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Infinite Differentiability of Hermitian and Positive C-Semigroups and C-Cosine Functions†

By

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Abstract

Let C be a bounded linear operator which is not necessarily injective. The following statements are proved: (1) hermitian C-semigroups are infinitely differentiable in operator norm on $(0, \infty)$; (2) hermitian C-cosine functions are norm continuous at either non or all of points in $[0, \infty)$; (3) positive C-semigroups which dominate C are infinitely differentiable in opetator norm on $[0, \infty)$; (4) positive C-cosine functions are infinitely differentiable in operator norm on $[0, \infty)$.

§1. Introduction

This paper is concerned with the differentiability of hermitian and positive C-semigroups and C-cosine functions. Let X be a Banach space and let $C \in B(X)$, the space of all bounded linear operators on X.

A strongly continuous family $S(\cdot) \equiv \{S(t); t \geq 0\}$ in $B(X)$ is called a *C-semigroup* (see [2], [3], [9], [10], [11], [13], [16]) on X if it satisfies:

(1.1) $$S(0) = C \quad \text{and} \quad S(s)S(t) = S(s+t)C \quad \text{for} \quad s, t \geq 0.$$

A strongly continuous family $C(\cdot) \equiv \{C(t); t \geq 0\}$ in $B(X)$ is called a *C-cosine function* (see [6], [7], [9], [10], [12], [14]) on X if it satisfies:

(1.2) $$C(0) = C \quad \text{and} \quad 2C(t)C(s) = [C(t+s) + C(|t-s|)]C \quad \text{for} \quad s, t \geq 0.$$

These are natural generalizations of the classical C_0-semigroups [5] and

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cosine operator functions (to be called C_0-cosine functions in this paper) [4, 15], which correspond to the special case $C=I$.

A C-semigroup $S(\cdot)$ is said to be *hermitian* if for each $t \geq 0$, the operator $S(t)$ has numerical range $V(S(t)) := \{f(S(t)); f \in B(X)^*, \|f\| = f(I) = 1\}$ contained in the real line $\boldsymbol{R}$, or equivalently, if $\|\exp(isS(t))\| = 1$ for all $s \in \boldsymbol{R}$[1]. If furthermore $V(S(t)) \subset [0, \infty)$ for all $t \geq 0$, $S(\cdot)$ is said to be *positive* [11]. Hermitian and positive C-cosine functions are similarly defined [12]. It is well-known that the numerical range $V(T)$ of a hermitian operator T is equal to the closed convex hull of its spectrum, and its norm and spectral radius are equal (cf. [1], §26).

$S(\cdot)$ (or $C(\cdot)$) is said to be *nondegenerate* if $S(t)x = 0$ (or $C(t)x = 0$) for all $t > 0$ implies $x = 0$. In order $S(\cdot)$ (or $C(\cdot)$) to be nondegenerate it is necessary and sufficient that C is injective.

When a C-semigroup or C-cosine function is nondegenerate, its generator is well-defined. But, when C is not injective, there is no way to define a generator for it.

In [11] and [12], we have discussed some interesting properties of hermitian and positive C-semigroups and C-cosine functions. Those nondegenerate ones are especially investigated through examination of their generators. In particular, from Theorems 2.7 and 3.1 of [11] and Theorem 3.2 of [12] we can observe the following properties:

(1) Every nondegenerate hermitian C-semigroup is infinitely differentiable in operator norm on $(0, \infty)$;

(2) Every nondegenerate positive C-semigroup which dominates C is infinitely differentiable in operator norm on $[0, \infty)$;

(3) Every nondegenerate positive C-cosine function is infinitely differentiable in operator norm on $[0, \infty)$.

Are the above statements (1)–(3) still true if one deletes the word "nondegenerate" (i.e., if C is not injective)? In this paper, we shall answer the question affirmatively. They will be proved in Theorems 2.4, 2.5, and 3.3, respectively.

