Publ. RIMS, Kyoto Univ.
35 (1999), 1-29

Existence Theorems for Ordered Variants of Weyl Quantization

By

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Abstract

We consider some mathematical properties of Weyl-like quantizations based on two families of orderings of $e^{i(aP+bQ)}$: the first family, $W_{(\lambda,0)}$, interpolates between Wick ($\lambda=1$) and antiWick ($\lambda=-1$) ordering, while the second family, $W_{(0,\mu)}$, interpolates between the Q- ($\mu=1$) and P- ($\mu=-1$) orderings. The ordering $W_{(0,0)}$ common to both families is the unordered Weyl system.

The most important property is that of the existence of quantizations. For all orderings $W_{(0,\mu)}$ and for $W_{(\lambda,0)}$ with $-1\leq\lambda\leq 0$ quantization is a well-defined map from the tempered distributions on phase space into the continuous linear operators from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})'$. For the orderings $W_{(\lambda,0)}$ with $0<\lambda\leq 1$ we have to restrict the class of wave functions from $\mathcal{S}(\mathbb{R})$ to a certain dense subset of it, and the resulting quantization procedure sends tempered distributions on phase space into sesquilinear forms on this subspace. For Wick ordering itself we have not been able to find any useable quantization scheme, and we doubt whether any one exists that is based on tempered distributions.

We also consider questions of boundedness, and determine the matrix coefficients for the quantizations of phase space functions of radius or of angle. In particular, we consider various quantizations of the angle function in phase space.

1. Introduction

In a series of earlier papers [6, 7, 14, 15, 20], we have considered the problem of Weyl quantization in polar coordinates. Our principal purpose has been the analysis of the properties of the Weyl quantized angle function in phase space, which we have proposed as a quantum phase operator. With that purpose in mind, our analysis was based on the function space of Schwartz and its dual, the space of tempered distributions. The phase space functions to be quantized are supposed to be tempered distributions, and their quantizations are linear maps from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$.

Communicated by October 28, 1997. Revised April 7, 1998
1991 Mathematics Subject Classification(s): 81S30, 46F10.
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The proposal to consider the Weyl quantized angle function as a phase operator was made independently by Royer [17, 18], who suggested at the same time that one ought to consider non-Weyl quantizations of the angle function as well. In support of this suggestion, he calculated the matrix elements (with respect to the standard Hermite basis) of some other quantizations.

What has not been discussed until now is the mathematical structure surrounding non-Weyl quantization of angle functions, based on Schwartz space as in the Weyl case. Thus, although non-Weyl quantizations have been discussed in a mathematically rigorous way before, the particular details needed for this application have not. The purpose of this paper is to fill some of that gap.

The non-Weyl quantizations we shall consider are obtained from the generalized two-parameter Weyl group

$$W_{(\lambda,\mu)}(a,\, b) = e^{\frac{1}{4}\lambda(a^2+b^2)} e^{\frac{1}{2}i(\mu+1)ab} e^{ibQ} e^{iaP}, \tag{1.1}$$

where $a,\, b \in \mathbb{R}$ and $\lambda,\ \mu \in [-1,\, 1]$, and where Q and P are the usual operators of position and momentum in the Schrödinger representation. We note that this two-parameter family contains the standard Weyl group

$$W_{(0,0)}(a,\, b) = e^{\frac{1}{2}iab} e^{ibQ} e^{iaP} = e^{i(aP+bQ)}. \tag{1.2}$$

The family of quantizations obtained from the one-parameter subfamily $\{W_{(0,\mu)}: -1 \leq \mu \leq 1\}$ is known as the **PQ-family** of quantizations. The P-ordering of an operator-valued function of P and Q is the operator obtained by writing it formally with all incidences of the P operators to the right of all incidences of the Q operators. The reverse is true for the Q-ordering. Using the Baker-Campbell-Hausdorff formula for exponentials, it is easy to show that the P- and Q-orderings of the Weyl group $W_{(0,0)}$ are

$$W_{(0,-1)}(a,\, b) = e^{ibQ}\, e^{iaP}, \tag{1.3.a}$$

$$W_{(0,1)}(a,\, b) = e^{iaP}\, e^{ibQ}, \tag{1.3.b}$$

respectively.

The family of quantizations obtained from the one-parameter subfamily $\{W_{(\lambda,0)}: -1 \leq \lambda \leq 1\}$ is known as the **Wick/anti-Wick family** of quantizations, or WAW-family for short. If we write P and Q in terms of the lowering and raising operators

$$A = \frac{1}{\sqrt{2}}(Q+iP), \qquad A^+ = \frac{1}{\sqrt{2}}(Q-iP), \tag{1.4}$$

then the Wick-ordering of an operator-valued function of P and Q is the operator obtained by writing it formally with all incidences of the A operators to the right of all incidences of the A^+ operators, with the reverse being true for the anti-Wick ordering. Again, simple considerations enable us to show that the

Wick and anti-Wick orderings of the Weyl group $W_{(0,0)}(a, b)$ are

$$W_{(1,0)}(a, b) = e^{zA^+}e^{-\bar{z}A}, \tag{1.5.a}$$

$$W_{(-1,0)}(a, b) = e^{-\bar{z}A}e^{zA^+}, \tag{1.5.b}$$

respectively, where $z = \frac{1}{\sqrt{2}}(-a+ib)$.

The first question we must address is the one of how to use these variations on the Weyl group to obtain viable quantization schemes. Our starting point is the formula

$$\boldsymbol{\Delta}_{(\lambda,\mu)}[T] = \frac{1}{2\pi}\iint_{\mathbb{R}^2}[\mathcal{F}T](a, b)\, W_{(\lambda,\mu)}(a, b)\, da\, db, \tag{1.6}$$

to obtain the quantization $\boldsymbol{\Delta}_{(\lambda,\mu)}[T]$ of the function T. Here $\mathcal{F}(T)$ denotes the Fourier transform (in two dimensions) of T. We shall be taking Fourier transforms in one and two dimensions frequently, and using the same symbol $\mathcal{F}$ each time — our convention for the Fourier transform in n dimensions is

$$[\mathcal{F}H](y_1, \dots, y_n) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} H(x_1, \dots, x_n)\, e^{-i(x_1y_1+\cdots+x_ny_n)}dx_1\cdots dx_n. \tag{1.7}$$

We shall use without comment the facts that $\mathcal{F}$ and its inverse are unitary operators on $L^2(\mathbb{R}^n)$ and topological isomorphisms $\mathcal{S}(\mathbb{R}^n)$ and (after a suitable extension) of its dual.

Equation (1.6), in the case $\lambda = \mu = 0$, is the formula given by Weyl, who points out that it is not to be taken literally. What this means is that it needs to be reworked into a form that is capable of rigorous mathematical interpretation. This process is done as follows. We define $\boldsymbol{\Delta}_{(\lambda,\mu)}[T]$ in terms of its matrix coefficients with respect to wave functions which are suitably smooth. The choice of the appropriate class of wave functions will be discussed later — for now we shall assume that a suitable choice has been made. Thus our definition is going to be *weak*, and the integral in (1.6) will be defined weakly — this is going to be necessary if we wish T to be a tempered distribution.

