ if $j\neq k$. Here ω denotes the standard volume element on the sphere S^{n-1}. Choosing $y_j\in\Gamma_j$ such that $|y_j|<\delta$, set

$$G_k(\tau,\ z):=\int_{\Delta_k{}^\circ\cap S^{n-1}}F(\tau,\ z;\ \xi)\,\omega(\xi).$$

Then, for any open subcone $\Delta_k'\Subset\Delta_k$, G_k are holomorphic on a neighborhood of $D(p_M(V\cap\mathrm{Cl}\Omega_M),\ \varepsilon_1,\ \Delta_k')$ for a neighborhood V of x^* in $\widetilde{M}_N$ and an $\varepsilon_1>0$. Moreover G_0 is holomorphic on a neighborhood of $0\in\mathbb{C}^{d+n}$. By virtue of the inverse formula of Radon transforms (see A. Kaneko [Kn], K. Kataoka [Kt 2])

$$\sum_{j=1}^{J}F_j(t,\ x+\sqrt{-1}\Gamma_j\, 0)=\sum_{k=0}^{K}G_k(t,\ x+\sqrt{-1}\Delta_k'\, 0)$$

holds on $\tilde{p}_M(U\cap\Omega_M)$. Since $\xi^*\notin\Delta_k{}^\circ$ for $1\leqslant k\leqslant K$ we obtain (2).

Conversely assume (2). Set $F_{J+k}:=-G_k$ and $\Gamma_{J+k}=\Xi_k$ for $1\leqslant j\leqslant K$. Lemma

3.9 entails that for any open convex cones $\Gamma'_j \Subset \Gamma_j$, there exist holomorphic functions $F_{jk}(\tau, z)$ on a neighborhood of $D_0(\varepsilon, \Gamma'_j+\Gamma'_k)$ with a positive constant ε such that

$$F_j=\sum_{j=1}^{J+K} F_{jk}, \qquad F_{jk}+F_{kj}=0 \qquad (1\leqslant j,\ k\leqslant J+K).$$

We may assume that $p_M(V\cap \mathrm{Cl}\Omega_M)\supset\{(0, x)\in N;\ |x|<\delta\}$ for a positive constant $\delta<\min\{\varepsilon, \varepsilon_2\}$. Choose $y_j\in\Gamma_j$ for $1\leqslant j\leqslant J+K$ and set

$$F(\tau, z; \zeta):=\sum_{j=1}^{J}\int_{C(\delta, y_j)} F_j(\tau, w)\, W(z-w; \zeta)\, dw,$$

$$G(\tau, z; \zeta):=\sum_{k=1}^{K}\int_{C(\delta, y_{J+k})} G_k(\tau, w)\, W(z-w;\zeta)\ dw.$$

Then we have

$$\begin{aligned} F(\tau, z; \zeta)-G(\tau, z; \zeta)&=\sum_{j=1}^{J+K}\sum_{k=1}^{J+K}\int_{C(\delta, y_j)} F_{jk}(\tau, w)\, W(z-w; \zeta)\, dw \\ &=\sum_{1\leqslant j<k\leqslant J+K}\int_{C_{jk}} F_{jk}(\tau, w)\, W(z-w; \zeta)\, dw, \end{aligned}$$

where

$$C_{jk}:=\left\{w\in\mathbb{C}^n;\ |\mathrm{Re}\ w|=\frac{\delta}{2},\ \mathrm{Im}\ w=(1-t)\, y_j+ty_k \text{ for some } 0\leqslant t\leqslant 1\right\}$$

with an appropriate orientation. Then we can easily see that the integrals of the last line are holomorphic on a neighborhood of $(0, \xi)\in\mathbb{C}^{d+n}\times(\mathbb{R}^n\backslash\{0\})$ for any non-zero ξ if each $|y_j|$ is small enough. Since $G(\tau, z; \zeta)$ is holomorphic on a neighborhood of $(0, \xi^*)\in\mathbb{C}^{d+n}\times\mathbb{R}^n$, so is $F(\tau, z; \zeta)$. This completes the proof. □

End of Proof of Theorem 3.8. Assume that the theorem is proved for $J-1$ $(J\geqslant 3)$. We may assume that $p_M(U\cap \mathrm{Cl}\Omega_M)\supset\{(0, x)\in N;\ |x|<\varepsilon\}$. Choose $y_J\in\Gamma_J$ and set

$$F(\tau, z; \zeta):=\int_{C(\varepsilon, y_J)} F_J(\tau, z)\, W(z-w; \zeta)\, dw.$$

Then since

$$-F_J(t, x+\sqrt{-1}\Gamma_J 0)=\sum_{j=1}^{J-1} F_j(t, x+\sqrt{-1}\Gamma_j 0),$$

by virtue of Proposition 3.10, $F(\tau, z; \zeta)$ is holomorphic on a neighborhood of

$\{0\} \times (\mathbb{R}^n \setminus (\Gamma_J{}^\circ \cap (\bigcup_{j=1}^{J-1} \Gamma_j{}^\circ)))$ if $|y_J|$ is sufficiently small. Let Γ''_j be convex open cones such that $\Gamma'_j \Subset \Gamma''_j \Subset \Gamma_j$, and $\Delta_j\,(1 \leqslant j \leqslant J-1)$ proper convex open cones of $\mathbb{R}^n$ such that

$$\Delta_j{}^\circ \Subset (\Gamma''_j + \Gamma''_J)^\circ, \qquad \bigcup_{j=1}^{J-1} \Delta_j{}^\circ \supset \Gamma_J{}^\circ \cap (\bigcup_{j=1}^{J-1} \Gamma_j{}^\circ),$$

and $\omega(\Delta_j{}^\circ \cap \Delta_k{}^\circ \cap S^{n-1}) = 0$ if $j \neq k$. Set

$$G_j(\tau, z) := \int_{\Delta_j{}^\circ \cap S^{n-1}} F(\tau, z; \xi)\, \omega(\xi)$$

for $1 \leqslant j \leqslant J-1$ and moreover set

$$G_0(\tau, z) := \int_{S^{n-1} \setminus \bigcup_{j=1}^{J-1} \Delta_j{}^\circ} F(\tau, z; \xi)\, \omega(\xi).$$

Then

$$F_J(\tau, z) = \sum_{j=0}^{J-1} G_j(\tau, z)$$

holds and $G_0(\tau, z)$ is holomorphic on a neighborhood of $0 \in \mathbb{C}^{d+n}$. Set

$$H_j(\tau, z) := \begin{cases} F_1(\tau, z) + G_0(\tau, z) + G_1(\tau, z) & (j=1), \\ F_j(\tau, z) + G_j(\tau, z) & (2 \leqslant j \leqslant J-1). \end{cases}$$

Then we have

$$\sum_{j=1}^{J-1} H_j(t, x + \sqrt{-1}\,\Gamma''_j 0) = 0$$

on $\tilde{p}_M(U' \cap \Omega_M)$ for a neighborhood U' of x^* in $\widetilde{M}_N$. By the induction hypothesis, we can find holomorphic functions H_{jk} defined on a neighborhood of $D(p_M(V \cap \mathrm{Cl}\Omega_M), \delta, \Gamma'_j + \Gamma'_k)$ with a neighborhood V of x^* in $\widetilde{M}_N$ and a positive constant δ such that

$$H_j = \sum_{k=1}^{J-1} H_{jk}, \qquad H_{jk} + H_{kj} = 0 \qquad (1 \leqslant j,\ k \leqslant J-1).$$

Thus set

$$F_{jk}(\tau, z) := \begin{cases} H_{jk}(\tau, z) & (1 \leqslant j,\ k \leqslant J-1), \\ -G_0(\tau, z) - G_1(\tau, z) & (j=1,\ k=J), \\ -G_j(\tau, z) & (2 \leqslant j \leqslant J-1,\ k=J). \end{cases}$$

Then F_{jk} satisfy the properties we need. □

3.11 Proposition. *There exists a morphism*

$$\gamma^F_{N|M}: \mathscr{B}^F_{N|M} \longrightarrow \tau_N^{-1}\mathscr{B}_N$$

defined by

$$\gamma^F_{N|M}(u)\,(x) := \sum_{j=1}^{J} F_j(0,\, x+\sqrt{-1}\Gamma,\, 0)$$

if a section $u(t, x)$ of $\mathscr{B}^F_{N|M}$ is expressed as in Definition 3.1. *In particular, $\gamma^F_{N|M}$ induces an isomorphism:*

$$\mathscr{B}^F_{N|M}\Big/\sum_{j=1}^{d} t_j\,\mathscr{B}^F_{N|M} \simeq \tau_N^{-1}\mathscr{B}_N$$

Proof. $\gamma^F_{N|M}(u)(x)$ does not depend on the choice of defining functions of $u(t, x)$ by virtue of Theorem 3.8. Moreover, it is easy to see that $\gamma^F_{N|M}$ is surjective and its kernel is $\sum_{j=1}^{d} t_j\,\mathscr{B}^F_{N|M}$. □

This proposition means that the boundary value $\gamma^F_{N|M}(u)(x)$ does not depend on the direction along which the boundary value is taken. More precisely, let $u(t, x)$ be a section of $\mathscr{B}^F_{N|M}$ on an open set U of T_NM with connected fibers. Then there exists a section $v(x)$ of $\mathscr{B}_N$ on $\tau_N(U)$ such that $\gamma^F_{N|M}(u|_V) = \tau_N^{-1}(v)|_V$ for any open subset V of U.

As a special case of F-mild hyperfunctions in relation to Lemma 3.3, let us consider F-mild hyperfunctions on a whole fiber of τ_N:

3.12 Proposition. *Let $u(t, x)$ be a hyperfunction on $M\setminus N$ which is F-mild at any point of $\dot{T}_NM$. Then there exists a unique hyperfunction $v(t, x)$ on M such that $v(t, x) = u(t, x)$ on $M\setminus N$ and $\mathrm{SS}_M(v) \cap \dot{T}^*_NM = \emptyset$.*

Proof. The uniqueness of $v(t, x)$ follows immediately from the Holmgren uniqueness theorem for hyperfunctions. Hence it suffices to prove the existence of $v(t, x)$ on a neighborhood of each point of N. Let $\mathring{x}$ be a point of N, and (τ, z) an admissible coordinate system such that $z(\mathring{x}) = 0$. Since $u(t, x)$ is F-mild at each point of $\tau_N(\mathring{x})$, there exist open cones U^k $(1 \leqslant k \leqslant K)$ of $\mathbb{R}^d$ such that $\bigcup_{k=1}^{K} U^k = \mathbb{R}^d\setminus\{0\}$, a positive constant ε, and holomorphic functions $F^k_j(\tau, z)$ $(1 \leqslant j \leqslant J_k)$ defined on a neighborhood of

$$\{(t, z) \in \mathbb{R}^d \times \mathbb{C}^n;\ |t|,\ |z| < \varepsilon,\ t \in U^k \cup \{0\},\ \operatorname{Im} z \in \Gamma_j^k\}$$

with open convex cones Γ_j^k of $\mathbb{R}^n$ such that

$$u(t, x) = \sum_{j=1}^{J_k} F_j^k(t, x + \sqrt{-1}\Gamma_j^k 0)$$

holds on $\{(t, x) \in M;\ |t|,\ |x| < \varepsilon,\ t \in U^k\}$. Choosing sufficiently small $y_j^k \in \Gamma_j^k$ we set

$$F^k(\tau, z; \zeta) := \sum_{j=1}^{J_k} \int_{C(\varepsilon, y_j^k)} F_j^k(\tau, w)\, W(z - w; \zeta)\, dw.$$

Then $F^k(\tau, z; \zeta)$ is holomorphic on a neighborhood of

$$\Big\{(t, z, \xi) \in \mathbb{R}^d \times \mathbb{C}^n \times (\mathbb{R}^n \backslash \{0\});\ |t| < \varepsilon,\ |z| < \frac{\varepsilon}{4},\ t \in U^k \cup \{0\},$$
$$\langle y, \xi\rangle > \langle y, y\rangle \sqrt{\langle \xi, \xi\rangle} - \frac{\langle y, \xi\rangle^2}{\sqrt{\langle \xi, \xi\rangle}}\Big\}$$

if ε is small enough. Proposition 3.10 implies that there exists a positive constant $\delta < \varepsilon/4$ such that $F^k(\tau, z; \zeta) - F^l(\tau, z; \zeta)$ is holomorphic on a neighborhood of $B_\delta \times (\mathbb{R}^n \backslash \{0\}) := \{(t, z, \xi) \in \mathbb{R}^d \times \mathbb{C}^n \times (\mathbb{R}^n \backslash \{0\});\ |t|,\ |z| < \delta\}$ if $U^k \cap U^l \neq \emptyset$. It is easy to see that $F^k(\tau, z; \zeta) - F^l(\tau, z; \zeta)$ is holomorphic on a neighborhood of $B_\delta \times (\mathbb{R}^n \backslash \{0\})$ for any k and l by taking chain of U^n's which connects U^k and U^l. Hence there exist holomorphic functions $G^k(\tau, z; \zeta)$ on a neighborhood of $B_\delta \times (\mathbb{R}^n \backslash \{0\})$ such that $G^k - G^l = F^k - F^l$. Then $F^k - G^k$ defines a holomorphic function $H(\tau, z; \zeta)$ on a neighborhood of

$$\{(t, z, \xi) \in \mathbb{R}^d \times \mathbb{C}^n \times (\mathbb{R}^n \backslash \{0\});\ |t|,\ |z| < \delta,\ \langle y, \xi\rangle > \langle y, y\rangle \sqrt{\langle \xi, \xi\rangle} - \frac{\langle y, \xi\rangle^2}{\sqrt{\langle \xi, \xi\rangle}}\}$$

Let Δ_j $(1 \leqslant j \leqslant J)$ be proper convex open cones of $\mathbb{R}^n$ such that $\bigcup_{j=1}^{J} \Delta_j^\circ = \mathbb{R}^n$ and $\omega(\Delta_j^\circ \cap \Delta_k^\circ \cap S^{n-1}) = 0$ if $j \neq k$. Set

$$H_j(\tau, z) := \int_{\Delta_j^\circ \cap S^{n-1}} H(\tau, z; \xi)\, \omega(\xi).$$

Then

$$w(t, x) := \sum_{j=1}^{J} H_j(t, x + \sqrt{-1}\Delta_j 0)$$

defines a hyperfunction on $\{(t, x) \in \mathbb{R}^{d+n};\ |t|,\ |x| < \delta\}$ with real analytic parameters t. In view of the inverse formula for Radon transforms, we have

$$u(t,x)-w(t,x)=\int_{S^{n-1}}G^k(t,x;\xi)\,\omega(\xi)$$

on $\{(t,x)\in\mathbb{R}^{d+n};\ |t|,\ |x|<\delta/2,\ t\in U^k\}$. Since the integral on the right-hand side is analytic on $\{(t,x)\in\mathbb{R}^{d+n};\ |t|,\ |x|<\delta/2\}$, $u(t,x)-w(t,x)$ is continued to an analytic function on $\{(t,x)\in\mathbb{R}^{d+n};\ |t|,\ |x|<\delta/2\}$. This completes the proof. $\square$

§4. F-Mild Microfunctions

In this section, we microlocalize the F-mildness property. To this end, we introduce new sheaves.