The following are some illustrative examples. The positive C_0-semigroup $S_1(\cdot)$, defined by $S_1(t)x := (e^{-nt}x_n)$ $(x = (x_n) \in l_2,\ t \geq 0)$, satisfies $0 \leq S_1(t) \leq I$ for all $t \geq 0$, and hence, by (1), is infinitely differentiable in operator norm on $(0, \infty)$. But it is not norm continuous at 0 because its generator A, defined as $Ax := (-nx_n)$ with natural domain, is unbounded. On the other hand, Theorem 2.5 asserts that the degenerate positive C-semigroup $S_2(\cdot)$, defined by $S_2(t)x := ((n-1)e^{nt-n^2}x_n)$ $(x = (x_n) \in l_2,\ t \geq 0)$, is infinitely differentiable in operator norm on $[0, \infty)$ because $S_2(t) \geq C \geq 0$. To illustrate Theorem 3.3, we see that the degenerate positive C-cosine function $C(\cdot)$, defined by $C(t)x := ((n-1)e^{-n^2} \cosh(nt)x_n)$ $(x = (x_n) \in l_2,\ t \geq 0)$, is infinitely differentiable in operator norm on $[0, \infty)$. Notice that, unlike the positive C-semigroup $S_1(\cdot)$, every positive C-cosine function $C(\cdot)$ has to satisfy $C(t) \geq C \geq 0$ (cf. [12, Lemma 3.1]).

The second question is: Does the conclusion of statement (1) also hold for hermitian C-cosine functions? The answer is "No". There are hermitian C_0-cosine functions which are not norm continuous at any $t > 0$. For example, the function $C(\cdot)$, defined by $C(t)x := (\cos(nt)x_n)$ $(x \in l_2)$ for $t \geq 0$, is a hermitian C_0-cosine function on l_2. Since its generator A, defined as $Ax := (-n^2 x_n)$ with natural domain, is unbounded, $C(\cdot)$ is not norm continuous at 0, which implies that it is not norm continuous at any $t \geq 0$. This is due to the fact that the norm continuity of a C_0-cosine function at any single point $t \geq 0$ implies its norm continuity on $[0, \infty)$. In fact, from the identities $C(2t+h) = 2C(t+h/2)^2 - I$ and $C(h) = 2C(t)C(t+h) - C(2t+h)$ we easily infer the norm continuity first at $2t$, and then at 0, and finally on the whole half line $[0, \infty)$ (cf. [8]).

Does a C-cosine function (with $C \neq I$) share the property that the norm continuity at a single point $t \geq 0$ implies the same on $[0, \infty)$? In Theorem 3.3, we shall prove this phenomenon for *hermitian* C-cosine functions. The answer of this question for general non-hermitian C-cosine functions is not clear yet.

§2. C-Semigroups

In this section, we discuss differentiability of hermitian and positive C-semigroups. We need the following elementary lemma [11, Lemma 2.1].

Lemma 2.1. *Let $f: [0, \infty) \to \boldsymbol{C}$ be a continuous function satisfying $f(t)f(s) = f(t+s)$ for $t, s \geq 0$. Then*

(i) *either $f \equiv 0$ or there is a complex number α such that $f(t) = e^{\alpha t}$ for all $t \geq 0$;*

(ii) *$f(0, \infty) \subset \boldsymbol{R}$ if and only if $f(0, \infty) \subset (0, \infty)$, if and only if $\alpha \in \boldsymbol{R}$;*

(iii) *$f(0, \infty) \subset [1, \infty)$ if and only if $\alpha \geq 0$; $f(0, \infty) = (0, 1)$ if and only if $\alpha < 0$.*

Proposition 2.2. *Let Ω be a nonempty set and let $B(\Omega)$ be the Banach algebra of all bounded complex-valued functions on Ω equipped with the sup-norm $\|f\|_\Omega := \sup\{|f(w)|;\ w \in \Omega\}$. Suppose $p, q: \Omega \to \boldsymbol{C}$ are two functions such that the* *function* $F:(0,\infty)\to B(\Omega)$, *defined by*

$$F(t)(w):=\exp(q(w)t)p(w), \quad w\in\Omega \text{ and } t>0,$$

is well-defined.