Before we proceed to show how this is to be done, we need to clarify some notation. We always use angular brackets to denote inner products, and choose the first (leftmost) factor to be antilinear. However we also need a notation for the (real) bilinear pairing between $\mathcal{S}(\mathbb{R}^n)$ and its dual, which we shall write as

$$[\![T; H]\!], \qquad T \in \mathcal{S}'(\mathbb{R}^n),\ H \in \mathcal{S}(\mathbb{R}^n). \tag{1.8}$$

We shall use this symbol most often when the domain is phase space, and we shall reserve the symbol Π for $\mathbb{R}^2$ so interpreted.

For suitable functions $f, g \in L^2(\mathbb{R})$, consider the function

$$(a, b) \mapsto \langle g, W_{(\lambda,\mu)}(a, b) f\rangle. \tag{1.9}$$

As we shall see below, by "suitable" we mean functions f, g such that the above function belongs to $\mathscr{S}(\mathbb{R}^2)$. We can then define a function $\mathscr{G}_{(\lambda,\mu)}(\bar{g}\otimes f) \in \mathscr{S}(\mathbb{I}\mathbb{I})$ such that

$$[\mathscr{F}^{-1}\mathscr{G}_{(\lambda,\mu)}(\bar{g}\otimes f)](a,b) = \frac{1}{2\pi}\langle g, W_{(\lambda,\mu)}(a,b)f\rangle. \tag{1.10}$$

We can then hope to define $\varDelta_{(\lambda,\mu)}[T]$ by setting

$$\langle g, \varDelta_{(\lambda,\mu)}[T]f\rangle = [\![T; \mathscr{G}_{(\lambda,\mu)}(\bar{g}\otimes f)]\!]. \tag{1.11}$$

At least formally, then, the above definitions are such that

$$\begin{aligned}\langle g, \varDelta_{(\lambda,\mu)}[T]f\rangle &= [\![T; \mathscr{G}_{(\lambda,\mu)}(\bar{g}\otimes f)]\!] \\ &= \frac{1}{2\pi}\iint_{\mathbb{R}^2}[\mathscr{F}(T)](a,b)\langle g, W_{(\lambda,\mu)}(a,b)f\rangle\, da\, db,\end{aligned}$$

and it is in this way that the formula (1.6) is interpreted weakly to define the operator $\varDelta_{(\lambda,\mu)}[T]$. In what follows we shall see that it is sufficient, in nearly all cases, to assume that f and g simply belong to $\mathscr{S}(\mathbb{R})$, in which case $\varDelta_{(\lambda,\mu)}[T]$ defines a linear map from $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}'(\mathbb{R})$. However, in some cases, it is necessary to restrict f and g to belonging to a much more restrictive space than $\mathscr{S}(\mathbb{R})$. For this reason, and others which will become clear, it is necessary to deal with the PQ-family and the WAW-family separately.

2. Quantization for the AW-Family

As we shall see, the PQ-family poses no problems as far as questions of existence of qtuantizations go. The WAW-family is different in this regard, for the WAW-family $\varDelta_{(\lambda,0)}$ is more regular than Weyl quantization for $-1\leq\lambda<0$, while it is less well-behaved than Weyl quantization for $0<\lambda<1$, while the quantization $\varDelta_{(1,0)}$ does not exist in any useful sense for a general tempered distribution. The subfamily of the WAW-family corresponding to $-1\leq\lambda\leq 0$ is called the antiWick-family (or AW-family), while the other half of the family (for $0<\lambda\leq 1$) is called the Wick-family (or W-family). With this in mind, we shall start our discussion with the best case, namely the AW-family.

At the end of the previous section, we showed how quantization can be defined rigorously, subject to certain conditions being satisfied. That these conditions can be satisfied in the case of the AW-family is dealt with by the next proposition.

Proposition 2.1. *For every $f, g\in\mathscr{S}(\mathbb{R})$ and all $-1\leq\lambda\leq 0$ the function*

$$[\mathscr{W}_{(\lambda,0)}(\bar{g}\otimes f)](a,b) = \langle g, W_{(\lambda,0)}(a,b)f\rangle \tag{2.1.a}$$

belongs to $\mathscr{S}(\mathbb{R}^2)$, and the map

$$(f, g) \ \mapsto \ \mathscr{G}_{(\lambda,0)}(\bar{g}\otimes f)=\frac{1}{2\pi}\mathscr{F}[\mathscr{W}_{(\lambda,0)}(\bar{g}\otimes f)] \tag{2.1.b}$$

is a jointly continuous sesquilinear map from $\mathscr{S}(\mathbb{R})\times\mathscr{S}(\mathbb{R})$ *to* $\mathscr{S}(\Pi)$. *Thus we can define* $\boldsymbol{\Delta}_{(\lambda,0)}[T]$ *as a linear map from* $\mathscr{S}(\mathbb{R})$ *to* $\mathscr{S}'(\mathbb{R})$ *for any* $T\in\mathscr{S}'(\Pi)$ *via the formula*

$$[\![\boldsymbol{\Delta}_{(\lambda,0)}[T]f; \bar{g}]\!] = [\![T; \mathscr{G}_{(\lambda,0)}(\bar{g}\otimes f)]\!]. \tag{2.1.c}$$

These quantization schemes are related to Weyl quantization through the convolution formula

$$\mathscr{G}_{(\lambda,0)}=Q_\lambda * \mathscr{G}_{(0,0)}, \tag{2.2.a}$$

where Q_λ *is the Gaussian*

$$Q_\lambda(x, y)=\frac{1}{\pi|\lambda|}e^{(x^2+y^2)/\lambda}. \tag{2.2.b}$$

Proof. Note that Q_{-4t} is the kernel of the semigroup generated by the negative of the Laplacian in two dimensions. We also need the one-dimensional heat kernel

$$q_\lambda(x)=\frac{1}{\sqrt{\pi|\lambda|}}e^{x^2/\lambda},$$

and for convenience we introduce the following notation for the unitary groups generated by the Schrödinger operators P and Q,

$$\mathscr{T}_a f(x)=f(x+a), \qquad \mathscr{M}_b f(x)=e^{ibx}f(x),$$

respectively. The proof involves a number of standard manipulations of integrals, which we shall omit — the reader can readily supply them.

Firstly we show that

$$\langle g, W_{(\lambda,0)}(a, b)f\rangle=\sqrt{2\pi}e^{\frac{1}{4}\lambda a^2+\frac{1}{2}iab}\{\mathscr{F}^{-1}(q\lambda * (\bar{g}\otimes\mathscr{T}_a f))\}(b).$$

Now we take the Fourier transform of this identity with respect to the second variable only, obtaining

$$\frac{1}{2\pi}\int_{\mathbb{R}}\langle g, W_{(\lambda,0)}(a, b)f\rangle e^{-iqb}db=e^{\frac{1}{4}\lambda a^2}\int_{\mathbb{R}}q_\lambda(q-k)\overline{g\left(k-\frac{a}{2}\right)}f\left(k+\frac{a}{2}\right)dk,$$

from which we deduce that $\mathscr{W}_{(\lambda,0)}(\bar{g}\otimes f)$ belongs to $\mathscr{S}(\mathbb{R}^2)$. This implies that $\mathscr{G}_{(\lambda,0)}(\bar{g}\otimes f)$ belongs to $\mathscr{S}(\Pi)$. It is now easy to obtain the convolution relationship between $\mathscr{G}_{(\lambda,0)}$ and $\mathscr{G}_{(0,0)}$. From this result follows the required continuity of the sesquilinear map $\mathscr{G}_{(\lambda,0)}$, and hence the quantization $\boldsymbol{\Delta}_{(\lambda,0)}$ is well-defined. ∎

This answers the question of existence. For the remainder of this section we shall consider the properties of the AW-quantizations in respect of marginal distributions and the taking of adjoints. These results are known in other contexts — these quantizations do not yield the correct marginals, but they do have desirable conjugation properties — what we are doing now is checking that the necessary manipulations are valid in the setting of tempered distributions.

