Publ. RIMS, Kyoto Univ.
34 (1998), 439-486

Integral and Theta Formulae for Solutions of sl_N Knizhnik-Zamolodchikov Equation at Level Zero

By

Atsushi NAKAYASHIKI

Abstract

The solutions of the sl_N Knizhnik-Zamolodchikov(KZ) equations at level 0 are studied. We present the integral formula which is obtained as a quasi-classical limit of the integral formula of the form factors of the $SU(N)$ invariant Thirring model due to F. Smirnov. A proof is given that those integrals satisfy sl_N KZ equation of level 0. The relation of the integral formulae with the chiral Szegö kernel is clarified. As a consequence the integral formula with the special choice of cycles is rewritten in terms of the Riemann theta functions associated with the Z_N curve. This formula gives a generalization of Smirnov's formula in the case of sl_2.

§ 0. Introduction

In [14] F. Smirnov derived a curious theta formula for the solution of the sl_2 Knizhnik-Zamolodchikov (KZ) equation at level 0. The aim of this paper is to generalize Smirnov's results to the case of sl_N. Before giving a more detail of our results let us summarize the reason why we are interested in the level 0 case of the KZ equation.

The KZ equation was introduced in [5] as one of the fundamental equations characterizing the correlation functions of the Wess-Zumino-Witten (WZW) model in conformal field theory. For the affine Lie algebra $\hat{\mathfrak{g}}$ and its highest weight representations $V_1,\cdots,V_m$ the KZ equation has the form

$$(k+g)\frac{\partial F}{\partial \lambda_i}=\sum_{j\neq i}\frac{\Omega_{ij}}{\lambda_i-\lambda_j}F,$$

where F is a $V_1\otimes\cdots\otimes V_m$ valued function in $\lambda_1, \cdots \lambda_m$, Ω_{ij} is the invariant tensor, with respect to the symmetric invariant bilinear form of $\mathfrak{g}$, acting on i-the and j-th tensor components, g is the dual Coxeter number of $\mathfrak{g}$ and k is a

Communicated by T. Miwa, March 24, 1998.
1991 Mathematics Subject Classification(s): 81T40, 14K25, 33C80

*Graduate School of Mathematics, Kyushu University, Ropponmatsu 4-2-1, Fukuoka 810, Japan and LPTHE, Université Pierre et Marie Curie, Tour 16 1er Etage, 4 Place Jussieu, 75252 Paris Cedex 05, France

parameter. The number k is called level. In the WZW models levels are positive integers which coinside with those of the integrable highest weight representation of $\hat{\mathfrak{g}}$.

The KZ equation acquires a new life from the study of the two dimensional integrable massive quantum field theories (IMQFT) and solvable lattice models (SLM). F. Smirnov formulated an axiom of locality for form factors and, for several models, obtained integral formulas of form factors [13]. In [12] the rational q deformed KZ (qKZ) equation was found as a consequence of the axiom. Hence the moment the qKZ equation is invented the integral formula for the solution is constructed. It is important to note that the qKZ equation appeared in this context is of level 0.

Around the same time I. Frenkel and N. Reshetikhin developped a general theory of vertex operators for quantum affine algebras [3]. They derived a qKZ equation of general level as the equation satisfied by the highest-highest matrix element of the vertex operators. This theory was successfully applied to the study of SLMs [4]. Although, in this application to SLM, the building blocks are the vertex operators of positive integer levels, the form factors and the corelation functions are shown to satisfy the level 0 and level $-2\times$ (dual Coxeter number) qKZ equation respectively [4][8]. Thus the level 0 qKZ equation and their degenerations are of special importance in the context of IMQFT and SLM.

In order to understand the nature of form factors F. Smirnov studied the quasi-classical limit [12][14]. He noticed that the period integral of the hyperelliptic curve $s^2=f(z)=\prod_{j=1}^{2n}(z-\lambda_j)$ appears as the limit of the integral formula for the form factors of $SU(2)$ invariant Thirring model. Then in [14] he rewrites them in terms of Riemann theta functions as

$$
\begin{aligned}
(1)\qquad & f_{\varepsilon_1,\cdots,\varepsilon_{2n}}(\lambda_1,\cdots,\lambda_{2n}) \\
&= \zeta_A(\det A)^{-3}\Delta^{-3/4}\theta[e_A](0)^4\det(\partial_i\partial_j\log\theta[e_A](0))_{1\leq i,j\leq g},
\end{aligned}
$$

where $\Lambda=(\varepsilon_1,\cdots,\varepsilon_{2n})$ is the sequence of $\pm$, the number of $+$ being equal to the number of $-$, ζ_Λ a certain fourth root of unity, e_Λ a nonsingular even half period corresponding to the partition $\{1,2,\cdots,2n\}=\{j\,|\,\varepsilon_j=+\}\sqcup\{j\,|\,\varepsilon_j=-\}$, $\Delta=\prod_{i<j}(\lambda_i-\lambda_j)$, $\partial_i=\partial/\partial z_i$, $g=n-1$ the genus of the curve, $\{A_i,B_j\}$ a canonical homology basis and $A=(\int_{A_i}z^{j-1}dz/s)_{1\leq i,j\leq n}$. The function $F=\sum f_{\varepsilon_1,\cdots,\varepsilon_{2n}}v_{\varepsilon_1}\otimes\cdots\otimes v_{\varepsilon_{2n}}$ gives a solution to the KZ equation taking the value in $V^{\otimes 2n}$ with $V=\mathbb{C}v_+\oplus\mathbb{C}v_-$ being the vector representation of sl_2.

Since the theta function of an algebraic curve is the tau function, modulo some factor, of a soliton equation, this result suggests an intimate relation of the level 0 KZ equation with the soliton equations. In spite of Smirnov's effort on this problem [14][15][16] this relation is not yet clearly understood.

Integral formulae are known for the solutions of the KZ equation with an arbitrary level associated with any Kac Moody Lie algebra [10][11]. In [9] it is shown that those general integral formulae have the exact forms as their integrands in the case of sl_2, level 0 and singlet solutions in the tensor product of vector representations. Taking this fact into consideration is crucial to give a complete correspondence between general formulae at level 0 and the Smirnov type formulae in [9]. A completely analogous structure exists in the case of sl_2 rational qKZ equation [9]. Thus Smirnov type formula is related with a subtle structure of level 0. In the sl_N case to find a similar structure to the sl_2 case in the formulae in [6] [10] is not yet succeeded.

One strategy to understand Smirnov type solutions more clearly will be to generalize it. This is the reason why we are interested in the generalization of the Smirnov's results to the other types of Lie algebras than sl_2.

Now let us describe our results. In [13] the integral formula for form factors of the $SU(N)$ invariant Thirring model is obtained. It is a solution to the sl_N rational qKZ equation of level zero. We take the quasi-classical limit of this integral formula. It is expressed as the determinant of the period integrals of a $\mathbf{Z}_N$ curve. A $\mathbf{Z}_N$ curve is a natural generalization of a hyperelliptic curve, which corresponds to $N=2$. Roughly speaking the integral formula obtained in this manner should give a solution to the sl_N KZ equation of level zero. From the mathematical point of view it is not very easy to prove rigorously that the asymptotics satisfies the KZ equation. On the other hand the formula for the quasi-classical limit is rather simple. Hence it is desirable and interesting to prove directly that it satisfies the KZ equation. We give a proof which is new even for the sl_2 case. Compared with the proof in the generic level case [6] [11] our proof looks more complicated. It will be related with the degenerate structure of Smirnov type solutions found in [9]. Since we have established a correct Smirnov type integral formula in the sl_N case it is an interesting problem to get them from the formulae in [6] [10] in the spirit of [9].

We rewrite the integral formula in terms of theta functions on a $\mathbf{Z}_N$ curve. A priori this is not a trivial task at all. In fact the following major problems are not obvious from the formula and arguments in the sl_2 case. The first one is what kind of rational periods parametrize the tensor components of the solution. The second one is whether we can expect the second order derivatives of the logarithm of theta functions or not in the sl_N case. The first problem is resolved with the help of the Thomae formula for $\mathbf{Z}_N$ curves which was discovered by Bershadsky and Radul [1] [7]. Namely the tensor component is parametrized by certain non-singular $1/N$ or $1/2N$ periods introduced in [1]. The second problem is solved by finding a relation of the integrand of the integral formula with the Szegö kernel. In fact the product of Szegö kernels is related with the second order derivatives of the logarithm of theta functions by the formula due to Fay [2].

Now the present paper is organized in the following manner. In section 1 the integral formula is given. The theta formula is given in section 2. It is proved in section 3. In section 4 a proof is given that the integral formula satisfies the KZ equation and belongs to the trivial representation of sl_N. A derivation of fundamental relations among differential forms used in section 4 is given in appendix.

§ 1. Integral Formulas

Let sl_N be the simple Lie algebra of type A_{N-1}, $(\ ,\ )$ the symmetric bilinear form on sl_N given by $(X, Y)=\mathrm{tr}(XY)$, $\{I_j\}$ a basis of sl_N and $\{I^j\}$ the dual basis with respect to $(\ ,\ )$. The invariant element Ω is given by

$$\Omega=\sum_j I_j \otimes I^j.$$

Let V be the N dimensional irreducible representation of sl_N and m a positive integer. The Knizhnik-Zamolodchikov (KZ) equation with values in the Nm fold tensor product $V^{\otimes Nm}$ of V is the differential equation for the $V^{\otimes Nm}$ valued function F

$$(k+N)\frac{\partial F}{\partial \lambda_i}=\sum_{j\neq i}\frac{\Omega_{ij}}{\lambda_i-\lambda_j}F, \tag{2}$$

where Ω_{ij} means the action of Ω on the i-th and j-th components of $V^{\otimes Nm}$, k is a complex number called level. The explicit form of KZ equation in terms of the vector components is given in section 4.

Let $v_j={}^t(0, \cdots, 1, \cdots, 0)$ in $\mathbb{C}^N$, where 1 is on the j-th place. Then we have $V=\oplus_{j=1}^N \mathbb{C}v_j$. We denote by $\Lambda=(\Lambda_1, \cdots, \Lambda_N)$ the ordered partition of $\{1, 2, \cdots, Nm\}$ such that the number $|\Lambda_i|$ of the elements of Λ_i is m for any i. To an ordered partition Λ we associate the weight zero vector v_Λ of $V^{\otimes Nm}$ by

$$v_\Lambda=v_{k_1}\otimes\cdots\otimes v_{k_{Nm}},$$

where

$$i\in\Lambda_j \text{ if and only if } k_i=j.$$

The set of $\{v_\Lambda\}$ forms a base of the weight zero subspace of $V^{\otimes Nm}$.

The operators $\sum_{j\neq i}(\lambda_i-\lambda_j)^{-1}\Omega_{ij}$ in the right hand side of (2) commute with the action of sl_N. Thus it has a sense to consider the KZ equation for a function taking values in a fixed weight subspace of $V^{\otimes Nm}$. In this paper we exclusively consider the solution F whose value is in the weight zero subspace of the tensor product $V^{\otimes Nm}$. Then we can define the component f_Λ of F by

$$F=\sum_\Lambda f_\Lambda v_\Lambda, \tag{3}$$

where the sum is over all ordered partition Λ.

We denote by C the compact Riemann surface defined from the equation $s^N=f(z)=\prod_{j=1}^{Nm}(z-\lambda_j)$. It is called a $\mathbb{Z}_N$ curve [1] [7]. The genus g of C is given by $2g=(N-1)(Nm-2)$. For Λ_r and $p\in\Lambda_r$ set

$$g^{(\Lambda_r)}(z)=\prod_{j\notin\Lambda_r}(z-\lambda_j),\quad g_{\Lambda_r}(z)=\prod_{j\in\Lambda_r}(z-\lambda_j),\quad g_{\Lambda_r}^{(p)}(z)=\prod_{j\in\Lambda_r,\, j\neq p}(z-\lambda_j)$$

and define the meromorphic differential form $\mu_p^A(z)$ on C by

$$\mu_p^A(z)=\frac{g^{(Ar)}(\lambda_p)g_{Ar}^{(p)}(z)}{(z-\lambda_p)s}dz.$$

We set $L=(N-1)m-1$. Then we have

Theorem 1. *Let $\{p_1, \cdots, p_L\}$ is an arbitrary subset of $\{1, 2, \cdots, Nm\}$. Define*

$$f(\lambda_1, \cdots, \lambda_{Nm})_A=\frac{\Delta^{\frac{N-1}{N^2}}}{\prod_{i<j}(A_iA_j)}\frac{\det(\int_{\gamma_i}\mu_{p_j}^A)_{1\le i,j\le L}}{\Delta(p_1, \cdots, p_L)} \tag{4}$$

where $(A_iA_j)=\prod_{r\in A_i, s\in A_j}(\lambda_r-\lambda_s)$, $\Delta(p_1, \cdots, p_L)=\det(\lambda_{p_j}^{L-i})_{1\le i,j\le L}$ and $\Delta=\prod_{i<j}(\lambda_i-\lambda_j)$. *Then*

0 . *The right hand side of* (4) *does not depend on the choice of* $\{p_1, \cdots, p_L\}$.
1 . *The function F given by* (3) *and* (4) *is a solution to the sl_N KZ equation of level zero for arbitrary set of L cycles $\{\gamma_1, \cdots, \gamma_L\}$ on C.*
2 . *For any $X\in sl_N$, $XF=0$.*

The first statement of Theorem 1 follows from another expression for f_A. Let us set

$$\zeta_j^A=\frac{dz}{s}\sum_{k=1}^{N}g_{A_k}(z)\left[\frac{d}{dz}\frac{g^{(A_k)}(z)}{z^{L-j+1}}\right]_0,$$

where $[\quad]_0$ denotes the polynomial part of a Laurent polynomial. It is obvious that d/dz can be out side of the symbol $[\quad]_0$.

Theorem 2. *The function f_A given by* (4) *is also written as*

$$f(\lambda_1, \cdots, \lambda_{Nm})_A=\frac{\Delta^{\frac{N-1}{N^2}}}{\prod_{i<j}(A_iA_j)}\det\left(\int_{\gamma_i}\zeta_j^A\right)_{1\le i,j\le L}. \tag{5}$$

We shall give some comments on the integral formula given here. In [13] F. Smirnov derived the integral formula of form factors of $SU(N)$ invariant Thirring model which satisfy the deformed Knizhnik-Zamolodchikov (dKZ) equation on level zero. Scaling the rapidity variables β_j as $\beta_j=\lambda_j/h$ and taking the quasi-classical limit $h\to 0$, we obtain the integral formula in Theorem 2 with some special choice of cycles $\{\gamma_i\}$.

§ 2. Theta Formula

We shall give another expression for the solution F given in Theorem 1. To give a precise statement we prepare necessary notations associated with the $\mathbf{Z}_N$ curve C [7]. The N-cyclic automorphism ϕ of C is defined by $\phi:(z, s)\mapsto(z, \omega s)$, where ω is the N-th primitive root of unity. There are Nm branch points $Q_1, \cdots, Q_{Nm}$, whose projection to z coordinate are $\lambda_1, \cdots, \lambda_{Nm}$. The basis of holomorphic

1-forms on C is given by

$$w_{\beta}^{(\alpha)}=\frac{z^{\beta-1}dz}{s^{\alpha}} \quad 1\leq\alpha\leq N-1, \quad 1\leq\beta\leq\alpha m-1.$$

We fix a canonical homology basis $\{\alpha_i, \beta_j\}$ whose intesection numbers are $\alpha_i\cdot\alpha_j=\beta_i\cdot\beta_j=0$, $\alpha_i\cdot\beta_j=\delta_{ij}$. Let Δ be a Riemann divisor for this choice of canonical basis. Let us define the divisor class D by $D=NQ_i$ which is independent of the choice of i. To each ordered partition Λ we associate the divisor class e_{Λ} [7] by

$$e_{\Lambda}\equiv\Lambda_1+2\Lambda_2+\cdots+(N-1)\Lambda_{N-1}-D-\Delta,$$

where for a subset S of $\{1, 2, \cdots, Nm\}$ we set

$$S=\sum_{j\in S}Q_j.$$

The divisor class e_{Λ} is a $1/N$ period for N even and is a $1/2N$ period for N odd. We consider the index of Λ_j by modulo N. In particular $\Lambda_0=\Lambda_N$.

Let $\{v_j(x)\}$ be the basis of the normalized abelian differentials of the first kind whose normalization is

$$\int_{A_j}v_k(x)=2\pi i\delta_{jk}.$$

We set $\tau_{jk}=\int_{B_j}v_k(x)$. Then the period matrix $\tau=(\tau_{jk})$ is symmetric and its real part is negative definite. The Jacobian variety $J(C)$ of C is described as $J(C)=\mathbb{C}^g/2\pi i\mathbb{Z}^g+\mathbb{Z}^g\tau$. For any element $e\in\mathbb{C}^g$, there exist unique elements $\delta, \varepsilon\in\mathbb{R}^g$ such that

$$e=\begin{Bmatrix}\delta\\ \varepsilon\end{Bmatrix}_{\tau}=2\pi i\varepsilon+\delta\tau.$$

We call δ, ε the characteristics of e. The Riemann theta function with characteristics δ, ε is defined by

$$\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(z)=\sum_{m\in\mathbb{Z}^g}\exp\Big(\frac{1}{2}(m+\delta)\tau(m+\delta)^t+(z+2\pi i\varepsilon)(m+\delta)^t\Big).$$

It satisfies the equation

$$\theta\begin{bmatrix}\delta+m\\ \varepsilon+n\end{bmatrix}(z)=\exp(2\pi i n\delta^t)\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(z), \tag{6}$$

for $m, n\in\mathbb{Z}^g$. For an ordered partition Λ let us take a representative $\bar{e}_{\Lambda}\in\mathbb{C}^g$ of e_{Λ} and let $\bar{e}_{\Lambda}=\begin{Bmatrix}\delta\\ \varepsilon\end{Bmatrix}_{\tau}$. Then the logarithmic derivatives

$$\partial^{\alpha}\log\theta\begin{bmatrix}\delta\\ \varepsilon\end{bmatrix}(z), \quad |\alpha|\geq1,$$

are independent of the choice of the representative $\bar{e}_\Lambda$ by (6), where $\alpha=(\alpha_1, \cdots, \alpha_g)$, $|\alpha|=\alpha_1+\cdots+\alpha_g$, $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_g^{\alpha_g}$ and $\partial_j=\partial/\partial z_j$. Hence we use the notation $\partial^\alpha \log\theta[e_\Lambda](z)$ for those logarithmic derivatives for the sake of simplicity.

Let us define the connection matrix between $\{w_\beta^{(\alpha)}\}$ and $\{v_j(x)\}$ by

$$v_j(x)=\sum_{\alpha,\beta}\sigma_{j(\alpha\beta)}w_\beta^{(\alpha)}.$$

With the aid of this matrix we define the vector field on $J(C)$ by

$$D_\beta=\sum_{j=1}^{g}\sigma_{j(N-1\beta)}\partial_j, \quad 1\le\beta\le L.$$

Now we can state the theta formula.