Recall the mapping defined in the beginning of Section 1. Further let us set $\iota_Y:=i_L\, i_Y\colon Y\to X$. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & N & T_NM \arrow[l, "\tau_N"'] & T^*_{T_NM}T_YL \arrow[l, "\pi_{N|M}"'] \\
 & T^*_NY \arrow[u, "\pi_N"] \arrow[ur, phantom, "\square"] & T_NM\underset{N}{\times}T^*_NY \arrow[l, "\tau_{Y\pi}"'] \arrow[u] \arrow[ur, "{}^t\iota'_Y", "\sim"'] & T_NM\underset{\widetilde{M}_N}{\times}T^*_{\widetilde{M}_N}\widetilde{L}_Y \arrow[u, "s_{L\pi}"', "\wr"] \arrow[d, "{}^ts'_L"'] \\
 & N\underset{M}{\times}T^*_MX \arrow[u, "{}^t\iota'_Y"] \arrow[r, "{}^ti'_L"'] \arrow[dl, "\iota_{Y\pi}"'] \arrow[d, hook, "i_{Y\pi}"] \arrow[dr, phantom, "\square"] & N\underset{M}{\times}T^*_ML \arrow[ul, "{}^t\iota''_Y"', "\sim"] \arrow[d, hook, "i_{Y\pi}"] & T^*_{\widetilde{M}_N}\widetilde{L}_Y \\
T^*_MX & M\underset{M}{\times}T^*_MX \arrow[l, "\sim", "i_{L\pi}"'] \arrow[r, "{}^ti'_L"'] & T^*_ML & \widetilde{M}_N\underset{M}{\times}T^*_ML. \arrow[l, "p_{L\pi}"'] \arrow[u, "{}^tp'_L"', "\wr"]
\end{tikzcd}$$

4.1 Definition. We set:

$$\widetilde{\mathscr{C}}^A_{N|M}:=\mathscr{H}^n(\mu_N(\iota_Y^{-1}\mathscr{O}_X\otimes or_{N/Y})),$$
$$\widetilde{\mathscr{B}}^A_{N|M}:=\widetilde{\mathscr{C}}^A_{N|M}|_N=\mathscr{H}^n_N(\iota_Y^{-1}\mathscr{O}_X)\otimes or_{N/Y},$$
$$\widetilde{\mathscr{A}}^A_{N|M}:=\mathscr{H}^0(\nu_N(\iota_Y^{-1}\mathscr{O}_X)).$$

Thus $\widetilde{\mathscr{C}}^A_{N|M}$, $\widetilde{\mathscr{B}}^A_{N|M}$ and $\widetilde{\mathscr{A}}^A_{N|M}$ are sheaves on T^*_NY, N, and T_NY respectively.

By the same arguments as in the theory of microfunctions, we can prove the following:

(1) There exist a natural monomorphism

$$\tilde{\mathfrak{b}}^A_{N|M}\colon \widetilde{\mathscr{A}}^A_{N|M}\rightarrowtail \tau^{-1}\widetilde{\mathscr{B}}^A_{N|M}$$

and a natural epimorphism

$$\widetilde{\mathrm{sp}}^A_{N|M}:\ \pi_N^{-1}\widetilde{\mathscr{B}}^A_{N|M} \twoheadrightarrow \widetilde{\mathscr{C}}^A_{N|M},$$

where τ: $T_N Y \to N$ is the canonical projection.

(2) Let us identify X with $\mathbb{C}^{d+n}$ by an admissible coordinate system (τ, z). Then we can represent any germ $f(t, x)$ of $\widetilde{\mathscr{B}}^A_{N|M}$ at the origin as

$$f(t, x) = \sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0)$$

for some natural number J. Here Γ_j are open cones of $\mathbb{R}^n$ and each $F_j(\tau, z)$ is holomorphic on a neighborhood of $D_0(\varepsilon, \Gamma_j)$ in X for a positive constant ε.

Moreover, we can prove the following:

4.2 Lemma. *Let x_0 be a point of N and (τ, z) an admissible local coordinate system of (τ, z) such that $x_0=0$ in this system. Then the following hold:*

(1) *Let $f(t, x)$ be a germ of $\widetilde{\mathscr{B}}^A_{N|M}$ at x_0. Then $(x_0; \sqrt{-1}\langle \xi^*, dx\rangle) \in T^*_N Y$ is not contained in* $\mathrm{supp}\,(\widetilde{\mathrm{sp}}^A_{N|M}(f))$ *if and only if there exist holomorphic functions $F_j(t, z)$ defined on a neighborhood of $D_0(\varepsilon, \Gamma_j)$ in X with a positive constant ε such that each Γ_j is an open cone of $\mathbb{R}^n$ with $\xi^* \notin \Gamma_j^\circ$ and that*

$$f(t, x) = \sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0)$$

(2) *Let ε be a positive constant and Γ_j an open convex cones of $\mathbb{R}^n$. Let $F_j(\tau, z)$ be holomorphic functions defined on a neighborhood of $D_0(\varepsilon, \Gamma_j)$ in X such that*

$$\sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0) = 0$$

holds as a germ of $\widetilde{\mathscr{B}}^A_{N|M}$ at x_0. Then for any open convex cones Γ'_j of $\mathbb{R}^n$ such that $\Gamma'_j \Subset \Gamma_j$, there exist a positive constant δ and holomorphic functions $F_{jk}(\tau, z)$ defined on a neighborhood of $D_0(\delta, \Gamma'_j+\Gamma'_k)$ in X such that

$$F_j(\tau, z) = \sum_{k=1}^{J} F_{jk}(\tau, z), \qquad F_{jk}(\tau, z) + F_{kj}(\tau, z) = 0 \qquad (1 \leqslant j, k \leqslant J).$$

The proof is similar to that of Lemma 2.1 of Oaku [O 3].

4.3 Lemma. *There exists a natural monomorphism*

$$\alpha^F_{N|M}\colon \widetilde{\mathscr{B}}^F_{N|M} \rightarrowtail \tau_N^{-1}\widetilde{\mathscr{B}}^A_{N|M}.$$

Proof. In view of Theorem 3.8, we can naturally define $\tilde{\alpha}_{N|M}$. Let us verify the injectivity. Let x^* be a point of $T_N M$ and (τ, z) an admissible local coordinate system of X around $x_0 := \tau_N(x^*)$ such that $z(x_0) = 0$. Suppose that $f(t, x) = \sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0)$ is a germ of $\mathscr{B}^F_{N|M}$ at x^* such that $\alpha^F_{N|M}(f) = 0 \in \tilde{\mathscr{B}}^A_{N|M, x_0}$. Here each $F_j(\tau, z)$ is holomorphic on a neighborhood of $D(p_M(U \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_j)$ in X for an open neighborhood of U of x^* in $\tilde{M}_N$, a positive constant ε and an open convex cone Γ_j in $\mathbb{R}^n$. Then by Lemma 4.2 (2), for any open convex cones Γ'_j of $\mathbb{R}^n$ such that $\Gamma'_j \Subset \Gamma_j$, there exist a positive constant δ and holomorphic functions $F_{jk}(\tau, z)$ defined on a neighborhood of $D_0(\delta, \Gamma'_j + \Gamma'_k)$ in X such that

$$F_j(\tau, z) = \sum_{k=1}^{J} F_{jk}(\tau, z), \qquad F_{jk}(\tau, z) + F_{kj}(\tau, z) = 0 \qquad (1 \leqslant j, k \leqslant J).$$

Now we shall use the same notation as in the proof of Proposition 3.10. Choose $y_j \in \Gamma'_j$ for $1 \leqslant j \leqslant J$ and set

$$\begin{aligned} F(\tau, z; \zeta) :&= \sum_{j=1}^{J} \int_{C(\delta, y_j)} F_j(\tau, w)\, W(z-w; \zeta)\, dw \\ &= \sum_{1 \leqslant j < k \leqslant J} \int_{C_{jk}} F_{jk}(\tau, w)\, W(z-w; \zeta)\, dw. \end{aligned}$$

Then by the second expression we can easily see that the right-hand side is holomorphic on a neighborhood of $(0, \xi) \in \mathbb{C}^{d+n} \times (\mathbb{R}^n \setminus \{0\})$ for any non-zero ξ if each $|y_j|$ is small enough. Let us set

$$\tilde{F}(\tau, z) := \int_{S^{n-1}} F(\tau, z; \xi)\, \omega(\xi).$$

Then $\tilde{F}(\tau, z)$ is holomorphic for $|\tau|, |z| < \delta'$. Moreover, by virtue of the inverse formula of Radon transformations we have

$$f(t, x) = \sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j 0) = \tilde{F}(t, x);$$

that is, $f(t, x)$ analytic at $(t, x) = (0, 0)$.

Hence by the injectivity of $\tilde{\mathrm{b}}^A_{N|M}: \tilde{\mathscr{A}}^A_{N|M} \rightarrowtail \tau^{-1}\tilde{\mathscr{B}}^A_{N|M}$, the condition $\alpha^F_{N|M}(f) = 0$ implies $f(t, x) = 0$ as an analytic function. This completes the proof. □

By this lemma, we can regard $\mathscr{B}^F_{N|M}$ as a subsheaf of $\tau_N^{-1} \tilde{\mathscr{B}}^A_{N|M}$.

4.4 Definition. The sheaf of *F-mild microfunctions on* $T^*_{T_N M} T_Y L$ *is defined* *by*

$$\widetilde{\mathscr{C}}^F_{N|M} := \mathrm{Image}\,(\widetilde{\mathrm{sp}}^A_{N|M}\,\alpha^F_{N|M}\colon \pi^{-1}_{N|M}\,\mathscr{B}^F_{N|M} \longrightarrow ({}^t\tau'_Y)_!\,\tau^{-1}_{Y\pi}\,\widetilde{\mathscr{C}}^A_{N|M}).$$

Sections of $\mathscr{C}^F_{N|M}$ are called *F-mild microfunctions*. The morphism $\widetilde{\mathrm{sp}}^A_{N|M}\,\alpha^F_{N|M}$ induces an epimorphism

$$\mathrm{sp}^F_{N|M}\colon \pi^{-1}_{N|M}\,\mathscr{B}^F_{N|M} \twoheadrightarrow \mathscr{C}^F_{N|M}.$$

For a section u of $\mathscr{B}^F_{N|M}$, $\mathrm{SS}^F_{N|M}(u)$ denotes $\mathrm{supp}\,(\mathrm{sp}^F_{N|M}(u))$.

We remark that by the definition $\mathscr{C}^F_{N|M}|_{T_NM} = \mathscr{B}^F_{N|M}$.

We also denote the natural inclusion $\mathscr{C}^F_{N|M} \rightarrowtail ({}^t\tau'_Y)_!\,\tau^{-1}_{Y\pi}\,\widetilde{\mathscr{C}}^A_{N|M}$ by $\alpha^F_{N|M}$.

4.5 Lemma. *Let $u(t, x)$ be a germ of $\mathscr{B}^F_{N|M}$ at a point x^* of T_NM. Then a point $p = (x^*; \sqrt{-1}\,\xi^*)$ of $T^*_{T_NM}T_YL$ is not contained in $\mathrm{SS}^F_{N|M}(u)$ if and only if $u(t, x)$ has an expression as in Definition 3.1 such that ξ^* does not contained in Γ_j° for any j.*

Proof. If $\xi^* = 0$, the proof is trivial. Thus we may assume that $\xi^* \neq 0$. Suppose that $u(t, x)$ has an expression as in Definition 3.1 and ξ^* does not contained in Γ_j° for any j. Then by Lemma 4.2 (1), we have $p \notin \mathrm{SS}^F_{N|M}(u)$. Conversely, suppose that u is a germ of $\mathscr{B}^F_{N|M}$ at a point x^* and $p \notin \mathrm{SS}^F_{N|M}(u)$. Then we may assume that by an admissible local coordinate system $x^* = 0 + \partial/\partial t_1$ and that

$$u(t, x) = F_0(t, x+\sqrt{-1}\Gamma_0\,0),$$

where $F_0(t, z)$ is holomorphic on a neighborhood of $D(p_M(U \cap \mathrm{Cl}\Omega_M), \varepsilon, \Gamma_0)$ with a neighborhood U of x^*, a positive constant ε and an open convex cone Γ_0 of $\mathbb{R}^n$ satisfying $\xi^* \in \Gamma_0^\circ$. By virtue of Lemma 4.2 (1), there exist holomorphic functions $F_j(t,z)$ defined on a neighborhood of $D_0(\delta, \Gamma_j)$ in X with a positive constant δ such that each Γ_j is an open cone of $\mathbb{R}^n$ with $\xi^* \notin \Gamma_j^\circ$ and that

$$F_0(t, x+\sqrt{-1}\Gamma_0\,0) + \sum_{j=1}^{J} F_j(t, x+\sqrt{-1}\Gamma_j\,0) = 0.$$

Choose open convex subcones $\Gamma'_j \Subset \Gamma_j$ with $\xi^* \in \mathrm{Int}\,(\Gamma'_0)^\circ$ and $\xi^* \notin (\Gamma'_j)^\circ$ $(1 \leqslant j \leqslant J)$. Then, by Lemma 4.2 (2), there exist a positive constant δ' and holomorphic functions F_{jk} defined on a neighborhood of $D_0(\delta', \Gamma'_j + \Gamma'_k)$ such that

$$F_j = \sum_{k=0}^{J} F_{jk}, \qquad F_{jk} + F_{kj} = 0 \qquad (1 \leqslant j,\ k \leqslant J).$$

Now we shall use the notation of the proof of Proposition 3.10. Choose $y_j \in \Gamma'_j$ for $0 \leqslant j \leqslant J$ and set

$$F(\tau, z; \zeta) := \int_{C(\delta', y_0)} F_0(\tau, w)\, W(z-w; \zeta)\, dw,$$

$$G(\tau, z; \zeta) := \sum_{j=1}^{J} \int_{C(\delta', y_j)} F_j(\tau, w)\, W(z-w; \delta)\, dw.$$