For the marginals, continuing to restrict our attention to the case $-1 \leq \lambda < 0$, we consider a tempered distribution $h \in \mathcal{S}'(\mathbb{R})$. Then the distributions $1 \otimes h$ and $h \otimes 1$ both belong to $\mathcal{S}'(\Pi)$, so we can calculate $\boldsymbol{\Delta}_{(\lambda,0)}[1 \otimes h]$ and $\boldsymbol{\Delta}_{(\lambda,0)}[h \otimes 1]$. If we do so, we obtain the following result.

Proposition 2.2. *For an arbitrary tempered distribution $h \in \mathcal{S}'(\mathbb{R})$ and $-1 \leq \lambda < 0$, we have the following identities between linear maps from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$:*

$$\boldsymbol{\Delta}_{(\lambda,0)}[1 \otimes h] = (h * q_\lambda)(Q), \tag{2.3.a}$$

$$\boldsymbol{\Delta}_{(\lambda,0)}[h \otimes 1] = (h * q_\lambda)(P). \tag{2.3.b}$$

These define the marginals for the AW-quantizations. While the AW-quantizations do map a function of q into a function of the operator Q, it does not map h to $h(Q)$. A similar statement holds for functions of p alone. As has been seen elsewhere, the Weyl quantization $\boldsymbol{\Delta}_{(0,0)}$ does not suffer from this defect.

To obtain the properties of the adjoint we need the following integral identity:

$$[Q_\lambda * \mathcal{G}(\bar{g} \otimes f)]^* = Q_\lambda * \mathcal{G}(\bar{f} \otimes g), \qquad f, g \in \mathcal{S}(\mathbb{R}).$$

From this result we obtain the following:

Proposition 2.3. *For any $T \in \mathcal{S}'(\Pi)$, its adjoint $T^* \in \mathcal{S}'(\Pi)$ is given by the formula*

$$[\![T^*; f]\!] = \overline{[\![T; \bar{f}]\!]}, \qquad f \in \mathcal{S}(\Pi). \tag{2.4.a}$$

We have that

$$[\![\boldsymbol{\Delta}_{(\lambda,0)}[T^*]f; \bar{g}]\!] = \overline{[\![\boldsymbol{\Delta}_{(\lambda,0)}[T]g; \bar{f}]\!]} \tag{2.4.b}$$

for $f, g \in \mathcal{S}(\mathbb{R})$ and $-1 \leq \lambda < 0$. If T is sufficiently regular, so that $\boldsymbol{\Delta}_{(\lambda,0)}[T]$ is a bounded operator on $L^2(\mathbb{R})$, then so is $\boldsymbol{\Delta}_{(\lambda,0)}[T^*]$, and

$$\boldsymbol{\Delta}_{(\lambda,0)}[T^*] = \boldsymbol{\Delta}_{(\lambda,0)}[T]^*. \tag{2.4.c}$$

Thus, if $T = T^*$ is real-valued and sufficiently regular, then $\boldsymbol{\Delta}_{(\lambda,0)}[T]$ is self-adjoint.

It is possible to investigate the behaviour of the marginals with respect to A and A^+ rather that P and Q. Indeed, it is possible to show that if $T\in\mathscr{S}'(\Pi)$ is such that $T(p,q)$ is an analytic function of $p+iq$, then

$$\boldsymbol{\Delta}_{(\lambda,0)}[T]=\boldsymbol{\Delta}_{(0,0)}[T]=T(i\sqrt{2}A^+) \tag{2.5.a}$$

for any $-1\leq\lambda<0$, while if $S\in\mathscr{S}'(\Pi)$ is such that $S(p,q)$ is an antianalytic function of $p+iq$, then

$$\boldsymbol{\Delta}_{(\lambda,0)}[S]=\boldsymbol{\Delta}_{(0,0)}[S]=S(-i\sqrt{2}A) \tag{2.5.b}$$

for any $-1\leq\lambda<0$. This is particularly easy to prove, since any $T\in\mathscr{S}'(\Pi)$ such that $T(p,q)=T(p+iq)$ is analytic (or antianalytic) is polynomially bounded as a function (since it belongs to $\mathscr{S}'(\Pi)$), and hence must in fact be a polynomial in $p+iq$ $(p-iq)$, and the required calculations for polynomials are straightforward. If we wish to deal with more complicated functions of $p+iq$, it would be necessary to reformulate our quantization scheme in a manner which did not concentrate on quantizing elements of $\mathscr{S}'(\Pi)$. As to the relevance of this to quantum theory, since measurements of P and Q are certainly possible we should expect the marginals for AW-quantization with respect to P and Q to appear whenever the classical limit was relevant. In an older terminology, AW-quantization is not wholly consonant with the correspondence principle. This point has been emphasized by Berezin and Shubin [3].

3. Polar AW-Quantization

We continue our discussion of AW-quantization by considering how it differs from Weyl quantization in respect of functions of the radius or of the angle in phase space. Since AW-quantization is obtained from Weyl quantization by the simple addition of an operation of convolution, it is to be expected that techniques which worked for Weyl quantization will also work for AW-quantization. This is indeed the case. In particular, we may calculate the matrix coefficients of such operators with respect to the standard Hermite basis in $L^2(\mathbb{R})$. Although our methods differ from those of Royer [17, 18], our results are the same.

We follow the methods we used in our paper on polar coordinates for Weyl quantization [6]. Matrix elements with respect to the Hermite-Gaussian functions $\{h_n: n\geq 0\}$ can be calculated using the generating function

$$G_t(x)=\sum_{n\geq 0}\frac{t^n}{\sqrt{2^n n!}}h_n(x)=\pi^{-\frac{1}{4}}\exp\left(-\frac{1}{4}t^2+tx-\frac{1}{2}x^2\right). \tag{3.1}$$

Taking the convolution of Q_λ with the known result

$$[\mathscr{G}_{(0,0)}(\overline{G_s}\otimes G_t)](p, q)=\frac{1}{\pi}\exp[-\frac{1}{2}st+(q+ip)s+(q-ip)t-p^2-q^2] \quad (3.2)$$

we deduce that

$$[\mathscr{G}_{(\lambda,0)}(\overline{G_s}\otimes G_t)](p, q)=\frac{1}{\pi(1-\lambda)}\exp\left[-\frac{1}{2}\frac{1+\lambda}{1-\lambda}st+\frac{(q+ip)s}{1-\lambda}+\frac{(q-ip)t}{1-\lambda}-\frac{p^2+q^2}{1-\lambda}\right]. \quad (3.3)$$

Now we introduce polar coordinates by substituting $p=r\cos\beta$, $q=r\sin\beta$ in the above. Using polar coordinates enables us to find relatively simple expressions for the matrix coefficients of the AW-quantizations of functions of the radius or of the angle with respect to the Hermite-Gaussian functions. While there is no *a priori* reason for us to expect the AW-quantization results to resemble the Weyl results closely, it turns out that they do so.