(i) *If* $q\le 0$, *then* F *is infinitely differentiable on* $(0,\infty)$ *and* $F^{(n)}(t)=e^{tq}q^n p$ *for* $t>0$, $n=1,2,\cdots$.

(ii) *If* $p\in B(\Omega)$ *and either* $q\in B(\Omega)$ *or* $q\ge 0$, *then* F, *with* $F(0):=p$, *is infinitely differentiable on* $[0,\infty)$, *and*

$$F(t)=\sum_{k=0}^{n-1}\frac{q^k}{k!}pt^k+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}e^{qt}q^n p\,ds,\ t\ge 0,\ n=1,2,\cdots.$$

(iii) *If* $q(\Omega)\subset\mathbb{R}$, *then* F *is infinitely differentiable on* $(0,\infty)$ *and* $F^{(n)}(t)=e^{tq}q^n p$ *for* $t>0$, $n=1,2,\cdots$.

Proof. Let $H(t)(w):=e^{tq(w)}q(w)p(w)$ for $w\in\Omega$ and $t\ge 0$.

(i) Let $s>0$ be arbitrary. Since $q\le 0$, we have for $\frac{1}{2}s<t<\frac{3}{2}s$ and $w\in\Omega$

$$\begin{aligned}
|H(t)(w)-H(s)(w)| &= |e^{tq(w)}q(w)p(w)-e^{sq(w)}q(w)p(w)| \\
&= \left|\int_s^t e^{rq(w)}q^2(w)p(w)dr\right| \\
&= \left|\int_{\frac{s}{2}}^{t-\frac{s}{2}} r^{-2}e^{rq(w)}[rq(w)]^2 e^{sq(w)/2}p(w)dr\right| \\
&\le \left|\int_{\frac{s}{2}}^{t-\frac{s}{2}} r^{-2}dr\right|\cdot\sup_{a\ge 0}(a^2e^{-a})\cdot\left\|F\left(\frac{s}{2}\right)\right\|_\Omega \\
&= \left|\left(\frac{s}{2}\right)^{-1}-\left(t-\frac{s}{2}\right)^{-1}\right|\cdot 4e^{-2}\cdot\left\|F\left(\frac{s}{2}\right)\right\|_\Omega .
\end{aligned}$$

So, we have

$$\|H(t)-H(s)\|_\Omega\le 8\,|s^{-1}-(2t-s)^{-1}|\cdot e^{-2}\cdot\left\|F\left(\frac{s}{2}\right)\right\|_\Omega\to 0\quad\text{as } t\to s.$$

Therefore $H(\cdot)$ is continuous at the point s. Since $s>0$ is arbitrary, $H(\cdot)$ is continuous on $(0,\infty)$.

Thus we have for every $t>0$ and for $h\ne 0$ with $t+h>0$

$$(2.1)\quad \|h^{-1}[F(t+h)-F(t)]-H(t)\|_\Omega=\left\|h^{-1}\int_t^{t+h}e^{rq}qpdr-H(t)\right\|_\Omega$$
$$=\left\|h^{-1}\int_t^{t+h}[H(r)-H(t)]dr\right\|_\Omega$$
$$\leq \sup\{\|H(r)-H(t)\|_\Omega;|r-t|\leq|h|,\ r>0\}$$

which tends to 0 as $h\to 0$. This proves that $F(\cdot)$ is continuously differentiable at t and $F'(t)=H(t)=e^{tq}qp$. Then, repeating the same argument inductively, we conclude that $F(\cdot)$ is infinitely differentiable on $(0,\infty)$ and $F^{(n)}(t)=e^{tq}q^np$ for $t>0$, $n=1,2,\cdots$.

(ii) Suppose $p\in B(\Omega)$ and let $F(0)=p$. If $q\in B(\Omega)$, then $F(t)=\exp(tq)p$ $=\Sigma_{n=0}^{\infty}\frac{(tq)^n}{n!}p, t\geq 0$, and hence $F(\cdot):[0,\infty)\to B(\Omega)$ is infinitely differentiable.