Then we have

$$F(\tau, z; \zeta) + G(\tau, z; \zeta) = \sum_{0 \leqslant j < k \leqslant J} \int_{C_{jk}} F_{jk}(\tau, w)\, W(z-w; \zeta)\, dw.$$

Each integral of the right-hand side is holomorphic on a neighborhood of $(0, \xi) \in \mathbb{C}^{d+n} \times (\mathbb{R}^n \setminus \{0\})$ for any non-zero ξ if each $|y_j|$ is small enough. Since $G(\tau, z; \zeta)$ is holomorphic on a neighborhood of $(0, \xi^*) \in \mathbb{C}^{d+n} \times \mathbb{R}^n$, so is $F(\tau, z; \zeta)$. Thus there exist a positive constant δ'' and a proper convex open cone Δ of $\mathbb{R}^n$ such that $\xi^* \in \operatorname{Int} \Delta^\circ$ and that $F(\tau, z; \zeta)$ is holomorphic on a neighborhood of

$$\{(\tau, z, \xi) \in \mathbb{C}^{d+n} \times (\mathbb{R}^n \setminus \{0\};\ |\tau|, |z| \leqslant \delta'', \xi \in \Delta^\circ\}.$$

Thus we can choose a natural number K and proper convex open cones Δ_k $(1 \leqslant k \leqslant K)$ such that $\mathbb{R}^n = \bigcup_{k=0}^{K} \Delta_k^\circ$ and $\omega(\Delta_j^\circ \cap \Delta_k^\circ \cap S^{n-1}) = 0$ if $j \neq k$.

$$G_k(\tau, z) := \int_{\Delta_k^\circ \cap S^{n-1}} F(\tau, z; \xi)\, \omega(\xi).$$

Then G_k are holomorphic on a neighborhood of $D(p_M(V \cap \mathrm{Cl}\,\Omega_M), \varepsilon_1, \Delta_k)$ for a neighborhood V of x^* in $\widetilde{M}_N$ and a constant $\varepsilon_1 > 0$. Moreover G_0 is holomorphic on a neighborhood of $0 \in \mathbb{C}^{d+n}$. By virtue of the inverse formula of Radon transforms

$$F_0(t, x + \sqrt{-1}\Gamma_0 0) = \sum_{k=0}^{K} G_k(t, x + \sqrt{-1}\Delta_k 0)$$

holds on $\tilde{p}_M(U \cap \Omega_M)$. Since $\xi^* \notin \Delta_k^\circ$ for $1 \leqslant k \leqslant K$ we prove the lemma. □

Let us set

$$\mathscr{C}^A_{N|M} := ({}^t\iota'_Y)_!\, \iota_{Y\pi}^{-1}\, \mathscr{C}_M.$$

Then we see that $\mathscr{C}^A_{N|M}|_N = \mathscr{B}^A_{N|M}$. Let us denote by $\mathrm{sp}^A_{N|M}$: $\pi_N^{-1} \mathscr{B}^A_{N|M} \twoheadrightarrow \mathscr{C}^A_{N|M}$ the spectral morphism. Therefore we have

$$\begin{aligned}
\mathscr{C}^A_{N|M} &\simeq \mathbb{R}({}^t\iota'_Y)_!\, \iota_{Y\pi}^{-1}\, \mu_M(\mathscr{O}_X) \otimes or_{M/X}[d+n] \\
&\to \mu_N(\iota_Y^{-1}\mathscr{O}_X) \otimes or_{M/X} \overset{L}{\otimes} \omega_{N/M}^{\otimes -1} \overset{L}{\otimes} \omega_{Y/X}[d+n] \\
&\simeq \mu_N(\iota_Y^{-1}\mathscr{O}_X) \otimes or_{N/Y}[n].
\end{aligned}$$

Thus we obtain a natural morphism

$$\alpha^A_{N|M}: \mathscr{C}^A_{N|M} \to \widetilde{\mathscr{C}}^A_{N|M}$$

4.6 Lemma. *The morphism $\alpha^A_{N|M}$ induces a natural monomorphism*

$$\alpha_{N|M}: ({}^t\tau'_Y)_! \, \tau_{Y\pi}^{-1} \mathscr{C}^A_{N|M} \rightarrowtail \mathscr{C}^F_{N|M}$$

such that the restriction of this morphism to the zero-section coincides with $\alpha_{N|M}$: $\tau_N^{-1}\mathscr{B}^A_{N|M} \rightarrowtail \mathscr{B}^F_{N|M}$ of Lemma 3.3 (2).

Proof. It is easy to see that we can define $\alpha_{N|M}$. The compatibility of this morphism and that of Lemma 3.3 (2) is clear by the definition. Let $p = (x^*; \sqrt{-1}\langle \xi^*, dx\rangle)$ be a point of $T^*_{T_NM} T_Y L$. In the case where $p \in T_NM$, since $({}^t\tau'_Y)_! \, \tau_{Y\pi}^{-1} \mathscr{C}^A_{N|M,p} = \mathscr{B}^A_{N|M,\tau_N(p)}$ and $\mathscr{C}^F_{N|M,p} = \mathscr{B}^F_{N|M,p}$, the injectivity is a consequence of Lemma 3.3 (2). In the case where $p \in \dot{T}^*_{T_NM} T_Y L$, we may assume that $\tau_N(x^*) = 0$. Then we can prove the injectivity at p in the same way as Lemma 4.5. This completes the proof. □

Now, we have

$$\begin{aligned}
&\boldsymbol{R}({}^t\tau'_Y)_! \, \tau_{Y\pi}^{-1} \mu_N(\iota_Y^{-1}\mathscr{O}_X \otimes or_{N/Y}) \\
&\Rightarrow \mu_{T_NM}(\tau_Y^{-1}\iota_Y^{-1}\mathscr{O}_X \overset{L}{\otimes} \omega^{\otimes -1}_{T_NM/N} \overset{L}{\otimes} \omega_{T_YL} \otimes or_{N/Y}) \\
&\simeq \mu_{T_NM}((i_L i_Y \tau_Y)^{-1}\mathscr{O}_X \otimes or_{M/L}) \simeq \mu_{T_NM}((i_L \, p_L s_L)^{-1}\mathscr{O}_X \otimes or_{M/L}) \\
&\to \mu_{T_NM}((p_L s_L)^{-1}\mathscr{B}\mathscr{O}_L \otimes or_{M/L}) \\
&\to \mu_{T_NM}(s_L^{-1}\boldsymbol{R}(j_L)_* \, j_L^{-1} p_L^{-1}\mathscr{B}\mathscr{O}_L \otimes or_{M/L}) \\
&\simeq \mu_{T_NM}(\mathscr{B}\mathscr{O}_{Y|L} \otimes or_{M/L}).
\end{aligned}$$

Hence taking the n-th cohomology, we obtain a natural morphism

$$\widetilde{\beta}_{N|M}: ({}^t\tau'_Y)_! \, \tau_{Y\pi}^{-1} \widetilde{\mathscr{C}}^A_{N|M} \to \widetilde{\mathscr{C}}_{N|M}.$$

Restricting this morphism to the zero-section, we have

$$\widetilde{\beta}_{N|M}: \tau_N^{-1} \widetilde{\mathscr{B}}^A_{N|M} \to \widetilde{\mathscr{B}}_{N|M}.$$

Similarly we can obtain natural morphisms

$$\begin{aligned}
&\beta^A_{N|M}: ({}^t\tau'_Y)_! \, \tau_{Y\pi}^{-1} \mathscr{C}^A_{N|M} \to \mathscr{C}_{N|M}, \\
&\beta^A_{N|M}: \tau_N^{-1} \mathscr{B}^A_{N|M} \to \mathscr{B}_{N|M}.
\end{aligned}$$

On the other hand, by Lemma 4.5, $\beta^F_{N|M}: \mathscr{B}^F_{N|M} \to \mathscr{B}_{N|M}$ induces a natural morphism

$$\beta^F_{N|M}: \mathscr{C}^F_{N|M} \to \mathscr{C}_{N|M}.$$

4.7 Lemma. *The following diagram is commutative:*

$$\begin{tikzcd}
& \pi_{N|M}^{-1}\mathscr{B}_{N|M}^{F} \arrow[rr, tail, "\alpha_{N|M}^{F}"] \arrow[dd, "\mathrm{sp}_{N|M}^{F}"] \arrow[dr, "\beta_{N|M}^{F}"] & & \pi_{N|M}^{-1}\tau_N^{-1}\widetilde{\mathscr{B}}_{N|M}^{A} \arrow[dd, "\widetilde{\mathrm{sp}}_{N|M}^{A}"] \arrow[dr, "\widetilde{\beta}_{N|M}"] & \\
\pi_{N|M}^{-1}\tau_N^{-1}\mathscr{B}_{N|M}^{A} \arrow[ur, "\alpha_{N|M}"] \arrow[rr, tail, "\beta_{N|M}^{A}"] \arrow[dd, "\mathrm{sp}_{N|M}^{A}"] & & \pi_{N|M}^{-1}\mathscr{B}_{N|M} \arrow[rr, tail, "\widetilde{\alpha}_{N|M}"] \arrow[dd, "\mathrm{sp}_{N|M}"] & & \pi_{N|M}^{-1}\widetilde{\mathscr{B}}_{N|M} \arrow[dd, "\widetilde{\mathrm{sp}}_{N|M}"] \\
& \mathscr{C}_{N|M}^{F} \arrow[rr, "\alpha_{N|M}^{F}"] \arrow[dr, "\beta_{N|M}^{F}"] & & ({}^t\tau'_Y)_!\tau_{Y\pi}^{-1}\widetilde{\mathscr{C}}_{N|M}^{A} \arrow[dr, "\widetilde{\beta}_{N|M}"] & \\
({}^t\tau_Y)_!\tau_{Y\pi}^{-1}\mathscr{C}_{N|M}^{A} \arrow[ur, "\alpha_{N|M}"] \arrow[rr, tail, "\beta_{N|M}^{A}"] & & \mathscr{C}_{N|M} \arrow[rr, tail, "\widetilde{\alpha}_{N|M}"] & & \widetilde{\mathscr{C}}_{N|M}.
\end{tikzcd}$$

The proof is straightforward.

Let $\gamma_{N|M}^{A}\colon \mathscr{C}_{N|M}^{A} \to \mathscr{C}_N$ and $\gamma_{N|M}^{A}\colon \mathscr{B}_{N|M}^{A} \to \mathscr{B}_N$ be the restriction morphisms. Then, these morphisms induce isomorphisms

$$\mathscr{C}_{N|M}^{A} \Big/ \sum_{j=1}^{d} t_j\, \mathscr{C}_{N|M}^{A} \simeq \mathscr{C}_N, \qquad \mathscr{B}_{N|M}^{A} \Big/ \sum_{j=1}^{d} t_j\, \mathscr{B}_{N|M}^{A} \simeq \mathscr{B}_N.$$

We shall define restriction and boundary value morphisms.

First, induced by a natural morphism $\iota_Y^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ there exists a natural morphism

$$\widetilde{\gamma}_{N|M}^{A}\colon \widetilde{\mathscr{C}}_{N|M}^{A} \to \mathscr{C}_N.$$

We also denote the restriction of this morphism to the zero-section by

$$\widetilde{\gamma}_{N|M}^{A}\colon \widetilde{\mathscr{B}}_{N|M}^{A} \to \mathscr{B}_N.$$

Then, these morphisms induce isomorphisms

$$\widetilde{\mathscr{C}}_{N|M}^{A} \Big/ \sum_{j=1}^{d} t_j\, \widetilde{\mathscr{C}}_{N|M}^{A} \simeq \mathscr{C}_N, \qquad \widetilde{\mathscr{B}}_{N|M}^{A} \Big/ \sum_{j=1}^{d} t_j\, \widetilde{\mathscr{B}}_{N|M}^{A} \simeq \mathscr{B}_N.$$

Next, by Lemma 4.5, we see that $\gamma_{N|M}^{F}\colon \mathscr{B}_{N|M}^{F} \to \tau_N^{-1}\mathscr{B}_N$ induces a morphism

$$\gamma_{N|M}^{F}\colon \mathscr{C}_{N|M}^{F} \to ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathscr{C}_N,$$

which in turn induces an isomorphism

$$\mathscr{C}^F_{N|M}\Big/\sum_{j=1}^{d} t_j\,\mathscr{C}^F_{N|M} \simeq ({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\,\mathscr{C}_N.$$

4.8 Lemma. *The following diagram is commutative:*

$$\begin{tikzcd}
\pi^{-1}_{N|M}\tau^{-1}_N\mathscr{B}^A_{N|M} \arrow[rr, two heads, "\mathrm{sp}^A_{N|M}"] \arrow[dr, hook, "\alpha_{N|M}"] \arrow[dd, "\gamma^A_{N|M}"'] & & ({}^t\tau'_Y)_!\tau^{-1}_{Y\pi}\mathscr{C}^A_{N|M} \arrow[dr, hook, "\alpha_{N|M}"] \arrow[dd, "\gamma^A_{N|M}"' near start] & \\
& \pi^{-1}_{N|M}\mathscr{B}^F_{N|M} \arrow[rr, two heads, "\mathrm{sp}^F_{N|M}" near start] \arrow[dl, "\gamma^F_{N|M}"] \arrow[dd, hook, "\alpha^F_{N|M}" near start] & & \mathscr{C}^F_{N|M} \arrow[dl, "\gamma^F_{N|M}"] \arrow[dd, hook, "\alpha^F_{N|M}"] \\
\pi^{-1}_{N|M}\tau^{-1}_N\mathscr{B}_N \arrow[rr, two heads, "\mathrm{sp}_N" near end] & & ({}^t\tau'_Y)_!\tau^{-1}_{Y\pi}\mathscr{C}_N & \\
& \pi^{-1}_{N|M}\tau^{-1}_N\widetilde{\mathscr{B}}^A_{N|M} \arrow[rr, two heads, "\widetilde{\mathrm{sp}}^A_{N|M}"] \arrow[ul, "\tilde{\gamma}^A_{N|M}"] & & ({}^t\tau'_Y)_!\tau^{-1}_{Y\pi}\widetilde{\mathscr{C}}^A_{N|M} \arrow[ul, "\tilde{\gamma}^A_{N|M}"]
\end{tikzcd}$$

The proof is straightforward.