As in the Weyl quantization case, radial quantization is particularly easy to deal with, so we consider it first. We have that

$$\int_{-\pi}^{\pi}[\mathscr{G}_{(\lambda,0)}(\overline{G_s}\otimes G_t)](r\cos\beta,\ r\sin\beta)\,d\beta$$

$$=\frac{2}{1-\lambda}\exp\left[-\frac{r^2}{1-\lambda}\right]\sum_{N=0}^{\infty}\frac{1}{N!}\left[-\frac{1}{2}\frac{1+\lambda}{1-\lambda}st\right]^N L_N\left(\frac{2r^2}{1-\lambda^2}\right) \quad (3.4)$$

for any $-1<\lambda<0$. We now integrate this expression against a tempered distribution which depends on the radius alone. In order to ensure that all quantities which occur are well-defined, it is sufficient that we restrict attention to functions $f:\mathbb{R}^+\to\mathbb{R}$ for which all the integrals

$$\rho_{\lambda,n}=\int_0^\infty e^{-u}L_n\left(\frac{2u}{1+\lambda}\right)f(\sqrt{(1-\lambda)u})\,du \quad (3.5)$$

exist for all $-1<\lambda<0$ and $n\geq 0$. Such a function defines a phase space function of the radius alone by setting

$$f_{\text{rad}}(p, q)=f(\sqrt{p^2+q^2}). \quad (3.6)$$

Then $\varDelta_{(\lambda,0)}[f_{\text{rad}}]$ exists as a densely defined operator on $L^2(\mathbb{R})$ whose domain contains all the Hermite-Gaussian functions. Moreover, the Hermite-Gaussians are its eigenfunctions, and

$$\varDelta_{(\lambda,0)}[f_{\text{rad}}]h_n=(-1)^n\left(\frac{1+\lambda}{1-\lambda}\right)^n\rho_{\lambda,n}h_n, \qquad n\geq 0,\ -1<\lambda<0. \quad (3.7)$$

Just as for Weyl quantization, AW-radial quantization leads to operators with a discrete spectrum of (in general) unit multiplicity and oscillator eigenfunctions.

We note that the case $\lambda=-1$ is not singular, although the above calculations seem to show that it is. This is because the eigenvalues of $\boldsymbol{\Delta}_{(\lambda,0)}[f_{\mathrm{rad}}]$ are

$$E_{\lambda,n}=(-1)^n\left(\frac{1+\lambda}{1-\lambda}\right)^n\rho_{\lambda,n}, \qquad n\geq 0, \tag{3.8.a}$$

and we can calculate that

$$E_{-1,n}=\lim_{\lambda\to-1}E_{\lambda,n}=\frac{1}{n!}\int_0^\infty e^{-u}f(\sqrt{2u})u^n du, \qquad n\geq 0, \tag{3.8.b}$$

so we deduce that $\boldsymbol{\Delta}_{(-1,0)}[f_{\mathrm{rad}}]$ is still diagonal, with

$$\boldsymbol{\Delta}_{(-1,0)}[f_{\mathrm{rad}}]h_n=\frac{1}{n!}\left(\int_0^\infty e^{-u}f(\sqrt{2u})u^n du\right)h_n, \qquad n\geq 0. \tag{3.8.c}$$

We have not considered placing sharper controls on the functions f in order that $\boldsymbol{\Delta}_{(\lambda,0)}[f_{\mathrm{rad}}]$ be of any particular operator class (for example mapping $\mathcal{S}(\mathbb{R})$ to itself), but this would not be particularly difficult to do. Summarizing these results, the quantization scheme $\boldsymbol{\Delta}_{(\lambda,0)}$ behaves well in respect of radial elements (when $-1\leq\lambda<0$) and the operators $\boldsymbol{\Delta}_{(\lambda,0)}[f_{\mathrm{rad}}]$ are of the same sort as the operators $\boldsymbol{\Delta}_{(0,0)}[f_{\mathrm{rad}}]$ of Weyl quantization.

Integrating $\mathscr{G}_{(\lambda,0)}(\bar{G}_s\otimes G_t)$ over the radial variable will enable us to find the matrix elements for the operators $\boldsymbol{\Delta}_{(\lambda,0)}[f_{\mathrm{ang}}]$, where for a function $f\in L^2[-\pi,\pi]$ we set

$$f_{\mathrm{ang}}(r\cos\beta, r\sin\beta)=\begin{cases} f(\beta) & r>0 \\ 0 & r=0. \end{cases} \tag{3.9}$$

This was done for the standard Weyl quantization scheme in [6], and we were able to find an expression for the matrix elements in terms of ratios of gamma functions and the Fourier coefficients of f, namely

$$\hat{f}_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\beta)e^{-ik\beta}\,d\beta \qquad k\in\mathbb{Z}. \tag{3.10}$$

In the case at hand, analogous calculations to those in [6] yield a similar result,

$$\int_0^\infty[\mathscr{G}_{(\lambda,0)}(\bar{G}_s\otimes G_t)](r\cos\beta, r\sin\beta)r dr=\frac{1}{2\pi}\sum_{m,n\geq 0}\frac{i^{m-n}A_{m,n}(\lambda)s^m t^n}{(1-\lambda)^{\frac{1}{2}(m+n)}}e^{i(n-m)\beta} \tag{3.11.a}$$

for any $-1\leq\lambda<0$, where

$$A_{m,n}(\lambda)=\sum_{j=0}^{\min(m,n)}\frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n-j+1)}{j!(m-j)!(n-j)!}\left(-\frac{1+\lambda}{2}\right)^j. \tag{3.11.b}$$

We can proceed to simplify this expression as in [6]. There are two special cases worth noting:

$$A_{n,n}(\lambda)=\frac{(1-\lambda)^n}{2^n n!}, \qquad n\geq 0, \tag{3.12.a}$$

$$A_{m,n}(-1)=\frac{\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)}{m!n!}, \qquad m,n\geq 0. \tag{3.12.b}$$

Our general solution is that

$$A_{m,n}(\lambda)=\frac{1}{2^{\max(m,n)}\min(m,n)!\,\Gamma(\frac{1}{2}|m-n|)} \times \sum_{j=0}^{\min(m,n)}\binom{m}{j}(-\lambda)^{\min(m,n)-j}B\left(\frac{1}{2}|n-m|,\frac{1}{2}j+s_j\right), \tag{3.13.a}$$

where

$$s_j=\begin{cases}\frac{1}{2} & j \text{ even}\\ 0 & j \text{ odd.}\end{cases} \tag{3.13.b}$$

From this we can readily deduce the matrix coefficients for $\Delta_{(\lambda,0)}[f_{\text{ang}}]$, namely

$$[\![\Delta_{(\lambda,0)}[f_{\text{ang}}]h_n;\overline{h_m}]\!]=2^{-\frac{1}{2}|m-n|}\sqrt{\frac{\max(m,n)!}{\min(m,n)!}}\frac{i^{m-n}\widehat{f}_{m-n}}{(1-\lambda)^{\frac{1}{2}(m+n)}} \times\left[\sum_{j=0}^{\min(m,n)}\binom{\min(m,n)}{j}(-\lambda)^{\min(m,n)-j}\frac{\Gamma(\frac{1}{2}j+s_j)}{\Gamma(\frac{1}{2}|m-n|+\frac{1}{2}j+s_j)}\right] \tag{3.14}$$

for $m,n\geq 0$ and $-1\leq\lambda<0$. We note that putting $\lambda=0$ in the above regains the matrix coefficients for Weyl quantization. Thus the AW-quantization coefficients are comparatively simple extensions of the Weyl quantization coefficients.