Theorem 3. *For any subset $\{i_1<\cdots<i_L\}$ of $\{1, 2, \cdots, g\}$ we take the cycles $\{\gamma_j\}$ as $\gamma_j=A_{i_j}$. Then the corresponding solution of the KZ equation in Theorem 1 is given by*

$$f(\lambda_1, \cdots, \lambda_{Nm})_\Lambda=c\frac{\Delta^{\frac{N-1}{N^2}}}{\prod_{i<j}(\Lambda_i\Lambda_j)}\det(\partial_{i_j}D_k\log\theta[e_\Lambda](0))_{1\le j,k\le L},$$

where c is the overall constant independent of λ_i's and Λ.

Using the Thomae formula for $\mathbf{Z}_N$ curves one can rewrite the $\prod_{i<j}(\Lambda_i\Lambda_j)$ in terms of theta constants. The result is

Theorem 4. *For the same choice of cycles as in Theorem 3, we have*

$$f(\lambda_1, \cdots, \lambda_{Nm})_\Lambda=C(\lambda)\zeta_\Lambda\prod_{\sigma\in S_{N-1}}\theta[e_{\Lambda^\sigma}](0)^{\frac{12N}{(N+1)!}}\det(\partial_{i_j}D_k\log\theta[e_\Lambda](0))_{1\le j,k\le L},$$

where ζ_Λ is some $N(N+1)!/3$-th root of unity, $C(\lambda)$, which is independent of the partition Λ, is given by

$$C(\lambda)=c(\det A)^{-\frac{6}{N+1}}\Delta^{-3\frac{N-1}{N+1}+\frac{N-1}{N^2}}.$$

Here c is a constant independent of λ_i's and $A=\det(\int_{A_i}w_\beta^{(\alpha)})$. For an element σ of the symmetric group S_{N-1} of degree $N-1$ we define

$$\Lambda^\sigma=(\Lambda_0, \Lambda_{\sigma(1)}, \cdots, \Lambda_{\sigma(N-1)}).$$

Remark. If $N=2$, then $L=m-1=g$ and $(\sigma_{j(1k)})=A^{-1}$, where g is the genus of the hyperelliptic curve C. In particular $(i_1, \cdots, i_L)=(1, \cdots, g)$. From the matrix relation

$$(\partial_i D_k\log\theta[e_\Lambda](0))=(\partial_i\partial_j\log\theta[e_\Lambda](0))A^{-1}$$

we have

$$f_\Lambda=c'\zeta_\Lambda(\det A)^{-3}\Delta^{-3/4}\theta[e_\Lambda](0)^4\det(\partial_i\partial_j\log\theta[e_\Lambda](0))_{1\le i,j\le g}$$

which is nothing but the Smirnov's formula (1) for sl_2.

§ 3. Proof of Theta Formulae

Let $\tilde{C}$ be the universal covering space of C. We identify a holomorphic one forms on C with those on $\tilde{C}$ which are invariant under the action of the fundamental group of C. We set $v=(v_1, \cdots, v_g)$, the vector of the normalized differentials of the first kind. Recall that the chiral Szegö kernel defined by e_A is

$$R(x, y|e_A)=\frac{\theta[e_A](y-x)}{\theta[e_A](0)E(x, y)} \qquad x, y\in\tilde{C},$$

where

$$y-x=\int_x^y v,$$

the integral being taken in $\tilde{C}$ and $E(x, y)$ is the prime form [7]. We remark that, as to the e_A dependence, $R(x, y|e_A)$ depends only on the divisor class of e_A.

Let $\omega(x, y)$ be the canonical symmetric differential, that is, $\omega(x, y)$ is the section of the canonical bundle of $C\times C$, symmetric in x and y, has a double pole at $x=y$, has the vanishing A period in each of the variables and has some normalization (for more precise definition see [7]). Theorem 3 is a corollary of the following proposition.

Proposition 1. *For* $1\leq p\leq Nm$ *we have*

$$\mu_p^A=Nf'(\lambda_p)^{\frac{N-1}{N}}\omega(x, Q_p)+N^2\sum_{i=1}^{g}\sum_{\beta=1}^{L}\lambda_p^{\beta-1}D_\beta\partial_i\log\theta[e_A](0)v_i(x).$$

As in [7] the value of a (half) differential form at the branch point Q_p is defined as the coefficient of dt (or $\sqrt{dt}$ in the half differential case) in the expansion of the form in the local coordinate $t=(z-\lambda_p)^{1/N}$. Assuming this proposition let us first prove Theorem 3.

Proof of Theorem 3. Since the integral of $\omega(x, Q_p)$ along the cycle A_i is zero for any i, we have

$$\int_{A_i}\mu_p^A=N^2\sum_{\beta=1}^{L}\lambda_p^{\beta-1}D_\beta\partial_i\log\theta[e_A](0),$$

where we use the normalization condition of $\{v_j\}$. Thus we have

$$\det\left(\int_{A_{i_j}}\mu_{p_k}^A\right)_{1\leq j,k\leq L}$$

$$=N^{2L}\det(\partial_{i_j}D_\beta\log\theta[e_A](0))_{1\leq j,\beta\leq L}\det(\lambda_{p_l}^{k-1})_{1\leq k,l\leq L}$$

$$=(-1)^{\frac{L(L-1)}{2}}N^{2L}\det(\partial_{i_j}D_\beta\log\theta[e_\Lambda](0))_{1\le j,\beta\le L}\det(\lambda_{p_l}^{L-k})_{1\le k,l\le L}$$
$$=(-1)^{\frac{L(L-1)}{2}}N^{2L}\det(\partial_{i_j}D_\beta\log\theta[e_\Lambda](0))_{1\le j,\beta\le L}\Delta(p_1,\cdots,p_L).$$

Substituting this equation into (4) we obtain the formula of Theorem 3. □

In order to prove Proposition 1 we first prove

Proposition 2. *For $1\le p\le Nm$ we have*

$$\mu_p^\Lambda=Nf'(\lambda_p)^{\frac{N-1}{N}}R(x,Q_p|e_\Lambda)R(x,Q_p|-e_\Lambda). \tag{7}$$

For the proof of this proposition let us recall the following notation [7] [1]:

$$\mathscr{L}=\left\{-\frac{N-1}{2},-\frac{N-1}{2}+1,\cdots,\frac{N-1}{2}\right\},$$
$$q_l(i)=\frac{1-N}{2N}+\left\{\frac{l+i+\frac{N-1}{2}}{N}\right\},$$

where $l\in\mathscr{L}+\mathbf{Z}$, $i\in\mathbf{Z}$ and $\{a\}=a-[a]$ is the fractional part of $a\in\mathbf{Q}$.

In [7] we have proved

Proposition 3. *For an ordered partition Λ we have*

$$R(x,y|e_\Lambda)=\frac{1}{N}\frac{\sum_{l\in\mathscr{L}}f_l(x,\Lambda)f_{-l}(y,\Lambda^-)}{z(y)-z(x)} \tag{8}$$

$$f_{\pm l}(x,\Lambda^\pm)=\prod_{i=1}^{Nm}(z(x)-\lambda_i)^{\pm q_l(k_i)}\sqrt{dz(x)},$$

where $\Lambda^-=(\Lambda_0,\Lambda_{N-1},\cdots,\Lambda_1)$, $\Lambda^+=\Lambda$ and $k_i=j$ is determined by $i\in\Lambda_j$.

By Proposition 4 in [7], for $l=-(N-1)/2+j$ with $j\in\mathbf{Z}$, we have

$$\operatorname{div}f_l(x,\Lambda^\pm)=\Lambda_{\pm1\mp j}+2\Lambda_{\pm2\mp j}+\cdots+(N-1)\Lambda_{\mp1\mp j}-\sum_{k=1}^{N}\infty^{(k)}, \tag{9}$$

where $\{\infty^{(k)}\}$ are N infinity points. The index k of Λ_k is considered by modulo N. The $\Lambda_{\mp j}$ is missing in the right hand side of (9). Let $p\in\Lambda_r$, $0\le r\le N-1$. We set $y=Q_p$ in (8). By (9) only the term $-l=-(N-1)/2+r$ in the sum of the right hand side of (8) is alive. Hence we have

$$R(x,Q_p|e_\Lambda)=\frac{1}{N}\frac{f_{-\frac{N-1}{2}+r^-}(x,\Lambda)f_{-\frac{N-1}{2}+r}(Q_p,\Lambda^-)}{\lambda_p-z(x)},$$

where $r^-=N-1-r$. Similarly

$$R(x,Q_p|-e_\Lambda)=R(x,Q_p|e_{\Lambda^-})=\frac{1}{N}\frac{f_{-\frac{N-1}{2}+r-1}(x,\Lambda^-)F_{-\frac{N-1}{2}+N-r}(Q_p,\Lambda)}{\lambda_p-z(x)}.$$

Therefore

(10) $R(x, Q_p|e_A)R(x, Q_p|-e_A)=$

$$\frac{1}{N^2(z(x)-\lambda_p)^2}\left(f_{-\frac{N-1}{2}+r^-}(x, A)f_{-\frac{N-1}{2}+r-1}(x, A^-)f_{-\frac{N-1}{2}+r}(Q_p, A^-)f_{-\frac{N-1}{2}+N-r}(Q_p, A)\right).$$

Let us calculate the right hand side of (10).

Lemma 1. *The following equations hold:*

$$f_{-\frac{N-1}{2}+r}(Q_p, A^-)f_{-\frac{N-1}{2}+N-r}(Q_p, A)=\frac{Nf'(\lambda_p)^{\frac{1}{N}}}{g_{A_r}^{(p)}(\lambda_p)}, \tag{11}$$

$$f_{-\frac{N-1}{2}+r^-}(x, A)f_{-\frac{N-1}{2}+r-1}(x, A^-)=\frac{g_{A_r}(z)}{s}dz. \tag{12}$$

Proof. Let $t=(z-\lambda_p)^{1/N}$ be the local coordinate around Q_p. Then

$$f_{-\frac{N-1}{2}+r}(x, A^-)=\sqrt{N}\prod_{j\neq p}^{Nm}(\lambda_p-\lambda_j)^{q_{-\frac{N-1}{2}+r}(N-k_j)}\sqrt{dt}\,(1+O(t^N)),$$

$$f_{-\frac{N-1}{2}+N-r}(x, A)=\sqrt{N}\prod_{j\neq p}^{Nm}(\lambda_p-\lambda_j)^{q_{-\frac{N-1}{2}+N-r}(k_j)}\sqrt{dt}\,(1+O(t^N)),$$

where we use the relation $-q_l(i)=q_{-l}(N-i)$ [7]. Therefore we need to calculate the number

$$q_{-\frac{N-1}{2}+r}(N-k_j)+q_{-\frac{N-1}{2}+N-r}(k_j).$$

By a direct calculation we have

$$q_{-\frac{N-1}{2}+r}(N-i)+q_{-\frac{N-1}{2}+N-r}(i)=\begin{cases}-1+\dfrac{1}{N} & \text{if } i=r \bmod N\\ \dfrac{1}{N} & \text{if } i\neq r \bmod N.\end{cases}$$

Hence

$$\begin{aligned}&f_{-\frac{N-1}{2}+r}(Q_p, A^-)f_{-\frac{N-1}{2}+N-r}(Q_p, A)\\ &=N\prod_{j\notin A_r}(\lambda_p-\lambda_j)^{1/N}\prod_{j\in A_r\setminus\{p\}}(\lambda_p-\lambda_j)^{-1+1/N}\\ &=\frac{Nf'(\lambda_p)^{1/N}}{\prod_{j\in A_r\setminus\{p\}}(\lambda_p-\lambda_j)}.\end{aligned}$$

Similarly, using

$$q_{-\frac{N-1}{2}+r^-}(i)+q_{-\frac{N-1}{2}+r-1}(N-i)=\begin{cases}1-\frac{1}{N} & \text{if } i=r \bmod N\\ -\frac{1}{N} & \text{if } i\neq r \bmod N\end{cases}$$

we have

$$f_{-\frac{N-1}{2}+r^-}(x,\Lambda)f_{-\frac{N-1}{2}+r-1}(x,\Lambda^-)=\frac{\Pi_{j\in\Lambda_r}(z-\lambda_j)}{s}dz.$$

Thus the lemma is proved. □

Multiplying (11) and (12) and substituting it into (10) we obtain the equation (7). □

Recall the Fay's formula ([2], Corollary 2.12, see also [7] section 4):

$$R(x,y|e_\Lambda)R(x,y|-e_\Lambda)=\omega(x,y)+\sum_{i,j=1}^{g}\frac{\partial^2\log\theta[e_\Lambda]}{\partial z_i\partial z_j}(0)v_i(x)v_j(y). \tag{13}$$

By calculation we have

$$v_j(Q_p)=\frac{N}{f'(\lambda_p)^{1-\frac{1}{N}}}\sum_{\beta=1}^{L}\sigma_{j(N-1\beta)}\lambda_p^{\beta-1}. \tag{14}$$

Substituting (14) into (13) and using Proposition 2 we have the equation in Proposition 1. □

In order to prove Theorem 4 let us recall the Thomae formula for $\mathbb{Z}_N$ curves [1] [7]:

$$\theta[e_\Lambda](0)^{2N}=C_\Lambda(\det A)^N\prod_{i\leq j}(\Lambda_i\Lambda_j)^{2Nq(i,j)+N\mu},$$

where C_Λ is a constant independent of λ_i's and

$$q(i,j)=\sum_{l\in\mathcal{L}}q_l(i)q_l(j),\quad \mu=\frac{(N-1)(2N-1)}{6N}, \tag{15}$$

$$(\Lambda_i\Lambda_j)=\prod_{r\in\Lambda_i,s\in\Lambda_j}(\lambda_r-\lambda_s)\ \text{for } i\neq j,$$

$$(\Lambda_i\Lambda_j)=\prod_{r,s\in\Lambda_i,r<s}(\lambda_r-\lambda_s).$$

The number $q(i,j)$ depends only on $|i-j|$ [7]. In particular $q(0,0)=q(i,i)$, $q(i,j)=q(j,i)$ for any i and j. We define the action of the symmetric group S_N of degree N on the set of ordered partitions by

$$\Lambda^\sigma=(\Lambda_{\sigma(0)},\cdots,\Lambda_{\sigma(N-1)}),\quad \sigma\in S_N.$$

The subgroup S_{N-1} acts on the index $1,2,\cdots,N-1$ as we already defined.

Proposition 4. *For an ordered partition Λ we have*

$$\prod_{i<j}(\Lambda_i\Lambda_j)=C\bar{\zeta}_\Lambda(\det A)^{\frac{6}{N+1}}\Delta^{3\frac{N-1}{N+1}}\theta[e_{\Lambda^\sigma}](0)^{-\frac{12N}{(N+1)!}},$$

where $\bar{\zeta}_\Lambda$ is some $N(N+1)!/3$-th root of unity and C is a constant independent of λ_i's and Λ.

Let us prove Proposition 4. By taking the product of theta function with the characteristics e_{Λ^σ} for all $\sigma\in S_N$ we have

$$\prod_{\sigma\in S_N}\theta[e_{\Lambda^\sigma}](0)^{2N}=\Big(\prod_{\sigma\in S_N}C_{\Lambda^\sigma}\Big)(\det A)^{N!N}\prod_{\sigma\in S_N}\prod_{i\le j}(\Lambda_{\sigma(i)}\Lambda_{\sigma(j)})^{2Nq(i,j)+N\mu}$$

$$=\pm\Big(\prod_{\sigma\in S_N}C_{\Lambda^\sigma}\Big)(\det A)^{N!N}\Delta^{N!N\mu}\prod_{i\le j}(\Lambda_i\Lambda_j)^{2N\Sigma_{\sigma\in S_N}q(\sigma(i),\sigma(j))}.$$

Set $\gamma=q(0,0)$ and

$$F:=\prod_{i\le j}(\Lambda_i\Lambda_j)^{2N\Sigma_{\sigma\in S_N}q(\sigma(i),\sigma(j))}$$

$$=\prod_{i=0}^{N-1}(\Lambda_i\Lambda_i)^{N!\cdot 2N\gamma}\prod_{i<j}(\Lambda_i\Lambda_j)^{2N\Sigma_{\sigma\in S_N}q(\sigma(i),\sigma(j))}.$$

Since, for $i\neq j$,

$$\sum_{\sigma\in S_N}q(\sigma(i),\sigma(j))=2\cdot(N-2)!\sum_{r<s}q(r,s),$$

we have

$$F=\prod_{i=0}^{N-1}(\Lambda_i\Lambda_i)^{N!\cdot 2N\gamma}\prod_{i<j}(\Lambda_i\Lambda_j)^{4N\cdot(N-2)!\Sigma_{i<j}q(i,j)}.$$

Using $q(i,j)=q(i+1,j+1)$ and Lemma 10 in [7] we have

$$\sum_{i<j}q(i,j)=-\frac{N^2-1}{24},\quad \gamma=\frac{N^2-1}{12N}.$$

Thus

$$F=\Delta^{\frac{(N+1)!(N-1)}{6}}\prod_{i<j}(\Lambda_i\Lambda_j)^{-\frac{(N+1)!N}{6}}.$$

From this we obtain

$$\prod_{\sigma\in S_N}\theta[e_{\Lambda^\sigma}](0)^{2N}=\pm\Big(\prod_{\sigma\in S_N}C_{\Lambda^\sigma}\Big)(\det A)^{N!N}\Delta^{\frac{N!N(N-1)}{2}}\prod_{i<j}(\Lambda_i\Lambda_j)^{-\frac{(N+1)!N}{6}}.$$

Recall that the ordered partitions which are obtained from Λ by the cyclic permutation of indices correspond to linear equivalent divisor e_Λ [7]. Therefore

$$\prod_{\sigma\in S_{N-1}}\theta[e_{\Lambda^\sigma}](0)^{2N^2}=\pm\Big(\prod_{\sigma\in S_{N-1}}C^N_{\Lambda^\sigma}\Big)(\det A)^{N!N}\Delta^{\frac{N!N(N-1)}{2}}\prod_{i<j}(\Lambda_i\Lambda_j)^{-\frac{(N+1)!N}{6}}.$$

Since C^{2N}_Λ does not depend on Λ, we have the equation in Proposition 4. □

§ 4. Proof of Integral Formurae

In this section we shall give a proof of Theorem 1 and Theorem 2.

§ 4.1. Proof of Theorem 2

Theorem 2 follows from the following proposition.

Proposition 5. *For any p*

$$\sum_{j=1}^{L}\zeta_j^\Lambda(z)\lambda_p^{L-j}=\mu_p^\Lambda(z)+Nd\frac{s^{N-1}}{z-\lambda_p}.$$

The proof of this proposition is totally similar to the case of sl_2 [14]. For the sake of making the paper selfcontained we give a proof.

Proof. Let $p\in\Lambda_r$. It is sufficient to prove the following equation, the coefficient of dz/s:

$$\sum_{j=1}^{L}\sum_{k=1}^{N}g_{\Lambda_k}(z)\left[\frac{d}{dz}\frac{g^{(\Lambda_k)}(z)}{z^{L-j+1}}\right]_0\lambda_p^{L-j}$$
$$=\frac{g^{(\Lambda_r)}(\lambda_p)g^{(p)}_{(\Lambda_r)}(z)}{z-\lambda_p}-\frac{Nf(z)}{(z-\lambda_p)^2}+\frac{(N-1)f'(z)}{z-\lambda_p}. \tag{16}$$

Since both hand sides are rational functions in z, it is sufficient to prove (16) for $|z|$ sufficiently large.