4.9 Proposition. *The morphism $\tilde{\alpha}_{N|M}$ defined in Theorem 2.8 induces a monomorphism*

$$\bar{\alpha}_{N|M}\colon \mathscr{C}_{N|M}\Big/\beta^F_{N|M}\,\mathscr{C}^F_{N|M} \rightarrowtail \widetilde{\mathscr{C}}_{N|M}\Big/\tilde{\beta}_{N|M}({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\,\widetilde{\mathscr{C}}^A_{N|M}.$$

Proof. At the zero-section of T_NM, the proof is similar to that of Proposition 2.3 of Oaku [O 3]. Let $p=(x^*;\sqrt{-1}\langle\xi^*, dx\rangle)$ be a point of $\dot{T}^*_{T_NM}T_YL$. Let $f(t, x)$ be a germ of $\mathscr{C}_{N|M}$ at p and suppose that $\tilde{\alpha}_{N|M}(f)\in\beta^F_{N|M}({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\,\widetilde{\mathscr{C}}^A_{N|M}$. There exist a positive constant ε, an open neighborhood U of x^* in $\tilde{L}_Y$ and an open cone Γ_0 of $\mathbb{R}^n$ with $\xi^*\in\Gamma_0{}^\circ$ such that the following hold: There exist a section $F_0(t, z)$ of $\mathscr{B}\mathscr{O}_{Y,L}$ on $\{(r, t, z)\in U;\ \mathrm{Im}\ z\in\Gamma_0\}$ and a holomorphic function $G_0(\tau, z)$ defined on a neighborhood of $D_0(\varepsilon, \Gamma_0)$ such that

$$f(t, x)=\mathrm{sp}_{N|M}(F_0(t, x+\sqrt{-1}\Gamma_0 0)),$$
$$\tilde{\alpha}_{N|M}(f)(t, x)=\tilde{\beta}_{N|M}\,\widetilde{\mathrm{sp}}^A_{N|M}(G_0(t, x+\sqrt{-1}\Gamma_0 0))$$

hold at p. By Lemma 4.7, we have

$$\begin{aligned}\tilde{\alpha}_{N|M}(f)(t, x)&=\widetilde{\mathrm{sp}}_{N|M}\,\tilde{\alpha}_{N|M}(F_0(t, x+\sqrt{-1}\Gamma_0 0))\\&=\widetilde{\mathrm{sp}}_{N|M}\,\tilde{\beta}_{N|M}(G_0(t, x+\sqrt{-1}\Gamma_0 0))\end{aligned}$$

at p. Let us set

$$H_0(t, z) := F_0(t, z) - G_0(t, z)$$

as a section of $\mathscr{BO}_{Y|L}$. By Lemma 2.6, there exist a natural number J and sections $H_j(t, z)$ of $\mathscr{BO}_{Y|L}$ on $\{(t, z) \in V; \operatorname{Im} z \in \Gamma_j\}$ $(1 \leqslant j \leqslant J)$ such that

$$H_0(t, x+\sqrt{-1}\Gamma_0 0) + \sum_{j=1}^{J} H_j(t, x+\sqrt{-1}\Gamma_j 0) = 0.$$

Here V is an open neighborhood of x^* in $T_Y L$ and each Γ_j is an open cone of $\mathbb{R}^n$ such that $\xi^* \notin \Gamma_j^{\circ}$. Choose subcones $\Gamma_j' \Subset \Gamma_j$ such that $\xi^* \in \operatorname{Int}(\Gamma_0')^{\circ}$ and $\xi^* \notin (\Gamma_j')^{\circ}$ for $1 \leqslant j \leqslant J$. Then, by Proposition 2.7 there exist sections $H_{jk}(t, z)$ of $\mathscr{BO}_{Y|L}$ on $\{(t, z) \in V'; \operatorname{Im} z \in \Gamma_j' + \Gamma_k'\}$ such that

$$H_j = \sum_{k=0}^{J} H_{jk}, \qquad H_{jk} + H_{kj} = 0 \qquad (0 \leqslant j,\ k \leqslant J),$$

where V' is an open neighborhood of x^* in $T_Y L$. Now we shall use the notation of the proof of Proposition 3.10. Choose $y_j \in \Gamma_j'$ $(0 \leqslant j \leqslant J)$ and sufficiently small positive constant δ and set

$$\begin{aligned}
F(t, z; \zeta) &:= \int_{C(\delta, y_0)} F_0(t, w)\, W(z-w; \zeta)\, dw, \\
G(t, z; \zeta) &:= \int_{C(\delta, y_0)} G_0(t, w)\, W(z-w; \zeta)\, dw, \\
H(t, z; \zeta) &:= \int_{C(\delta, y_0)} H_0(t, w)\, W(z-w; \zeta)\, dw, \\
H'(\tau, z; \zeta) &:= \sum_{j=1}^{J} \int_{C(\delta, y_j)} H_j(t, w)\, W(z-w; \zeta)\, dw.
\end{aligned}$$

Then

$$H(t, z; \zeta) + H'(t, z; \zeta) = \sum_{0 \leqslant j < k \leqslant J} \int_{C_{jk}} H_{jk}(t, w)\, W(z-w; \zeta)\, dw.$$

The right-hand side of the integral above defines a section of $\mathscr{BO}_{Y \times \mathbb{C}^n | L \times \mathbb{C}^n}$ on a neighborhood of $W \times (\mathbb{R}^n \setminus \{0\})$. Moreover we see that $H'(t, z; \zeta)$ defines a section of $\mathscr{BO}_{Y \times \mathbb{C}^n | L \times \mathbb{C}^n}$ on a neighborhood of $W \times \Delta^{\circ}$. Here W is a open neighborhood of x^* in $T_Y L$ and Δ is a proper convex open cone of $\mathbb{R}^n$ such that $\xi^* \in \operatorname{Int} \Delta^{\circ}$. Thus $H(t, z; \zeta) = F(t, z; \zeta) - G(t, z; \zeta)$ defines a section of $\mathscr{BO}_{Y \times \mathbb{C}^n | L \times \mathbb{C}^n}$ on a neighborhood of $W \times \Delta^{\circ}$. Choose an open subcone $\Xi \Subset \Delta$ such that $\xi^* \in \operatorname{Int} \Xi^{\circ}$ and $H(t, z; \zeta)$ defines a section of $\mathscr{BO}_{Y \times \mathbb{C}^n | L \times \mathbb{C}^n}$ on a neighborhood of $W \times \Xi^{\circ}$. Let us set

$$\tilde{F}(t,z):=\int_{\Xi^\circ\cap S^{n-1}}F(t,z;\zeta)\,\omega(\xi),$$
$$\tilde{G}(t,z):=\int_{\Xi^\circ\cap S^{n-1}}G(t,z;\zeta)\,\omega(\xi),$$
$$\tilde{H}(t,z):=\int_{\Xi^\circ\cap S^{n-1}}H(t,z;\zeta)\,\omega(\xi).$$

Choose a proper convex open subcone $\Xi'\Subset\Xi$ of $\mathbb{R}^n$ such that $\xi^*\in\operatorname{Int}(\Xi')^\circ$. Then, $\tilde{F}(t,z)$ defines a section of $\mathscr{B}\mathscr{O}_{Y,L}$ on $\{(r,t,z)\in U';\ \operatorname{Im} z\in\Xi'\}$, $\tilde{G}(\tau,z)$ defines a holomorphic function on a neighborhood of $D_0(\varepsilon',\Xi')$, and $\tilde{H}(t,z)$ defines a section of $\mathscr{B}\mathscr{O}_{Y,L}$ on a neighborhood of $\{(0,t,z)\in\tilde{L}_Y;(t,z)\in W',\ \operatorname{Im} z\in\Xi'\}$. Here U' is an open neighborhood of x^* in $\tilde{L}_Y$, W' is an open neighborhood of x^* in T_YL and ε is a positive constant. Since $\tilde{H}=\tilde{F}-\tilde{G}$ as a section of $\mathscr{B}\mathscr{O}_{Y|L}$, $\tilde{G}(\tau,z)$ is holomorphic on a neighborhood of $D(p_M(V'\cap \mathrm{Cl}\,\Omega_M),\varepsilon',\Xi')$ by virtue of the unique continuation property of $\mathscr{C}\mathscr{O}$. Here V' is a neighborhood of x^* in $\tilde{M}_N$ and ε' is a positive constant. Hence $\tilde{G}$ defines an F-mild hyperfunction. Since

$$0=\tilde{\alpha}_{N|M}\,\mathrm{sp}_{N|M}(\tilde{H})(t,x+\sqrt{-1}\,\Xi'0)=\tilde{\alpha}_{N|M}\,\mathrm{sp}_{N|M}(\tilde{F}-\tilde{G})(t,x+\sqrt{-1}\,\Xi'0)$$

and $\tilde{\alpha}_{N|M}$ is injective by Theorem 2.8, it follows that

$$\begin{aligned}f&=\mathrm{sp}_{N|M}(\tilde{F})=\mathrm{sp}_{N|M}(\tilde{G})=\mathrm{sp}_{N|M}\,\beta^F_{N|M}(\tilde{G})\\&=\beta^F_{N|M}\,\mathrm{sp}^F_{N|M}(\tilde{G}).\end{aligned}$$

The proof is complete. □

§5. Non-Characteristic Higher-Codimensional Boundary Value Problem

Let $\mathscr{D}_X$ be *the sheaf on X of rings of linear partial differential operators* (*of finite order*) *with holomorphic coefficients*. Let $\mathscr{M}$ be a coherent (left) $\mathscr{D}_X$-Module on X; that is, a system of linear partial differential equation with holomorphic coefficients. Recall that Y is *non-characteristic* for $\mathscr{M}$ if

$$T^*_YX\cap\operatorname{char}(\mathscr{M})\subset T^*_XX,$$

where $\operatorname{char}(\mathscr{M})$ denotes the characteristic variety of $\mathscr{M}$. We denote *the inverse image of $\mathscr{M}$ by ι_Y* in $\mathscr{D}$-Modules by $\underline{\iota_Y^{-1}}\mathscr{M}$; that is,

$$\underline{\iota_Y^{-1}}\mathscr{M}:=\mathscr{D}_{Y\to X}\overset{L}{\underset{\iota_Y^{-1}\mathscr{D}_X}{\otimes}}\iota_Y^{-1}\mathscr{M}\simeq\mathscr{O}_Y\overset{L}{\underset{\iota_Y^{-1}\mathscr{O}_X}{\otimes}}\iota_Y^{-1}\mathscr{M}.$$

Let us set $\mathscr{M}_Y:=\mathscr{H}^0(\underline{\iota_Y^{-1}}\mathscr{M})$. Note that if Y is non-characteristic for $\mathscr{M}$, then $\underline{\iota_Y^{-1}}\mathscr{M}$ is concentrated in degree zero; that is, we can identify $\underline{\iota_Y^{-1}}\mathscr{M}$ with $\mathscr{M}_Y$, and that $\mathcal{M}_Y$ is a coherent $\mathcal{D}_Y$-Module.

5.1 Proposition. *Suppose that Y is non-characteristic for $\mathcal{M}$. Then there exists an isomorphism:*

$$\tilde{\gamma}_{N|M}\colon \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) \xrightarrow{\sim} \mathbb{R}({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}}\mathcal{M}, \mathcal{C}_N).$$

To prove Proposition 5.1, we shall prove the following lemma:

5.2 Lemma. *If Y is non-characteristic for $\mathcal{M}$, then there exist natural isomorphisms:*

$$\begin{aligned}\tau_Y^{-1}\iota_Y^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) &\xrightarrow{\sim} \tau_Y^{-1}\iota_Y^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_L)\\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|L}).\end{aligned}$$

Proof. Since $\mathcal{B}\mathcal{O}_{Y|L} = \nu_Y(\mathcal{B}\mathcal{O}_L)$, we have a canonical morphism

$$\tau_Y^{-1}\, i_Y^{-1}\, \mathcal{B}\mathcal{O}_L \simeq \tau_Y^{-1}\mathbb{R}(\tau_Y)_*\, \mathcal{B}\mathcal{O}_{Y|L} \to \mathcal{B}\mathcal{O}_{Y|L}.$$

Thus we have a natural morphism

$$\tau_Y^{-1} i_Y^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_L) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{B}\mathcal{O}_{Y|L})$$

Define an object $\mathcal{K}$ by a distinguished triangle:

$$\mathcal{K} \to \tau_Y^{-1}\, i_Y^{-1}\, \mathcal{B}\mathcal{O}_L \to \mathcal{B}\mathcal{O}_{Y|L} \xrightarrow{+1}.$$

Let z^* be a point of $T_Y L$. Since the proof is trivial in the case where $z^* \in Y$, we assume that $z^* \in \dot{T}_Y L$. Further the question being local, we may assume that $z^* = 0 + \partial/\partial t_1$ by an admissible local coordinate system. Then by the definition for any integer k we have

$$\mathcal{H}^k(\mathcal{K})_{z^*} = \varinjlim_{\varepsilon \to +0} H^k_{Z_\varepsilon \cap U_\varepsilon}(U_\varepsilon; \mathcal{B}\mathcal{O}_L),$$

where we set $U_\varepsilon := \{(t, z) \in L;\ |t|, |z| < \varepsilon\}$ and $Z_\varepsilon := \{(t, z) \in L;\ \varepsilon t_1 \leqslant \sum_{j=2}^{d} |t_j|\}$.

Let $L^{\mathbb{R}}$ be the real manifold underlying L. Set $\tilde{X} := X \times \bar{Y} = \{(t, z, \bar{w})\}$. Then we can regard $\tilde{X}$ as a complexification of $L^{\mathbb{R}}$ by $L^{\mathbb{R}} \ni (t, z) \mapsto (t, z, \bar{z}) \in \tilde{X}$. We denote by $\underset{\mathcal{D}}{\boxtimes}$ the external tensor products of $\mathcal{D}$-Modules. Then we have

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_L) &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\bar{Y}}}(\mathcal{D}_X \underset{\mathcal{D}}{\boxtimes} \mathcal{O}_{\bar{Y}}, \mathcal{B}_{L^{\mathbb{R}}}))\\ &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{M} \underset{\mathcal{D}}{\boxtimes} \mathcal{O}_{\bar{Y}}, \mathcal{B}_{L^{\mathbb{R}}}).\end{aligned}$$

Here we remark that $\operatorname{char}(\mathcal{M} \underset{\mathcal{D}}{\boxtimes} \mathcal{O}_{\bar{Y}}) = \operatorname{char}(\mathcal{M}) \times \operatorname{char}(\mathcal{O}_{\bar{Y}}) = \operatorname{char}(\mathcal{M}) \times T^*_{\bar{Y}}\bar{Y}$.

