Publ. RIMS, Kyoto Univ.
34 (1998), 591–599

Atomic Positive Linear Maps in Matrix Algebras

By

Kil-Chan HA*

Abstract

We show that all of the known generalizations of the Choi maps are atomic maps.

§1. Introduction

Let M_n be the C^*-algebra of all $n \times n$ matrices over the complex field and $\boldsymbol{P}_s[M_n]$ (respectively, $\boldsymbol{P}^s[M_n]$) the convex cone of all s-positive (respectively, s-copositive) linear maps between M_n. One of the basic problems about the structures of the positive cone $\boldsymbol{P}_1[M_n]$ is if this cone can be decomposed as the algebraic sum of some simpler subcones. It is well known [22, 25] that every positive linear map between M_2 is *decomposable*, that is, it can be written as the sum of a completely positive linear map and a completely copositive linear map. But, this is not the case for higher dimensional matrix algebras.

The first example of an indecomposable positive linear map in M_3 was given by Choi [5]. Choi and Lam [7] also gave an example of an indecomposable extremal positive linear map in M_3 (see also [6]). Another examples of indecomposable extremal positive linear maps are found in [9, 16, 21]. These maps are neither 2-positive nor 2-copositive, and so they become *atomic* maps in the sense in [24], that is, they can not be decomposed into the sums of 2-positive linear maps and 2-copositive linear maps. Several authors [1, 2, 10, 15, 17, 24] considered indecomposable positive linear maps as extensions of the Choi's example. The first two examples [2, 10] are generalizations of the Choi's map [5, 6] in M_3 and the other maps $\tau_{n,k}$ in [1, 15, 17, 24] (see Section 2 for the definition) are extensions of the Choi map [7] in

Communicated by T. Miwa, June 19, 1998.
1991 Math. Subject Classifications: 46L05, 15A30.
partially supported by GARC-KOSEF
* Department of Mathematics, Seoul National University, Seoul 151–742, KOREA.

higher dimensional matrix algebra M_n. Among them, examples in [10] and the map $\tau_{n,1}$ $(n\geq 3)$ [17] are known to be atomic maps. But the atomic properties of the other indecomposable maps are not determined. Even decomposabilities of the maps $\tau_{n,k}$ are remained open except for some special cases [17]. The usual method to show the atomic property of a positive linear map depends on a tedious matrix manipulation.

The purpose of this note is to show that all of the above mentioned examples are atomic maps, using the recent result of Eom and Kye [8]. Generalizing the Woronowicz's argument [25], they considered the duality between the space $M_n\otimes M_m(=M_{nm})$ of all $nm\times nm$ matrices over the complex field and the space $\mathscr{L}(M_m, M_n)$ of all linear maps from M_m into M_n, which is given by

$$(1.1)\qquad \langle A,\phi\rangle=\mathrm{Tr}\left[\sum_{i,j=1}^{m}(\phi(e_{i,j})\otimes e_{i,j})A^{t}\right]=\sum_{i,j=1}^{m}\langle\phi(e_{i,j}),a_{i,j}\rangle,$$