We note in passing that a number of authors have proposed the (Toeplitz) operator X, with matrix elements

$$\langle h_m, Xh_n\rangle=[1-\delta_{mn}]\frac{i^{m-n+1}}{n-m}, \qquad m,n\geq 0,$$

as a phase operator. We have previously shown that X is not the Weyl quantization of any function f_{ang}. The same result holds here, since it is not difficult to show from its matrix elements that no function f exists for which $\Delta_{(\lambda,0)}[f_{\text{ang}}]=X$ for some $-1\leq\lambda\leq 0$.

4. The AW-Phase Operator Kernel

The question as to which quantization scheme is "preferred" by nature is, in a certain sense, empty of content. Quantum theory tells us that all self-adjoint operators containing the Hermite-Gaussian functions in their domain are measurable in principle. There is a caveat about the possible accuracy of the measurements that can be made, especially for operators with a continuous spectrum, but the important point is that no mention is made of how the operator was originally obtained. Thus, if different quantization schemes lead to different self-adjoint operators for a given distribution T, these different operators can all be measured in principle. What is problematical is determining what observables they represent, and to this question there is no general answer. No more is there an answer to the question of what operator is being measured when faced with a jumble of laboratory equipment with banks of dials and flashing lights attended by people we know are scientists since — in the best traditions of cinematic science fiction— they are wearing white lab coats.

Nonetheless, any quantization of a function of the angle has some connection to what might loosely be called quantum phase phenomena. This statement can be strengthened somewhat by a result concerning coherent light which we have proved and will publish elsewhere [8]. Suppose that we take it that the laser model originally suggested by Dicke [4] and by Graham and Haken [11], upgraded and treated rigorously by Hepp and Lieb [16], Sewell [19] and recently by Alli and Sewell [1], does indeed describe the "ideal" coherent radiation (as the thermodynamic limit of the Bose gas describes "ideal" condensation, that of the strong coupling BCS model describes an idealized type of superconductivity, and so on). Our result is that the operator $\boldsymbol{\Delta}_{(0,0)}[T]$ may be treated in this model in the same way as the base photon operators A and A^+ are. Moreover, if we do so, the intensive global observable it determines is $T(p, q)$.

This means that if the Weyl dequantization of an operator is a function of angle alone, it must have something to do with the phase of the coherent light. If we first (λ, μ)-quantize a function of angle alone, and then Weyl-dequantize the resulting operator, we get a function of both the angle and the radius, so such operators determine intensive variables which depend both on the intensity and the phase of the laser light. However, in many cases, the limit as r tends to ∞ can be taken, and will yield a well-defined function of angle. This will be a global observable associated with the (λ, μ)-quantization at high (infinite) intensities.

We do not mean for this discussion to be definitive, but only to suggest the importance of considering the operators $\mathbf{\Delta}_{(\lambda,\mu)}[f_{\mathrm{ang}}]$, and $\mathbf{\Delta}_{(\lambda,\mu)}[\varphi]$ in particular, where $\varphi=\Theta_{\mathrm{ang}}$, where

$$\Theta(\beta)=\beta, \qquad -\pi\leq\beta<\pi, \tag{4.1.a}$$

so that

$$\varphi(r\cos\beta, r\sin\beta)=\begin{cases}\beta, & r>0,\ -\pi\leq\beta<\pi,\\ 0, & r=0.\end{cases} \tag{4.1.b}$$

Note that we give values to φ along the negative real axis (the cut associated with the principal branch of the arctangent), even at the origin. Thus we have defined φ on the whole plane Π, even though it is not continuous everywhere there. Clearly, the definition of the value of φ in the negative real axis is somewhat arbitrary, but does not make any difference since, as a tempered distribution, such a level of uncertainty has no effect on the definition of φ.

Of course, we could use the results of the previous section to obtain explicit expressions for the matrix coefficients of $\mathbf{\Delta}_{(\lambda,0)}[\varphi]$ with respect to the Hermite-Gaussian functions when $-1\leq\lambda\leq 0$, but we shall not do so here. What we shall do is derive an expression for the integral kernel of $\mathbf{\Delta}_{(\lambda,0)}[\varphi]$ when $-1\leq\lambda<0$. The derivation of this kernel requires some integral identities.

Lemma 4.1. *The following identities hold*:

$$\int_{\mathbb{R}}\operatorname{sgn}(q)\,q_\lambda(q-y)\,dq=\operatorname{erf}\left(\frac{y}{\sqrt{|\lambda|}}\right), \tag{4.2.a}$$

$$\int_{\mathbb{R}}\operatorname{sgn}(q)\,q_\lambda(q-y)\,e^{-|xq|}dq=e^{-\frac{1}{4}\lambda x^2}\operatorname{sgn}(y)\left\{e^{-|xy|}-\frac{2}{\sqrt{\pi}}\int_{|y/\lambda|}^{\infty}\exp\left(-\xi^2-\frac{x^2y^2}{4\xi^2}\right)d\xi\right\}, \tag{4.2.b}$$

for any $-1\leq\lambda<0$.

Proof. The first identity is straightforward to establish, and we omit the details. Consider now the quantity

$$A(\lambda,x,y)=e^{-|x|y}\int_{\frac{1}{2}\sqrt{|\lambda|}|x|-\frac{y}{\sqrt{|\lambda|}}}^{\infty}e^{-q^2}\,dq-e^{|x|y}\int_{\frac{1}{2}\sqrt{|\lambda|}|x|+\frac{y}{\sqrt{|\lambda|}}}^{\infty}e^{-q^2}\,dq.$$

It may be verified that A satisfies the differential equation

$$\frac{\partial A}{\partial\lambda}=\frac{y}{\lambda\sqrt{\pi|\lambda|}}e^{\frac{1}{4}\lambda x^2+\frac{y^2}{\lambda}}$$

and the boundary condition

$$\lim_{\lambda\to 0+} A(\lambda, x, y) = \operatorname{sgn}(y)e^{-|xy|}.$$

From this it follows that

$$\int_{\mathbb{R}} \operatorname{sgn}(q) q_\lambda (q-y) e^{-|xq|} dq = e^{-\frac{1}{4}\lambda x^2} A(\lambda, x, y)$$

$$= e^{-\frac{1}{4}\lambda x^2} \operatorname{sgn}(y) \left\{ e^{-|xy|} - \frac{2}{\sqrt{\pi}} \int_{|y/\lambda|}^{\infty} \exp\left(-\xi^2 - \frac{x^2y^2}{4\xi^2}\right) d\xi \right\},$$

as required. ■

We can now use these results to derive the integral kernel representation of $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ for $-1 \leq \lambda < 0$.