Next, we suppose that $q\geq 0$. Let $b>0$ be arbitrary. We have for $0\leq s,t\leq b$ and $w\in\Omega$

$$(2.2)\quad |H(t)(w)-H(s)(w)|=|e^{tq(w)}q(w)p(w)-e^{sq(w)}q(w)p(w)|$$
$$=\left|\int_s^t e^{rq(w)}q^2(w)p(w)dr\right|$$
$$\leq\left|\int_s^t 2\,|\,e^{(r+1)q(w)}p(w)\,|\,dr\right|,$$

so that $\|H(t)-H(s)\|\leq 2|t-s|\|F(b+1)\|_\Omega\to 0$ as $|t-s|\to 0$ $(0\leq t,s\leq b)$. Therefore $H(\cdot)$ is continuous on $[0,b]$. Since $b>0$ is arbitrary, $H(\cdot)$ is continuous on $[0,\infty)$. Using (2.1) we see that $F(\cdot)$ is continuously differentiable on $[0,\infty)$. Then, repeating the same argument inductively, we conclude that $F(\cdot)$ is infinitely differentiable on $[0,\infty)$ and $F^{(n)}(t)=e^{tq}q^np$ for all $t\geq 0$, $n=1,2,\cdots$.

Let $\Omega_-:=\{w\in\Omega;q(w)\leq 0\}$ and $\Omega_+:=\Omega\backslash\Omega_-$. Then (iii) follows by applying (i) and (ii) on $B(\Omega_-)$ and $B(\Omega_+)$, respectively.

Lemma 2.3. *Let* **A** *be a commutative unital Banach algebra and let* $f:[a,b]\to\mathbf{A}$ *be such that* $f(t)$ *is hermitian for all* $t\in[a,b]$. *Define* $F:[a,b]\to C(m)$ *by* $F(t)(\phi):=\phi(f(t))$ *for* $t\geq 0$ *and* $\phi\in m$, *where* m *is the state (or maximal ideal) space of* **A**. *That is,* $F(t)=\widehat{f(t)}$, *the Gelfand transform of* $f(t)$. *Then*

(i) *f is continuous if and only if F is continuous.*

(ii) *f is continuously differentiable if and only if F is continuously differentiable. In this case, each $f'(t), t\in[a,b]$, is also hermitian.*

Proof. Since $f(t)$ is hermitian, we have for $t, s \in [a,b]$

$$\begin{aligned}\|F(t)-F(s)\|_m &= \sup_{\phi\in m}|\phi(f(t))-\phi(f(s))| \\ &= \sup_{\phi\in m}|\phi(f(t)-(f(s))| \\ &= \|f(t)-f(s)\|.\end{aligned}$$

This proves (i).

(ii) If f is continuously differentiable, then $\hat{f'}$ is continuous by part (i) and we have for $t\in[a,b]$ and $h\neq 0$ with $t+h\in[a,b]$

$$\begin{aligned}&\sup_{\phi\in m}|h^{-1}[F(t+h)-F(t)](\phi)-\widehat{[f'(t)]}(\phi)| \\ &= \sup_{\phi\in m}\left|h^{-1}\phi\left(\int_t^{t+h} f'(r)dr\right)-\phi(f'(t))\right| \\ &= \left\|h^{-1}\int_t^{t+h}(f'(r)-f'(t))dr\right\|_m \to 0\end{aligned}$$

uniformly in t on $[a,b]$ as $h\to 0$. Therefore F is continuously differentiable.

Conversely, if F is continuously differentiable, then f is continuous by (i) and we have for $|k|\geq|h|>0$

$$\begin{aligned}&\|h^{-1}[f(t+h)-f(t)]-k^{-1}[f(t+k)-f(t)]\| \\ &= \sup_{\phi\in m}|\phi\{h^{-1}[f(t+h)-f(t)]-k^{-1}[f(t+k)-f(t)]\}| \\ &\leq \sup_{\phi\in m}|h^{-1}[F(t+h)-F(t)](\phi)-[F'(t)](\phi)| \\ &\quad + \sup_{\phi\in m}|k^{-1}[F(t+k)-F(t)](\phi)-[F'(t)](\phi)| \\ &= \left\|h^{-1}\int_t^{t+h}(F'(s)-F'(t))ds\right\|_m + \left\|k^{-1}\int_t^{t+k}(F'(s)-F'(t))ds\right\|_m \\ &\leq 2\sup\{\|F'(s)-F'(t)\|_m; |s-t|\leq|k|\}\to 0\end{aligned}$$

uniformly in t on $[a,b]$ as $|k|\to 0$. Therefore $\{h^{-1}[f(\cdot+h)-f(\cdot)]\}_{|h|\to 0}$ is a