for $A=\Sigma_{i,j=1}^{m}a_{i,j}\otimes e_{i,j}\in M_n\otimes M_m$ and a linear map $\phi:M_m\to M_n$, where $\{e_{i,j}\}$ is the matrix units of M_m and the bilinear form in the right-side is given by $\langle X,Y\rangle=\mathrm{Tr}(YX^t)$ for $X,Y\in M_n$ with the usual trace Tr. For the convenience of readers, we briefly explain the results in [8]. For a matrix $A=\Sigma_{i,j=1}^{m}x_{i,j}\otimes e_{i,j}\in M_n\otimes M_m$, we denote by A^{T} the *block-transpose* $\Sigma_{ij=1}^{m}x_{j,i}\otimes e_{i,j}$ of A. We say that a vector $z=\Sigma_{i=1}^{m}z_i\otimes e_i\in \mathbf{C}^n\otimes\mathbf{C}^m$ is an *s-simple* if the linear span of $\{z_1,\cdots,z_m\}$ has the dimension $\leq s$, where $\{e_1,\cdots,e_m\}$ is the usual orthonormal basis of $\mathbf{C}^m$. Let $V_s[M_n]$ (respectively $V^s[M_n]$) denote the convex cone in $M_n\otimes M_n$ generated by $zz^*\in M_n\otimes M_n$ (respectively $(zz^*)^{\mathrm{T}}\in M_n\otimes M_n$) with all s-simple vectors $z\in\mathbf{C}^n\otimes\mathbf{C}^n$. It turns out that $V_s[M_n]$ (respectively $V^s[M_n]$) is the dual cone of $\mathbf{P}_s[M_n]$ (respectively $\mathbf{P}^s[M_n]$) with respect to the pairing (1.1). With this machinery, the maximal faces of $\mathbf{P}_s[M_n]$ and $\mathbf{P}^s[M_n]$ are characterized in terms of s-simple vectors (see also [12, 13, 14]). Another consequence is a characterization of the cone $\mathbf{P}_s[M_n]+\mathbf{P}^t[M_n]$: For a linear map $\phi:M_n\to M_n$, the map ϕ is the sum of an s-positive linear map and a t-copositive linear map if and only if $\langle A,\phi\rangle\geq 0$ for each $A\in V_s[M_n]\cap V^t[M_n]$. This result provides us a useful method to examine the atomic property for the generalizations of the Choi maps mentioned before.

Throughout this note, every vector in the space $\mathbf{C}^r$ will be considered as an $r\times 1$ matrix. The usual orthonormal basis of $\mathbf{C}^r$ and matrix units of M_r will be denoted by $\{e_i:i=1,\cdots,r\}$ and $\{e_{i,j}:i,j=1,\cdots,r\}$ respectively, regardless of the dimension r.

§2. The Maps $\tau_{n,k}$

Let ε be the canonical projection of M_n to the diagonal part and S be the rotation matrix in M_n which sends e_i to e_{i+1} (mod n) for $i=1,\cdots,n$. The map $\tau_{n,k}:M_n\to M_n$ is defined by

$$\tau_{n,k}(X)=(n-k)\varepsilon(X)+\sum_{i=1}^{k}\varepsilon(S^iXS^{*i})-X, \qquad X\in M_n,$$

for $k=1,2,\cdots,n-1$. The map $\tau_{n,0}:M_n\to M_n$ is also defined by

$$\tau_{n,0}(X)=n\varepsilon(X)-X, \qquad X\in M_n.$$

It is easy to see that $\tau_{n,0}$ is completely positive and $\tau_{n,n-1}$ is completely copositive. The positivity of $\tau_{n,k}$ is equivalent [1] to a certain cyclic inequality, which was shown by Yamagami [26]. The map $\tau_{3,1}$ is the Choi and Lam's example mentioned in the introduction. For $n\geq 4$, Osaka showed that $\tau_{n,n-2}$ is not the sum of a 3-positive linear map and a 3-copositive linear map [15], and $\tau_{n,1}$ is atomic [17]. In this section, we show that the map $\tau_{n,k}$ is an atomic map for each $n\geq 3$ and $k=1,2,\cdots,n-2$.

For each fixed natural number $n=1,2,\cdots$, let $\{\omega_i:1\leq i\leq 3^n\}$ be the 3^n-th roots of unity. Then we have

$$\sum_{i=1}^{3^n}\omega_i^k=0, \qquad 1\leq k\leq 3^n-1. \tag{2.1}$$

For each $k=1,2,\cdots,n$, we define $m_k\in \boldsymbol{Z}$ by $m_k=\frac{3}{2}(3^{k-1}-1)$. Then it is easy to see the following:

$$m_k-m_l=m_i-m_j \quad \text{if and only if } (k,i)=(l,j) \text{ or } (k,l)=(i,j). \tag{2.2}$$