Theorem 4.1. *The operator* $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$, *where* $-1 \leq \lambda < 0$, *has integral kernel representation given by*

$$[\![\boldsymbol{\Delta}_{(\lambda,0)}[\varphi] f; \bar{g}]\!] = \frac{\pi}{2} \int_{\mathbb{R}} \operatorname{erf}\left(\frac{y}{\sqrt{|\lambda|}}\right) \overline{g(y)} f(y) dy \tag{4.3}$$

$$- \frac{i}{2} \lim_{L\to\infty} \iint_{\mathbb{R}^2} \operatorname{sgn}(y) \left\{ e^{-|xy|} - \frac{2}{\sqrt{\pi}} \int_{|y/\lambda|}^{\infty} \exp\left(-\xi^2 - \frac{x^2y^2}{4\xi^2}\right) d\xi \right\}$$

$$g_{I(L)}(x) \overline{g\left(y+\frac{x}{2}\right)} f\left(y-\frac{x}{2}\right) dx dy$$

for any $f, g \in \mathcal{S}(\mathbb{R})$. *Here* $g_{I(L)}$ *is the convergence factor*

$$g_{I(L)}(x) = \begin{cases} x^{-1}, & L^{-1} \leq |x| \leq L, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4}$$

and the integrals are meant in the sense of Lebesgue.

Proof. It is convenient notationally to introduce the function $H_{f,g} \in \mathcal{S}(\mathbb{R}^2)$, where

$$H_{f,g}(x, y) = \overline{g\left(y+\frac{x}{2}\right)} f\left(y-\frac{x}{2}\right),$$

so that

$$\sqrt{2\pi}\, \mathcal{G}_{(0,0)}(\bar{g} \otimes f) = \mathcal{F}_1^{-1} H_{f,g},$$

where the subscript on the Fourier transform indicates that it is a one-dimensional Fourier transform taken with respect to the first variable only. We shall adopt a similar notational convention with the one-dimensional Fourier transform with respect to the second variable, and with respect to one-dimensional convolutions. With this understanding we note that

$$\mathscr{G}_{(\lambda,0)}(\bar{g}\otimes f)=\mathscr{F}_1^{-1}[(\mathscr{F}q_\lambda\otimes 1)\cdot(q_\lambda *_2 H_{f,g})].$$

We now use the technical result found in the Appendix in [6] to deduce that

$$\int_{\mathbb{R}}\varphi(p,q)\,[\mathscr{G}_{(\lambda,0)}(\bar{g}\otimes f)]\,(p,q)\,dp$$

$$=\frac{\pi}{2}\operatorname{sgn}(q)\,[q *_2 H_{f,g}]\,(0,q)-\frac{i}{2}\operatorname{sgn}(q)\lim_{L\to\infty}\int_{\mathbb{R}} g_{I(L)}(x)\,[q_\lambda *_2 H_{f,g}]\,(x,q)\,e^{\frac{1}{4}\lambda x^2-|xq|}dx,$$

and the preceding Lemma can now be used to identify the two expressions given above with the desired two expressions in the result. ∎

5. The AW-Phase Operator Is Bounded

In the previous section, we performed the convolution of Q_λ with $\mathscr{G}_{(0,0)}(\bar{g}\otimes f)$ before integrating against the angle function φ. It is clearly equivalent to calculate the convolution of Q_λ with φ first, since then $\Delta_{(\lambda,0)}[\varphi]$ is the standard Weyl quantization of the resulting distribution. Our first task will be to identify this convolution.

Proposition 5.1. *If we define the scaled beta function $B_\lambda(x,y)$ by the formula*

$$B_\lambda(x,y)=x\int_0^y\frac{1}{t^2+x^2}e^{\frac{t^2+x^2}{\lambda}}dt \qquad x\neq 0, \tag{5.1}$$

then

$$[\varphi * Q_\lambda]\,(x,y)=\varphi(x,y)-\frac{\pi}{2}\left[\operatorname{sgn}(y)-\operatorname{erf}\left(\frac{y}{\sqrt{|\lambda|}}\right)\right]+B_\lambda(y,x). \tag{5.2}$$

The proof, which we shall omit, is a fairly straightforward consequence of the lemma to be found in the Appendix of [6]. It is clear that the Weyl quantizations of the first two terms on the right-hand side of equation (5.2) yield bounded operators on $L^2(\mathbb{R})$, and so we need to concentrate on the third term. By first differentiating, and then integrating, with respect to λ, we can show that

$$B_\lambda(x,y)=\frac{x\sqrt{\pi}}{2}\int_0^{|\lambda|}e^{-\frac{x^2}{\mu}}\operatorname{erf}\left(\frac{y}{\sqrt{\mu}}\right)d\mu,$$

from which we can deduce that

$$[\![\Delta_{(0,0)}[B_\lambda]f;\bar{g}]\!]=\frac{i}{8}\int_0^{|\lambda|}\left(\iint_{\mathbb{R}^2}xe^{-\frac{1}{4}\mu x^2}\left[\operatorname{erf}\left(\frac{y}{\sqrt{\mu}}\right)-\operatorname{sgn}(y)\right]H_{f,g}(x,y)\,dxdy\right)d\mu$$

$$+\frac{i}{2}\iint_{\mathbb{R}^2} x^{-1}(1-e^{\frac{1}{4}\lambda x^2})\,\mathrm{sgn}(y) H_{f,g}(x,y)\,dxdy.$$

Let us investigate these two terms separately. We need to develop a generalization of the technique used in [6].

For any $h\in L^\infty(0,\infty)$ and $\alpha>0$, consider the function $\kappa_{h,\alpha}$: $(0,\infty)\times(0,\infty)\to\mathbb{R}$ defined by

$$\kappa_{h,\alpha}(x,y)=\begin{cases}(x+y)^{-1}h(x+y), & 0<\alpha x\le y,\\ 0, & 0<y\le\alpha x.\end{cases}\tag{5.3}$$

Then direct calculation shows us that

$$\int_0^\infty\frac{1}{\sqrt{x}}|\kappa_{h,\alpha}(x,y)|dx\le\frac{2}{\sqrt{y}}\|h\|_\infty\tan^{-1}(\alpha^{-\frac{1}{2}})\le\frac{\pi}{\sqrt{y}}\|h\|_\infty,\tag{5.4.a}$$

$$\int_0^\infty\frac{1}{\sqrt{y}}|\kappa_{h,\alpha}(x,y)|dy\le\frac{2}{\sqrt{x}}\|h\|_\infty\left[\frac{\pi}{2}-\tan^{-1}(\alpha^{\frac{1}{2}})\right]\le\frac{\pi}{\sqrt{x}}\|h\|_\infty.\tag{5.4.b}$$

We can thus use the Schur test [13] to define a bounded operator $K_h(\alpha)$ from $L^\infty(0,\infty)$ to itself, such that $\|K_h(\alpha)\|\le\pi\|h\|_\infty$ (and, in particular, $\|K_h(1)\|\le\frac{\pi}{2}\|h\|_\infty$), by setting

$$[K_h(\alpha)u](x)=\int_0^\infty\kappa_{h,\alpha}(x,y)u(y)dy=\int_{\alpha x}^\infty\frac{h(x+y)u(y)}{x+y}dy\tag{5.5}$$

for any $u\in L^\infty(0,\infty)$. Next we introduce the continuous linear maps $P_\pm$: $L^2(\mathbb{R})\to L^2(0,\infty)$ given by

$$[P_+g](x)=g(x),\tag{5.6.a}$$

$$[P_-g](x)=g(-x),\tag{5.6.b}$$

for $g\in L^2(\mathbb{R})$ and $x>0$. Putting these various maps together, we obtain a continuous linear map $\mathcal{K}_h(\alpha)$ from $L^2(\mathbb{R})$ to itself, with $\|\mathcal{K}_h(\alpha)\|\le\pi\|h\|_\infty$ (and $\|\mathcal{K}_h(1)\|\le\frac{\pi}{2}\|h\|_\infty$), by setting

$$[\mathcal{K}_h(\alpha)g](x)=\begin{cases}[K_h(\alpha)P_-g](x), & x>0,\\ 0, & x=0,\\ [K_h(\alpha)P_+g](-x), & x<0,\end{cases}\qquad g\in L^2(\mathbb{R}).\tag{5.7}$$