Hence for any integer k we have

$$\begin{aligned}\mathscr{H}^k(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{K}))_{z^*} &\simeq \varinjlim_{\varepsilon\to+0} H^k(\boldsymbol{R}\Gamma_{Z_\varepsilon\cap L_\varepsilon}(U_\varepsilon; \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}\mathscr{O}_L)))\\ &\simeq \varinjlim_{\varepsilon\to+0} H^k(U_\varepsilon; \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\tilde{X}}}(\mathscr{M}\underset{\mathscr{D}}{\boxtimes}\mathscr{O}_{\tilde{Y}}, \boldsymbol{R}\Gamma_{Z_\varepsilon}(\mathscr{B}_{L^{\mathbb{R}}})))\\ &\simeq \varinjlim_{\varepsilon\to+0} \mathscr{H}^k(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_{\tilde{X}}}(\mathscr{M}\underset{\mathscr{D}}{\boxtimes}\mathscr{O}_{\tilde{Y}}, \boldsymbol{R}\Gamma_{Z_\varepsilon}(\mathscr{B}_{L^{\mathbb{R}}})))_0.\end{aligned}$$

We shall show that if ε is sufficiently small, then any conormal $\theta=(0;\ \theta_t,\ 0)$ of Z_ε at the origin of $T^*L^{\mathbb{R}}$ is hyperbolic for $\mathscr{M}\underset{\mathscr{D}}{\boxtimes}\mathscr{O}_{\tilde{Y}}$; that is,

$$\theta\notin C_{T^*_{L^{\mathbb{R}}}\tilde{X}}(\mathrm{char}(\mathscr{M}\underset{\mathscr{D}}{\boxtimes}\mathscr{O}_{\tilde{Y}})).$$

Here we remark that there exists a canonical embedding $T^*L^{\mathbb{R}}\hookrightarrow T_{T^*_{L^{\mathbb{R}}}\tilde{X}}T^*\tilde{X}$ (cf. Chapter VI of [K-S 2]). Suppose that $\theta\in C_{T^*_{L^{\mathbb{R}}}\tilde{X}}(\mathrm{char}(\mathscr{M}\underset{\mathscr{D}}{\boxtimes}\mathscr{O}_{\tilde{Y}}))$. Then by the definition, there exists a sequence $\{(c_j;(\tau_j,\ z_j;\ \langle\eta_j,\ d\tau\rangle+\langle\zeta_j,\ dz\rangle);\ \bar{w}_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}_{>0}\times\mathrm{char}(\mathscr{M})\times\bar{Y}$ such that $\lim_{j\to\infty}(\tau_j,\ z_j,\ \bar{w}_j;\langle\eta_j,\ d\tau\rangle+\langle\zeta_j,\ dz\rangle)=0\in T^*_{L^{\mathbb{R}}}\tilde{X}$ and $\lim_{j\to\infty} c_j(\mathrm{Im}\ \tau_j,\ z_j-\bar{w}_j,\ \mathrm{Re}\ \eta_j,\ \zeta_j)=(0,\ 0,\ \theta_t,\ 0)$. For a positive constant δ, let us set

$$V_\delta:=\{(\tau,\ z;\ \langle\eta,\ d\tau\rangle+\langle\zeta,\ dz\rangle)\in T^*X;\ |\tau|,\ |z|<\delta,\ |\zeta|<\delta|\eta|\}.$$

Since $\theta_t\neq0$ and $c_j>0$, for any $\delta>0$, there exists a $J\in\mathbb{N}$ such that for any $j\geqslant J$, we have $(\tau_j,\ z_j;\ \langle\eta_j,\ d\tau\rangle+\langle\zeta_j,\ dz\rangle)\in V_\delta\cap\mathrm{char}(\mathscr{M})$.

On the other hand, by the non-characteristic condition and the fact that $\mathrm{char}(\mathscr{M})$ is $\mathbb{C}^\times$-conic, we have $V_\delta\cap\mathrm{char}(\mathscr{M})=\emptyset$, if δ is sufficiently small. This is a contradiction.

Hence by applying Corollary 2.2.2 of Kashiwara-Schapia [K-S 1] we obtain

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}\underset{\mathscr{D}}{\boxtimes}\mathscr{O}_{\tilde{Y}},\ \boldsymbol{R}\Gamma_{Z_\varepsilon}(\mathscr{B}_{L^{\mathbb{R}}}))_0=0.$$

This yields an isomorphism

$$\tau_Y^{-1}\ i_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\ \mathscr{B}\mathscr{O}_L)\xrightarrow{\sim}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\ \mathscr{B}\mathscr{O}_{Y|L}).$$

Next let us consider an exact sequence:

$$0\longrightarrow i_L^{-1}\mathscr{O}_X\longrightarrow\mathscr{B}\mathscr{O}_L\longrightarrow(\dot{\pi}_Y)_*\mathscr{C}\mathscr{O}_L\longrightarrow0,$$

where $\dot{\pi}_Y$: $\dot{T}^*_YX\longrightarrow Y$. Since

$$\mathrm{supp}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\ \mathscr{C}\mathscr{O}_L))\subset T^*_LX\cap\mathrm{char}(\mathscr{M}),$$

we have

$$\iota_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\ (\dot{\pi}_Y)_*\mathscr{C}\mathscr{O}_L)=0.$$

Therefore we obtain

$$i_L^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_L).$$

This completes the proof. □

Proof of Proposition 5.1. By the Cauchy-Kovalevskaja-Kashiwara theorem (see for example Theorem 11.3.5 of [K-S 2]) and Lemma 5.2, we have

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mu_{T_NM}(\tau_Y^{-1}\mathcal{O}_Y)) &\simeq \mu_{T_NM}(\tau_Y^{-1}(\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{O}_Y)))\\ &\simeq \mu_{T_NM}(\tau_Y^{-1}\iota_Y^{-1}(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)))\\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})).\end{aligned}$$

Then, by the non-characteristic condition, we have

$$\mathrm{SS}(\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{O}_Y)) = \mathrm{char}(\underline{\iota_Y^{-1}\mathcal{M}}) = {}^t\iota_Y'(\iota_{Y\pi}^{-1}(\mathrm{char}(\mathcal{M}))),$$

where SS($*$) denotes the micro-support in the sense of Kashiwara-Schapira (see [K-S 2]). Then by Corollary 6.7.3 of Kashiwara-Schapira [K-S 2] we obtain

$$\begin{aligned}&\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathbb{R}({}^t\tau_Y')_!\,\tau_{Y\pi}^{-1}\,\mu_N(\mathcal{O}_Y))\\ &\quad\simeq \mathbb{R}({}^t\tau_Y')_!\,\tau_{Y\pi}^{-1}\,\mu_N(\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{O}_Y))\\ &\quad\xrightarrow{\sim} \mu_{T_NM}(\tau_Y^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{O}_Y)\overset{L}{\otimes}\omega_{T_NM/N}^{\otimes -1}\overset{L}{\otimes}\omega_{T_YL/Y})\\ &\quad\simeq \mu_{T_NM}(\tau_Y^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{O}_Y))\\ &\quad\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{T_NM}(\mathcal{B}\mathcal{O}_{Y|L})).\end{aligned}$$

Hence applying the functor $\otimes or_{N/Y}$, we obtain an isomorphism $\tilde{\gamma}_{N|M}$. □

By Proposition 5.1 and Theorem 2.8, we have the following:

5.3 Theorem. *Suppose that Y is non-characteristic for $\mathcal{M}$. Then, the morphisms $\tilde{\alpha}_{N|M}$: $\mathcal{C}_{N|M} \rightarrowtail \tilde{\mathcal{C}}_{N|M}$ and $\tilde{\gamma}_{N|M}$ of Proposition* 5.1 *induce a natural morphism:*

$$\gamma_{N|M} := \tilde{\gamma}_{N|M}\,\tilde{\alpha}_{N|M}\colon \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \to \mathbb{R}({}^t\tau_Y')_!\,\tau_{Y\pi}^{-1}\,\mathbb{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{C}_N).$$

In particular, there exists a natural monomorphism:

$$\gamma_{N|M}\colon \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrowtail ({}^t\tau_Y')_!\,\tau_{Y\pi}^{-1}\,\mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{C}_N).$$

Now, let us give the explicit and concrete expression of the morphism of Theorem 5.3 using the boundary value morphism of F-mild microfunctions.

5.4 Lemma. *Suppose that Y is non-characteristic for $\mathcal{M}$. Then the morphism $\tilde{\gamma}^A_{N|M}$: $\tilde{\mathcal{C}}^A_{N|M} \to \mathcal{C}_N$ induces a natural isomorphism:*

$$\tilde{\gamma}^A_{N|M}\colon \mathcal{H}om_{\tau_Y^{-1}\mathcal{D}_X}(\iota_Y^{-1}\mathcal{M}, \tilde{\mathcal{C}}^A_{N|M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}(\underline{\iota_Y^{-1}\mathcal{M}}, \mathcal{C}_N).$$

Proof. By the Cauchy-Kovalevalevskaja-Kashiwara theorem, we have

$$\iota_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \xrightarrow{\sim} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\underline{\iota_Y^{-1}}\mathscr{M}, \mathscr{O}_Y).$$

By applying the functor $\mu_N(*)\otimes or_{N/Y}$ we have

$$\boldsymbol{R}\mathscr{H}om_{\iota_Y^{-1}\mathscr{D}_X}(\iota_Y^{-1}\mathscr{M}, \mu_N(\iota_Y^{-1}\mathscr{O}_X)\otimes or_{N/Y}) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\underline{\iota_Y^{-1}}\mathscr{M}, \mu_N(\mathscr{O}_Y)\otimes or_{N/Y}).$$

Hence taking the n-th cohomology, we have an isomorphism. It is easy to see that this morphism coincides with $\gamma_{N|M}^A$ in view of the construction and the fact that $\tilde{\gamma}_{N|M}^A$ induces the isomorphism $\underline{\iota_Y^{-1}}\,\tilde{\mathscr{C}}_{N|M}^A \simeq \tilde{\mathscr{C}}_{N|M}^A \Big/ \sum_{j=1}^{d} t_j\tilde{\mathscr{C}}_{N|M}^A \xrightarrow{\sim} \mathscr{C}_N$. This completes the proof. □

5.5 Proposition. *Suppose that Y is non-characteristic for $\mathscr{M}$.*

(1) *The morphism $\tilde{\beta}_{N|M}: ({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\tilde{\mathscr{C}}_{N|M}^A \to \tilde{\mathscr{C}}_{N|M}$ induces a natural isomorphism:*

$$\tilde{\beta}_{N|M}: ({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\mathscr{H}om_{\iota_Y^{-1}\mathscr{D}_X}(\iota_Y^{-1}\mathscr{M}, \tilde{\mathscr{C}}_{N|M}^A) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \tilde{\mathscr{C}}_{N|M}).$$

Moreover $\tilde{\beta}_{N|M}$ is compatible with $\tilde{\gamma}_{N|M}^A$ and $\tilde{\gamma}_{N|M}$; that is, the following diagram is commutative:

$$\begin{tikzcd}[column sep=small]
({}^t\tau'_Y)_!\tau_{Y\pi}^{-1}\mathscr{H}om_{\iota_Y^{-1}\mathscr{D}_X}(\iota_Y^{-1}\mathscr{M}, \tilde{\mathscr{C}}_{N|M}^A) \arrow[rr, "\sim", "\tilde{\gamma}_{N|M}^A"'] \arrow[dr, "\sim", "\tilde{\beta}_{N|M}"'] & & ({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\underline{\iota_Y^{-1}}\mathscr{M}, \mathscr{C}_N) \\
& \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \tilde{\mathscr{C}}_{N|M}). \arrow[ur, "\sim", "\tilde{\gamma}_{N|M}"'] &
\end{tikzcd}$$

(2) *The morphism $\beta_{N|M}^F: \mathscr{C}_{N|M}^F \to \mathscr{C}_{N|M}$ induces an isomorphism:*

$$\beta_{N|M}^F: \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}^F) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}).$$

Proof. (1) The method of the proof is similar to that of Proposition 5.1. By applying functior. $\mu_{T_NM}(*)\otimes or_{N/M}$ to the isomorphism of Lemma 5.2, we have

$$\mu_{T_NM}(\tau_Y^{-1}\iota_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) \xrightarrow{\sim} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_{T_NM}(\mathscr{B}\mathscr{O}_{Y|L})).$$

Then, since Y is non-characteristic for $\mathscr{M}$ we have

$$\begin{aligned} \mathrm{SS}(\iota_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) &\subset {}^t\iota'_Y(\iota_{Y\pi}^{-1}(\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}, \mathscr{O}_X)))) \\ &= {}^t\iota'_Y(\iota_{Y\pi}^{-1}(\mathrm{char}(\mathscr{M}))), \end{aligned}$$

Hence by using the non-characteristic condition again, we can apply Corollary 6.7.3 of Kashiwara-Schapira [K-S 2] to obtain

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \boldsymbol{R}({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\mu_N(\iota_Y^{-1}\mathscr{O}_X))$$

$$\begin{aligned}
&\simeq \mathbb{R}({}^t\tau'_Y)_! \tau_{Y\pi}^{-1} \mu_N(\iota_Y^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) \\
&\xrightarrow{\sim} \mu_{T_NM}(\tau_Y^{-1}\iota_Y^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \overset{L}{\otimes} \omega_{T_NM/N}^{\otimes -1} \overset{L}{\otimes} \omega_{T_YL/Y}) \\
&\simeq \mu_{T_NM}(\tau_Y^{-1}\iota_Y^{-1}\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)) \\
&\xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mu_{T_NM}(\mathscr{B}\mathscr{O}_{Y|L})).
\end{aligned}$$

Hence applying the functor $\otimes\, or_{N/Y}$ and taking the n-th cohomology, we obtain an isomorphism. We easily see that this isomorphism is induced by $\beta_{N|M}$. The proof of the commutativity is straightforward.

(2) follows from (1) and Proposition 4.9. □

By Propositions 1.11 and 5.5(2), we obtain the following:

5.6 Corollary ([O 4]). *Suppose that Y is non-characteristic for $\mathscr{M}$. Then the morphism $\beta_{N|M}^F$: $\mathscr{B}_{N|M}^F \to \mathscr{B}_{N|M}$ induces an isomorphism:*

$$\beta_{N|M}^F\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{N|M}^F) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_{N|M}).$$

In particular, all the hyperfunction solutions to $\mathscr{M}$ defined an a wedge domain with edge N are always F-mild.