For any $\gamma>0$, we define $a_{ik},\ c_r\in \boldsymbol{C}^n$ by

$$a_{i1}=\sum_{j=1}^{n}\omega_i^{m_j}e_j, \qquad 1\leq i\leq 3^n,$$

$$a_{ik}=\omega_i^{-m_k}a_{i1}, \qquad 1\leq i\leq 3^n, \quad 2\leq k\leq n,$$

$$c_1=e_1+\gamma e_2+\sum_{k=3}^{n-1}e_k+\frac{1}{\gamma}e_n,$$

$$c_r=S^{r-1}c_1, \qquad 2\leq r\leq n.$$

For each $r=1,2,\cdots,n$, $i=1,2,\cdots,3^n$ and $j=1,2,\cdots,n$, we define $b_{rij}\in C^n$ by

$$b_{rij}=\begin{cases} a_{ij}, & j\neq r \\ c_j\circ a_{ij}, & j=r, \end{cases}$$

where $\circ$ is the Schur product of $n\times 1$ matrices c_j and a_{ij}. We also define $z_{ri}\in C^n\otimes C^n$ and $A_r\in M_n\otimes M_n$ by

$$z_{ri}=\sum_{j=1}^{n} b_{rij}\otimes e_j, \qquad 1\leq r\leq n,\ 1\leq i\leq 3^n,$$

$$A_r=\frac{1}{3^n}\sum_{i=1}^{3^n} z_{ri}z_{ri}^*, \qquad 1\leq r\leq n.$$

Then we see that each z_{ri} is a 2-simple vector and so $A_r\in V_2[M_n]$. If we write A_r by $A_r=\Sigma_{p,q=1}^n(A_r)_{p,q}\otimes e_{p,q}\in M_n\otimes M_n$, then it is easy to see that

$$(2.3)\qquad (A_r)_{p,q}=\begin{cases} e_{p,q}, & p\neq q, \\ \Sigma_{i=1}^n e_{i,i}, & p=q, p\neq r, \\ S^{r-1}(e_{1,1}+\gamma^2 e_{2,2}+\Sigma_{i=3}^{n-1}e_{i,i}+\frac{1}{\gamma^2}e_{n,n})S^{*r-1}, & p=q=r, \end{cases}$$

by (2.1) and (2.2).

Now we define $A\in V_2[M_n]$ by $A=\frac{1}{n}\Sigma_{r=1}^n A_r=\Sigma_{p,q=1}^n A_{p,q}\otimes e_{p,q}\in M_n\otimes M_n$. Then, by (2.3), we have

$$A_{p,q}=\begin{cases} e_{p,q}, & p\neq q, \\ e_{1,1}+\frac{1}{n}(\gamma^2+(n-1))e_{2,2}+\Sigma_{i=3}^{n-1}e_{i,i}+\frac{1}{n}\left(\frac{1}{\gamma^2}+(n-1)\right)e_{n,n}, & p=q=1, \\ S^{p-1}A_{1,1}S^{*p-1}, & p=q, p\neq 1. \end{cases}$$

In order to show that $A\in V^2[M_n]$, define u_i, v_i, α_i and $\beta_{ij}\in C^n$ by

$$u_i=\frac{\gamma}{\sqrt{n}}e_{i+1}\otimes e_i+\frac{1}{\sqrt{n\gamma}}e_i\otimes e_{i+1}, \qquad 1\leq i\leq n,$$

$$v_i=\sqrt{\frac{n-1}{n}}e_{i+1}\otimes e_i+\sqrt{\frac{n-1}{n}}e_i\otimes e_{i+1}, \qquad 1\leq i\leq n,$$

$$\alpha_i = e_i \otimes e_i \qquad 1 \le i \le n,$$

$$\beta_{1j} = e_j \otimes e_1 + e_1 \otimes e_j, \qquad 3 \le j \le n-1,$$

$$\beta_{ij} = e_j \otimes e_i + e_i \otimes e_j, \qquad 2 \le i \le (n-2), (i+2) \le j \le n,$$