Let t be a complex parameter. More generally we calculate the right hand side of (16) replaced λ_p by t. We have

$$\sum_{j=1}^{L}\left[\frac{d}{dz}\frac{g^{(\Lambda_k)}(z)}{z^{L-j+1}}\right]_0 t^{L-j}=\frac{d}{dz}\left[\sum_{j=0}^{\infty}(tz^{-1})^jz^{-1}g^{(\Lambda_k)}(z)\right]_0$$
$$=\frac{d}{dz}\left[\frac{g^{(\Lambda_k)}(z)}{z-t}\right]_0$$
$$=\frac{d}{dz}\frac{g^{(\Lambda_k)}(z)-g^{(\Lambda_k)}(t)}{z-t}.$$

Thus

$$\sum_{j=1}^{L}\sum_{k=1}^{N} g_{\Lambda_k}(z)\left[\frac{d}{dz}\frac{g^{(\Lambda_k)}(z)}{z^{L-j+1}}\right]_0 t^{L-j} \tag{17}$$

$$=-\frac{Nf(z)}{(z-t)^2}+\frac{(N-1)f'(z)}{z-t}+\sum_{k=1}^{N}\frac{g_{\Lambda_k}(z)g^{(\Lambda_k)}(t)}{(z-t)^2}.$$

In this calculation we use

$$\sum_{k=1}^{N} g_{\Lambda_k}(z)\frac{d}{dz}g^{(\Lambda_k)}(z)=(N-1)f'(z).$$

If we set $t=\lambda_p$, $p\in\Lambda_r$ in (17), then we get the right hand side of (16). □

§4.2. Some Notations

For an ordered partition Λ let us denote by $\Lambda^{(ij)}$ the ordered partition which is obtained from Λ by exchanging i and j. For example if $\Lambda_i=\{(i-1)m+1,\cdots,im\}$, $1\le i\le N$, then $\Lambda^{(12)}=\Lambda$ (assuming $m\ge 2$), $\Lambda_1^{(1m+1)}=\{m+1,2,\cdots,m\}$, $\Lambda_2^{(1m+1)}=\{1,m+2,\cdots,2m\}$, $\Lambda_k^{(1m+1)}=\Lambda_k$ for $k\ge 3$.

In terms of the components f_Λ, if $p\in\Lambda_i$, the KZ equation of level zero is

$$N\frac{\partial f_\Lambda}{\partial\lambda_p}=\left(\left(1-\frac{1}{N}\right)\sum_{j\in\Lambda_i, j\ne p}\frac{1}{\lambda_p-\lambda_j}-\frac{1}{N}\sum_{j\notin\Lambda_i}\frac{1}{\lambda_p-\lambda_j}\right)f_\Lambda+\sum_{j\notin\Lambda_i}\frac{1}{\lambda_p-\lambda_j}f_{\Lambda^{(pj)}}.$$

If we define the function $\bar{f}_\Lambda$ by

$$\bar{f}_\Lambda=\Delta^{-\frac{N-1}{N^2}}f_\Lambda,$$

the KZ equation above is equivalent to

$$\frac{\partial\bar{f}_\Lambda}{\partial\lambda_p}=-\frac{1}{N}\sum_{j\notin\Lambda_i}\frac{\bar{f}_\Lambda}{\lambda_p-\lambda_j}+\frac{1}{N}\sum_{j\notin\Lambda_i}\frac{1}{\lambda_p-\lambda_j}\bar{f}_{\Lambda^{(pj)}}. \tag{18}$$

For a nonnegative integer r and a subset $\{p_1,\cdots,p_r\}\subset\{1,\cdots,Nm\}$, set

$$\begin{pmatrix}\Lambda & \cdots & \Lambda\\ p_1 & & p_r\end{pmatrix}=\det\left(\mu_{p_j}^{\Lambda}(z_i)\right)_{1\le i,j\le r}.$$

Then Theorem 1 is equivalent to the following proposition.

Proposition 6. *Let*

$$\bar{f}_\Lambda=\frac{\Delta(p_1,\cdots,p_L)^{-1}}{\prod_{i<j}(\Lambda_i\Lambda_j)}\begin{pmatrix}\Lambda & \cdots & \Lambda\\ p_1 & & p_L\end{pmatrix}. \tag{19}$$

Then, modulo exact forms, $\bar{f}_\Lambda$ *satisfies the equation* (18) *and* $X\bar{f}_\Lambda=0$ *for any* $X\in sl_N$.

Our aim is to prove this proposition. Let us set

$$\Lambda_r=\{i_1^r, \cdots, i_m^r\} \qquad \text{for } 1\leq r\leq N.$$

For the sake of simple exposition we shall prove the equation (18) for $p=i_m^N$. Other cases are similarly proved.

In the following sections we use the usual equality symbol $=$ for the equality modulo exact forms. We remark that all the modulo exact relations, which we use, follow from the relation in Proposition 5.

§ 4.3. Fundamental Relations

Now let us give all the relations which we need for our purpose. For the sake of simplicity, in the formulae below, we denote λ_i by i. For instance $i-j=\lambda_i-\lambda_j$. We set $\mathcal{K}=\{i_l^r | r<N,\ (r, l)\neq(N-1, m)\}$ and

$$A_r=\frac{\prod_{s=1}^m (i_m^N-i_s^r)}{\prod_{s=1}^{m-1}(i_m^N-i_s^N)}.$$

For $p\neq i_m^N$ and $1\leq j\leq Nm$ we also set

$$\begin{pmatrix} p \\ j \end{pmatrix}=\begin{pmatrix} \Lambda^{(i_m^N p)} \\ j \end{pmatrix}$$

for the sake of simplicity. Now the relations are given as follows.

1. For $p\in\Lambda_r$, $r\neq N$

$$\frac{\partial}{\partial\lambda_{i_m^N}}\begin{pmatrix} \Lambda \\ p \end{pmatrix}=\left(1-\frac{1}{N}\right)\frac{1}{i_m^N-p}\begin{pmatrix} \Lambda \\ p \end{pmatrix}-\frac{1}{N}\frac{1}{i_m^N-p}\prod_{j\notin\Lambda_r, j\neq i_m^N}\frac{p-j}{i_m^N-j}\begin{pmatrix} p \\ i_m^N \end{pmatrix}. \tag{20}$$

2. For $l\neq l'$, (r, l), $(r, l')\neq(N, m)$,

$$\begin{aligned}\begin{pmatrix} i_{l'}^r \\ i_l^r \end{pmatrix}&=\begin{pmatrix} \Lambda \\ i_l^r \end{pmatrix}\\ &+\frac{i_m^N-i_{l'}^r}{i_m^N-i_m^{N-1}}\sum_{k=1}^{m-1}\frac{\prod_{j\notin\Lambda_r, j\neq i_m^N}(i_l^r-j)\prod_{s\neq l,l'}^m (i_k^{N-1}-i_s^r)}{\prod_{j\neq i_m^N, i_m^{N-1}, i_k^{N-1}}(i_k^{N-1}-j)}\left[-\begin{pmatrix} \Lambda \\ i_k^{N-1} \end{pmatrix}+\begin{pmatrix} i_m^{N-1} \\ i_k^{N-1} \end{pmatrix}\right].\end{aligned} \tag{21}$$

3. For $l\neq l'$,

$$\begin{pmatrix} i_{l'}^{N-1} \\ i_l^{N-1} \end{pmatrix}=\frac{(i_m^{N-1}-i_{l'}^{N-1})(i_m^N-i_l^{N-1})}{(i_l^{N-1}-i_{l'}^{N-1})(i_m^N-i_m^{N-1})}\begin{pmatrix} \Lambda \\ i_l^{N-1} \end{pmatrix}+\frac{(i_l^{N-1}-i_m^{N-1})(i_m^N-i_{l'}^{N-1})}{(i_l^{N-1}-i_{l'}^{N-1})(i_m^N-i_m^{N-1})}\begin{pmatrix} i_m^{N-1} \\ i_l^{N-1} \end{pmatrix} \tag{22}$$

$$+\frac{(i_{l'}^{N-1}-i_m^{N-1})\,(i_m^N-i_{l'}^{N-1})}{(i_{l'}^{N-1}-i_l^{N-1})\,(i_m^N-i_m^{N-1})}\prod_{j\notin\Lambda_{N-1},\,j\neq i_m^N}\frac{i_l^{N-1}-j}{i_{l'}^{N-1}-j}\left[-\begin{pmatrix}\Lambda\\ i_{l'}^{N-1}\end{pmatrix}+\begin{pmatrix}i_m^{N-1}\\ i_{l'}^{N-1}\end{pmatrix}\right].$$

4. For $r\neq N$,

$$\begin{pmatrix}i_l^r\\ i_m^N\end{pmatrix}=\sum_{k\in\mathcal{H}}\prod_{j\in\mathcal{H},j\neq k}\frac{i_m^N-j}{k-j}\begin{pmatrix}\Lambda\\ k\end{pmatrix}\tag{23}$$

$$+\sum_{k=1}^{m-1}\prod_{j\in\mathcal{H},\,j\neq i_k^{N-1}}\frac{i_m^N-j}{i_k^{N-1}-j}B(r,l,k)\left[-\begin{pmatrix}\Lambda\\ i_k^{N-1}\end{pmatrix}+\begin{pmatrix}i_m^{N-1}\\ i_k^{N-1}\end{pmatrix}\right],$$

where we set

$$B(r,l,k)=1-\frac{i_m^N-i_l^r}{A_r}\,\frac{\prod_{s\neq l}^m(i_k^{N-1}-i_s^r)}{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}.$$

5. For $(r,l)\neq(N,m)$,

$$\begin{pmatrix}i_l^r\\ i_l^r\end{pmatrix}=\sum_{k\in\mathcal{H},\,k\neq i_l^r}\frac{i_m^N-i_l^r}{i_m^N-k}\prod_{j\in\mathcal{H},\,j\neq k,\,i_l^r}\frac{i_l^r-j}{k-j}\begin{pmatrix}i_l^r\\ k\end{pmatrix}+\prod_{j\in\mathcal{H},\,j\neq i_l^r}\frac{i_l^r-j}{i_m^N-j}\begin{pmatrix}i_l^r\\ i_m^N\end{pmatrix}.\tag{24}$$

The proofs of these relations are given in appendix.

§ 4.4. Equation for $\bar{f}_\Lambda$

Let us take $(p_1,\cdots,p_L)=(i_1^1,\cdots,i_{m-1}^{N-1})$ in the expression of $\bar{f}_\Lambda$. By differentiating the defining equation (19) in $\lambda_{i_m^N}$ we have

$$\frac{\partial\bar{f}_\Lambda}{\partial\lambda_{i_m^N}}=\sum_{j\notin\Lambda_N}\frac{-1}{i_m^N-j}\bar{f}_\Lambda+\frac{\Delta(i_1^1,\cdots,i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)}\frac{\partial}{\partial\lambda_{i_m^N}}\begin{pmatrix}\Lambda\cdots\Lambda\\ i_1^1\cdots i_{m-1}^{N-1}\end{pmatrix}.\tag{25}$$

Substituting (20) into (25) we have

$$\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\Delta(i_1^1,\cdots,i_{m-1}^{N-1})^{-1}}\frac{\partial\bar{f}_\Lambda}{\partial\lambda_{i_m^N}}=\left(-\frac{1}{i_m^N-i_m^{N-1}}-\frac{1}{N}\sum_{j\in\mathcal{H}}\frac{1}{i_m^N-j}\right)\begin{pmatrix}\Lambda\cdots\Lambda\\ i_1^1\cdots i_{m-1}^{N-1}\end{pmatrix}\tag{26}$$

$$-\frac{1}{N}\sum_{i_l^r\in\mathcal{H}}\frac{1}{i_m^N-i_l^r}\prod_{j\notin\Lambda_r,\,j\neq i_m^N}\frac{i_l^r-j}{i_m^N-j}\begin{pmatrix}\Lambda\cdots i_l^r\cdots\Lambda\\ i_1^1\cdots i_m^N\cdots i_{m-1}^{N-1}\end{pmatrix}$$

Using (23) in the second term of (26) we obtain

$$(-N)\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\Delta(i_1^1,\cdots,i_{m-1}^{N-1})^{-1}}\frac{\partial\bar{f}_\Lambda}{\partial\lambda_{i_m^N}}\tag{27}$$

$$
=\Bigg[\frac{-m+3}{i_m^N-i_m^{N-1}}+\sum_{l=1}^{m-1}\frac{2}{i_m^N-i_l^{N-1}}+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{1}{i_m^N-i_l^r}
$$

$$
+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}\begin{pmatrix}\Lambda & \cdots & \Lambda\\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix}
$$

$$
+\sum_{i_l^r\in\mathcal{K}}\frac{A_r}{(i_m^N-i_l^r)\,(i_m^N-i_m^{N-1})}\sum_{k=1}^{m-1}\frac{1}{i_m^N-i_k^{N-1}}\frac{\prod_{j\notin\Lambda_r,\, j\neq i_m^N}(i_l^r-j)}{\prod_{j\in\mathcal{K},\, j\neq i_k^{N-1}}(i_k^{N-1}-j)}B(r,l,k)
$$

$$
\times\left(\begin{matrix}\Lambda\\ i_1^1\end{matrix}\cdots\left|\begin{matrix}\Lambda\\ i_1^r\end{matrix}\cdots\begin{matrix}i_m^{N-1}\\ i_k^{N-1}\end{matrix}\cdots\begin{matrix}\Lambda\\ i_m^r\end{matrix}\right|\cdots\begin{matrix}\Lambda\\ i_{m-1}^{N-1}\end{matrix}\right)
$$

where in the second term $\begin{pmatrix} i_m^{N-1}\\ i_k^{N-1}\end{pmatrix}$ is on the l-th position counted from $\begin{pmatrix}\Lambda\\ i_1^r\end{pmatrix}$. In the derivation of (27) we have used

$$
\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{i_l^r-i_m^{N-1}}{i_m^N-i_m^{N-1}}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}
$$

$$
=-\frac{N-2}{i_m^N-i_m^{N-1}}+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}
$$

which follows from

$$
\frac{i_l^r-i_m^{N-1}}{(i_m^N-i_l^r)\,(i_m^N-i_m^{N-1})}=\frac{1}{i_m^N-i_l^r}-\frac{1}{i_m^N-i_m^{N-1}}
$$

and

$$
\sum_{l=1}^{m}\frac{\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{(i_m^N-i_l^r)\prod_{s\neq l}^{m}(i_l^r-i_s^r)}=\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s=1}^{m}(i_m^N-i_s^r)}=A_r^{-1}. \tag{28}
$$

The equation (28) follows from the residue theorem for the function

$$
\frac{\prod_{s=1}^{m-1}(z-i_s^N)}{(z-i_m^N)\prod_{s=1}^{m}(z-i_s^r)}.
$$

This is the typical argument to prove an identity in the proof below.

§ 4.5. KZ Equation

We shall rewrite the KZ equation in a similar manner to (27). We take $(p_1, \cdots, p_L)=(i_1^1, \cdots, i_{m-1}^{N-1})$ in the expression (19) for $\bar{f}_{\Lambda^{(j,i_m^N)}}$ for any $j\notin\Lambda_N$. Then the KZ equation substituted by (19) is

$$\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\Delta(i_1^1,\cdots,i_{m-1}^{N-1})^{-1}}\frac{\partial\bar{f}_\Lambda}{\partial\lambda_{i_m^N}}=-\frac{1}{N}\sum_{j\notin\Lambda_N}\frac{1}{i_m^N-j}\begin{pmatrix}\Lambda & \cdots & \Lambda\\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix}$$
$$+\frac{1}{N}\sum_{r=1}^{N-1}\sum_{l=1}^{m}\frac{1}{i_m^N-i_l^r}\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\prod_{t<u}(\Lambda_t^{(i_m^N i_l^r)}\Lambda_u^{(i_m^N i_l^r)})}\begin{pmatrix}i_l^r & \cdots & i_l^r\\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix} \tag{29}$$

Note the relations

$$\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\prod_{t<u}(\Lambda_t^{(i_m^N i_l^r)}\Lambda_u^{(i_m^N i_l^r)})}=-\frac{A_r}{i_m^N-i_l^r}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}. \tag{30}$$

Thus we have

$$\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\Delta(i_1^1,\cdots,i_{m-1}^{N-1})^{-1}}\frac{\partial\bar{f}_\Lambda}{\partial\lambda_{i_m^N}}=-\frac{1}{N}\sum_{j\notin\Lambda_N}\frac{1}{i_m^N-j}\begin{pmatrix}\Lambda & \cdots & \Lambda\\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix}$$
$$-\frac{1}{N}\sum_{r=1}^{N-1}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}\begin{pmatrix}i_l^r & \cdots & i_l^r\\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix}. \tag{31}$$

By the equation (24), if $r\neq N$ and $i_l^r\neq i_m^{N-1}$, we have

$$\begin{pmatrix}i_l^r & \cdots & i_l^r & \cdots & i_l^r\\ i_1^1 & & i_l^r & & i_{m-1}^{N-1}\end{pmatrix}=\prod_{j\in\mathcal{K},\, j\neq i_l^r}\frac{i_l^r-j}{i_m^N-j}\begin{pmatrix}i_l^r & \cdots & i_l^r & \cdots & i_l^r\\ i_1^1 & & i_m^N & & i_{m-1}^{N-1}\end{pmatrix}. \tag{32}$$

Using the relation (23) we have, for $r\leq N-2$,

$$\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & i_l^r & \cdots & i_l^r & \cdots & \Lambda\\ i_1^1 & & i_1^r & & i_m^N & & i_m^r & & i_{m-1}^{N-1}\end{array}\right) \tag{33}$$
$$=\sum_{k=1}^{m}\prod_{j\in\mathcal{K}\ j\neq i_k^r}\frac{i_m^N-j}{i_k^r-j}\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & \Lambda & \cdots & i_l^r & \cdots & \Lambda\\ i_1^1 & & i_1^r & & i_k^r & & i_m^r & & i_{m-1}^{N-1}\end{array}\right)$$
$$+\sum_{k=1}^{m-1}\prod_{j\in\mathcal{K}\ j\neq i_k^{N-1}}\frac{i_m^N-j}{i_k^{N-1}-j}B(r,l,k)\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & i_m^{N-1} & \cdots & i_l^r & \cdots & \Lambda\\ i_1^1 & & i_1^r & & i_k^{N-1} & & i_m^r & & i_{m-1}^{N-1}\end{array}\right),$$

and, for $l\neq m$,

$$\left(\begin{array}{cc|ccccc}\Lambda & \cdots & i_l^{N-1} & \cdots & i_l^{N-1} & \cdots & i_l^{N-1}\\ i_1^1 & & i_l^{N-1} & & i_m^N & & i_{m-1}^{N-1}\end{array}\right) \tag{34}$$
$$=\prod_{j\in\mathcal{K},\, j\neq i_l^{N-1}}\frac{i_m^N-j}{i_l^{N-1}-j}\,\frac{i_m^N-i_l^{N-1}}{A_{N-1}}\,\frac{\prod_{s\neq l}^{m}(i_l^{N-1}-i_s^{N-1})}{\prod_{s=1}^{m-1}(i_l^{N-1}-i_s^N)}\left(\begin{array}{cc|ccccc}\Lambda & \cdots & i_l^{N-1} & \cdots & \Lambda & \cdots & i_l^{N-1}\\ i_1^1 & & i_1^{N-1} & & i_l^{N-1} & & i_{m-1}^{N-1}\end{array}\right)$$

$$+\sum_{k=1}^{m-1}\prod_{j\in\mathcal{K}\ j\neq i_k^{N-1}}\frac{i_m^N-j}{i_k^{N-1}-j}B(N-1,l,k)\left(\begin{array}{cc|ccccc} \Lambda & \cdots & i_l^{N-1} & \cdots & i_m^{N-1} & \cdots & i_l^{N-1} \\ i_1^1 & & i_1^{N-1} & & i_k^{N-1} & & i_{m-1}^{N-1}\end{array}\right).$$