Cauchy net in $C([a,b]; \mathbf{A})$, which implies that f is differentiable and, as a uniform limit of continuous functions, the derivative f' is continuous on $[a,b]$. Since the space of all hermitian elements of $\mathbf{A}$ is closed, $f'(t)$ is hermitian for all $t \in [a,b]$.

We are ready to prove the following main results of this section.

Theorem 2.4. *If $S(\cdot)$ is a hermitian C-semigroup, then $S(\cdot)$ is infinitely differentiable in operator norm on $(0,\infty)$ and $S^{(n)}(t)$ is hermitian for all $t>0$ and $n \geq 1$.*

Proof. Let $\mathbf{A}$ be the unital Banach algebra generated by $S(\cdot)$ and C and let $m_{S(\cdot)}$ be its state space. We set $\Omega = m'_{S(\cdot)} := \{\phi \in m_{S(\cdot)}; \phi(S(\cdot)) \neq 0\}$ and define

$$F(t)(\phi) := \phi(S(t)), t \geq 0 \text{ and } \phi \in \Omega.$$

Since $S(\cdot)$ is hermitian, C is hermitian and each $F(t)$ is real-valued. For each $\phi \in \Omega$, applying Lemma 2.1 to the function $f_\phi(t) := \phi(S(t))/\phi(C)$, $t \in [0,\infty)$, shows that, there is a corresponding number $\alpha_\phi \in \boldsymbol{R}$ such that $F(t)(\phi) = \exp(\alpha_\phi t)\phi(C)$ for all $t \geq 0$. Then application of Proposition 2.2 (iii) and Lemma 2.3 yields the infinite differentiability of $S(\cdot)$ on $(0,\infty)$. Since the space of all hermitian operators in $B(X)$ is closed, $S^{(n)}(t)$ is hermitian for all $t>0$ and $n \geq 1$.

Similarly, we can deduce from Proposition 2.2(ii) and Lemma 2.3 the next theorem.

Theorem 2.5. *If $S(\cdot)$ is a C-semigroup such that $S(t) \geq C \geq 0$, then $S(\cdot)$ is infinitely differentiable in operator norm on $[0,\infty)$ and $S^{(n)}(t)$ is positive for all $t \geq 0$ and $n \geq 1$.*

§3. C-Cosine Functions

This section is concerned with differentiability of hermitian and positive C-cosine functions.

Lemma 3.1. ([12, Lemma 2.1]) *Let $g: [0,\infty) \to \boldsymbol{C}$ be a continuous function satisfying $2g(t)g(s) = g(t+s) + g(|t-s|)$ for $t, s \geq 0$. Then*

(i) *either $g \equiv 0$ or there is an $\alpha \in \boldsymbol{C}$ such that $g(t) = \cosh(\alpha t)$ for all $t \geq 0$;*

(ii) $g(0,\infty)\subset \boldsymbol{R}$ *if and only if* $g(0,\infty)\subset[-1,\infty)$, *if and only if* $\alpha\in \boldsymbol{R}\cup i\boldsymbol{R}$;
(iii) $g(0,\infty)\subset[1,\infty)$ *if and only if* $\alpha\in\boldsymbol{R}$; $g(0,\infty)=[-1,1]$ *if and only if* $\alpha\in i\boldsymbol{R}\backslash\{0\}$.