where suffixes are understood in $\mathrm{mod}\, n$. A direct calculation shows

$$A^{\mathrm{T}} = \sum_{i=1}^{n} (u_i u_i^* + v_i v_i^* + \alpha_i \alpha_i^*) + \sum_{j=3}^{n-1} \beta_{1j} \beta_{1j}^* + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \beta_{ij} \beta_{ij}^*,$$

and so, we have $A \in V^2[M_n]$. Furthermore, we also have $\langle A, \tau_{n,k} \rangle = \gamma^2 - 1$, for each $n = 3, 4, \cdots$ and $k = 1, 2, \cdots, n-2$, and so we see that $\langle A, \tau_{n,k} \rangle < 0$ for $0 < \gamma < 1$. By the result in [8] mentioned in the introduction, we conclude the following:

Theorem 2.1. *For $n \ge 3$ and $1 \le k \le n-2$, the map $\tau_{n,k} : M_n \to M_n$ is an atomic positive linear map.*

§3. The Generalized Choi Map

The other generalization of the Choi map is given in [2]. For nonnegative real numbers a, b, and c, the map $\Phi[a,b,c] : M_3 \to M_3$ is defined by

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{1,1} + bx_{2,2} + cx_{3,3} & 0 & 0 \\ 0 & ax_{2,2} + bx_{3,3} + cx_{1,1} & 0 \\ 0 & 0 & ax_{3,3} + bx_{1,1} + cx_{2,2} \end{pmatrix} - X$$

for each $X = (x_{i,j}) \in M_3$. The map $\Phi[2,0,\mu]$ with $\mu \ge 1$ is the example of an indecomposable positive linear map given by Choi [6], and $\Phi[2,0,2]$ and $\Phi[2,0,1]$ are the examples given in [5] and [7] respectively. Choi and Lam [7] showed that $\Phi[2,0,1]$ is an extremal positive linear map using the theory of biquadratic forms. Later, Tanahashi and Tomiyama [24] showed the atomic property of the map $\Phi[2,0,1]$ which is same as the map $\tau_{3,1}$. It was shown [2] that the map $\Phi[a,b,c]$ is an indecomposable positive linear map if and only if the following conditions are satisfied:

$$\text{(i)}\quad 1\le a<3,$$

$$\text{(ii)}\quad a+b+c\ge 3,$$

$$(3.1)\qquad \text{(iii)}\quad \begin{cases} (2-a)^2\le bc<\left(\dfrac{3-a}{2}\right)^2 & \text{if } 1\le a\le 2 \\[2ex] 0\le bc<\left(\dfrac{3-a}{2}\right)^2 & \text{if } 2\le a<3. \end{cases}$$

In this section we show that these conditions imply that $\Phi[a,b,c]$ is an atom.

Let $\{\omega_i : i=1,2,3\}$ be the cube roots of unity and s be any positive real number. Define $a_{ik}\in \boldsymbol{C}^3$, $z_i, u_i\in \boldsymbol{C}^3\otimes\boldsymbol{C}^3$ and $B\in V_2[M_3]$ by

$$a_{i1}=(\omega_i,0,0)^{\mathrm{t}},\qquad a_{i2}=(0,\omega_i,\frac{\omega_i^2}{s})^{\mathrm{t}},\qquad a_{i3}=(\frac{s}{\omega_i})a_{i2},\qquad i=1,2,3,$$

$$z_i=\sum_{k=1}^{3} a_{ik}\otimes e_k,\qquad i=1,2,3,$$

$$u_1=e_2\otimes e_1,\qquad u_2=e_1\otimes e_3,\qquad u_3=e_3\otimes e_1,\qquad u_4=e_1\otimes e_2,$$

$$B=\frac{1}{3}\left(\sum_{i=1}^{3} z_i z_i^*\right)+\frac{1}{s^2}\left(\sum_{i=1}^{2} u_i u_i^*\right)+s^2\left(\sum_{i=3}^{4} u_i u_i^*\right).$$