If we substitute (32), (33) and (34) into (31) we have

$$(35)\quad (-N)\frac{\prod_{t<u}(\Lambda_t\Lambda_u)}{\Delta(i_1^1,\cdots,i_{m-1}^{N-1})^{-1}}\frac{\partial\bar{f}_\Lambda}{\partial\lambda_{i_m^N}}=\sum_{j\notin\Lambda_N}\frac{1}{i_m^N-j}\begin{pmatrix}\Lambda & \cdots & \Lambda \\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix}$$

$$+\frac{A_{N-1}}{(i_m^N-i_m^{N-1})^2}\prod_{s=1}^{m-1}\frac{i_m^{N-1}-i_s^N}{i_m^{N-1}-i_s^{N-1}}\left(\begin{array}{cc|ccc}\Lambda & \cdots & i_m^{N-1} & \cdots & i_m^{N-1} \\ i_1^1 & & i_1^{N-1} & & i_{m-1}^{N-1}\end{array}\right)$$

$$+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{1}{i_m^N-i_l^r}\sum_{k=1}^{m}\frac{A_r}{i_m^N-i_k^r}\cdot\frac{\prod_{j\notin\Lambda_r,\,j\neq i_m^N,\,i_m^{N-1}}(i_l^r-j)}{\prod_{j\in\mathcal{K},\,j\neq i_k^r}(i_k^r-j)}\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & \Lambda & \cdots & i_l^r & \cdots & \Lambda \\ i_1^1 & & i_1^r & & i_k^r & & i_m^r & & i_{m-1}^{N-1}\end{array}\right)$$

$$+\sum_{l=1}^{m-1}\frac{1}{i_m^N-i_l^{N-1}}\left(\begin{array}{cc|ccccc}\Lambda & \cdots & i_l^{N-1} & \cdots & \Lambda & \cdots & i_l^{N-1} \\ i_1^1 & & i_1^{N-1} & & i_l^{N-1} & & i_{m-1}^{N-1}\end{array}\right)$$

$$+\sum_{i_l^r\in\mathcal{K}}\frac{1}{(i_m^N-i_l^r)(i_l^r-i_m^{N-1})}\sum_{k=1}^{m-1}\frac{A_r}{i_m^N-i_k^{N-1}}\frac{\prod_{j\notin\Lambda_r,\,j\neq i_m^N}(i_l^r-j)}{\prod_{j\in\mathcal{K},\,j\neq i_k^{N-1}}(i_k^{N-1}-j)}B(r,l,k)$$

$$\times\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & i_m^{N-1} & \cdots & i_l^r & \cdots & \Lambda \\ i_1^1 & & i_1^r & & i_k^{N-1} & & i_m^r & & i_{m-1}^{N-1}\end{array}\right)$$

where as in the previous case $\begin{pmatrix}\Lambda \\ i_k^r\end{pmatrix}$ etc. are all on the l-th place counted from $\begin{pmatrix}i_l^r \\ i_1^r\end{pmatrix}$ etc. For the economy of space we set

$$\Omega(r,l,\Lambda,k)=\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & \Lambda & \cdots & i_l^r & \cdots & \Lambda \\ i_1^1 & & i_1^r & & i_k^r & & i_m^r & & i_{m-1}^{N-1}\end{array}\right),$$

$$\Omega(r,l,N-1,k)=\left(\begin{array}{cc|ccccc|cc}\Lambda & \cdots & i_l^r & \cdots & i_m^{N-1} & \cdots & i_l^r & \cdots & \Lambda \\ i_1^1 & & i_1^r & & i_k^{N-1} & & i_m^r & & i_{m-1}^{N-1}\end{array}\right)$$

where the positions of $\begin{pmatrix}\Lambda \\ i_k^r\end{pmatrix}$ and $\begin{pmatrix}i_m^{N-1} \\ i_k^{N-1}\end{pmatrix}$ are as above. If $r=N-1$ then i_m^r should be replaced by i_{m-1}^r.

The forms

$$\begin{pmatrix}\Lambda \\ i_1^1\end{pmatrix},\ \cdots,\ \begin{pmatrix}\Lambda \\ i_{m-1}^{N-1}\end{pmatrix},\ \begin{pmatrix}i_m^{N-1} \\ i_1^{N-1}\end{pmatrix},\ \cdots,\ \begin{pmatrix}i_m^{N-1} \\ i_{m-1}^{N-1}\end{pmatrix}$$

are linearly independent in the cohomology group $H^1(C,\mathbf{C})$. Since we do not use this fact in this paper, we do not give a proof of it. But the fact helps to understand the strategy of the proof below. The right hand side of (27) is written using these forms only. We shall rewrite the right hand side of (35) in terms of these forms.

§ 4.6. Reduction of Expressions of Fundamental Determinants

For $1\leq r\leq N-1$, $I=\{s_1<\cdots<s_p\}$, $J=\{t_1<\cdots<t_q\}$, $p+q=m\,(r<N-1)$, $p+q=m-1\,(r=N-1)$, we set

$$\Omega_{IJ}^{r}=\begin{pmatrix} \Lambda & \cdots & \Bigl| \prod_{s\in I}\begin{pmatrix} \Lambda \\ i_s^r \end{pmatrix} \prod_{t\in J}\begin{pmatrix} i_m^{N-1} \\ i_t^{N-1} \end{pmatrix} \Bigr| & \cdots & \Lambda \\ i_1^1 & & & & i_{m-1}^{N-1} \end{pmatrix}=\begin{pmatrix} \Lambda & \cdots & \Lambda & \cdots & \Lambda & i_m^{N-1} & \cdots & i_m^{N-1} & \cdots & \Lambda \\ i_1^1 & \cdots & i_{s_1}^r & \cdots & i_{s_p}^r & i_{t_1}^{N-1} & \cdots & i_{t_q}^{N-1} & \cdots & i_{m-1}^{N-1} \end{pmatrix},$$

where $\begin{pmatrix} \Lambda \\ i_{s_1}^r \end{pmatrix}$ is on the $(r-1)m+1$-th position counted from $\begin{pmatrix} \Lambda \\ i_1^1 \end{pmatrix}$.

∘ The coefficient of Ω_{IJ}^{r} in $\Omega(r, l, \Lambda, k)$.

We assume $r<N-1$. Let us denote this coefficient by $\det_{IJ}^{1}$. We set

$$F_{tu}^{r}=\frac{1}{i_u^{N-1}-i_t^r},\quad G_t^r=\frac{1}{\prod_{j\notin\Lambda,\ j\neq i_m^N}(i_t^r-j)}$$

and

$$A_{tu}^{rl}=\frac{i_m^N-i_l^r}{i_m^N-i_m^{N-1}}\,\frac{\prod_{s\neq l}^{m}(i_u^{N-1}-i_s^r)}{\prod_{j\neq i_m^N,\, i_m^{N-1},\, i_u^{N-1}}(i_u^{N-1}-j)}\,\frac{F_{tu}^r}{G_t^r}.$$

Then (21) can be written as

$$\begin{pmatrix} i_l^r \\ i_{l'}^r \end{pmatrix}=\begin{pmatrix} \Lambda \\ i_{l'}^r \end{pmatrix}+\sum_{k=1}^{m-1}A_{lk}^{rl}\left[-\begin{pmatrix} \Lambda \\ i_k^{N-1} \end{pmatrix}+\begin{pmatrix} i_m^{N-1} \\ i_k^{N-1} \end{pmatrix}\right]. \tag{36}$$

Using this equation we have

$$\det_{IJ}^{1}=\left|\begin{array}{ccccccccc|ccc}
1 & & & & & & & & 0 & A_{11}^{rl} & \cdots & A_{1m-1}^{rl} \\
 & \cdots & & & & & & & \circ & \circ & & \circ \\
 & & 1 & & & & & & 0 & \circ & & \circ \\
 & & & 0 & \circ & \circ & 1 & & 0 & 0 & \cdots & 0 \\
 & & & & 1 & & \vdots & & 0 & \circ & & \circ \\
 & & & & & \cdots & \vdots & & \circ & \circ & & \circ \\
 & & & & & & 1 & & 0 & \circ & & \circ \\
 & & & & & & & \cdots & \circ & \circ & & \circ \\
 & & & & & & & & 1 & A_{m1}^{rl} & \cdots & A_{mm-1}^{rl}
\end{array}\right|.$$

$$=\left(\frac{i_m^N-i_l^r}{i_m^N-i_m^{N-1}}\right)^{|J|}\prod_{u\in J}\left(\frac{\prod_{s\neq l}^{m}(i_u^{N-1}-i_s^r)}{\prod_{j\neq i_m^N,\, i_m^{N-1},\, i_u^{N-1}}(i_u^{N-1}-j)}\right)\frac{1}{\prod_{t=1}^{m}G_t^r}$$

$$
\times\begin{vmatrix}
\begin{array}{ccccccccc|ccc}
G_1^r & & & & & & & & 0 & F_{11}^r & \cdots & F_{1m-1}^r \\
 & \cdots & & & & & & & \cdot & \circ & & \cdot \\
 & & G_{l-1}^r & & & & & & 0 & \cdot & & \circ \\
 & & & 0 & \cdot & \cdot & G_l^r & & 0 & 0 & \cdots & 0 \\
 & & & & G_{l+1}^r & & \vdots & & 0 & \circ & & \cdot \\
 & & & & & \cdots & \vdots & & \cdot & \cdot & & \circ \\
 & & & & & & G_k^r & & 0 & \circ & & \cdot \\
 & & & & & & & \cdots & \circ & \cdot & & \circ \\
 & & & & & & & & G_m^r & F_{m1}^r & \cdots & F_{mm-1}^r
\end{array}
\end{vmatrix}.
$$

The meaning of the determinant symbol of the matrix above is the following. The matrix consists of two matrices, say, the left matrix and the right matrix. We take I-th columns from the left matrix and J-th columns from the right matrix. Then form the determinant of the resulting matrix of degree $|I|+|J|$. We shall use similar notations from now on.

Notice that, by definition, $\det^1_{IJ}=0$ unless $k\in I$.

∘ The coefficient of Ω^r_{IJ} in $\Omega(r, l, N-1, k)$.

We assume $r<N-1$. Let us denote this coefficient by $\det^2_{IJ}$. Then, again by (36),

$$
\det{}^2_{IJ}=\left(\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\right)^{|J|-1}\prod_{u\in J\setminus\{k\}}\left(\frac{\prod^m_{s\neq l}(i^{N-1}_u-i^r_s)}{\prod_{j\neq i^N_m, i^{N-1}_m, i^{N-1}_u}(i^{N-1}_u-j)}\right)\frac{1}{\prod^m_{t\neq l}G^r_t}
$$

$$
\times\begin{vmatrix}
\begin{array}{cccccccc|ccccc}
G_1^r & & & \circ & & & & & F_{11}^{rl} & \cdots & \bar{A}_{1k}^{rl} & \cdots & F_{1m-1}^{rl} \\
 & \cdots & & \circ & & & & & \circ & & \circ & & \circ \\
 & & G_{l-1}^r & \circ & & & & & \circ & & \circ & & \circ \\
\circ & \cdot & \cdot & 0 & \cdot & \circ & \cdot & & 0 & & 1 & & 0 \\
 & & & & G_{l+1}^r & & & & \circ & & \circ & & \cdot \\
 & & & & & \cdots & & & \cdot & & \cdot & & \circ \\
 & & & & & & G_m^r & & F_{m1}^{rl} & \cdots & \bar{A}_{mk}^{rl} & \cdots & F_{mm-1}^{rl}
\end{array}
\end{vmatrix},
$$

where

$$
\bar{A}^{rl}_{tk}=A^{rl}_{tk}G^r_t.
$$

∘ The coefficient of Ω^{N-1}_{IJ} in $\Omega(N-1, l, \Lambda, l)$.

This coefficient is denoted by $\det^3_{IJ}$. Let us set, for $s\neq l$,

$$
C^l_{ss}=\frac{(i^{N-1}_l-i^{N-1}_m)(i^N_m-i^{N-1}_s)}{(i^N_m-i^{N-1}_m)(i^{N-1}_l-i^{N-1}_s)},\ C^l_{sl}=-\frac{(i^N_m-i^{N-1}_l)(i^{N-1}_l-i^{N-1}_m)}{(i^N_m-i^{N-1}_m)(i^{N-1}_l-i^{N-1}_s)}\prod_{j\notin\Lambda_{N-1}, j\neq i^N_m}\frac{i^{N-1}_s-j}{i^{N-1}_l-j},
$$

$$
D^l_{ss}=\frac{(i^N_m-i^{N-1}_l)(i^{N-1}_s-i^{N-1}_m)}{(i^N_m-i^{N-1}_m)(i^{N-1}_s-i^{N-1}_l)},\ D^l_{sl}=-C^l_{sl}.
$$

Then, (22) is written as

$$\begin{pmatrix} i_l^{N-1} \\ i_s^{N-1} \end{pmatrix} = C_{ss}^l \begin{pmatrix} A \\ i_s^{N-1} \end{pmatrix} + C_{sl}^l \begin{pmatrix} A \\ i_l^{N-1} \end{pmatrix} + D_{ss}^l \begin{pmatrix} i_m^{N-1} \\ i_s^{N-1} \end{pmatrix} + D_{sl}^l \begin{pmatrix} i_m^{N-1} \\ i_l^{N-1} \end{pmatrix}. \tag{37}$$

Using (37) we have

$\det_{IJ}^3 =$

$$\left|\begin{array}{ccccccc|ccccc}
C_{11}^l & & & C_{1l}^l & & & 0 & D_{11}^l & \cdots & D_{1l}^l & \cdots & 0 \\
 & \cdots & & \circ & & & \circ & \circ & & \circ & & \circ \\
 & & C_{l-1l-1}^l & \circ & & & \circ & \circ & & \circ & & \circ \\
0 & \circ & \circ & 1 & \circ & \circ & 0 & 0 & & 0 & & 0 \\
\circ & & & \circ & C_{l+1l+1}^l & & \circ & \circ & & \circ & & \circ \\
\circ & & & \circ & & \cdots & \circ & \circ & & \circ & & \circ \\
0 & & & C_{m-1l}^l & & & C_{m-1m-1}^l & 0 & \cdots & D_{m-1l}^l & \cdots & D_{m-1m-1}^l
\end{array}\right|$$

∘ The coefficient of Ω_{IJ}^{N-1} in $\Omega(N-1, l, N-1, k)$.
This coefficient is denoted by $\det_{IJ}^4$. Then

$\det_{IJ}^4 =$

$$\left|\begin{array}{ccccccc|ccccccc}
C_{11}^l & & & C_{1l}^l & & & 0 & D_{11}^l & \circ & 0 & \circ & D_{1l}^l & \cdots & 0 \\
 & \cdots & & \circ & & & \circ & \circ & \cdots & \circ & & \circ & & \circ \\
 & & C_{l-1l-1}^l & \circ & & & \circ & \circ & & \circ & \cdots & \circ & & \circ \\
0 & \circ & \circ & 0 & \circ & \circ & 0 & 0 & \cdots & 1 & \cdots & 0 & & 0 \\
\circ & & & \circ & C_{l+1l+1}^l & & \circ & \circ & & \circ & & \circ & \cdots & \circ \\
\circ & & & \circ & & \cdots & \circ & \circ & & \circ & & \circ & \cdots & \circ \\
0 & & & C_{m-1l}^l & & & C_{m-1m-1}^l & 0 & \cdots & 0 & \cdots & D_{m-1l}^l & \cdots & D_{m-1m-1}^l
\end{array}\right|.$$

In the right matrix 1 is in the (l, k) component.

§ 4.7. Comparison of Two Equations

Now let us calculate the reduced expression of the right hand side of (35) and compare it with (27). We shall calculate the coefficient of Ω_{IJ}^r by dividing the case into nine as

(I). $r=N-1$, $I=\phi$, $J=\{1, 2, \cdots, m-1\}$,
(II). $r=N-1$, $I=\{1, 2, \cdots, m-1\}$, $J=\phi$, or $r<N-1$, $I=\{1, 2, \cdots, m\}$, $J=\phi$,
(III). $r=N-1$, $I=\{1, 2, \cdots, m-1\}\backslash\{t\}$, $J=\{t\}$,
(IV). $r=N-1$, $I=\{1, 2, \cdots, m-1\}\backslash\{u\}$, $J=\{t\}$, $u\neq t$,
(V). $r<N-1$, $I=\{1, 2, \cdots, m-1\}\backslash\{u\}$, $J=\{t\}$,
(VI). $r=N-1$, $I\cap J=\phi$, $|I|\geq 1$, $|J|\geq 2$,

(VII). $r=N-1$, $|I\cap J|=1$, $|J|\geq 2$,
(VIII). $r=N-1$, $|I\cap J|\geq 2$,
(IX). $r<N-1$, $|J|\geq 2$.

(I). The coefficient of Ω_{IJ}^{N-1} with $I=\phi$, $J=\{1, 2, \cdots, m-1\}$.
Let us calculate the contribution from the term which contains $\Omega(N-1, l, N-1, k)$ $(1\leq l\leq m-1)$. We have, for $k\neq l$,

$$\det{}_{IJ}^{4}=-\frac{D_{kl}^{l}}{D_{kk}^{l}}\prod_{s\neq l}^{m-1}D_{ss}^{l}$$
$$=\frac{i_l^{N-1}-i_m^{N-1}}{i_k^{N-1}-i_m^{N-1}}\left(\frac{i_m^N-i_l^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\prod_{s\neq l}^{m-1}\frac{i_m^{N-1}-i_s^{N-1}}{i_l^{N-1}-i_s^{N-1}}\prod_{j\notin\Lambda_{N-1}, j\neq i_m^N}\frac{i_k^{N-1}-j}{i_l^{N-1}-j}.$$

For $k=l$, $\det_{IJ}^4=\prod_{s\neq l}^{m-1}D_{ss}^l$. If we set $k=l$ in the right hand side of

$$\frac{D_{kl}^l}{D_{kk}^l}=-\frac{i_l^{N-1}-i_m^{N-1}}{i_k^{N-1}-i_m^{N-1}}\prod_{j\notin\Lambda_{N-1}, j\neq i_m^N}\frac{i_k^{N-1}-j}{i_l^{N-1}-j},$$

we have -1. Thus the formula for $\det_{IJ}^4$ given above is valid for all $1\leq k$, $l\leq m-1$. The contribution to (35) is

$$-A_{N-1}\frac{\prod_{s=1}^{m-1}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l=1}^{m-1}\frac{(i_m^N-i_l^{N-1})^{m-3}}{\prod_{s\neq l}^{m}(i_l^{N-1}-i_s^{N-1})}$$
$$\times\sum_{k=1}^{m-1}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\neq k}^{m}(i_k^{N-1}-i_s^{N-1})}B(N-1, l, k). \tag{38}$$

By the residue theorem we have

$$\sum_{k=1}^{m-1}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\neq k}^{m}(i_k^{N-1}-i_s^{N-1})}=\frac{1}{A_{N-1}}-\frac{\prod_{s=1}^{m-1}(i_m^{N-1}-i_s^N)}{(i_m^N-i_m^{N-1})\prod_{s=1}^{m-1}(i_m^{N-1}-i_s^{N-1})},$$

$$\frac{i_m^N-i_l^{N-1}}{A_{N-1}}\sum_{k=1}^{m-1}\frac{\prod_{s\neq l}(i_k^{N-1}-i_s^{N-1})}{(i_m^N-i_k^{N-1})\prod_{s\neq k}^{m}(i_k^{N-1}-i_s^{N-1})}=\frac{1}{A_{N-1}}.$$

Thus, substituting the definition of $B(N-1, l, k)$ to (38),

$$(38)=A_{N-1}\frac{\prod_{s=1}^{m-1}(i_m^{N-1}-i_s^N)}{(i_m^N-i_m^{N-1})^{m-1}}\sum_{l=1}^{m-1}\frac{(i_m^N-i_l^{N-1})^{m-3}}{\prod_{s\neq l}^{m}(i_l^{N-1}-i_s^{N-1})}$$
$$=-\frac{A_{N-1}}{(i_m^N-i_m^{N-1})^2}\prod_{s=1}^{m-1}\frac{i_m^{N-1}-i_s^N}{i_m^{N-1}-i_s^{N-1}}. \tag{39}$$

Here we again use the residue theorem to evaluate the summation in l. Hence the coefficient of Ω in the right hand side of (35) is zero. This is the case for (27).