Since the restriction morphism $\gamma_{N|M}^A$: $\mathscr{C}_{N|M}^A \to \mathscr{C}_N$ induces an isomorphism

$$\underline{\iota_Y^{-1}}\mathscr{C}_{N|M}^A \simeq \mathscr{C}_{N|M}^A \Big/ \sum_{j=1}^{d} t_j\, \mathscr{C}_{N|M}^A \xrightarrow{\sim} \mathscr{C}_N,$$

we easily see that this morphism induces a natural morphism

$$\gamma_{N|M}^A\colon \mathscr{H}om_{\iota_Y^{-1}\mathscr{D}_X}(\iota_Y^{-1}\mathscr{M}, \mathscr{C}_{N|M}^A) \to \mathscr{H}om_{\mathscr{D}_Y}(\underline{\iota_Y^{-1}}\mathscr{M}, \mathscr{C}_N).$$

For the same reason, the boundary value morphism $\gamma_{N|M}^F$: $\mathscr{C}_{N|M}^F \to ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathscr{C}_N$ induces a natural morphism

$$\gamma_{N|M}^F\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}^F) \to ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathscr{H}om_{\mathscr{D}_Y}(\underline{\iota_Y^{-1}}\mathscr{M}, \mathscr{C}_N).$$

On the other hand, induced by

$$\alpha_{N|M}\colon ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathscr{C}_{N|M}^A \rightarrowtail \mathscr{C}_{N|M}^F, \qquad \alpha_{N|M}^F\colon \mathscr{C}_{N|M}^F \rightarrowtail ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \widetilde{\mathscr{C}}_{N|M}^A,$$

there exist natural monomorphisms

$$\begin{aligned}
&\alpha_{N|M}\colon ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathscr{H}om_{\iota_Y^{-1}\mathscr{D}_X}(\iota_Y^{-1}\mathscr{M}, \mathscr{C}_{N|M}^A) \rightarrowtail \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}^F), \\
&\alpha_{N|M}^F\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}^F) \rightarrowtail ({}^t\tau'_Y)_!\, \tau_{Y\pi}^{-1}\, \mathscr{H}om_{\iota_Y^{-1}\mathscr{D}_X}(\iota_Y^{-1}\mathscr{M}, \widetilde{\mathscr{C}}_{N|M}^A).
\end{aligned}$$

5.7 Proposition. *The preceding morphisms are compatible; that is, the following diagram is commutative:*

$$
\begin{tikzcd}
& ({}^t\tau'_Y)_!\tau_{Y\pi}^{-1}\mathcal{H}om_{\iota_Y^{-1}\mathcal{D}_X}(\iota_Y^{-1}\mathcal{M},\mathcal{C}^A_{N|M}) \arrow[dl, "\alpha_{N|M}"'] \arrow[dr] \arrow[dd, "\gamma^t_{N|M}"] & \\
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}^F_{N|M}) \arrow[rr, equal] \arrow[dr, "\gamma^F_{N|M}"] \arrow[dd, "\alpha^F_{N|M}"'] & & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_{N|M}) \arrow[dl, "\gamma_{N|M}"'] \arrow[dd, "\tilde{\alpha}_{N|M}"] \\
& ({}^t\tau'_Y)_!\tau_{Y\pi}^{-1}\mathcal{H}om_{\mathcal{D}_X}(\iota_Y^{-1}\mathcal{M},\mathcal{C}_N) & \\
({}^t\tau'_Y)_!\tau_{Y\pi}^{-1}\mathcal{H}om_{\iota_Y^{-1}\mathcal{D}_X}(\iota_Y^{-1}\mathcal{M},\mathcal{C}^A_{N|M}) \arrow[ur, "\tilde{\gamma}^A_{N|M}", "\sim"'] \arrow[rr, "\sim", "\tilde{\beta}_{N|M}"'] & & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\tilde{\mathcal{C}}_{N|M}) \arrow[ul, "\tilde{\gamma}_{N|M}"', "\sim"]
\end{tikzcd}
$$

The proof is straightforward.

5.8 Example (cf. S. Tajima [Tj]). Let X be a complex manifold and N a real analytic submanifold of X. Assume that N is generic; that is, $TN\underset{N}{+}\sqrt{-1}\,TN = N\underset{X}{\times} TX$. Let Y be a complexification of N in $X\times\bar{X}$ and $\mathcal{D}_{C-R}:=\mathcal{D}_X\underset{\mathcal{D}}{\boxtimes}\mathcal{O}_{\bar{X}}$ the Cauchy-Riemann system. Let f be a holomorphic function defined on a wedge domain with edge N; that is, a section of $\nu_N(\mathcal{O}_X)=\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{C-R},\mathcal{B}_{N|M})$. Then, f is well-defined as an F-mild hyperfunction along N since Y is non-characteristic for $\mathcal{D}_{C-R}$. In particular, the boundary value morphism of f to N as a hyperfunction is well-defined and injective by Proposition 5.7.

§6. Fuchsian Systems of Partial Differential Equations

In this section, we shall prove the uniqueness theorem in the boundary value problems for $\mathcal{D}$-Modules of Fuchsian type and of the Fuchs-Goursat type in the framework of F-mild microfunctions.

First, assume that $\mathcal{M}$ is a Fuchsian system along Y in the sense of Laurent-Monteiro Fernandes [L-MF]. Recall that a coherent $\mathcal{D}_X$-Module $\mathcal{M}$ is a Fuchsian system along Y if and only if for any (local) section $u\in\mathcal{M}$, there exists a differential operator P such that $Pu=0$ and that P can be written in a coordinate system (τ, z) with $Y=\{(\tau,z);\,\tau=0\}$ as follows:

$$P(\tau, z; \partial_\tau, \partial_z)=\sum_{0\leq|\alpha|=|\beta|\leq\operatorname{ord}P} P_{\alpha\beta}(z)\,\tau^\alpha\partial_\tau^\beta+Q(\tau, z; \partial_\tau, \partial_z),$$

where ord denotes the (usual) order of a differential operator, and the conditions below hold:

(a) For any $\eta\in\mathbb{C}^d\setminus\{0\}$, one has $\sum_{|\alpha|=|\beta|=\operatorname{ord}P} P_{\alpha\beta}(z)\,\eta^\alpha\bar{\eta}^\beta\neq0$;

(b) For any $j\in\mathbb{Z}$, one has $Q\mathscr{I}_Y{}^j\subset\mathscr{I}_Y{}^{j+1}$. Here $\mathscr{I}_Y$ denotes the defining ideal of Y in X with a convention $\mathscr{I}_Y{}^j=\mathscr{O}_X$ for $j\leqslant 0$.

Note that all the cohomologies of $\underline{\iota_Y^{-1}}\mathscr{M}$ are coherent $\mathscr{D}_Y$-Modules by Théorème 3.3 of Laurent-Schapira [L-S] and that we may choose the coordinate system above as admissible.

6.1 Theorem. *Let $\mathscr{M}$ be a Fuchsian system along Y. Then, the boundary value morphism $\gamma^F_{N|M}\colon \mathscr{C}^F_{N|M}\to({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\mathscr{C}_N$ induces a monomorphism on $T^*_{T_NM}T_YL$*

$$\gamma^F_{N|M}\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}^A_{N|M})\rightarrowtail({}^t\tau'_Y)_!\,\tau_{Y\pi}^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N).$$

In particular, the boundary value morphism $\gamma^F_{N|M}\colon \mathscr{B}^F_{N|M}\to\tau_N^{-1}\mathscr{B}_N$ on T_NM induces a monomorphism

$$\gamma^F_{N|M}\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}^F_{N|M})\rightarrowtail\tau_N^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{B}_N).$$

Note that not all the hyperfunction solutions to $\mathscr{M}$ are necessarily F-mild, contrary to the non-characteristic case studied in the previous section.

Proof. By Théorème 3.2.2 of [L-MF], we have the Cauchy-Kovalevskaja type theorem:

$$\iota_Y^{-1}\mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)\xrightarrow{\sim}\mathbf{R}\mathscr{H}om_{\mathscr{D}_Y}(\underline{\iota_Y^{-1}}\mathscr{M},\mathscr{O}_Y).$$

Applying the functor $\mu_N(*)\otimes or_{N/Y}$ and taking the n-th cohomology, we have

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\widetilde{\mathscr{C}}^A_{N|M})\xrightarrow{\sim}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N).$$

In view of Definition 4.4, this isomorphism induces a morphism $\gamma^F_{N|M}$. □

By virtue of Lemma 4.6 we have the following corollary:

6.2 Corollary. *Let $\mathscr{M}$ be a Fuchsian system along Y. Then, the restriction morphism $\gamma^A_{N|M}\colon\mathscr{C}^A_{N|M}\to\mathscr{C}_N$ induces a monomorphism*

$$\gamma^A_{N|M}\colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}^A_{N|M})\rightarrowtail\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N).$$

6.3 Example. Let P be a differential operator of order m of the following form:

$$P(\tau,z;\partial_\tau,\partial_z)=b\Big(\sum_{j=1}^{d}\tau_j\partial_{\tau_j}\Big)+Q(\tau,z;\partial_\tau,\partial_z),$$

where $b(s)$ is a polynomial of degree m and Q satisfies $Q\mathscr{I}_Y{}^j\subset\mathscr{I}_Y{}^{j+1}$ for any $j\in$

$\mathbb{Z}$. Put $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X P$. Let μ, $\nu\in\mathbb{Z}$ be the minimum and the maximum integral roots of the equation $b(s)=0$ respectively (set $\mu=\nu=-1$ if $b(s)$ has no integral root). Let $u(t, x)$ be an F-mild hyperfunction (or microfunction) solution to $Pu=0$ and assume that for any $\alpha\in\mathbb{N}_0{}^d$ with $\mu\leqslant|\alpha|\leqslant\nu$

$$\gamma^F_{N|M}(\partial_t^\alpha u)=0.$$

Then, it follows that $u=0$ on a neighborhood of N. In fact, $\mathcal{M}_Y$ is generated by $\partial_t^\alpha\bar{u}$ for $\alpha\in\mathbb{N}_0{}^d$ satisfying $\mu\leqslant|\alpha|\leqslant\nu$ over $\mathcal{D}_Y$, where $\bar{u}$ denotes the residue class of 1 in $\mathcal{M}_Y$ (cf. Laurent-Schapira [L-S]).

Next, we shall give similar theorems for a matrix of Fuchs-Goursat type introduced by Madi [M] and Yamazaki [Y]. To state the results, we define boundary value morphisms, which we shall regard as Goursat data, as follows: By an admissible coordinate system, we may assume that $X=\mathbb{C}^d_\tau\times\mathbb{C}^n_z$, $Y=\mathbb{C}^n_z$, $L=\mathbb{R}^d_t\times\mathbb{C}^n_z$, $M=\mathbb{R}^d_t\times\mathbb{R}^n_x$ and $N=\mathbb{R}^n_x$. For $1\leqslant i\leqslant d$, let us set $L_i:=\{(t, z)\in L;\ t_i=0\}$ and $M_i:=M\cap L_i$. Then the inclusion $L_i\hookrightarrow L$ induces mappings

$$\begin{tikzcd}
T_Y L_i \arrow[r, hook, "\varphi_i"] \arrow[dr, phantom, "\circlearrowleft"] & T_Y L \\
T_N M_i \arrow[u, hook] \arrow[r, hook, "\varphi_{iN}"] & T_N M. \arrow[u, hook]
\end{tikzcd}$$

Moreover, we have the following commutative diagram:

$$\begin{tikzcd}
 & T_N M \arrow[dl, "\tau_N"'] & T^*_{T_N M} T_Y L \arrow[l, "\pi_{N|M}"'] \\
N & T_N M_i \arrow[l, "\tau_{iN}"'] \arrow[u, "\varphi_{iN}"] \arrow[ur, phantom, "\square"] & T_N M_i \underset{T_N M}{\times} T^*_{T_N M} T_Y L \arrow[l] \arrow[u, "\varphi_{i\pi}"'] \arrow[d, "{}^t\varphi'_i", "\wr"'] \\
T_N^* Y \arrow[u, "\pi_N"] \arrow[ur, phantom, "\square"] & T_N M_i \underset{N}{\times} T_N^* Y \arrow[l, "\tau_{iY\pi}"'] \arrow[u] \arrow[r, "{}^t\varphi'_{iY}"'] & T^*_{T_N M_i} T_Y L_i. \arrow[ul, "\pi_{N|M_i}"]
\end{tikzcd}$$

Then we have the following:

6.4 Lemma. *There exist natural morphisms*

$$\tilde{\gamma}^{A,i}_{N|M}\colon \widetilde{\mathcal{C}}^A_{N|M}\longrightarrow\widetilde{\mathcal{C}}^A_{N|M_i},$$
$$\tilde{\gamma}^{A,i}_{N|M}\colon \widetilde{\mathcal{B}}^A_{N|M}\longrightarrow\widetilde{\mathcal{B}}^A_{N|M_i},$$

and

$$\gamma^{F,t}_{N|M}\colon ({}^t\varphi'_t)_!\, \varphi^{-1}_{t\pi}\,\mathscr{C}^F_{N|M} \longrightarrow \mathscr{C}^F_{N|M_t},$$
$$\gamma^{F,t}_{N|M}\colon \varphi^{-1}_{tN}\,\mathscr{B}^F_{N|M} \longrightarrow \mathscr{B}^F_{N|M_t}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\pi^{-1}_{N|M_t}\varphi^{-1}_{tN}\mathscr{B}^F_{N|M} \arrow[rr, two heads, "\mathrm{sp}^F_{N|M}"] \arrow[ddd, "\gamma^{F,t}_{N|M}"'] \arrow[dr, "\alpha^F_{N|M}"] & & ({}^t\varphi'_t)_!\varphi^{-1}_{t\pi}\mathscr{C}^F_{N|M} \arrow[ddd, "\gamma^{F,t}_{N|M}"'] \arrow[dr, "\alpha^F_{N|M}"] & \\
& \pi^{-1}_{N|M_t}\pi^{-1}_{tN}\widetilde{\mathscr{B}}^A_{N|M} \arrow[rr, two heads, "\mathrm{sp}^A_{N|M}"] \arrow[ddd, "\tilde{\gamma}^{A,t}_{N|M}"] & & ({}^t\varphi'_{tY})_!\tau^{-1}_{tY\pi}\widetilde{\mathscr{C}}^F_{N|M} \arrow[ddd, "\tilde{\gamma}^{A,t}_{N|M}"] \\
& & & \\
\pi^{-1}_{N|M_t}\mathscr{B}^F_{N|M_t} \arrow[rr, two heads, "\mathrm{sp}^F_{N|M_t}"] \arrow[dr, "\alpha^F_{N|M_t}"] & & \mathscr{C}^F_{N|M_t} \arrow[dr, "\alpha^F_{N|M_t}"] & \\
& \pi^{-1}_{N|M_t}\tau^{-1}_{tN}\widetilde{\mathscr{B}}^A_{N|M_t} \arrow[rr, two heads, "\widetilde{\mathrm{sp}}^A_{N|M_t}"] & & ({}^t\varphi'_{tY})_!\tau^{-1}_{tY\pi}\widetilde{\mathscr{C}}^A_{N|M}.
\end{tikzcd}$$

The proof is straightforward.