It is now a piece of lengthy, but elementary, analysis to show the following result:

Proposition 5.2. *If, for any* $-1\le\lambda<0$, *we consider the function* $h(\lambda)\in$

$L^\infty(0,\infty)$ given by

$$[h(\lambda)](x)=1-e^{\frac{1}{4}\lambda x^2}, \qquad x>0, \tag{5.8}$$

then

$$\iint_{\mathbb{R}^2} x^{-1}(1-e^{\frac{1}{4}\lambda x^2})\operatorname{sgn}(y)H_{f,g}(x,y)\,dxdy$$

$$=\pi i\langle g, \operatorname{sgn}(Q)\circ[\operatorname{sgn}(P)-\operatorname{erf}\Big(\frac{P}{\sqrt{|\lambda|}}\Big)]f\rangle-2\langle g, \mathcal{K}_{h(\lambda)}(1)f\rangle. \tag{5.9}$$

From these above results, it is now straightforward to show the following result.

Theorem 5.1. *The linear operator $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ is bounded on $L^2(\mathbb{R})$ for any $-1\le\lambda<0$, with*

$$\|\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]\|\le 3\pi+\sqrt{|\lambda|}\,2^{-\frac{5}{4}}, \tag{5.10}$$

and we have the integral representation

$$\langle g, \boldsymbol{\Delta}_{(\lambda,0)}[\varphi]f\rangle$$

$$=\langle g, \boldsymbol{\Delta}_{(0,0)}[\varphi]f\rangle-\pi\langle g, \chi_{(0,\infty)}(P)\circ\Big[\operatorname{sgn}(Q)-\operatorname{erf}\Big(\frac{Q}{\sqrt{|\lambda|}}\Big)\Big]f\rangle+i\langle\mathcal{F}g, \mathcal{K}_{h(\lambda)}(1)\mathcal{F}f\rangle$$

$$+\frac{i}{8}\int_0^{|\lambda|}\Big(\iint_{\mathbb{R}^2} xe^{-\frac{1}{4}\mu x^2}\Big[\operatorname{sgn}(y)-\operatorname{erf}\Big(\frac{y}{\sqrt{\mu}}\Big)\Big]\overline{(\mathcal{F}g)\Big(y+\frac{x}{2}\Big)}(\mathcal{F}f)\Big(y-\frac{x}{2}\Big)dxdy\Big)d\mu. \tag{5.11}$$

We note finally that the map $\lambda\mapsto h(\lambda)$ is continuous from $[-1, 0)$ to $L^\infty(0,\infty)$, and hence the map $\lambda\mapsto\mathcal{K}_{h(\lambda)}(1)$ from $[-1,0)$ to $\mathcal{L}(L^2(\mathbb{R}))$ is norm-continuous. We can also show that the map $\lambda\mapsto\operatorname{erf}\Big(\frac{Q}{\sqrt{|\lambda|}}\Big)$ is norm-continuous on $[-1,0)$, and so we deduce that the map $\lambda\mapsto\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ is norm-continuous on $[-1,0)$, in addition to being weakly continuous on $[-1,0]$. It is not clear whether the map $\lambda\mapsto\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ is norm-continuous at 0, since two of the operators which add together in equation (5.11) to form $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ are, while the other two are not, cancellation of these discontinuities could conceivably occur.
It has taken some considerable analysis to show that $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ is bounded when $-1\le\lambda<0$, and the consequence is that it is highly likely that the upper bound we have established on the norms of these operators is not sharp. This mirrors the currently-known situation for the Weyl quantization $\boldsymbol{\Delta}_{(0,0)}[\varphi]$ itself, where the best that is known at present is that $\pi\le\|\boldsymbol{\Delta}_{(0,0)}[\varphi]\|\le\frac{3\pi}{2}$, whereas we

have good reason to believe that the actual norm of this operator is π.

6. W-Quantization

Having considered the AW-family at some length, it is now the turn of the W-family. The first thing that we have to establish is the framework for a useful quantization scheme. In the case of the AW-family, we could show that $\mathcal{W}_{(\lambda,0)}(\bar{g}\otimes f)$ belonged to $\mathcal{S}(\mathbb{R}^2)$ for all $f, g\in\mathcal{S}(\mathbb{R})$ whenever $-1\leq\lambda\leq 0$. However this is no longer necessarily the case for the W-family, since we now have to deal with increasing Gaussian functions. What we must do, therefore, is define an appropriate space $\Sigma(\mathbb{R})$ of functions, which contains all Hermite-Gaussian functions, such that $\mathcal{W}_{(\lambda,0)}(\bar{g}\otimes f)$ belongs to $\mathcal{S}(\mathbb{R}^2)$ whenever f and g belong to $\Sigma(\mathbb{R})$. To make the correct definition, we introduce the following operators. For any $\alpha\geq 0$, define the linear map e_α: $\mathcal{S}(\mathbb{R})\rightarrow\mathscr{C}^\infty(\mathbb{R})$ by the formula

$$[e_\alpha f](x)=e^{\frac{1}{2}\alpha x^2}f(x),\qquad f\in\mathcal{S}(\mathbb{R}). \tag{6.1}$$

Note that operator e_α leaves a test function smooth, only affecting its fall-off properties. We make the following definition.

Definition 6.1. *By the space $\Sigma(\mathbb{R})$ we shall mean the following subset of $\mathcal{S}(\mathbb{R})$:*

$$\Sigma(\mathbb{R})=\{f\in\mathcal{S}(\mathbb{R}): e_\beta\mathcal{F}e_\alpha f\in\mathcal{S}(\mathbb{R})\ \textit{whenever}\ 0\leq\alpha<1,\ \beta\geq 0,\ (1-\alpha)\beta<1\}. \tag{6.2}$$

The utility of the space $\Sigma(\mathbb{R})$ is based upon the following result.

Lemma 6.1. *The function $\mathcal{W}_{(\alpha,0)}(\bar{g}\otimes f)$ belongs to $\mathcal{S}(\mathbb{R}^2)$ for all $f, g\in\Sigma(\mathbb{R})$ and $0<\alpha<1$.*

Proof. After a number of standard calculations we can show that

$$[\mathcal{W}_{(0,0)}(\bar{g}\otimes f)](a,b)=\sqrt{2\pi}e^{\frac{1}{4}\alpha a^2}\Big[q_{-4\alpha}*_1[(\mathcal{F}q_{-\beta}\otimes 1)\cdot (q_{-4\beta}*_2\mathcal{W}_{(0,0)}(\overline{e_\beta\mathcal{F}e_\alpha f}\otimes e_\beta\mathcal{F}e_\alpha g))]\Big](-b,a)$$

for any $f, g\in\Sigma(\mathbb{R})$, $0<\alpha<1$ and $0<\beta<(1-\alpha)^{-1}$. Since $W_{(0,0)}(a,b)$ is a unitary map, it follows that

$$|(q_{-4\beta}*_2\mathcal{W}_{(0,0)}(\overline{e_\beta\mathcal{F}e_\alpha g}\otimes e_\beta\mathcal{F}e_\alpha f))(p,q)|\leq\|e_\beta\mathcal{F}e_\alpha f\|\cdot\|e_\beta\mathcal{F}e_\alpha g\|,$$

and hence

$$|(\mathcal{W}_{(0,0)}(\bar{g}\otimes f))(p,q)| \leq \frac{1}{\sqrt{1+\alpha\beta}}\exp\left[-\frac{\alpha a^2}{4}-\frac{b^2}{4(\alpha+\beta^{-1})}\right]\|e_\beta\mathcal{F}e_\alpha f\|\cdot\|e_\beta\mathcal{F}e_\alpha g\|.$$

In particular, by setting $\beta=(\alpha^{-1}-\alpha)^{-1}$, we see that

$$(\mathcal{W}_{(\alpha,0)}(\bar{g}\otimes f))(a,b)=e^{\frac{1}{4}\alpha(a^2+b^2)}(\mathcal{W}_{(0,0)}(\bar{g}\otimes f))(a,b)$$

is a bounded function for any $0<\alpha<1$.