(II). The coefficient of $\Omega_{IJ}^{r}=\begin{pmatrix} A & \cdots & A \\ i_1^1 & & i_{m-1}^{N-1} \end{pmatrix}$ with $r=N-1$, $I=\{1, 2, \cdots, m-1\}$ and $J=\phi$ or $r<N-1$, $I=\{1, 2, \cdots, m\}$ and $J=\phi$.

(II - I). The contribution to (35) from the term which contains $\Omega(r, l, A, k)$ $(1\leq l\leq m)$.

We have

$$\det{}_{IJ}^{1}=\begin{cases} 0 & k\neq l \\ 1 & k=l \end{cases}$$

Hence the contribution to the rhs of (35) from the term containing these determinants is

$$\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{j\notin A_r, j\neq i_m^N, i_m^{N-1}}(i_l^r-j)}{\prod_{j\in\mathcal{K}, j\neq i_l^r}(i_l^r-j)} \tag{40}$$

$$=\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}.$$

(II - II). The contribution to (35) from the term which contains $\Omega(N-1, l, A, l)$ $(1\leq l\leq m-1)$.

We have

$$\det{}_{IJ}^{3}=\prod_{s\neq l}^{m-1}C_{ss}^{l}=\left(\frac{i_l^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\prod_{s\neq l}^{m-1}\frac{i_m^N-i_s^{N-1}}{i_l^{N-1}-i_s^{N-1}}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$\frac{\prod_{s=1}^{m-1}(i_m^N-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l=1}^{m-1}\frac{(i_l^{N-1}-i_m^{N-1})^{m-2}}{(i_m^N-i_l^{N-1})^2\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})} \tag{41}$$

$$=-\frac{\prod_{s=1}^{m-1}(i_m^N-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\operatorname*{Res}_{z=i_m^N}\frac{(z-i_m^{N-1})^{m-2}}{(z-i_m^N)^2\prod_{s=1}^{m-1}(z-i_s^{N-1})}$$

$$=-\frac{m-2}{i_m^N-i_m^{N-1}}+\sum_{s=1}^{m-1}\frac{1}{i_m^N-i_s^{N-1}}.$$

From (40) and (41) the coefficient of Ω in the rhs of (35) is

$$\sum_{j\notin A_N}\frac{1}{i_m^N-j}-\frac{m-2}{i_m^N-i_m^{N-1}}+\sum_{s=1}^{m-1}\frac{1}{i_m^N-i_s^{N-1}}+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}$$

$$=\frac{-m+3}{i_m^N-i_m^{N-1}}+\sum_{s=1}^{m-1}\frac{2}{i_m^N-i_s^{N-1}}+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{1}{i_m^N-i_l^r}+\sum_{r=1}^{N-2}\sum_{l=1}^{m}\frac{A_r}{(i_m^N-i_l^r)^2}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)}.$$

This coincides with the coefficient of Ω in the rhs of (27)

(III). The coefficient of Ω_{IJ}^{N-1} for which $I=\{1, 2, \cdots, m-1\}\backslash\{t\}$ and $J=\{t\}$.

(III-I). The contribution to (35) from the term which contains $\Omega(N-1, l, \Lambda, l)$.

It is obvious that $\det^3_{IJ}=0$ for $t=l$. We have, for $t\neq l$,

$$\det{}^3_{IJ}=(-1)^{m+t+1}D^l_{tt}\prod_{s\neq t,l}^{m-1}C^l_{ss}$$

$$=(-1)^{m+t+1}\frac{(i_m^{N-1}-i_t^{N-1})(i_l^{N-1}-i_m^{N-1})^{m-3}}{(i_m^N-i_m^{N-1})^{m-2}}\frac{\prod_{s\neq t}^{m-1}(i_m^N-i_s^{N-1})}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$(42)\quad (-1)^{m+t+1}\frac{(i_m^{N-1}-i_t^{N-1})\prod_{s\neq t}^{m-1}(i_m^N-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l\neq t}^{m-1}\frac{(i_l^{N-1}-i_m^{N-1})^{m-3}}{(i_m^N-i_l^{N-1})\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

(III-II). The contribution from the term which contains $\Omega(N-1, l, N-1, l)$ $(1\leq l\leq m-1)$.

If $l\neq t$ then $\det^4_{IJ}=0$. We have, for $l=t$,

$$\det{}^4_{IJ}=(-1)^{m+t+1}\prod_{s\neq t}^{m-1}C^t_{ss}=(-1)^{m+t+1}\left(\frac{i_t^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\prod_{s\neq t}^{m-1}\frac{i_m^N-i_s^{N-1}}{i_t^{N-1}-i_s^{N-1}}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$(43)\quad \frac{(-1)^{m+t+1}A_{N-1}(i_t^{N-1}-i_m^{N-1})^{m-3}}{(i_t^{N-1}-i_m^N)^2(i_m^N-i_m^{N-1})^{m-2}}\frac{\prod_{s=1}^{m-1}(i_t^{N-1}-i_s^N)}{\prod_{s\neq t}^{m-1}(i_t^{N-1}-i_s^{N-1})}\prod_{s\neq t}^{m-1}\frac{i_m^N-i_s^{N-1}}{i_t^{N-1}-i_s^{N-1}}$$

$$+\frac{(-1)^{m+t+1}}{i_t^{N-1}-i_m^N}\left(\frac{i_t^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\prod_{s\neq t}^{m-1}\frac{i_m^N-i_s^{N-1}}{i_t^{N-1}-i_s^{N-1}}.$$

Note that the second term of (43) is the $l=t$ case of the summand of (42).

(III - III). The contribution from the term which contains $\Omega(N-1, l, N-1, k)$ with $1\leq k, l\leq m-1$ and $k\neq l$.

If $k\neq t$ then $\det_{IJ}^{4}=0$. We have, for $k=t$,

$$\det{}_{IJ}^{4}=(-1)^{m+t}C_{tl}^{l}\prod_{s\neq t,l}^{m-1}C_{ss}^{l}$$

$$=(-1)^{m+t+1}\left(\frac{i_{l}^{N-1}-i_{m}^{N-1}}{i_{m}^{N}-i_{m}^{N-1}}\right)^{m-2}\frac{\prod_{s\neq t}^{m-1}(i_{m}^{N}-i_{s}^{N-1})}{\prod_{s\neq l}^{m-1}(i_{l}^{N-1}-i_{s}^{N-1})}\prod_{j\notin A_{N-1},j\neq i_{m}^{N}}\frac{i_{t}^{N-1}-j}{i_{l}^{N-1}-j}.$$

The contribution to the rhs of (35) is

$$\frac{(-1)^{m+t+1}A_{N-1}\prod_{s\neq t}^{m-1}(i_{m}^{N}-i_{s}^{N-1})\prod_{s=1}^{m-1}(i_{t}^{N-1}-i_{s}^{N})}{(i_{m}^{N}-i_{t}^{N-1})(i_{m}^{N}-i_{m}^{N-1})^{m-2}\prod_{s\neq t}^{m-1}(i_{t}^{N-1}-i_{s}^{N-1})} \tag{44}$$

$$\times\sum_{l\neq t}^{m-1}\frac{(i_{l}^{N-1}-i_{m}^{N-1})^{m-3}}{(i_{m}^{N}-i_{l}^{N-1})\prod_{s\neq l}^{m-1}(i_{l}^{N-1}-i_{s}^{N-1})}.$$

In deriving (44) we use

$$\frac{i_{m}^{N}-i_{l}^{N-1}}{A_{N-1}}\frac{\prod_{s\neq l}^{m-1}(i_{t}^{N-1}-i_{s}^{N-1})}{\prod_{s=1}^{m-1}(i_{t}^{N-1}-i_{s}^{N})}=0$$

which is a consequence of $t\neq l$. Note that the first term in (43) is the $l=t$ case of the summand of (44).

We add (42), (43), (44) and obtain

$$\frac{(-1)^{m+t+1}A_{N-1}\prod_{s=1}^{m-1}(i_{t}^{N-1}-i_{s}^{N})}{(i_{m}^{N}-i_{t}^{N-1})(i_{m}^{N}-i_{m}^{N-1})^{m-2}}\prod_{s\neq t}^{m-1}\frac{(i_{m}^{N}-i_{s}^{N-1})}{(i_{t}^{N-1}-i_{s}^{N-1})}B(N-1, t, t) \tag{45}$$

$$\times\sum_{l=1}^{m-1}\frac{(i_{l}^{N-1}-i_{m}^{N-1})^{m-3}}{(i_{m}^{N}-i_{l}^{N-1})\prod_{s\neq l}^{m-1}(i_{l}^{N-1}-i_{s}^{N-1})}$$

$$=\frac{(-1)^{m+t+1}A_{N-1}}{(i_{m}^{N}-i_{t}^{N-1})^{2}(i_{m}^{N}-i_{m}^{N-1})}\frac{\prod_{s=1}^{m-1}(i_{t}^{N-1}-i_{s}^{N})}{\prod_{s\neq t}^{m-1}(i_{t}^{N-1}-i_{s}^{N-1})}B(N-1, t, t).$$

In deriving (45) we use the identity

$$\sum_{l=1}^{m-1}\frac{(i_{l}^{N-1}-i_{m}^{N-1})^{m-3}}{(i_{m}^{N}-i_{l}^{N-1})\prod_{s\neq l}^{m-1}(i_{l}^{N-1}-i_{s}^{N-1})}=\frac{(i_{m}^{N}-i_{m}^{N-1})^{m-3}}{\prod_{s=1}^{m-1}(i_{m}^{N}-i_{s}^{N-1})}. \tag{46}$$

The equation (45) is nothing but the corresponding coefficient in the rhs of

(27).

(IV). The coefficient of Ω_{IJ}^{N-1} with $I=\{1, 2, \cdots, m-1\}\setminus\{u\}$ and $J=\{t\}$, $u\neq t$.

(IV-I). The contribution to (35) from the term which contains $\Omega(N-1, l, \Lambda, l)$.

If $l=u$ then $\det_{IJ}^3=0$. We assume $l\neq u$. Then $\det_{IJ}^3=0$ for $t\neq l$, since u-th and l-th rows are proportional. Thus we assume $l=t$. We have

$$\det{}_{IJ}^3=(-1)^{u+m+1}D_{ut}^t\prod_{s\neq u,t}^{m-1}C_{ss}^t$$

$$=(-1)^{u+m+1}\left(\frac{i_t^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\frac{\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})}{\prod_{s\neq t}^{m-1}(i_t^{N-1}-i_s^{N-1})}\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_u^{N-1}-j}{i_t^{N-1}-j}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$\frac{(-1)^{u+m+1}}{i_m^N-i_t^{N-1}}\left(\frac{i_t^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\frac{\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})}{\prod_{s\neq t}^{m-1}(i_t^{N-1}-i_s^{N-1})}\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_u^{N-1}-j}{i_t^{N-1}-j}. \tag{47}$$

(IV-II). The contribution to (35) from the term which contains $\Omega(N-1, l, N-1, l)$ with $1\leq l\leq m-1$.

If $l\neq t$ then $\det_{IJ}^4=0$. We have, for $l=t$,

$$\det{}_{IJ}^4=(-1)^{u+m}C_{ut}^t\prod_{s\neq u,t}^{m-1}C_{ss}^t=(-1)^{u+m+1}D_{ut}^t\prod_{s\neq u,t}^{m-1}C_{ss}^t$$

$$=(-1)^{u+m+1}\left(\frac{i_t^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\frac{\prod_{s\neq u}^{m-1}(i_m^N-i_m^{N-1})}{\prod_{s\neq t}^{m-1}(i_t^{N-1}-i_s^{N-1})}\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_u^{N-1}-j}{i_t^{N-1}-j}.$$

The contribution to the rhs of (35) from the terms containing these determinants is

$$\frac{(-1)^{u+m+1}A_{N-1}(i_t^{N-1}-i_m^{N-1})^{m-3}}{(i_m^N-i_t^{N-1})^2(i_m^N-i_m^{N-1})^{m-2}}\frac{\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_u^{N-1}-j)}{\prod_{s\neq t}^{m-1}(i_t^{N-1}-i_s^{N-1})\prod_{j\notin\Lambda_N,j\neq i_t^{N-1},i_m^{N-1}}(i_t^{N-1}-j)} \tag{48}$$

$$+\frac{(-1)^{u+m}}{i_m^N-i_t^{N-1}}\left(\frac{i_t^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\frac{\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})}{\prod_{s\neq t}^{m-1}{}_{s\neq t}(i_t^{N-1}-i_s^{N-1})}\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_u^{N-1}-j}{i_t^{N-1}-j}.$$

Note that the second term of this equation is the minus of (47).

(IV-III). The contribution to (35) from the term which contains $\Omega(N-1, l, N-1, k)$ with $1\leq k, l\leq m-1$ and $k\neq l$.

If $k \neq t$ then $\det^4_{IJ}=0$. We have, for $k=t$,

$$\det{}^4_{IJ}=(-1)^{u+m}\frac{C^l_{ul}}{C^l_{uu}}\prod_{s\neq l}^{m-1}C^l_{ss}$$

$$=(-1)^{u+m+1}\left(\frac{i_l^{N-1}-i_m^{N-1}}{i_m^N-i_m^{N-1}}\right)^{m-2}\frac{\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_u^{N-1}-j}{i_l^{N-1}-j}.$$

Here we understand $C^l_{ul}/C^l_{uu}=-1$ for $u=l$. This follows from the equation

$$\frac{C^l_{ul}}{C^l_{uu}}=-\frac{i_m^N-i_l^{N-1}}{i_m^N-i_u^{N-1}}\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_u^{N-1}-j}{i_l^{N-1}-j}.$$

The contribution to the rhs of (35) is

$$\frac{(-1)^{u+m+1}A_{N-1}\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_u^{N-1}-j)}{(i_m^N-i_t^{N-1})(i_m^N-i_m^{N-1})^{m-2}\prod_{j\notin\mathcal{K}\ j\neq i_t^{N-1}}(i_t^{N-1}-j)}\tag{49}$$

$$\times\sum_{l\neq t}^{m-1}\frac{(i_l^{N-1}-i_m^{N-1})^{m-3}}{(i_m^N-i_l^{N-1})\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

Note that the $l=t$ term of this equation is precisely the first term of (48).

Thus, using (46), we have

$$(47)+(48)+(49)$$

$$=\frac{(-1)^{u+m+1}A_{N-1}\prod_{s\neq u}^{m-1}(i_m^N-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_u^{N-1}-j)}{(i_m^N-i_t^{N-1})(i_m^N-i_m^{N-1})^{m-2}\prod_{j\in\mathcal{K}\ j\neq i_t^{N-1}}(i_t^{N-1}-j)}$$

$$\times\sum_{l=1}^{m-1}\frac{(i_l^{N-1}-i_m^{N-1})^{m-3}}{(i_m^N-i_l^{N-1})\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

$$=\frac{(-1)^{u+m+1}A_{N-1}}{(i_m^N-i_u^{N-1})(i_m^N-i_m^{N-1})(i_m^N-i_t^{N-1})}\frac{\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_u^{N-1}-j)}{\prod_{j\in\mathcal{K},j\neq i_t^{N-1}}(i_t^{N-1}-j)}$$

which coincides with the coefficient of Ω^{N-1}_{IJ} in the rhs of (27).

(V). The coefficient of Ω^r_{IJ} for which $I=\{1,2,\cdots,m\}\setminus\{u\}$ and $J=\{t\}$, $r<N-1$.

(V-I). The contribution from the term which contains $\Omega(r,l,\Lambda,l)$.

If $l=u$ then $\det^1_{IJ}=0$. We assume $l\neq u$. Then

$$\det{}^1_{IJ}=(-1)^{u+m}\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\frac{\prod^m_{s\neq l}(i^{N-1}_t-i^r_s)}{\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_t}(i^{N-1}_t-j)}\frac{1}{\prod^m_{s=1}G^r_s}F^r_{ut}\prod^m_{s\neq u}G^r_s$$

$$=(-1)^{u+m}\frac{i^N_m-i^r_l}{(i^N_m-i^{N-1}_m)(i^{N-1}_t-i^r_u)}\frac{\prod^m_{s\neq l}(i^{N-1}_t-i^r_s)\prod_{j\notin\Lambda_r,j\neq i^N_m}(i^r_u-j)}{\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_t}(i^{N-1}_t-j)}.$$

The contribution to the rhs of (35) is

$$\frac{(-1)^{u+m}A_r\prod^m_{s\neq u}(i^{N-1}_t-i^r_s)\prod_{j\notin\Lambda_r,j\neq i^N_m}(i^r_u-j)}{(i^N_m-i^{N-1}_m)\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_t}(i^{N-1}_t-j)}\sum^m_{l\neq u}\frac{\prod^{m-1}_{s=1}(i^r_l-i^N_s)}{(i^N_m-i^r_l)(i^{N-1}_t-i^r_l)\prod^m_{s\neq l}(i^r_l-i^r_s)} \tag{50}$$

(V - II). The contribution from the term which contains $\Omega(r,l,\Lambda,k)$ with $k\neq l$.