Proposition 3.2. *Let* Ω *be a nonempty set and let* $p,q:\Omega\to\boldsymbol{C}$ *be two functions such that the function* $F:[0,\infty)\to B(\Omega)$, *where* $B(\Omega)$ *is the Banach algebra of all bounded complex-valued functions on* Ω *equipped with the sup-norm* $\|f\|_\Omega:=\sup\{|f(w)|\,|\,w\in\Omega\}$, *defined by*

$$F(t)(w):=\cosh(q(w)t)p(w)(=\cos(iq(w)t)p(w)),\quad w\in\Omega \text{ and } t\geq 0,$$

is well-defined (*in particular*, $p\in B(\Omega)$). *Then*

(i) *If* $q(\Omega)\subset i\boldsymbol{R}$ *and* $F(\cdot)$ *is continuous at some* $t_0\geq 0$, *then* F *is continuous on* $[0,\infty)$.
(ii) *If* $q\in B(\Omega)$, *then* $F(\cdot)$ *is infinitely differentiable on* $[0,\infty)$.
(iii) *If* $q(\Omega)\subset\boldsymbol{R}$ *and* $p\geq 0$ (*resp.* $p\leq 0$), *then* $F(\cdot)$ *is infinitely differentiable on* $[0,\infty)$ *and*

$$F(t)=\sum_{k=0}^{n-1}\frac{t^{2k}}{(2k)!}q^{2k}p+\int_0^t\frac{(t-s)^{2n-1}}{(2n-1)!}\cosh(qs)q^{2n}p\,ds,\quad t\geq 0 \text{ and } n=1,2,\cdots.$$

Furthermore, $F^{(n)}(t)\geq 0$ (*resp.* ≤ 0) *on* Ω *for all* $n=0,1,\cdots$ *and* $t\geq 0$.

Proof. (i) Suppose $q(\Omega)\subset i\boldsymbol{R}$. Fix an $\varepsilon_1>0$ and let $\Omega_0:=\{w\in\Omega;|p(w)|\geq\varepsilon_1\}$. If F is continuous at a point $t_0\geq 0$, then for any $\varepsilon_2>0$ there exists $\delta_1>0$ such that

$$t\geq 0,\ |t-t_0|<\delta_1 \text{ implies } \|F(t)-F(t_0)\|_\Omega\leq\varepsilon_1\varepsilon_2.$$

Then we have for every $t\geq 0$, $|t-t_0|<\delta_1$, and all $w\in\Omega_0$

$$\begin{aligned}|\cos(iq(w)t)-\cos(iq(w)t_0)|&=\frac{1}{|p(w)|}|\cosh(q(w)t)p(w)-\cosh(q(w)t_0)p(w)|\\&\leq\frac{1}{\varepsilon_1}\|F(t)-F(t_0)\|_\Omega\leq\frac{1}{\varepsilon_1}\cdot\varepsilon_2\cdot\varepsilon_1=\varepsilon_2.\end{aligned}$$

So, the $B(\Omega_0)$-valued cosine function $G:[0,\infty)\to B(\Omega_0)$ defined by

$$G(t)(w):=\cos(iq(w)t),\quad w\in\Omega_0 \text{ and } t\geq 0,$$

is continuous at t_0, and hence, as remarked in Section 1, it must be continuous on $[0, \infty)$.

Now, if $s \geq 0$, then there is $\delta_2 > 0$ such that

$$t \geq 0, \ |t-s| < \delta_2 \text{ implies } \|G(t) - G(s)\|_{\Omega_0} \leq \frac{\varepsilon_1}{\|p\|_\Omega + 1}.$$

So, we have for $t \geq 0$, $|t-s| < \delta_2$

$$\begin{aligned} |F(t)(w) - F(s)(w)| &= |\cos(iq(w)t)p(w) - \cos(iq(w)s)p(w)| \\ &\leq |G(t)(w) - G(s)(w)| \cdot \|p\|_\Omega \\ &\leq \frac{\varepsilon_1}{\|p\|_\Omega + 1} \cdot \|p\|_\Omega < \varepsilon_1 \end{aligned}$$

for all $w \in \Omega_0$, and

$$\begin{aligned} |F(t)(w) - F(s)(w)| &= |\cos(iq(w)t)p(w) - \cos(iq(w)s)p(w)| \\ &\leq 2|p(w)| < 2\varepsilon_1 \end{aligned}$$

for all $w \in \Omega \backslash \Omega_0$. Therefore F is continuous at s.