It is clear that $B\in V_2[M_3]$. To show that $B\in V^2[M_3]$, we define z_i and $u_i\in \boldsymbol{C}^3\otimes\boldsymbol{C}^3$ by

$$z_i=\frac{1}{s}(e_{i+1}\otimes e_i)+s(e_i\otimes e_{i+1}),\qquad i=1,2,3,$$

$$u_i=e_i\otimes e_i,\qquad i=1,2,3,$$

where suffixes are understood in mod 3. A direct calculation show that

$$B^{\mathrm{T}}=\sum_{i=1}^{3}(z_i z_i^*+u_i u_i^*)\in V_2[M_3].$$

It is also easy to calculate

$$(3.2)\qquad \langle B,\Phi[a,b,c]\rangle=3((a-3)+\frac{c}{s^2}+s^2 b).$$

We proceed to show the conditions in (3.1) imply that the pairing in (3.2) is negative. We first consider the case $bc=0$. If $b=0$ then the pairing (3.2) becomes negative for $s>\sqrt{\frac{c}{3-a}}$. If $c=0$ then (3.2) is negative for $0<s<\sqrt{\frac{3-a}{b}}$. When $bc\neq 0$, we take $s=\left(\frac{c}{b}\right)^{1/4}$, then the pairing (3.2) is reduced to

$$\langle B, \Phi[a,b,c]\rangle = 3((a-3)+2\sqrt{bc}),$$

which is also negative since $\sqrt{bc}<\frac{3-a}{2}$ in (3.1). Therefore we have Theorem 3.1

Theorem 3.1. *The map $\Phi[a,b,c]$ is an indecomposable positive linear map if and only if it is an atomic positive linear map.*

For the Choi's map $\Phi[2,0,\mu]$, the condition (3.1) is reduced to the condition $\mu\geq 1$. Therefore, we see that the Choi's map $\Phi[2,0,\mu]$ is atomic whenever $\mu\geq 1$.

§4. The Robertson's Map

An example of an indecomposable positive linear map on M_4 was given by Robertson [18] by considering an extension of an automorphism on a certain spin factors. To describe this map, let $\sigma: M_2 \to M_2$ be the symplectic involution defined by

$$\sigma\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}.$$

The Robertson's map $\Psi: M_4 \to M_4$ is defined by

$$\Psi\begin{pmatrix}X & Y\\ Z & W\end{pmatrix}=\begin{pmatrix}\mathrm{tr}(W)I_2 & \frac{1}{2}(Y+\sigma(Z))\\ \frac{1}{2}(Z+\sigma(Y)) & \mathrm{tr}(X)I_2\end{pmatrix}$$

for $X, Y, Z, W\in M_2$, where tr is the normalized trace on M_2. The indecomposability of this map was shown in [20] by using the Størmer's characterization [23] of decomposability.
It turns out that this map is an extremal positive linear map which is neither 2-positive nor 2-copositive [19,21]. So this map is an atomic map. We

provide a simple proof.

Define $z_i \in \mathbb{C}^4 \otimes \mathbb{C}^4$ and $D \in V_2[M_4]$ by

$$z_1 = e_1 \otimes e_1, \quad z_2 = e_1 \otimes e_3, \quad z_3 = e_2 \otimes e_1, \quad z_4 = e_2 \otimes e_4,$$

$$z_5 = e_3 \otimes e_1, \quad z_6 = e_3 \otimes e_3, \quad z_7 = e_3 \otimes e_4,$$

$$D = (z_1 - z_6)(z_1 - z_6)^* + (z_5 + z_4)(z_5 + z_4)^* + z_2 z_2^* + z_3 z_3^* + z_7 z_7^*.$$

Then we see that

$$D^{\mathrm{T}} = (z_5 - z_2)(z_5 - z_2)^* + (z_3 + z_7)(z_3 + z_7)^* + z_1 z_1^* + z_6 z_6^* + z_4 z_4^* \in V_2[M_4],$$

and so $D \in V_2[M_4] \cap V^2[M_4]$. Furthermore, we can show that the pairing $\langle D, \Psi \rangle = -\frac{1}{2}$ by an easy calculation. Consequently, we conclude that Ψ is an atomic map.