If $l\neq u$ then $\det^1_{IJ}=0$. In fact if further $k\neq u$ then k-th, l-th, and u-th rows are proportional and if $k=u$ then l-th row is a null vector. Thus we assume $l=u$. Then

$$\det{}^1_{IJ}=(-1)^{u+m-1}\frac{i^N_m-i^r_u}{i^N_m-i^{N-1}_m}\frac{\prod^m_{s\neq u}(i^{N-1}_t-i^r_s)}{\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_t}(i^{N-1}_l-j)}\frac{1}{\prod^m_{s=1}G^r_s}F^r_{kt}\prod^m_{s\neq k}G^r_s$$

$$=(-1)^{u+m-1}\frac{i^N_m-i^r_u}{(i^N_m-i^{N-1}_m)(i^{N-1}_t-i^r_k)}\frac{\prod^m_{s\neq u}(i^{N-1}_t-i^r_s)\prod_{j\notin\Lambda_r,j\neq i^N_m}(i^r_k-j)}{\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_t}(i^{N-1}_t-j)}.$$

The contribution to the rhs of (35) is

$$\frac{(-1)^{u+m-1}A_r\prod^m_{s\neq u}(i^{N-1}_t-i^r_s)\prod_{j\notin\Lambda_r,j\neq i^N_m,i^{N-1}_m}(i^r_u-j)}{(i^N_m-i^{N-1}_m)\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_t}(i^{N-1}_t-j)}$$
$$\times\sum^m_{k\neq u}\frac{(i^r_k-i^{N-1}_m)\prod^{m-1}_{s=1}(i^r_k-i^N_s)}{(i^N_m-i^r_k)(i^{N-1}_t-i^r_k)\prod^m_{s\neq k}(i^r_k-i^r_s)}. \tag{51}$$

Note that the $k=u$ term of this equation is equal to the minus of the $l=u$ term in (50).

(V - III). The contribution from the term which contains $\Omega(r,l,N-1,k)$.

If $k\neq t$, $\det^2_{IJ}=0$. We assume $k=t$. Then $\det^2_{IJ}=0$ for $l\neq u$, since l-th and u-th rows are proportional. Thus we assume $l=u$. Then

$$\det{}^2_{IJ}=(-1)^{u+m}.$$

The contribution to the rhs of (35) is

$$\text{(52)} \qquad \frac{(-1)^{u+m}A_r\prod_{j\in A_r, j\neq i_m^N}(i_u^r-j)}{(i_m^N-i_u^r)(i_u^r-i_m^{N-1})(i_m^N-i_t^{N-1})\prod_{j\in\mathcal{K}, j\neq i_t^{N-1}}(i_t^{N-1}-j)}B(r,u,t)$$

$$=\frac{(-1)^{u+m}A_r\prod_{j\in A_r, j\neq i_m^N}(i_u^r-j)}{(i_m^N-i_u^r)(i_m^N-i_m^{N-1})(i_m^N-i_t^{N-1})\prod_{j\in\mathcal{K}, j\neq i_t^{N-1}}(i_t^{N-1}-j)}B(r,u,t)$$

$$+\frac{(-1)^{u+m}A_r\prod_{j\in A_r, j\neq i_m^N}(i_u^r-j)}{(i_u^r-i_m^{N-1})(i_m^N-i_m^{N-1})(i_m^N-i_t^{N-1})\prod_{j\in\mathcal{K}, j\neq i_t^{N-1}}(i_t^{N-1}-j)}B(r,u,t).$$

where we have used

$$\frac{1}{(i_m^N-i_u^r)(i_u^r-i_m^{N-1})}=\frac{1}{(i_m^N-i_u^r)(i_m^N-i_m^{N-1})}+\frac{1}{(i_u^r-i_m^{N-1})(i_m^N-i_m^{N-1})}.$$

Note that the first term of (52) is nothing but the coefficient of Ω_{IJ}^r in (27).

Let us calculate (50)+(51). We have

$$(50)+(51)$$

$$=\frac{(-1)^{u+m}A_r\prod_{s\neq u}^{m}(i_t^{N-1}-i_s^r)\prod_{j\in A_r, j\neq i_m^N}(i_u^r-j)}{(i_m^N-i_m^{N-1})\prod_{j\neq i_m^N, i_m^{N-1}, i_t^{N-1}}(i_t^{N-1}-j)}$$

$$\times\sum_{l=1}^{m}\left[1+\frac{i_l^r-i_m^{N-1}}{i_u^r-i_m^{N-1}}\right]\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{(i_m^N-i_l^r)(i_t^{N-1}-i_l^r)\prod_{s\neq l}^{m}(i_l^r-i_s^r)}.$$

By the residue theorem

$$\sum_{l=1}^{m}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{(i_m^N-i_l^r)(i_t^{N-1}-i_l^r)\prod_{s\neq l}^{m}(i_l^r-i_s^r)}$$

$$=\frac{\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{(i_t^{N-1}-i_m^N)\prod_{s=1}^{m}(i_m^N-i_s^r)}+\frac{\prod_{s=1}^{m-1}(i_t^{N-1}-i_s^N)}{(i_m^N-i_t^{N-1})\prod_{s=1}^{m}(i_t^{N-1}-i_s^r)},$$

$$\sum_{k=1}^{m}\frac{(i_k^r-i_m^{N-1})\prod_{s=1}^{m-1}(i_k^r-i_s^N)}{(i_m^N-i_k^r)(i_t^{N-1}-i_k^r)\prod_{s\neq k}^{m}(i_k^r-i_s^r)}$$

$$=\frac{(i_m^N-i_m^{N-1})\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{(i_t^{N-1}-i_m^N)\prod_{s=1}^{m}(i_m^N-i_s^r)}+\frac{(i_t^{N-1}-i_m^{N-1})\prod_{s=1}^{m-1}(i_t^{N-1}-i_s^N)}{(i_m^N-i_t^{N-1})\prod_{s=1}^{m}(i_t^{N-1}-i_s^r)}.$$

Hence

$$(50)+(51)$$

$$=\frac{(-1)^{u+m}A_r\prod_{s\neq u}(i_t^{N-1}-i_s^r)\prod_{j\in A_r, j\neq i_m^N}(i_u^r-j)}{(i_m^N-i_m^{N-1})(i_m^N-i_t^{N-1})\prod_{j\neq i_m^N, i_m^{N-1}, i_t^{N-1}}(i_t^{N-1}-j)}\times$$

$$\times\left[\frac{i_m^N-i_u^r}{i_u^r-i_m^{N-1}}\frac{\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{\prod_{s=1}^{m}(i_m^N-i_s^r)}+\frac{i_u^r-i_t^{N-1}}{i_u^r-i_m^{N-1}}\frac{\prod_{s=1}^{m-1}(i_t^{N-1}-i_s^N)}{\prod_{s=1}^{m}(i_t^{N-1}-i_s^r)}\right]$$

$$=\frac{(-1)^{u+m+1}A_r\prod_{j\notin\Lambda_r, j\neq i_m^N}(i_u^r-j)}{(i_u^r-i_m^{N-1})(i_m^N-i_m^{N-1})(i_m^N-i_t^{N-1})\prod_{j\in\mathcal{H}, j\neq i_t^{N-1}}(i_t^{N-1}-j)}B(r, u, t).$$

This is the minus of the second term of (52). Hence

$$(50)+(51)+(52)=\text{the first term of (52)}$$

which is equal to the coefficient of Ω_{IJ}^r in (27).

(VI). The coefficient of Ω_{IJ}^{N-1} for which $I\cap J=\emptyset$, $|I|\geq 1$ and $|J|\geq 2$.

Let us set $I=\{p_1<\cdots<p_u\}$ and $J=\{q_1<\cdots<q_t\}$, u, $t\leq m-1$, $u+t=m-1$.

(VI-I). The contribution from the term which contains $\Omega(N-1, l, \Lambda, l)$.

If $l\notin I$ then $\det_{IJ}^3=0$. We assume $l\in I$. We have

$$\det{}_{IJ}^3=\mathrm{sgn}\cdot\prod_{s\in I\setminus\{l\}}C_{ss}^l\prod_{s\in J}D_{ss}^l$$

$$=\mathrm{sgn}\cdot\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^t}{(i_m^N-i_m^{N-1})^{m-2}}\prod_{s\in I\setminus\{l\}}\frac{i_m^N-i_s^{N-1}}{i_l^{N-1}-i_s^{N-1}}\prod_{s\in J}\frac{i_m^{N-1}-i_s^{N-1}}{i_l^{N-1}-i_s^{N-1}},$$

where $\mathrm{sgn}=\mathrm{sgn}(p_1, \cdots, p_u, q_1, \cdots, q_t)$ is the sign of the permutation. The contribution to the rhs of (35) is

$$\text{(53)}\quad \mathrm{sgn}\frac{\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l\in I}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

(VI-II). The contribution from the term which contains $\Omega(N-1, l, N-1, l)$ with $1\leq l\leq m-1$.

If $l\notin J$, $\det_{IJ}^4=0$. We assume $l\in J$. We have

$$\det{}_{IJ}^4=\mathrm{sgn}\cdot\prod_{s\in I}C_{ss}^l\prod_{s\in J\setminus\{l\}}D_{ss}^l$$

$$=\mathrm{sgn}\cdot\frac{(i_l^{N-1}-i_m^{N-1})^u(i_m^N-i_l^{N-1})^{t-1}}{(i_m^N-i_m^{N-1})^{m-2}}\prod_{s\in I}\frac{i_m^N-i_s^{N-1}}{i_l^{N-1}-i_s^{N-1}}\prod_{s\in J\setminus\{l\}}\frac{i_m^{N-1}-i_s^{N-1}}{i_l^{N-1}-i_s^{N-1}}.$$

The contribution to the rhs of (35) is

$$\text{(54)}\quad -\mathrm{sgn}\frac{A_{N-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}$$

$$\times\sum_{l\in J}\frac{(i_l^{N-1}-i_m^{N-1})^{u-2}(i_m^N-i_l^{N-1})^{t-3}\prod_{s=1}^{m-1}(i_l^{N-1}-i_s^N)}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})^2}$$

$$+\mathrm{sgn}\frac{\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l\in J}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

Note that the sum of the second term of this equation and (53) is zero by the residue theorem and the conditions $u\geq 1$, $t\geq 2$.

(VI-III). The contribution from the term which contains $\Omega(N-1, l, N-1, k)$ with $1\leq k$, $l\leq m-1$, $k\neq l$.

If $k\notin J$ then $\det_{IJ}^4=0$. We assume $k\in J$.

(VI-III-I). $l\in I$ case.

We have

$$\det{}_{IJ}^4=-\mathrm{sgn}\cdot C_{kl}^l\prod_{s\in I\setminus\{l\}}C_{ss}^l\prod_{s\in J\setminus\{k\}}D_{ss}^l$$

$$=\mathrm{sgn}\frac{(i_l^{N-1}-i_m^{N-1})^u(i_m^N-i_l^{N-1})^t\prod_{s\in I\setminus\{l\}}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{k\}}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

$$\times\prod_{j\notin A_{N-1},j\neq i_m^N}\frac{i_k^{N-1}-j}{i_l^{N-1}-j}.$$

The contribution to the rhs of (35) is

$$\mathrm{sgn}\frac{A_{N-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l\in I}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}\tag{55}$$

$$\times\sum_{k\in J}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_k^{N-1}-i_m^N)\prod_{s\neq k}^m(i_k^{N-1}-i_s^{N-1})}.$$

(VI-III-II). $l\in J$ case.

We have

$$\det{}_{IJ}^4=-\mathrm{sgn}\cdot D_{kl}^l\prod_{s\in I}C_{ss}^l\prod_{s\in J\setminus\{k,l\}}D_{ss}^l$$

$$=-\mathrm{sgn}\frac{(i_l^{N-1}-i_m^{N-1})^{u+1}(i_m^N-i_l^{N-1})^{t-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{k,l\}}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

$$\times\prod_{j\notin A_{N-1},j\neq i_m^N}\frac{i_k^{N-1}-j}{i_l^{N-1}-j}.$$

The contribution to the rhs of (35) is

$$
(56)\quad \mathrm{sgn}\frac{A_{N-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l\in J}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}
$$

$$
\times\sum_{k\in J,k\neq l}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_k^{N-1}-i_m^N)\prod_{s\neq k}^{m}(i_k^{N-1}-i_s^{N-1})}.
$$

If we set $k=l$ in this equation, then it is equal to the first term of (54).

Thus we have

$$
(53)+(54)+(55)+(56)
$$

$$
=\mathrm{sgn}\frac{A_{N-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}}\sum_{l\in 1}^{m-1}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}
$$

$$
\times\sum_{k\in J}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_k^{N-1}-i_m^N)\prod_{s\neq k}^{m}(i_k^{N-1}-i_s^{N-1})}
$$

$$
=0,
$$

by applying the residue theorem to the summation in l.

(VII). The coefficient of Ω_{IJ}^{N-1} for which $|I\cap J|=1$, $|J|\geq 2$.

We set $I=\{p_1<\cdots<p_u\}$, $J=\{q_1<\cdots<q_t\}$ $(u+t=m-1)$ and $I\cap J=\{\bar{k}\}$.

(VII-I). The contribution from the term which contains $\Omega(N-1, l, \Lambda, l\}$.

If $k\neq l$ then $\det_{IJ}^3=0$. In fact if $\bar{k}\neq l$, either the l-th row is a null vector or there exists a row proportional to the l-th row. We assume $l=\bar{k}$. Let us define v, w, y by $p_v=q_w=l$, $I\cup J=\{1, 2, \cdots, m-1\}\backslash\{y\}$. Then we have

$$
\det{}_{IJ}^3=\mathrm{sgn}\cdot D_{yl}^l\prod_{s\in I\backslash\{l\}}C_{ss}^l\prod_{s\in J\backslash\{l\}}D_{ss}^l
$$

$$
=\mathrm{sgn}\cdot\frac{(i_l^{N-1}-i_m^{N-1})^u(i_m^N-i_l^{N-1})^t\prod_{s\in I\backslash\{l\}}(i_m^N-i_s^{N-1})\prod_{s\in J\backslash\{l\}}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}
$$

$$
\times\prod_{j\in\Lambda_{N-1},j\neq i_m^N}\frac{i_y^{N-1}-j}{i_l^{N-1}-j},
$$

where $\mathrm{sgn}=\mathrm{sgn}(p_1, \cdots, p_u, q_1, \cdots, y, \cdots, q_t)$, y being on the place of q_w. The contribution to the rhs of (35) is

$$(57)\quad -\mathrm{sgn}\cdot\frac{(i_{\bar{k}}^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_{\bar{k}}^{N-1})^{t-2}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq k}^{m-1}(i_{\bar{k}}^{N-1}-i_s^{N-1})}$$

$$\times\prod_{j\notin A_{N-1},j\neq i_m^N}\frac{i_y^{N-1}-j}{i_{\bar{k}}^{N-1}-j}.$$

(VII-II). The contribution from the term which contains $\Omega(N-1, l, N-1, l)$ with $1\leq l\leq m-1$.

If $\bar{k}\neq l$ then $\det_{IJ}^4=0$ by the same reson as (VII-I). We assume $l=\bar{k}$. Let us define v, w, y as in (VII-I). We have

$$\det{}_{IJ}^4=-\mathrm{sgn}\cdot C_{yl}^l\prod_{s\in I\backslash\{l\}}C_{ss}^l\prod_{s\in J\backslash\{l\}}D_{ss}^l$$

$$=\mathrm{sgn}\cdot D_{yl}^l\prod_{s\in I\backslash\{l\}}C_{ss}^l\prod_{s\in J\backslash\{l\}}D_{ss}^l$$

which is same as $\det_{IJ}^3$ in (VII-I). The contribution to the rhs of (35) is

$$(58)\quad -\mathrm{sgn}\frac{A_{N-1}(i_{\bar{k}}^{N-1}-i_m^{N-1})^{u-2}(i_m^N-i_{\bar{k}}^{N-1})^{t-3}}{(i_m^N-i_m^{N-1})^{m-2}}$$

$$\times\frac{\prod_{s=1}^{m-1}(i_{\bar{k}}^{N-1}-i_s^N)\prod_{s\in I}(i_m^{N-1}-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{\prod_{s\neq\bar{k}}^{m-1}(i_{\bar{k}}^{N-1}-i_s^{N-1})^2}\prod_{j\notin A_{N-1},j\neq i_m^N}\frac{i_y^{N-1}-j}{i_{\bar{k}}^{N-1}-j}$$

$$+\mathrm{sgn}\cdot\frac{(i_{\bar{k}}^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_{\bar{k}}^{N-1})^{t-2}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq\bar{k}}^{m-1}(i_{\bar{k}}^{N-1}-i_s^{N-1})}$$

$$\times\prod_{j\notin A_{N-1},j\neq i_m^N}\frac{i_y^{N-1}-j}{i_{\bar{k}}^{N-1}-j}.$$

The second term of this equation is equal to the minus of (57).

(VII-III). The contribution from the term which contains $\Omega(N-1, l, N-1, k)$ with $1\leq k, l\leq m-1, k\neq l$.

If $\bar{k}\neq k$ then $\det^4{}_{IJ}=0$. In fact if $\bar{k}\neq k$ then $\bar{k}$-th column in the right matrix and that in the left matrix are proportional. We assume $k=\bar{k}$. Let us define v, w, y by $p_v=q_w=k$, $I\cup J=\{1, 2, \cdots, m-1\}\backslash\{y\}$.

(VII-III-I). $l\in I$ case.

We have

$$\det{}_{IJ}^4=-\mathrm{sgn}\cdot C_{yl}^l\prod_{s\in I\backslash\{l\}}C_{ss}^l\prod_{s\in J\backslash\{k\}}D_{ss}^l$$

$$=\mathrm{sgn}\cdot\frac{(i_l^{N-1}-i_m^{N-1})^u(i_m^N-i_l^{N-1})^t\prod_{s\in I\setminus\{l\}}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{k\}}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

$$\times\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_y^{N-1}-j}{i_l^{N-1}-j}.$$

The contribution to the rhs of (35) is

$$\text{(59)}\quad \mathrm{sgn}\cdot A_{N-1}\frac{\prod_{s\in I\setminus\{\bar{k}\}}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{\bar{k}\}}(i_m^{N-1}-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_y^{N-1}-j)}{(i_m^N-i_m^{N-1})^{m-2}\prod_{j\in\mathcal{K},j\neq i_{\bar{k}}^{N-1}}(i_{\bar{k}}^{N-1}-j)}.$$

$$\times\sum_{l\in I,l\neq\bar{k}}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

(VII-III-II). $l\in J$ case.

We have

$$\det{}_{IJ}^4=-\mathrm{sgn}\cdot D_{yl}^l\prod_{s\in I}C_{ss}^l\prod_{s\in J\setminus\{k,l\}}D_{ss}^l$$

$$=\mathrm{sgn}\cdot\frac{(i_l^{N-1}-i_m^{N-1})^u(i_m^N-i_l^{N-1})^{t-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{k\}}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

$$\times\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}\frac{i_y^{N-1}-j}{i_l^{N-1}-j}.$$

The contribution to the rhs of (35) is

$$\text{(60)}\quad \mathrm{sgn}\cdot A_{N-1}\frac{\prod_{s\in I\setminus\{\bar{k}\}}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{\bar{k}\}}(i_m^{N-1}-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_y^{N-1}-j)}{(i_m^N-i_m^{N-1})^{m-2}\prod_{j\in\mathcal{K},j\neq i_{\bar{k}}^{N-1}}(i_{\bar{k}}^{N-1}-j)}$$

$$\times\sum_{l\in J,l\neq\bar{k}}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

Note that the first term of (58) is equal to the $l=\bar{k}$ term of (60).

(VII-III-III). $l=y$ case.