(ii) Suppose $q \in B(\Omega)$. Then $F(t) = \cosh(tq)p = \Sigma_{n=0}^{\infty} \frac{(tq)^{2n}}{(2n)!} p$, $t \geq 0$, and hence $F(\cdot) : [0, \infty) \to B(\Omega)$ is infinitely differentiable.

(iii) Suppose $q(\Omega) \subset \boldsymbol{R}$ and $p \geq 0$. Then $F(t) = \cosh(qt)p \geq 0$ for $t \geq 0$. Define

$$E(t)(w) := \int_0^t F(r)(w)dr \text{ for } t \geq 0 \text{ and } w \in \Omega.$$

We have for $0 \leq s, t \leq h$

$$\begin{aligned} \|E(t) - E(s)\|_\Omega &= \left\| \int_s^t \cosh(qr)p dr \right\|_\Omega \\ &\leq \|\cosh(hq)p\|_\Omega \cdot |t-s| \\ &= \|F(h)\|_\Omega |t-s|. \end{aligned}$$

This implies that $E(\cdot)$ is a $B(\Omega)$-valued continuous function on $[0, \infty)$. It is clear that $E(t)(\Omega) \subset [0, \infty)$ and hence

$$V(E(t)) = \overline{co}(\sigma(E(t))) = \overline{co}(E(t)(\Omega)) \subset [0, \infty)$$

for all $t \geq 0$, that is, each $E(t)$ is a positive element of $B(\Omega)$. Therefore the function $L_{E(\cdot)}$ of left multiplication operators on $B(\Omega)$ is a positive integrated C-cosine function on $B(\Omega)$, with $C = L_p$. Since every positive integrated C-cosine function is continuously differentiable in operator norm (see Theorem 2.5 of [12]), $\frac{d}{ds}L_{E(s)}$ is norm continuous in $B(B(\Omega))$ on $[0, \infty)$. Since the mapping $E(s) \to L_{E(s)}$ is an isometry from $B(\Omega)$ into $B(B(\Omega))$ and $E'(\cdot) = F(\cdot)$, it follows that $F(\cdot)$ is continuous on $[0, \infty)$.

Next, let $G_n(t) := \cosh(qt)q^{2n}p, t \geq 0$. We show the following two inequalities for $n \geq 0$ and $t \geq 0$:

$$\|q^n p\|_\Omega \leq 2M_n \|F(1)\|_\Omega,$$
$$\|G_n(t)\|_\Omega \leq [2\|F(1)\|_\Omega \|F(2t)\|_\Omega M_{4n}]^{1/2},$$

where $M_n := \sup\{a^n e^{-a}; a \geq 0\} = n^n e^{-n}$. Indeed, we have for $n = 0, 1, \cdots$ and $w \in \Omega$

$$\begin{aligned} |q^n(w)p(w)| &\leq |q(w)|^n e^{-|q(w)|} \cdot e^{|q(w)|} p(w)| \\ &\leq M_n \cdot 2\cosh(|q(w)|)p(w) \\ &\leq 2M_n \|F(1)\|_\Omega \end{aligned}$$

and

$$\begin{aligned} |G_n(t)(w)| &\leq |q^{4n}(w)p(w)|^{1/2} \cdot |\cosh^2(q(w)t)p(w)|^{1/2} \\ &\leq (2M_{4n}\|F(1)\|_\Omega)^{1/2} \cdot \left|\frac{1+\cosh(2q(w)t)}{2} \cdot p(w)\right|^{1/2} \\ &\leq (2M_{4n}\|F(1)\|_\Omega)^{1/2} \|F(2t)\|_\Omega^{1/2} \\ &= [2\|F(1)\|_\Omega \|F(2t)\|_\Omega M_{4n}]^{1/2}. \end{aligned}$$