References

[1] Ando, T., Positivity of certain maps, *Seminar Notes*, 1985.

[2] Cho, S. J., Kye, S.-H. and Lee, S. G., Generalized Choi maps in 3-dimensional matrix algebras, *Linear Alg. Appl.*, **171** (1992), 213–224.

[3] Choi, M.-D., Positive linear maps on C^*-algebras, *Canad. Math. J.*, **24** (1972), 520–529.

[4] ———, Completely positive linear maps on complex matrices, *Linear Alg. Appl.*, **10** (1975), 285–290.

[5] ———, Positive semidefinite biquadratic forms, *Linear Alg. Appl.*, **12** (1975), 95–100.

[6] ———, Some assorted inequalities for positive linear maps on C^*-algebras, *J. Operator Theory*, **4** (1980), 271–285.

[7] Choi, M.-D. and Lam, T. Y., Extremal positive semidefinite forms, *Math. Ann.*, **231** (1977), 1–18.

[8] Eom, M. H. and Kye, S.-H., Duality for positive linear maps in matrix algebras, *Math. Scand.*, to appear.

[9] Kim, H.-J. and Kye, S.-H., Indecomposable extreme positive linear maps in matrix algebras, *Bull. London Math. Soc.*, **26** (1994), 575–581.

[10] Kye, S.-H., *Elementary Operators and Applications*, World-Scientific, 1992.

[11] ———, Positive linear maps between Matrix algebras which fix diagonals, *Linear Alg. Appl.*, **216** (1995), 239–256.

[12] ———, Facial structures for positive linear maps between matrix algebra, *Canad. Math. Bull.*, **39** (1996), 74–82.

[13] ———, Boundaries of the cone of positive linear maps and its subcones in matrix algebras, *J. Korean Math. Soc.*, **33** (1996), 669–677.

[14] ———, On the convex set of all completely positive linear maps in matrix algebras, *Math. Proc. Camb. Phil. Soc.*, **122** (1997), 45–54.

[15] Osaka, H., Indecomposable positive maps in low dimensional matrix algebras, *Linear Alg. Appl.*, **153** (1991), 73–83.

[16] ———, A class of extremal positive maps in 3×3 matrix algebras, *Publ. RIMS. Kyoto Univ.*, **28** (1992), 747–756.

[17] Osaka, H., A series of absolutely positive maps in matrix algebras, *Linear Alg. Appl.*, **186** (1993), 45–53.

[18] Robertson, A. G., Automorphisms of spin factors and the decomposition of positive maps, *Quart. J. Math. Oxford* (2), **34** (1983), 87–96.

[19] ———, Positive extensions of automorphisms of spin factors, *Proc. Roy. Soc. Edinburgh Sect. A*, **94** (1983), 71–77.

[20] ———, Schwarz inequalities and the decomposition of positive maps on C^*-algebras, *Math. Proc. Camb. Phil. Soc.*, **94** (1983).

[21] ———, Positive projections on C^*-algebra and an extremal positive map, *J. London Math. Soc.*, **32** (1985), 133–140.

[22] Størmer, E., Positive linear maps of operator algebras, *Acta Math.*, **110** (1963), 233–278.

[23] ———, Decomposable positive maps on C^*-algebras, *Proc. Amer. Math. Soc.*, **86** (1982), 402–404.

[24] Tanahashi, K. and Tomiyama, J., Indecomposable positive maps in matrix algebras, *Canad. Math. Bull.*, **31** (1988), 308–317.

[25] Woronowicz, S. L., Positive maps of low dimensional matrix algebras, *Rep. Math. Phys.*, **10** (1976), 165–183.

[26] Yamagami, S., Cyclic inequalities, *Proc. Amer. Math. Soc.*, **118** (1993), 521–527.