We have

$$\det{}_{IJ}^4=\mathrm{sgn}\cdot\prod_{s\in I}C_{ss}^l\prod_{s\in J\setminus\{k\}}D_{ss}^l$$

$$=\mathrm{sgn}\cdot\frac{(i_l^{N-1}-i_m^{N-1})^u(i_m^N-i_l^{N-1})^{t-1}\prod_{s\in I}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{k\}}(i_m^{N-1}-i_s^{N-1})}{(i_m^N-i_m^{N-1})^{m-2}\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}.$$

The contribution to the rhs of (35) is

$$(61)\quad \operatorname{sgn}\cdot A_{N-1}\frac{\prod_{s\in I\setminus\{\bar{k}\}}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{\bar{k}\}}(i_m^{N-1}-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_y^{N-1}-j)}{(i_m^N-i_m^{N-1})^{m-2}\prod_{j\in\mathcal{K},j\neq i_{\bar{k}}^{N-1}}(i_{\bar{k}}^{N-1}-j)}$$

$$\times\frac{(i_y^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_y^{N-1})^{t-2}}{\prod_{s\neq y}^{m-1}(i_y^{N-1}-i_s^{N-1})}.$$

This equation coincides with that obtained from (59) or (60) by setting $l=y$.

Now we have

$$(57)+(58)+(59)+(60)+(61)$$

$$=\operatorname{sgn}\cdot A_{N-1}\frac{\prod_{s\in I\setminus\{\bar{k}\}}(i_m^N-i_s^{N-1})\prod_{s\in J\setminus\{\bar{k}\}}(i_m^{N-1}-i_s^{N-1})\prod_{j\notin\Lambda_{N-1},j\neq i_m^N}(i_y^{N-1}-j)}{(i_m^N-i_m^{N-1})^{m-2}\prod_{j\in\mathcal{K}\ j\neq i_{\bar{k}}^{N-1}}(i_{\bar{k}}^{N-1}-j)}$$

$$\times\sum_{l=1}^{m-1}\frac{(i_l^{N-1}-i_m^{N-1})^{u-1}(i_m^N-i_l^{N-1})^{t-2}}{\prod_{s\neq l}^{m-1}(i_l^{N-1}-i_s^{N-1})}$$

$$=0.$$

The last equality follows from the residue theorem.

(VIII). The coefficient of Ω_{IJ}^{N-1} for which $|I\cap J|\geq 2$.

(VIII-I). The contribution from the term which contains $\Omega(N-1, l, \Lambda, l)$ is zero.

In fact $\det_{IJ}^3=0$ since at least one pair of common column is linearly dependent.

(VIII-II). The contribution from the term which contains $\Omega(N-1, l, N-1, l)$ with $1\leq l\leq m-1$ is zero by the same reason as (VIII-I).

(VIII-III). The contribution from the term which contains $\Omega(N-1, l, N-1, k)$ with $1\leq k, l\leq m-1, k\neq l$ is zero by the following reason. Since $C_{kl}^l=-D_{kl}^l$ for $k\neq l$, the common column except k-th column is linearly dependent. Hence $\det_{IJ}^4=0$.

As a whole the coefficient of Ω_{IJ}^{N-1} in the rhs of (35) is zero.

(IX). The coefficient of Ω_{IJ}^r for which $|J|\geq 2$.

Let us set $I=\{p_1<\cdots<p_u\}$, $J=\{q_1<\cdots<q_t\}$, $\bar{I}=\{1, 2, \cdots, m\}\setminus I=\{\bar{p}_1<\cdots<\bar{p}_t\}$ with $u+t=m$.

(IX-I). The contribution from the term which contains $\Omega(r, l, \Lambda, l)$.

If $l\notin I$, $\det_{IJ}^1=0$ We assume $l\in I$. Then

$$\det{}^1_{IJ}=(-1)^{\Sigma p_t+\frac{1}{2}u(u+1)}\prod_{s\in J}G^r_s\det(F_{\bar{p}_t q_t})\left(\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\right)^t\frac{E_J}{\prod_{s\in J}(i^{N-1}_s-i^r_l)}\frac{1}{\prod^m_{s=1}G^r_s}$$

$$=\operatorname{sgn}\frac{E_JD_{IJ}}{G^r(\bar{I})}\left(\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\right)^t\frac{1}{\prod_{s\in J}(i^{N-1}_s-i^r_l)},$$

where $\operatorname{sgn}=(-1)^{\Sigma p_t+\frac{1}{2}u(u+1)+\frac{1}{2}t(t+1)}$, $G^r(\bar{I})=\prod_{s\in\bar{I}}G^r_s$ and

$$D_{IJ}=(-1)^{\frac{1}{2}t(t+1)}\det(F_{\bar{p}_t q_t})=\frac{\prod_{\alpha<\beta}(i^r_{\bar{p}_\alpha}-i^r_{\bar{p}_\beta})\prod_{\alpha<\beta}(i^{N-1}_{q_\alpha}-i^{N-1}_{q_\beta})}{\prod_{s\in\bar{I}}\prod_{s'\in J}(i^r_s-i^{N-1}_{s'})},$$

$$E_J=\prod_{y\in J}\left(\frac{\prod^m_{s=1}(i^{N-1}_y-i^r_s)}{\prod_{j\neq i^N_m,i^{N-1}_m,i^{N-1}_y}(i^{N-1}_y-j)}\right).$$

The contribution to the rhs of (35) is

$$\frac{\operatorname{sgn}\cdot A_rE_JD_{IJ}}{(i^N_m-i^{N-1}_m)^tG^r(\bar{I})}\sum_{l\in I}\frac{(i^N_m-i^r_l)^{t-2}\prod^{m-1}_{s=1}(i^r_l-i^N_s)}{\prod^m_{s\neq l}(i^r_l-i^r_s)\prod_{s\in J}(i^{N-1}_s-i^r_l)}. \tag{62}$$

(IX-II). The contribution from the term which contains $\Omega(r,l,\Lambda,k)$ with $k\neq l$.

If $l\in I$ or $k\notin I$ then $\det^1_{IJ}=0$. In fact if $l\in I$ then l-th column in the left matrix is a null vector and if $k\notin I$ then l-th row is a null vector. We assume $l\notin I$ and $k\in I$. Let us set $\tilde{I}=\{\tilde{p}_1<\cdots<\tilde{p}_t\}=\bar{I}\setminus\{l\}\cup\{k\}$ and $p^*_t=\bar{p}_t\ (\bar{p}_t\neq l)$, $p^*_t=k\ (\bar{p}_t=l)$. Let us define v, w by $p_v=k$, $p_{w-1}<l<p_w$. Then

$$\det{}^1_{IJ}=(-1)^{\Sigma p_t+\frac{1}{2}u(u+1)+l-k+v-w}\left(\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\right)^t\frac{E_JG^r_l\prod_{s\in I\setminus\{k\}}G^r_s\cdot\det(F_{\tilde{p}_t q_t})}{\prod_{s\in J}(i^{N-1}_s-i^r_l)\prod^m_{s=1}G^r_s}$$

$$=(-1)^{\Sigma p_t+\frac{1}{2}u(u+1)-1}\frac{E_J}{G^r(\bar{I})}\left(\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\right)^t\frac{\det(F_{p^*_t q_t})}{\prod_{s\in J}(i^{N-1}_s-i^r_l)}\frac{G^r_l}{G^r_k}$$

$$=-\operatorname{sgn}\frac{E_JD_{IJ}}{G^r(\bar{I})}\left(\frac{i^N_m-i^r_l}{i^N_m-i^{N-1}_m}\right)^t\frac{1}{\prod_{s\in J}(i^{N-1}_s-i^r_k)}\frac{\prod_{j\notin\Lambda_r,j\neq i^N_m}(i^r_k-j)}{\prod_{j\notin\Lambda_r,j\neq i^N_m}(i^r_l-j)}\frac{\prod_{s\in\bar{I}\setminus\{l\}}(i^r_k-i^r_s)}{\prod_{s\in\bar{I}\setminus\{l\}}(i^r_l-i^r_s)},$$

where we use

$$\det(F_{\tilde{p}_t q_t})=(-1)^{k-l+w-v-1}\det(F_{p^*_t q_t}).$$

The contribution to the rhs of (35) is

$$-\frac{\operatorname{sgn}\cdot A_rE_ID_{IJ}}{G^r(\bar{I})(i^N_m-i^{N-1}_m)^t}\sum_{l\in\bar{I}}\frac{(i^N_m-i^r_l)^{t-1}}{(i^r_l-i^{N-1}_m)\prod_{s\in\bar{I}\setminus\{l\}}(i^r_l-i^r_s)} \tag{63}$$

$$\times\sum_{k\in I}\frac{(i^r_k-i^{N-1}_m)\prod^{m-1}_{s=1}(i^r_k-i^N_s)}{(i^N_m-i^r_k)\prod_{s\in(I\cup\{l\})\setminus\{k\}}(i^r_k-i^r_s)\prod_{s\in J}(i^{N-1}_s-i^r_k)}.$$

If we set $k=l$ in this equation then it equals to the minus of (62).

(IX-III). The contribution from the term which contains $\Omega(r, l, N-1, k)$.

If $l\in I$ or $k\notin J$, $\det_{IJ}^{2}=0$. In fact if $l\in I$ then l-th column in the left matrix is zero and if $k\notin J$ then l-th row is zero. We assume $l\in\bar{I}$ and $k\in J$. We define v, w here by $q_v=k$ and $p_w<l<p_{w+1}$.
We note that

$$\#\{s|\bar{p}_s<l\}=l-1-w, \quad \#\{s|q_s<k\}=v-1.$$

Using these relations we have

$$(64) \quad \det{}_{IJ}^{2}=(-1)^{\Sigma p_i+\frac{1}{2}u(u+1)+l-w+v}\left(\frac{i_m^N-i_l^r}{i_m^N-i_m^{N-1}}\right)^{t-1}\frac{E_J\prod_{j\neq i_m^N, i_m^{N-1}, i_k^{N-1}}^{m}(i_k^{N-1}-j)}{\prod_{s\neq l}^{m}(i_k^{N-1}-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_l^r)\prod_{s\neq l}^{m}G_s^r}$$

$$\times\prod_{s\in I}G_s^r\det_{(\bar{I}\setminus\{l\})\times(J)\setminus\{k\})}(F_{\bar{p}_iq_j})$$

$$=\operatorname{sgn}\frac{E_JD_{IJ}}{G^r(\bar{I})}\left(\frac{i_m^N-i_l^r}{i_m^N-i_m^{N-1}}\right)^{t-1}\frac{\prod_{j\neq i_m^N, i_m^{N-1}, i_k^{N-1}}^{m}(i_k^{N-1}-j)}{\prod_{j\notin\Lambda_r, j\neq i_m^N}(i_l^r-j)\prod_{s\neq l}^{m}(i_k^{N-1}-i_s^r)}$$

$$\times\frac{\prod_{s\in\bar{I}\setminus\{l\}}(i_s^r-i_k^{N-1})}{\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}.$$

The contribution to the rhs of (35) is

$$(65) \quad \frac{(-1)^{t-1}\operatorname{sgn}\cdot A_rE_JD_{IJ}}{G^r(\bar{I})(i_m^N-i_m^{N-1})^{t-1}}\sum_{l\in\bar{I}}\frac{(i_m^N-i_l^r)^{t-2}}{(i_l^r-i_m^{N-1})\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}$$

$$\times\sum_{k\in J}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\in I}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}$$

$$-\frac{\operatorname{sgn}\cdot E_JD_{IJ}}{G^r(\bar{I})(i_m^N-i_m^{N-1})^{t-1}}\sum_{l\in\bar{I}}\frac{(i_m^N-i_l^r)^{t-1}}{(i_l^r-i_m^{N-1})\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}$$

$$\times\sum_{k\in J}\frac{\prod_{s\in\bar{I}\setminus\{l\}}(i_s^r-i_k^{N-1})}{(i_m^N-i_k^{N-1})\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}$$

$$=\frac{(-1)^t\operatorname{sgn}\cdot A_rE_JD_{IJ}}{G^r(\bar{I})(i_m^N-i_m^{N-1})\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)}$$

$$\times\sum_{k\in J}\frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\in I}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}$$

$$-\frac{\mathrm{sgn}\cdot E_J D_{IJ}\prod_{s\in\bar{I}}(i_s^r-i_m^N)}{G^r(\bar{I})\,(i_m^N-i_m^{N-1})\prod_{s\in J}(i_m^N-i_s^{N-1})\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)}.$$

In deriving the last equation we use

$$(66)\qquad \sum_{l\in\bar{I}}\frac{(i_m^N-i_l^r)^{t-2}}{(i_l^r-i_m^{N-1})\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}=-\frac{(i_m^N-i_m^{N-1})^{t-2}}{\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)},$$

$$\sum_{k\in J}\frac{\prod_{s\in\bar{I}\setminus\{l\}}(i_s^r-i_k^{N-1})}{(i_m^N-i_k^{N-1})\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}=\frac{\prod_{s\in\bar{I}\setminus\{l\}}(i_s^r-i_m^N)}{\prod_{s\in J}(i_m^N-i_s^{N-1})}.$$

Let us name the first and the second term of (65) Z and W respectively.

Now we have

$$\begin{aligned}&(62)+(63)+(65)\\
&=\frac{\mathrm{sgn}\cdot A_r E_J D_{IJ}}{(i_m^N-i_m^{N-1})^t G^r(\bar{I})}\sum_{l=1}^{m}\frac{(i_m^N-i_l^r)^{t-2}\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_l^r)}\\
&\quad-\frac{\mathrm{sgn}\cdot A_r E_I D_{IJ}}{G^r(\bar{I})\,(i_m^N-i_m^{N-1})^t}\sum_{l\in I}\frac{(i_m^N-i_l^r)^{t-1}}{(i_l^r-i_m^{N-1})\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}\\
&\quad\times\sum_{k\in I\cup\{l\}}\frac{(i_k^r-i_m^{N-1})\prod_{s=1}^{m-1}(i_k^r-i_s^N)}{(i_m^N-i_k^r)\prod_{s\in(I\cup\{l\})\setminus\{k\}}(i_k^r-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_k^r)}+Z+W.\end{aligned}$$

Let us name the first and the second term of this equation X and Y respectively. We shall rewrite X and Y. Using

$$\sum_{l=1}^{m}\frac{(i_m^N-i_l^r)^{t-2}\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^{m}(i_l^r-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_l^r)}=\sum_{l\in J}\frac{(i_m^N-i_l^{N-1})^{t-2}\prod_{s=1}^{m-1}(i_l^{N-1}-i_s^N)}{\prod_{s=1}^{m}(i_l^{N-1}-i_s^r)\prod_{s\in J\setminus\{l\}}(i_s^{N-1}-i_l^{N-1})}$$

we have

$$X=\frac{\mathrm{sgn}\cdot A_r E_J D_{IJ}}{(i_m^N-i_m^{N-1})^t G^r(\bar{I})}\sum_{l\in J}\frac{(i_m^N-i_l^{N-1})^{t-2}\prod_{s=1}^{m-1}(i_l^{N-1}-i_s^N)}{\prod_{s=1}^{m}(i_l^{N-1}-i_s^r)\prod_{s\in J\setminus\{l\}}(i_s^{N-1}-i_l^{N-1})}.$$

Using

$$\begin{aligned}&\sum_{k\in I\cup\{l\}}\frac{(i_k^r-i_m^{N-1})\prod_{s=1}^{m-1}(i_k^r-i_s^N)}{(i_m^N-i_k^r)\prod_{s\in(I\cup\{l\})\setminus\{k\}}(i_k^r-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_k^r)}\\
&=\frac{(i_m^N-i_m^{N-1})\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{(i_m^N-i_l^r)\prod_{s\in I}(i_m^N-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_m^N)}\end{aligned}$$

$$+\sum_{k\in J}\frac{(i_k^{N-1}-i_m^{N-1})\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})(i_k^{N-1}-i_l^r)\prod_{s\in I}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_s^{N-1}-i_k^{N-1})}$$

we have

$$Y=-\frac{\mathrm{sgn}\cdot A_rE_JD_{IJ}\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{G^r(\bar{I})(i_m^N-i_m^{N-1})^{t-1}\prod_{s\in I}^m(i_m^N-i_s^r)\prod_{s\in J}(i_s^{N-1}-i_m^N)}\sum_{l\in\bar{I}}\frac{(i_m^N-i_l^r)^{t-2}}{(i_l^r-i_m^{N-1})\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}$$

$$-\frac{\mathrm{sgn}\cdot A_rE_JD_{IJ}}{G^r(\bar{I})(i_m^N-i_m^{N-1})^t}\sum_{l\in\bar{I}}\frac{(i_m^N-i_l^r)^{t-1}}{(i_l^r-i_m^{N-1})(i_k^{N-1}-i_l^r)\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}$$

$$\times\sum_{k\in J}\frac{(i_k^{N-1}-i_m^{N-1})\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\in I}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_s^{N-1}-i_k^{N-1})}.$$

Using further (66) and

$$\sum_{l\in\bar{I}}\frac{(i_m^N-i_l^r)^{t-1}}{(i_l^r-i_m^{N-1})(i_k^{N-1}-i_l^r)\prod_{s\in\bar{I}\setminus\{l\}}(i_l^r-i_s^r)}$$

$$=-\frac{(i_m^N-i_m^{N-1})^{t-1}}{(i_k^{N-1}-i_m^{N-1})\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)}+\frac{(i_m^N-i_k^{N-1})^{t-1}}{(i_k^{N-1}-i_m^{N-1})\prod_{s\in\bar{I}}(i_k^{N-1}-i_s^r)}$$

we have $Y=-W-Z-X$. Thus

$$(62)+(63)+(65)=0. \tag{67}$$

This completes the proof of (1) of Theorem 1.

§ 4.8. Proof of (2) of Theorem 1

Let us prove the remaining part of Theorem 1. Let $\{E_{ij}\}$ be the standard basis of gl_N where E_{ij} is the matrix unit with 1 in ij component. Set $h_i=E_{ii}-E_{i+1i+1}$ $(1\leq i\leq N-1)$. By the definition of $f=\sum f_\Lambda v_\Lambda$ it has weight zero, $h_if=0$ for any i. Hence it is sufficient to prove $E_{ij}f=0$ for any $i\neq j$. First we assume $N\geq 3$. Then it is sufficient to prove

$$E_{rN}f=0,\quad r=1,\cdots,N-2. \tag{68}$$

In fact by the following reason the proof for an arbitrary E_{ij} case is reduced to the above case. In our description of our basis of differential forms the index N and $N-1$ play a special role. For E_{ij} we replace the role of N by j and that of $N-1$ by k with $k\neq i,j$. This is possible because $N\geq 3$. Then the following proof is totally the same in this modified situation. Thus let us prove (68).

Let $\Lambda=(\Lambda_1,\cdots,\Lambda_N)$ with $\Lambda_j=(i_1^j,\cdots,i_m^j)$ $(1\leq j\leq N)$. We consider $\Lambda'=$

$(\Lambda_1', \cdots, \Lambda_N')$ with

$$\Lambda_j'=\Lambda_j\,(j\neq r, N),\ \Lambda_r'=(i_1^r, \cdots, i_m^r, i_m^N),\ \Lambda_N'=(i_1^N, \cdots, i_{m-1}^N).$$

Define $v_{\Lambda'}$ in an obvious way. Then the coefficient of $v_{\Lambda'}$ of $E_{rN}f$ is

$$f_\Lambda+\sum_{l=1}^m f_{\Lambda^{(i_l^r i_m^N)}}.$$

Hence it is sufficient to prove

$$\bar{f}_\Lambda+\sum_{l=1}^m \bar{f}_{\Lambda^{(i_l^r i_m^N)}}=0. \tag{69}$$

We shall devide the case into two for the proof of (69).