6.5 Lemma. *Let $l=(l_1,\dots,l_d)$ be a d-tuple of non-negative integers and $f(t,x)$ a germ of $\widetilde{\mathscr{C}}^A_{N|M}$ at $p=(x_0;\sqrt{-1}\langle\xi^*,dx\rangle)\in T^*_N Y$. Then the following conditions are equivalent:*

(1) *There exists a germ $g(t,x)$ of $\widetilde{\mathscr{C}}^A_{N|M}$ at p such that $f(t,x)=t^l g(t,x)$.*

(2) *For any $0\leqslant k_i\leqslant l_i-1$ $(1\leqslant i\leqslant d)$*

$$\tilde{\gamma}^{A,t}_{N|M}(\partial_{t_i}{}^{k_i} f(t,x))=0.$$

Moreover in this case, $g(t,x)$ is unique.

Proof. By an admissible coordinate system, we may assume that $x_0=0$. Since the proof is similar, we may assume that $\xi^*\neq 0$. First, let us prove the uniqueness. Suppose that $g(t,x)\in\widetilde{\mathscr{C}}^A_{N|M,p}$ satisfies $t^l g(t,x)=0$. Then, there exists a holomorphic function $G(\tau,z)$ defined on a neighborhood of $D_0(\varepsilon,\Gamma_0)$ such that

$$g(t,x)=\widetilde{\mathrm{sp}}_{N|M}(G(t,x+\sqrt{-1}\Gamma_0\,0))$$

at p, where ε is a positive constant and Γ_0 is an open cone of $\mathbb{R}^n$ such that $\xi^*\in\operatorname{Int}\Gamma_0^\circ$. Since $t^l g(t,x)=0$, by Lemma 4.2 (1) we can find a natural number J and holomorphic functions $G_j(\tau,z)$ defined on a neighborhood of $D_0(\delta,\Gamma_j)$ such that

$$t^l G(t,\ x+\sqrt{-1}\Gamma_0\, 0)=\sum_{j=1}^{J} G_j(t,\ x+\sqrt{-1}\Gamma_j\, 0),$$

where δ is a positive constant and each Γ_j is an open cone of $\mathbb{R}^n$ such that $\xi^*\notin \Gamma_j^\circ$. Set $G_0(\tau,\ z):=-t^l G(t,\ z)$. Choose proper convex open subcones $\Gamma'_j \Subset \Gamma_j$ $(0\leqslant j\leqslant J)$ such that $\xi^*\in \mathrm{Int}(\Gamma'_0)^\circ$ and $\xi^*\notin \mathrm{Int}(\Gamma'_j)^\circ (1\leqslant j\leqslant J)$. Then, by Lemma 4.2 (2) there exist a positive constant δ' and holomorphic functions $G_{jk}(\tau,\ z)$ defined on a neighborhood of $D_0(\delta',\ \Gamma'_j+\Gamma'_k)$ such that

$$G_j=\sum_{k=0}^{J} G_{jk}, \qquad G_{jk}+G_{kj}=0 \qquad (0\leqslant j,\ k\leqslant J).$$

By the Taylor expansion, let us write $G_{jk}(\tau,\ z)=\sum_{\alpha\in\mathbb{N}_0^d} a_{jk,\alpha}(z)\tau^\alpha$ and set

$$G'_{jk}(\tau,\ z):=\sum_{\alpha\geqslant l} a_{jk,\alpha}(z)\tau^{\alpha-l}$$
$$H_{jk}(\tau,\ z):=G_{jk}(\tau,\ z)-\tau^l G'_{jk}(\tau,\ z).$$

Since

$$\tau^l G(\tau,\ z)=-G_0(\tau,\ z)=-\tau^l\sum_{k=1}^{J} G_{0k}(\tau,\ z)-\sum_{k=1}^{J} H_{0k}(\tau,\ z)$$

holds, we have

$$G(\tau,\ z)=-\sum_{k=1}^{J} G_{0k}(\tau,\ z).$$

This implies

$$G(t,\ x+\sqrt{-1}\Gamma_0\, 0)=-\sum_{k=1}^{J} G_{0k}(t,\ x+\sqrt{-1}(\Gamma'_0+\Gamma'_k)0).$$

Since ξ^* is not contained in $(\Gamma'_0+\Gamma'_k)^\circ$, we have $g(t,\ x)=\widetilde{\mathrm{sp}}_{N|M}(G(t,\ x+\sqrt{-1}\Gamma_0\, 0))=0$. This proves the uniqueness.

It is clear that (1) implies (2). Suppose (2). By virtue of the uniqueness, we can argue by induction on $|l|:=\sum_{i=1}^{d} l_i$. Thus, we may assume that $l=(1,\ 0,\dots,\ 0)$. Let $f(t,\ x)$ be represented by a germ $F(t,\ x+\sqrt{-1}\,\Gamma_0)$ of $\widetilde{\mathscr{B}}^A_{N|M}$, where $F(\tau,\ z)$ is holomorphic on a neighborhood of $D_0(\varepsilon,\ \Gamma)$ for $\varepsilon>0$ and open convex cone Γ of $\mathbb{R}^n$ with $\xi^*\in \mathrm{Int}\,\Gamma^\circ$. Then there exist holomorphic functions $G(\tau,\ z)$ and $H(\tau',\ z)$ defined on a neighborhood of $D_0(\varepsilon,\ \Gamma)$ such that

$$F(\tau, z) = \tau_1\, G(\tau, z) + H(\tau', z),$$

where $\tau' := (\tau_2, \ldots, \tau_d)$. Let us set

$$g(t, x) := \tilde{\mathrm{sp}}_{N|M}(G(t, x + \sqrt{-1}\Gamma\, 0)),$$
$$h(t', x) := \tilde{\mathrm{sp}}_{N|M}(h(t', x + \sqrt{-1}\Gamma\, 0)).$$

Then since

$$0 = \tilde{\gamma}^{A,1}_{N|M}(f(t, x)) = h(t', x) \in \tilde{\mathscr{C}}^{A}_{N|M_1,p},$$

we easily see that

$$h(t', x) = 0 \in \tilde{\mathscr{C}}^{A}_{N|M,p}.$$

The proof is complete.

For a vector $l = (l_1, \ldots, l_d) \in \mathbb{R}^d$, we set $[l]_+ := ([l_1]_+, \ldots, [l_d]_+)$, where $[l_j]_+ = \max\{l_j, 0\}$. We fix $J \in \mathbb{N}$, $m^{(\nu)} = (m_1^{(\nu)}, \ldots, m_d^{(\nu)})$ and $k^{(\nu)} = (k_1^{(\nu)}, \ldots, k_d^{(\nu)}) \in \mathbb{N}_0{}^d$ with $m^{(\nu)} \geqslant k^{(\nu)}$ $(1 \leqslant \nu \leqslant J)$ and set $m = (m^{(1)}, \ldots, m^{(J)})$ and $k = (k^{(1)}, \ldots, k^{(J)}) \in (\mathbb{N}_0{}^d)^J$. Set $\mathbb{1}_d := (1, \ldots, 1) \in \mathbb{N}^d$.

6.6 Definition. Let $P(\tau, z; \partial_\tau, \partial_z) = (P^{(\mu,\nu)}(\tau, z; \partial_\tau, \partial_z))^J_{\mu,\nu=1}$ be a matrix of size $J \times J$ whose components is in $\mathscr{D}_X$ defined in a neighborhood of the origin. Then, P is said to be *of Fuchs-Goursat type with weight* (k, m) (*with respect to* τ*-variables*) if it can be written in a form

$$P^{(\mu,\nu)}(\tau, z; \partial_\tau, \partial_z) = \sum_{0 \leqslant \alpha \leqslant m^{(\nu)}} P_\alpha^{(\mu,\nu)}(\tau, z; \partial_z)\, \partial_\tau{}^\alpha,$$

where each $P_\alpha^{(\mu,\nu)}$ is a differential operator satisfying the following:

(1) The order $\mathrm{ord}\, P_\alpha^{(\mu,\nu)}$ of $P_\alpha^{(\mu,\nu)}$ is at most $|m^{(\nu)}| - |\alpha|$;

(2) There exist $P_\alpha^{1,(\mu,\nu)}(\tau, z)$ and $P_\alpha^{2,(\mu,\nu)}(\tau, z; \partial_z)$ $(0 \leqslant \alpha \leqslant m^{(\nu)})$ such that

$$P_\alpha^{(\mu,\nu)}(\tau, z; \partial_z) = \tau^{[\alpha - m^{(\nu)} + k^{(\nu)}]_+} P_\alpha^{1,(\mu,\nu)}(\tau, z) + \tau^{[\alpha - m^{(\nu)} + k^{(\nu)} + \mathbb{1}_d]_+} P_\alpha^{2,(\mu,\nu)}(\tau, z; \partial_z).$$

Let $T^{(\nu)} = (T_1^{(\nu)}, \ldots, T_d^{(\nu)})$ $(1 \leqslant \nu \leqslant J)$ be indeterminates and set

$$\tilde{T} := (T^{(1)}, \ldots, T^{(J)}).$$

If P is of Fuchs-Goursat type with weight (k, m), we define *the indicial polynomial of P* by

$$\mathscr{I}_P(z; \tilde{T}) := \det\Bigl(\sum_{m^{(\nu)} - k^{(\nu)} \leqslant \alpha \leqslant m^{(\nu)}} P_\alpha^{1,(\mu,\nu)}(0, z)\, \mathscr{I}_\alpha(T^{(\nu)}) \Bigr)$$

where $\mathscr{I}_\alpha(T^{(\nu)}) := \prod_{j=1}^{d} \mathscr{I}_{\alpha_j}(T_j^{(\nu)})$ with

$$\mathscr{I}_{\alpha_j}(T_j^{(\nu)}):=\begin{cases} T_j^{(\nu)}(T_j^{(\nu)}-1)\cdots(T_j^{(\nu)}-\alpha_j+1) & (\alpha_j\geqslant 1), \\ 1 & (\alpha_j=0). \end{cases}$$

Consider the following condition:

(A). *There exist a positive constant $C>0$ and a neighborhood W of the origin in $\mathbb{C}^n$ such that for any $z\in W$ and $\beta\in\mathbb{N}_0{}^d$*

$$|\mathscr{I}_P(z;\beta+m^{(1)}-k^{(1)},\dots,\beta+m^{(J)}-k^{(J)})|\geqslant C\prod_{\nu=1}^{J}(\beta+\mathbf{1}_d)^{m^{(\nu)}}.$$

Under the notation above, we can prove the following theorems:

6.7 Theorem. *Let P be a matrix of Fuchs-Goursat type of size $J\times J$ with weight (k, m). Let $p=(x^*;\sqrt{-1}\langle\xi^*, dx\rangle)$ be a point of $T^*_{T_NM}T_YL$ with $\tau_N(x^*)=0$. Assume that P satisfies* (A). *Let $u(t, x)={}^t(u_1(t, x),\dots, u_J(t, x))$ be a germ of $(\mathscr{C}^F_{N|M})^{\oplus J}$ at p. Suppose that $u(t, x)$ satisfies*

$$\begin{cases} P(t, x;\partial_\tau,\partial_x)\ u(t, x)=0, \\ \tilde{\gamma}^{A_i,j_i}_{N|M}\,\alpha^F_{N|M}(\partial_{t_i}{}^{j_i}u_\nu(t, x))=0 \quad (1\leqslant\nu\leqslant J,\ 1\leqslant i\leqslant d,\ 0\leqslant j_i\leqslant m_i^{(\nu)}-k_i^{(\nu)}-1). \end{cases}$$

Then it follows that $u(t, x)=0$ at p.

6.8 Corollary. *Let P be a matrix of Fuchs-Goursat type of size $J\times J$ with weight (k, m). Let x^* be a point of T_NM with $\tau_N(x^*)=0$. Assume that P satisfies* (A). *Let $u(t, x)={}^t(u_1(t, x),\dots, u_J(t, x))$ be a germ of $(\mathscr{B}^F_{N|M})^{\oplus J}$ at x^*. Suppose that $u(t, x)$ satisfies.*

$$\begin{cases} P(t, x;\partial_t,\partial_x)\,u(t, x)=0, \\ \tilde{\gamma}^{A_i,j_i}_{N|M}\,\alpha^F_{N|M}(\partial_{t_i}{}^{j_i}u_\nu(t, x))=0 \quad (1\leqslant\nu\leqslant J,\ 1\leqslant i\leqslant d,\ 0\leqslant j_i\leqslant m_i^{(\nu)}-k_i^{(\nu)}-1). \end{cases}$$

Then it follows that $u(t, x)=0$ at x^*.

6.9 Theorem. *Let U be an open set of $T^*_{T_NM}T_YL$ such that each fiber of*

$$\tau_{Y\pi}:({}^t\tau'_Y)^{-1}(U)\to T^*_NY$$

is connected and intersects with $T^*_{T_NM_i}T_YL_i$ for any $1\leqslant i\leqslant d$. Let P be a matrix of Fuchs-Goursat type of size $J\times J$ with weight (k, m). Assume that P satisfies (A) *and that a section $u(t, x)\in\Gamma(U;\mathscr{C}^F_{N|M})^{\oplus J}$ satisfies*

$$\begin{cases} P(t, x;\partial_t,\partial_x)\,u(t, x)=0, \\ \gamma^{F,j_i}_{N|M}(\partial_{t_i}{}^{j_i}u_\nu(t, x))=0\in\Gamma(U\cap T^*_{T_NM_i}T_YL_i;\mathscr{C}^F_{N|M_i}) \quad \begin{pmatrix} 1\leqslant\nu\leqslant J,\ 1\leqslant i\leqslant d, \\ 0\leqslant j_i\leqslant m_i^{(\nu)}-k_i^{(\nu)}-1 \end{pmatrix}. \end{cases}$$

Then it follows that $u(t, x)=0$.