Now, since for each $n = 1, 2, \cdots$, $q^{2n}p \in B(\Omega)$, $G_n(t) \in B(\Omega)$ for all $t \geq 0$, and $q^{2n}p$ is positive, we can replace p in $F(\cdot)$ by $q^{2n}p$ and hence assert that G_n is continuous on $[0, \infty)$ for all $n = 1, 2, \cdots$. Since

$$F(t) = \sum_{k=0}^{n-1} \frac{t^{2k}}{(2k)!} q^{2k} p + \int_0^t \frac{(t-s)^{2n-1}}{(2n-1)!} G_n(s) ds$$

for $t \geq 0$, $n = 1, 2, \cdots$, it follows that F is infinitely differentiable on $[0, \infty)$. Since $q(\Omega) \subset \boldsymbol{R}$ and $p \geq 0$, clearly $F^{(n)}(t) \geq 0$ for all $t \geq 0$ and $n = 0, 1, \cdots$.

When $q(\Omega) \subset \boldsymbol{R}$ and $p \leq 0$, we can apply the above result to $-F(\cdot)$. This completes the proof.

We are ready to prove the following main results of this section.

Theorem 3.3. *Let $C(\cdot)$ be a hermitian C-cosine function. Then either $C(\cdot)$ is not norm continuous at any point in $[0, \infty)$, or $C(\cdot)$ is norm continuous on $[0, \infty)$.*

Proof. Let **A** be the unital Banach algebra generated by $C(\cdot)$ and C and let $m_{C(\cdot)}$ be its state space. We set $\Omega := m'_{C(\cdot)}$ and define

$$F(t)(\phi) := \phi(C(t)), \ t \geq 0 \text{ and } \phi \in \Omega.$$

Since $C(\cdot)$ is hermitian, C is hermitian and each $F(t)$ is real-valued. For each $\phi \in \Omega$, applying Lemma 3.1 to the function $g_\phi(t) := \phi(C(t))/\phi(C)$, $t \in [0, \infty)$ shows that there is a corresponding number $\alpha_\phi \in \boldsymbol{R} \cup i\boldsymbol{R}$ such that $F(t)(\phi) = \cosh(\alpha_\phi t)\phi(C)$ for all $t \geq 0$. Therefore we can decompose $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where $\Omega_1 := \{\phi \in \Omega; \alpha_\phi \in i\boldsymbol{R}\}$, $\Omega_2 := \{\phi \in \Omega; \alpha_\phi \in \boldsymbol{R}$ and $\phi(C) \geq 0\}$, and $\Omega_3 := \{\phi \in \Omega; \alpha_\phi \in \boldsymbol{R}$ and $\phi(C) < 0\}$.

Suppose that $C(\cdot)$ is norm continuous at some point $t_0 \geq 0$, then $F(\cdot)|_{\Omega_1}$ is continuous at t_0. Hence $F(\cdot)|_{\Omega_1}$ is continuous on $[0, \infty)$ by Proposition 3.2(i) with Ω replaced by Ω_1. On the other hand, if we replace Ω by Ω_2 or Ω_3, $F(\cdot)|_\Omega$ is always continuous on $[0, \infty)$ by Proposition 3.2(iii). Combining these arguments, we have that $F(\cdot)$ is $\|\cdot\|_\Omega$-continuous on $[0, \infty)$. Hence, by Lemma 2.3(i), $C(\cdot)$ must be norm continuous on $[0, \infty)$.

Theorem 3.4. *If $C(\cdot)$ is a positive C-cosine function, then $C(\cdot)$ is infinitely differentiable in operator norm on $[0, \infty)$ and $C^{(n)}(t)$ is positive for all $n \geq 1$.*

Proof. It follows from Lemma 3.1(iii) that $[F(t)](\phi) = \cosh(\alpha_\phi t)\phi(C)$ for all $t \geq 0$ and $\phi \in m'_{C(\cdot)}$ and some $\alpha_\phi \in \boldsymbol{R}$. Thus we obtain from Proposition 3.2(iii) that $F(\cdot)$ is infinitely differentiable and $G^{(n)}(t) \geq 0$ for all $t \geq 0$ and $n = 0, 1, \cdots$. Therefore, by Lemma 2.3, $C(\cdot)$ is infinitely differentiable and $C^{(n)}(t) \geq 0$ for all $t \geq 0$ and $n = 0, 1, \cdots$. This completes the proof.

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