(i). The coefficient of $\Omega=\begin{pmatrix}\Lambda & \cdots & \Lambda \\ i_1^1 & & i_{m-1}^{N-1}\end{pmatrix}$ of the left hand side of (69).

In (II) of the proof of 1 of Theorem 1 we have calculated the coefficient of Ω in

$$\frac{\Delta(i_1^1, \cdots, i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)}\sum_{l=1}^m \frac{1}{i_m^N-i_l^r}\bar{f}_{\Lambda^{(i_l^r i_m^N)}}.$$

From the calculation there we can easily read off the coefficient of Ω in

$$\frac{\Delta(i_1^1, \cdots, i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)}\sum_{l=1}^m \bar{f}_{\Lambda^{(i_l^r i_m^N)}}.$$

It is

$$-\sum_{l=1}^m \frac{A_r}{i_m^N-i_l^r}\frac{\prod_{s=1}^{m-1}(i_l^r-i_s^N)}{\prod_{s\neq l}^m(i_l^r-i_s^r)}=-A_r\frac{\prod_{s=1}^{m-1}(i_m^N-i_s^N)}{\prod_{s=1}^m(i_m^N-i_s^r)}=-1.$$

Thus the coefficient of Ω in the lhs of (69) is 0, since

$$\frac{\Delta(i_1^1, \cdots, i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)}\bar{f}_\Lambda=\Omega.$$

(ii). The coefficient of Ω_{IJ}^r with $|J|\geq 1$ in the lhs of (69).

In (IX) we have proved that the coefficient of Ω_{IJ}^r in

$$\frac{\Delta(i_1^1, \cdots, i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)}\sum_{l=1}^m \frac{1}{i_m^N-i_l^r}\bar{f}_{\Lambda^{(i_l^r i_m^N)}}$$

is zero. There the condition $|J|\geq 2$ is used only when the residue theorem is applied. Taking care of it we can again easily read off the coefficient of Ω_{IJ}^{r} of

$$\frac{\Delta(i_1^1, \cdots, i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)} \sum_{l=1}^{m} \bar{f}_{\Lambda^{(i_l^r i_m^N)}}$$

from the calculation in (IX). We used the notation (62), (63), (65), X, Y to denote the equation apeared there. We shall use the prime of them for the corresponding equation like (62)$'$, X' etc. Then

$$Z'=\frac{(-1)^t \mathrm{sgn}\cdot A_r E_J D_{IJ}}{G^r(\bar{I})\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)} \sum_{k\in J} \frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\in I}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}$$

$$W'=-\frac{\mathrm{sgn}\cdot E_J D_{IJ}\prod_{s\in\bar{I}}(i_s^r-i_m^N)}{G^r(\bar{I})\prod_{s\in J}(i_m^N-i_s^{N-1})\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)}.$$

Also we have

$$X'=\frac{\mathrm{sgn}\cdot A_r E_J D_{IJ}}{(i_m^N-i_m^{N-1})^t G^r(\bar{I})} \sum_{l\in J} \frac{(i_m^N-i_l^{N-1})^{t-1}\prod_{s=1}^{m-1}(i_l^{N-1}-i_s^N)}{\prod_{s=1}^{m}(i_l^{N-1}-i_s^r)\prod_{s\in J\setminus\{l\}}(i_s^{N-1}-i_l^{N-1})}$$

and

$$\begin{aligned}
(70)\quad Y'=&\frac{\mathrm{sgn}\cdot E_J D_{IJ}\prod_{s\in\bar{I}}(i_s^r-i_m^N)}{G^r(\bar{I})\prod_{s\in J}(i_m^N-i_s^{N-1})\prod_{s\in\bar{I}}(i_m^{N-1}-i_s^r)}\\
&+\frac{(-1)^{t-1}\mathrm{sgn}\cdot A_r E_J D_{IJ}}{G^r(\bar{I})\prod_{s\in I}(i_m^{N-1}-i_s^r)} \sum_{k\in J} \frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{(i_m^N-i_k^{N-1})\prod_{s\in I}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_k^{N-1}-i_s^{N-1})}\\
&-\frac{\mathrm{sgn}\cdot A_r E_J D_{IJ}}{(i_m^N-i_m^{N-1})^t G^r(\bar{I})} \sum_{k\in J} \frac{(i_m^N-i_k^{N-1})^{t-1}\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{\prod_{s=1}^{m}(i_k^{N-1}-i_s^r)\prod_{s\in J\setminus\{k\}}(i_s^{N-1}-i_k^{N-1})}.\\
=&-W'-Z'-X'.
\end{aligned}$$

Hence

$$(62)'+(63)'+(65)'=Z'+W'+X'+Y'=0.$$

Thus the equation (69) is proved.

In the $N=2$ case we can similarly read off easily the coefficient of

$$\frac{\Delta(i_1^1, \cdots, i_{m-1}^{N-1})^{-1}}{\prod_{t<u}(\Lambda_t\Lambda_u)}\left(\bar{f}_\Lambda+\sum_{l=1}^{m}\bar{f}_{\Lambda^{(i_l^r i_m^N)}}\right)$$

from (I), (II), (III), (IV), (VI), (VII), (VIII) in the proof of 1 of Theorem 1. They are all zero as we expect.

§ 5. Discussion

In this paper we give integral and theta formulae for the solutions of sl_N Knizhnik-Zamolodchikov (KZ) equations of level 0 with the value in the trivial representation in the tensor product of the vector representations of sl_N. The formula generalizes the Smirnov's formula in the case of sl_2. We have found that the differential form μ_p^A, which is a building block of the integral formula, is obtained by evaluating one of the variables to the branch point Q_p in the product of chiral Szegö kernels. This is a key for the proof of the theta formula.

Let us discuss remaining problems and related subjects.

In $N=2$ case it is conjectured that Smirnov type solutions span the singlet solution space [16]. On the other hand the dimension of the vector space spanned by our integral formulae is less than the multiplicity of the trivial representation in $V^{\otimes Nm}$ for $N\geq 3$ and $m\geq 2$. In fact the multiplicity is given by

$$\mathrm{mult}(0, V^{\otimes Nm}) = \frac{(Nm)!}{\prod_{k=0}^{N-1}\prod_{j=0}^{m-1}(m+k-j)}.$$

On the other hand the demension $D(N, m)$ of the vector space spanned by integral formulae satisfies

$$D(N, m) \leq I(N, m) = \binom{Nm-2}{(N-1)m-1},$$

where the right hand side is the binomial coefficient. The number $Nm-2$ is the dimension of an eigenspace of the N-cyclic automorphism ϕ on the first homology group of a Z_N curve. Then

$$\frac{\mathrm{mult}(0, V^{\otimes Nm})}{I(N, m)} = \frac{N-\frac{1}{m}}{N-1}\prod_{k=1}^{N-1}\prod_{j=0}^{m-2}\frac{(N-1)(m-1-j)+k}{m-j+k}. \tag{71}$$

Since

$$(N-1)(m-1-j)+k-(m-j+k) = (N-2)\left(m-2-j+\frac{N-3}{N-2}\right),$$

(71) is greater than 1 if $N\geq 3$ and $m\geq 2$. Note that mult $(0, V^{\otimes N}) = I(N, 1) = 1$. For $N=2$ we have

$$\mathrm{mult}(0, V^{\otimes 2m}) = \mathrm{I}(2, m) - \binom{2m-2}{m-3},$$

where the second term in the right hand side comes from the Riemann's bilinear identity [16].

This structure of solution space should be same in the qKZ case. To construct remaining solutions for both KZ and qKZ equations is an interesting and important problem. In the qKZ case to study a relation of these missing solutions with form factors is also interesting.

We still do not understand the relation between the integral formula given here and those given in [6,10,11] in the case of sl_N, $N \geq 3$. In $N=2$ case the relation is given in [9]. If we understand this structure then it will help to find the missing solution discussed above.

The relation of the solution to the KZ equation of level 0 with a classical integrable system is still to be clarified. The relation with the Szegö kernel will give some hint to understand this problem since the Szegö kernel is related with the tau function of the KP hierarchy. Anyway it is true that we can introduce a Jacobian variable in the theta formula for the solutions to the KZ equation. Hence it is natural to ask what kind of equation governs the dependence on the Jacobian variables and what the zero value means for that equation.

Once we introduce the Jacobian variable we can ask what is the difference analogue, q analogue of the theta function? As to the abelian integral, Smirnov [16,17] discussed its difference analogue.

Since the Smirnov type formula is related with the algebraic curves in the case of sl_N, it is interesting to study Smirnov type solutions for other type of Lie algebras and whether they are related with algebraic curves.

The determinantal structure of Smirnov type solution is still lacking an understanding from the representation theoretical view point.

Acknowledgement

We would like to thank Koji Cho, Fedor Smirnov, Yasuhiko Yamada for the stimulating discussions and useful comments. We also benefited from the discussion with Masaki Kashiwara.

§ Appendix

In this section we give a derivation of the formula (20) - (24). We recall the definition of $\begin{pmatrix} \Lambda \\ p \end{pmatrix}$

$$\begin{pmatrix} \Lambda \\ p \end{pmatrix} = \frac{g^{(\Lambda_r)}(p) g^{(p)}_{\Lambda_r}(z)\, dz}{(z-p)s} \quad p \in \Lambda_r,$$

where as in the main text $z-p$ means $z-\lambda_p$ and $g^{(\Lambda_r)}(p)$ means $g^{(\Lambda_r)}(\lambda_p)$ etc.

(I). a derivation of the formula (20):

By differentiating the defining formula of $\begin{pmatrix} \Lambda \\ p \end{pmatrix}$ we have

$$\frac{\partial}{\partial \lambda_{i_m^N}} \begin{pmatrix} \Lambda \\ p \end{pmatrix} = -\frac{1}{p-i_m^N} \begin{pmatrix} \Lambda \\ p \end{pmatrix} + \frac{1}{N} \frac{g^{(\Lambda_r)}(p) g^{(p)}_{(\Lambda_r)}(z)\, dz}{(z-p)(z-i_m^N)s} \tag{72}$$

$$=\left(1-\frac{1}{N}\right)\frac{1}{i_m^N-p}\begin{pmatrix}\Lambda\\ p\end{pmatrix}+\frac{1}{N}\frac{1}{i_m^N-p}\frac{g^{(\Lambda r)}(p)g_{\Lambda r}^{(p)}(z)\,dz}{(z-i_m^N)\,s} \tag{73}$$

$$=\left(1-\frac{1}{N}\right)\frac{1}{i_m^N-p}\begin{pmatrix}\Lambda\\ p\end{pmatrix}-\frac{1}{N}\frac{1}{i_m^N-p}\prod_{j\notin\Lambda_r,\,j\neq i_m^N}\frac{p-j}{i_m^N-j}\begin{pmatrix}p\\ i_m^N\end{pmatrix}. \tag{74}$$

Here to obtain (73) from (72) we use

$$\frac{1}{(z-p)\,(z-i_m^N)}=\frac{1}{i_m^N-p}\left[-\frac{1}{z-p}+\frac{1}{z-i_m^N}\right],$$

and to get (74) from (73) we use

$$g_{\Lambda r}^{(p)}(z)=g_{\Lambda_r(p;i_m^N)}^{(i_m^N)}(z),$$

$$\frac{g^{(\Lambda r)}(p)}{g^{(\Lambda_r(p;i_m^N))}(i_m^N)}=-\prod_{j\notin\Lambda_r,\,j\neq i_m^N}\frac{p-j}{i_m^N-j}.$$

(II). a derivation of the formula (21):

Since

$$\begin{pmatrix}i_{l'}^r\\ i_l^r\end{pmatrix}=\frac{g^{(\Lambda_r(i_m^N;i_{l'}^r))}(i_l^r)\,g_{\Lambda_r(i_m^N;i_{l'}^r)}^{(i_l^r)}(z)\,dz}{(z-i_l^r)\,s}$$

we have

$$\begin{pmatrix}\Lambda\\ i_l^r\end{pmatrix}-\begin{pmatrix}i_{l'}^r\\ i_l^r\end{pmatrix}=\frac{g^{(\Lambda r)}(i_l^r)\,g_{\Lambda_r}^{(i_l^r i_{l'}^r)}(z)}{(z-i_l^r)\,s}\,dz\left[z-i_{l'}^r-\frac{i_l^r-i_{l'}^r}{i_l^r-i_m^N}(z-i_m^N)\right]$$

$$=\frac{i_m^N-i_{l'}^r}{i_m^N-i_l^r}\,\frac{g^{(\Lambda r)}(i_l^r)\,g_{\Lambda_r}^{(i_l^r i_{l'}^r)}(z)}{s}\,dz,$$

where we set

$$g_{\Lambda_r}^{(i_l^r i_{l'}^r)}(z)=\prod_{j\in\Lambda_r\,j\neq i_l^r,\,i_{l'}^r}(z-j).$$

Hence

$$\frac{g_{\Lambda_r}^{(i_l^r i_{l'}^r)}(z)}{s}dz=\frac{i_m^N-i_l^r}{i_m^N-I_{l'}^r}\,\frac{1}{g^{(\Lambda r)}(i_l^r)}\left[\begin{pmatrix}\Lambda\\ i_l^r\end{pmatrix}-\begin{pmatrix}i_{l'}^r\\ i_l^r\end{pmatrix}\right]. \tag{75}$$

If we set $r=N-1$ and $l'=$m in this equation we get

$$\frac{g_{\Lambda_{N-1}}^{(i_l^{N-1}i_m^{N-1})}(z)}{s}\,dz=\frac{i_m^N-i_l^{N-1}}{i_m^N-i_m^{N-1}}\,\frac{1}{g^{(\Lambda_{N-1})}(i_l^{N-1})}\left[\begin{pmatrix}\Lambda\\ i_l^{N-1}\end{pmatrix}-\begin{pmatrix}i_m^{N-1}\\ i_l^{N-1}\end{pmatrix}\right]. \tag{76}$$

Now let $F(z)$ be an arbitrary polynomial of degree at most $m-2$ then

$$F(z)=\sum_{k=1}^{m-1}\frac{F(i_k^{N-1})}{\prod_{s\neq k}^{m-1}(i_k^{N-1}-i_s^{N-1})}g_{\Lambda_{N-1}}^{(i_k^{N-1}i_m^{N-1})}(z). \tag{77}$$

Thus we have

$$\frac{g_{\Lambda_r}^{(i_h^r i_l^r)}(z)}{s}dz=\sum_{k=1}^{m-1}\frac{g_{\Lambda_r}^{(i_h^r i_l^r)}(i_k^{N-1})}{\prod_{s\neq k}^{m-1}(i_k^{N-1}-i_s^{N-1})}\frac{g_{\Lambda_{N-1}}^{(i_k^{N-1}i_m^{N-1})}(z)}{s}dz. \tag{78}$$

Substituting (75) and (76) into (78) we get (21). The relation (22) is a special case $r=N-1$ of (21).

(III). Here we prove the formula

$$\binom{A}{i_m^N}=\sum_{i_l^r\in\mathcal{K}}\prod_{j\in\mathcal{K},j\neq i_l^r}\frac{i_m^N-j}{i_l^r-j}\binom{A}{i_l^r}. \tag{79}$$

For the sake of simplicity we set $i_j^r=i_{(r-1)m+j}$. By Proposition 5 we have

$$[\binom{A}{i_1},\cdots,\binom{A}{i_L}]=[\zeta_1^{(A)},\cdots,\zeta_L^{(A)}]A,$$

where A is the L by L matrix whose kl component A_{kl} is given by $A_{kl}=\lambda_{i_l}^{k-1}$. Let d_{kl} be the kl cofactor of A.Then we have

$$\zeta_k^{(A)}=\frac{1}{\det A}\sum_{j=1}^{L}\binom{A}{i_j}d_{kj}.$$

Again by Proposition 5

$$\binom{A}{i_m^N}=\sum_{k=1}^{L}\zeta_k^{(A)}\lambda_{i_m^N}^{k-1}=\frac{1}{\det A}\sum_{j=1}^{L}\sum_{k=1}^{L}\lambda_{i_m^N}^{k-1}d_{kj}\binom{A}{i_j}. \tag{80}$$

Using the expansion of the Vandermond determinant in a column we have

$$\frac{1}{\det A}\sum_{k=1}^{L}\lambda_{i_m^N}^{k-1}d_{kj}=\prod_{s\neq j}^{L}\frac{i_m^N-i_s}{i_j-i_s}. \tag{81}$$

Substituting (81) into (80) we have (79).

The relation (24) is proved in an exactly similar manner.

(IV). a derivation of the formula (23):

The derivation is similar to (II). We have

$$(82)\qquad \begin{pmatrix} \Lambda \\ i_m^N \end{pmatrix} - \begin{pmatrix} i_l^r \\ i_m^N \end{pmatrix} = g^{(\Lambda_N)}(i_m^N)\frac{dz}{(z-i_m^N)s}\left[g_{\Lambda_N}^{(i_m^N)}(z) - \frac{g_{\Lambda_N}^{(i_m^N)}(i_m^N)}{g_{\Lambda_r}^{(i_l^r)}(i_m^N)}g_{\Lambda_r}^{(i_l^r)}(z)\right].$$

Since the polynomial in [ ] is devided by $z-i_m^N$ we can define the polynomial $G^{rl}(z)$ by

$$G^{rl}(z) = \frac{1}{z-i_m^N}\left[g_{\Lambda_N}^{(i_m^N)}(z) - \frac{g_{\Lambda_N}^{(i_m^N)}(i_m^N)}{g_{\Lambda_r}^{(i_l^r)}(i_m^N)}g_{\Lambda_r}^{(i_l^r)}(z)\right].$$

Then

$$(83)\qquad G^{rl}(i_k^{N-1}) = \frac{\prod_{s=1}^{m-1}(i_k^{N-1}-i_s^N)}{i_k^{N-1}-i_m^N}\left[1-\prod_{s=1}^{m-1}\frac{i_m^N-i_s^N}{i_k^{N-1}-i_s^N}\prod_{s\neq l}^{m}\frac{i_k^{N-1}-i_s^r}{i_m^N-i_s^r}\right].$$

Using (83), (77) and (76) we have, from (82),

$$(84)\qquad \begin{pmatrix} \Lambda \\ i_m^N \end{pmatrix} - \begin{pmatrix} i_l^r \\ i_m^N \end{pmatrix}$$

$$= \sum_{k=1}^{m-1}\prod_{j\in\mathcal{K},\, j\neq i_k^N}\frac{i_m^N-j}{i_k^{N-1}-j}\left[1-\prod_{s=1}^{m-1}\frac{i_m^N-i_s^N}{i_k^{N-1}-i_s^N}\prod_{s\neq l}^{m}\frac{i_k^{N-1}-i_s^r}{i_m^N-i_s^r}\right]\left[\begin{pmatrix} \Lambda \\ i_k^{N-1} \end{pmatrix} - \begin{pmatrix} i_m^{N-1} \\ i_k^{N-1} \end{pmatrix}\right].$$

Substituting (79) into (84) we have (23).

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