6.10 Corollary. *Let* U *be an open set of* $T_N M$ *such that each fiber of* τ_N: $U \to N$ *is connected and intersects with* $T_N M_i$ *for any* $1 \leqslant i \leqslant d$. *Let* P *be a matrix of Fuchs-Goursat type of size* $J \times J$ *with weight* (k, m). *Assume that* P *satisfies* (A) *and that a section* $u(t, x) \in \Gamma(U; \mathscr{B}^F_{N|M})^{\oplus J}$ *satisfies*

$$\begin{cases} P(t, x; \partial_t, \partial_x)\, u(t, x)=0, \\ \gamma^{F, i}_{N|M}(\partial_{t_i}{}^{j_i} u_\nu(t, x))=0 \in \Gamma(U \cap T_N M_i; \mathscr{B}^F_{N|M_i}) \quad \begin{pmatrix} 1 \leqslant \nu \leqslant J,\ 1 \leqslant i \leqslant d, \\ 0 \leqslant j_i \leqslant m_i^{(\nu)} - k_i^{(\nu)} - 1 \end{pmatrix}. \end{cases}$$

Then it follows that $u(t, x)=0$.

By Lemma 4.6 we have also the following corollary:

6.11 Corollary. *Let* P *be a matrix of Fuchs-Goursat type of size* $J \times J$ *with weight* (k, m). *Assume that* P *satisfies* (A) *and a germ* $u(t, x)$ *of* $(\mathscr{C}^A_{N|M})^{\oplus J}$ *at* $(0; \sqrt{-1}\langle \xi^*, dx\rangle) \in T^*_M X$ *satisfies*

$$\begin{cases} P(t, x; \partial_t, \partial_x)\ u(t, x)=0, \\ \qquad \partial_{t_i}{}^{j_i} u_\nu|_{t_i=0}=0 \quad (1 \leqslant \nu \leqslant J,\ 1 \leqslant i \leqslant d,\ 0 \leqslant j_i \leqslant m_i^{(\nu)} - k_i^{(\nu)} - 1). \end{cases}$$

Then it follows that $u(t, x)=0$ *at* $(0; \sqrt{-1}\langle \xi^*, dx\rangle)$.

6.12 Remark. (1) Since the induced system $\mathscr{M}_Y$ is not necessarily a coherent $\mathscr{D}_Y$-Module in cases of Theorems 6.7 and 6.9, we must impose boundary (or rather initial) conditions on each hypersurface M_i, rather than the boundary conditions on N. This might be regarded as a hyperfunction (or microfunction) version of the Goursat problem, rather than the higher-codimensional boundary value problem.

(2) In [Y], we discussed the solvability of the Goursat problem for a Fuchs-Goursat type operator in the framework of microfunctions. By Corollary 6.11, in the differential case we can conclude a uniqueness of each solution in Theorem 4.2 and Corollary 4.5 of [Y].

Proof of Theorems 6.7 *and* 6.9. Assume that the conditions of Theorems 6.7 or 6.9 are satisfied. Then we can find a $v(t, x)={}^t(v_1(t, x), \dots, v_J(t, x)) \in (\widetilde{\mathscr{C}}^A_{N|M})^{\oplus J}$ such that for any $1 \leqslant \nu \leqslant J$

$$\alpha^F_{N|M}(u_\nu(t, x))=t^{m^{(\nu)}-k^{(\nu)}} v_\nu(t, x).$$

Indeed, in case of Theorem 6.7, we may apply Lemmas 6.4 and 6.5. Next let us consider the case of Theorem 6.9. By Lemma 6.4 and the injectivity of $\alpha^F_{N|M}$, on the set $U\cap T^*_{T_NM_t}T_YL_t$ we have

$$\tilde{\gamma}^{A,t}_{N|M}\,\alpha^F_{N|M}(\partial_{t_i}{}^{j_i}\,u_\nu(t,x))=0 \qquad (1\leqslant\nu\leqslant J,\ 1\leqslant i\leqslant d,\ 0\leqslant j_i\leqslant m_i^{(\nu)}-k_i^{(\nu)}-1).$$

On the other hand, since $\alpha^F_{N|M}(u_\nu(t,x))\in\Gamma(\tau_Y(({}^t\tau'_Y)^{-1}(U));\tilde{\mathscr{C}}^A_{N|M})$ by virtue of Lemma 4.6 and the assumption on U, the equalities above hold at each point of U. Hence applying Lemma 6.5 at each point of U, we can obtain the desired result. Define P_1 by

$$P_1:=P\cdot\begin{pmatrix} t^{m^{(1)}-k^{(1)}} & & 0\\ & \ddots & \\ 0 & & t^{m^{(J)}-k^{(J)}}\end{pmatrix}.$$

Then we see that P_1 is of Fuchs-Goursat type with weight (m,m) and satisfies (A). Moreover since $P\,u=P_1\,v$, the proof of Theorems 6.7 and 6.9 are reduced to the following proposition:

6.13 Proposition. *Let P be a matrix of Fuchs-Goursat type of size $J\times J$ with weight (m,m). Let $p=(0;\sqrt{-1}\langle\xi^*,dx\rangle)$ be a point of T^*_NY. Suppose that P satisfies* (A) *and that a J-tuple $u(t,x)\in(\tilde{\mathscr{C}}^A_{N|M})^{\oplus J}$ at p satisfies*

$$P(t,x;\partial_t,\partial_x)\;u(t,x)=0.$$

Then it follows that $u(t,x)=0$ at p.

To prove Proposition 6.13, we need the following lemma:

6.14 Lemma. *Under the same assumption as in Proposition* 6.13, *the morphism*

$$\psi\colon \mathscr{D}_X{}^J\ni(Q_1,\dots,Q_J)\mapsto(Q_1,\dots,Q_J)\cdot P\in\mathscr{D}_X{}^J$$

is injective on Y.

Proof. It suffices to prove that ψ is injective at the origin of X. In general, for a germ Q of $\mathscr{D}_X$ at the origin, write Q in the form

$$Q(\tau,z;\partial_\tau,\partial_z)=\sum_{\alpha,\beta\in\mathbb{N}_0^d}Q_{\alpha\beta}(z,\partial_z)\,\tau^\alpha\partial_\tau{}^\beta$$

Then we define $\mathrm{ord}_Y(Q)$ and $\hat{\sigma}_Y(Q)$ by

$$\mathrm{ord}_Y(Q):=\max\{|\beta-\alpha|;\ Q_{\alpha\beta}\not\equiv0\}$$

and

$$\widehat{\sigma}_Y(Q)(\tau, z; \partial_\tau, \partial_z) = \widehat{\sigma}_{Y,\mathrm{ord}_Y(Q)}(Q)(\tau, z; \partial_\tau, \partial_z) := \sum_{|\beta-\alpha|=\mathrm{ord}_Y(Q)} Q_{\alpha\beta}(z, \partial_z)\tau^\alpha \partial_\tau^\beta.$$

Moreover, let us denote by $\mathrm{ord}(\widehat{\sigma}_Y(Q))$ the usual order (with respect to ∂_τ and ∂_z) of $\widehat{\sigma}_Y(Q)$ and by $\sigma_l(\widehat{\sigma}_Y(Q))$ the (principal) symbol of $\widehat{\sigma}_Y(Q)$ of order l if $\mathrm{ord}(\widehat{\sigma}_Y(Q)) \leqslant l$. Note that $\mathrm{ord}_Y(P^{(\mu,\nu)}) \leqslant 0$ and

$$\widehat{\sigma}_{Y,0}(P^{(\mu,\nu)})(\tau, z; \partial_\tau) = \sum_{0\leqslant\alpha\leqslant m^{(\nu)}} P_\alpha^{1,(\mu,\nu)}(0, z)\mathscr{I}_\alpha(\tau\partial_\tau)$$

with $\tau\partial_\tau := (\tau_1\partial_{\tau_1},\dots,\tau_d\partial_{\tau_d})$. We denote by η and ζ the dual variables of τ and z respectively. Hence we have $\mathrm{ord}(\widehat{\sigma}_{Y,0}(P^{(\mu,\nu)})) \leqslant |m^{(\nu)}|$ and

$$\sigma_{|m^{(\nu)}|}(\widehat{\sigma}_{Y,0}(P^{(\mu,\nu)}))(\tau, z; \eta) = P_{m^{(\nu)}}^{1,(\mu,\nu)}(0, z)\tau^{m^{(\nu)}}\eta^{m^{(\nu)}}.$$

Now assume that a J-tuple $(Q_1,\dots,Q_J) \neq 0$ satisfies

$$\psi(Q_1,\dots,Q_J) = \Big(\sum_{\mu=1}^{J} Q_\mu P^{(\mu,1)},\dots,\sum_{\mu=1}^{J} Q_\mu P^{(\mu,J)}\Big) = 0.$$

Let us set

$$j_0 := \max\{\mathrm{ord}(\widehat{\sigma}_Y(Q_\mu));\ 1\leqslant\mu\leqslant J\},$$
$$l_0 := \max\{\mathrm{ord}(\widehat{\sigma}_{Y,j_0}(Q_\mu));\ 1\leqslant\mu\leqslant J\}.$$

Then we have

$$\sum_{\mu=1}^{J} \widehat{\sigma}_{Y,j_0}(Q_\mu)\,\widehat{\sigma}_{Y,0}(P^{(\mu,\nu)}) = 0 \qquad (1\leqslant\nu\leqslant J).$$

Thus it follows that

$$\sum_{\mu=1}^{J} \sigma_{l_0}(\widehat{\sigma}_{Y,j_0}(Q_\mu))(\tau, z; \eta, \zeta)\, P_{m^{(\nu)}}^{1,(\mu,\nu)}(0, z)\tau^{m^{(\nu)}}\eta^{m^{(\nu)}} = 0 \quad (1\leqslant\nu\leqslant J).$$

Let us set

$$\mathscr{I}_0(z; \beta) := \det(P_{m^{(\nu)}}^{1,(\mu,\nu)}(0, z)\beta^{m^{(\nu)}})$$

with $\beta \in \mathbb{C}^d$. Then $\mathscr{I}_0(z; \beta)$ is written in the form $\mathscr{I}_0(z; \beta) = \beta^{\tilde{m}}c(z)$ with a holomorphic function $c(z)$ and $\tilde{m} := \sum_{\iota=1}^{d} m^{(\iota)}$. On the other hand, we have

$$\lim_{\beta\to\infty}\beta^{-\tilde{m}}\mathscr{I}_P(z; \beta) = c(z),$$

where $\beta \to \infty$ means that each component β_i of β tends to infinity. Thus the condition (A) implies, in particular, that $c(0)\neq 0$. It follows; that $\mathscr{I}_0(z;\beta)$ never vanishes if $|z|$ is small enough and $\beta_i\in\mathbb{C}\setminus\{0\}$. Hence we have $\sigma_{l_0}(\hat{\sigma}_{Y,j_0}(Q_\mu))=0$ as a function of τ, z, η and ζ for $1\leqslant\mu\leqslant J$, which is a contradiction. Hence we have $Q_\mu=0$ for $1\leqslant\mu\leqslant J$. This completes the proof. □

Proof of Proposition 6.13. Let us set $\mathcal{M}:=\mathscr{D}_X{}^J/$ Image $\psi=\mathscr{D}_X{}^J/\mathscr{D}_X{}^J P$. Then by Lemma 6.14, we have an exact sequence:

$$0\longrightarrow\mathscr{D}_X{}^J\xrightarrow{\cdot P}\mathscr{D}_X{}^J\longrightarrow\mathcal{M}\longrightarrow 0.$$

Hence $\iota_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{M},\mathscr{O}_X)$ can be represented by the complex:

$$0\longrightarrow\iota_Y^{-1}\mathscr{O}_X{}^{\oplus J}\xrightarrow{P\cdot}\iota_Y^{-1}\mathscr{O}_X{}^{\oplus J}\longrightarrow 0.$$

This complex is exact by virtue of the Cauchy-Kovalevskaja type theorem (Theorem 1.3 of [Y] which is an extension of Théorème (1.1) of Madi [M]). Thus we have

$$\iota_Y^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathcal{M},\mathscr{O}_X)=0.$$

Applying the functor $\mu_N(*)\otimes or_{N/Y}$ and taking the n-th cohomology, we have

$$\mathscr{H}om_{\mathscr{D}_X}(\mathcal{M},\widetilde{\mathscr{C}}^A_{N|M})_{(0,\sqrt{-1}\langle\xi^*,dx\rangle)}=0.$$

This completes the proof. □

6.15 Example. Let us choose $J=d$ and set $m^{(\nu)}:=(0,\dots,0,\overset{\nu}{\check{2}},0\dots,0)$ and $k^{(\nu)}:=(0,\dots,0,\overset{\nu}{\check{1}},0,\dots,0)\in\mathbb{N}_0{}^d$ $(1\leqslant\nu\leqslant d)$. Consider the following matrix of differential operators:

$$\begin{aligned}P(\tau,z;\partial_\tau,\partial_z)&=(P^{(i,j)}(\tau,z;\partial_\tau,\partial_z))_{i,j=1}^d\\&=(\tau_j\partial_{\tau_j}{}^2\delta_{ij}+(A_{ij}(\tau,z)+\tau^{1_d}B_{ij}(\tau,z;\partial_z))\partial_{\tau_j}\\&\qquad+C_{ij}(\tau,z)+\tau^{1_d}D_{ij}(\tau,z;\partial_z))_{i,j=1}^d,\end{aligned}$$

where $(A_{ij}(\tau,z))_{i,j=1}^d$ is an upper triangular matrix, $B_{ij}(\tau,z;\partial_z)$ and $D_{ij}(\tau,z;\partial_z)$ are differential operators with ord $B_{ij}\leqslant 1$ and ord $D_{ij}\leqslant 2$ respectively. Then, P is of Fuchs-Goursat type with weight $((k^{(1)},\dots,k^{(d)}),(m^{(1)},\dots,m^{(d)}))$ and

$$\mathscr{I}_P(z;\beta+m^{(1)}-k^{(1)},\dots,\beta+m^{(d)}-k^{(d)})=\det((\beta_i+1)\beta_j\delta_{ij}+A_{ij}(0,z)(\beta_j+1)).$$

Hence if each eigenvalue of $(A_{ij}(0,0))_{i,j=1}^d$ is not in $\{l\in\mathbb{Z};\,l\leqslant 0\}$, we see that P satisfies the condition (A).

Let $u(t,x)={}^t(u_1(t,x),\dots,u_d(t,x))$ be a d-tuple of F-mild hyperfunctions (or F-mild microfunctions). Suppose that $u(t,x)$ satisfies $P\,u(t,x)=0$ and

$$\gamma^{F, i}_{N|M}(u_i(t, x)) = 0 \qquad (1 \leqslant i \leqslant d)$$

Then it follows that $u(t, x) = 0$.

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