By simple generalizations of the results for Weyl quantization, it is possible to show that any function $\mathcal{W}_{(\alpha,0)}(\bar{g}\otimes f)$ is smooth, and moreover any multi-derivative of such a function, and any polynomial times such a function, is a linear combination of functions of the same type. Since we have seen that all functions of this type are bounded, it now follows that every function of this type belongs $\mathcal{S}(\mathbb{R}^2)$, as required. ∎

Consequently we deduce that $\mathcal{G}_{(\lambda,0)}(\bar{g}\otimes f)$ belongs to $\mathcal{S}(\Pi)$ for any $f, g\in\Sigma(\mathbb{R})$ and $0<\lambda<1$. Thus any $T\in\mathcal{S}'(\Pi)$ defines a *sesquilinear form* $\Delta_{(\lambda,0)}[T]$ on $\Sigma(\mathbb{R})$ by the formula

$$[\Delta_{(\lambda,0)}[T]](f,g)=[\![T;\mathcal{G}_{(\lambda,0)}(\bar{f}\otimes g)]\!], \qquad f,g\in\Sigma(\mathbb{R}), \tag{6.3}$$

for any $0<\lambda<1$.

Note that we can only define $\Delta_{(\lambda,0)}[T]$ as a form on $\Sigma(\mathbb{R})$, and not as an operator. This is mainly because we have not attempted to impose a topology on $\Sigma(\mathbb{R})$. Although our definition of $\Sigma(\mathbb{R})$ seems rather restrictive, this space is a dense linear subspace of $\mathcal{S}(\mathbb{R})$, and does contain a large number of useful functions.

Proposition 6.1. *The space $\Sigma(\mathbb{R})$ is a dense linear subspace of $\mathcal{S}(\mathbb{R})$ which contains all the Hermite-Gaussian functions and their translates. In particular, $\Sigma(\mathbb{R})$ contains the generating function G_t of the Hermite-Gaussian functions.*

Proof. For any $n\geq 0$ and $a\in\mathbb{R}$ the translate $(\mathcal{T}_a h_n)(x)$ of the n^{th} Hermite-Gaussian function is equal to $e^{-\frac{1}{2}(x+a)^2}$ times a polynomial of degree n in x. Thus, if $0\leq\alpha<1$, the function $(e_\alpha\mathcal{T}_a h_n)(x)$ is a linear combination of the functions

$$h_j\left(\sqrt{1-\alpha}\left(x+\frac{a}{1-\alpha}\right)\right), \qquad 0\leq j\leq n.$$

Consequently $(\mathcal{F}e_\alpha\mathcal{T}_a h_n)(x)$ is a linear combination of the functions

$$e^{-\frac{iax}{1-\alpha}}h_j\left(\frac{x}{\sqrt{1-\alpha}}\right), \qquad 0\leq j\leq n,$$

and so, if $0<\beta<(1-\alpha)^{-1}$, then

$$(e_\beta \mathscr{F} e_\alpha \mathscr{T}_a h_n)(x) = e^{-\frac{1}{2}[(1-\alpha)^{-1}-\beta]x^2} e^{-\frac{iax}{1-\alpha}} Q_{n,\alpha,\beta}(x),$$

where $Q_{n,\alpha,\beta}(x)$ is a polynomial of degree n in x, and hence $e_\beta \mathscr{F} e_\alpha \mathscr{T}_a h_n$ belongs to $\mathscr{S}(\mathbb{R})$ for all such α, β. Thus we deduce that $\mathscr{T}_a h_n$ belongs to $\Sigma(\mathbb{R})$ for all $a \in \mathbb{R}$ and all $n \geq 0$, and hence $\Sigma(\mathbb{R})$ is a dense linear subspace of $\mathscr{S}(\mathbb{R})$, as required. ■

We can now apply this result to the case of angular quantization. Because $\mathscr{G}_{(\lambda,0)}(\overline{G_s} \otimes G_t)$ exists and belongs to $\mathscr{S}(\Pi)$ for all $s, t \in \mathbb{R}$ and $0<\lambda<1$, we can calculate the numbers

$$\left[\Delta_{(\lambda,0)}[f_{\text{ang}}]\right](G_s, G_t)$$

for any $f \in L^2[-\pi, \pi]$ and $s, t \in \mathbb{R}$, and we can use this result to obtain the matrix coefficients of the sesquilinear form $\Delta_{(\lambda,0)}[f_{\text{ang}}]$ with respect to the Hermite-Gaussian functions. When we perform these calculations, we obtain the following result:

Corollary 6.1. *If $f \in L^2[-\pi, \pi]$ and $0<\lambda<1$ then*

$$\left[\Delta_{(\lambda,0)}[f_{\text{ang}}]\right](h_m, h_n) = \frac{2^{\frac{1}{2}(m+n)}\sqrt{m!n!}\, i^{m-n} A_{m,n}(\lambda)}{(1-\lambda)^{\frac{1}{2}(m+n)}} \hat{f}_{m-n} \tag{6.4}$$

for all $m, n \geq 0$. This is the natural extension of our previous results for the matrix coefficients in the anti Wick ordering case.

However, the fact that these matrix coefficients diverge as λ tends to 1 would indicate that we cannot create an appropriate sesquilinear form $\Delta_{(1,0)}[\varphi]$ on $\Sigma(\mathbb{R})$. Consequently, although we can approach the Wick ordering case arbitrarily closely, we cannot find a successful quantization scheme of this sort which works exactly for Wick ordering.

Recall that the quantization of the phase angle φ was a bounded operator on $L^2(\mathbb{R})$ for all orderings between Weyl and anti-Wick ordering. Assume that for some $0<\lambda<1$ the form $\Delta_{(\lambda,0)}[\varphi]$ defines a linear operator from $\Sigma(\mathbb{R})$ to $L^2(\mathbb{R})$ (which, by an abuse of notation, we shall also denote by $\Delta_{(\lambda,0)}[\varphi]$) such that

$$[\Delta_{(\lambda,0)}[\varphi]](f, g) = \langle f, \Delta_{(\lambda,0)}[\varphi] g\rangle, \qquad f, g \in \Sigma(\mathbb{R}).$$

Then we would deduce that

$$\langle h_n, \Delta_{(\lambda,0)}[\varphi] h_0\rangle = [\Delta_{(\lambda,0)}[\varphi]](h_n, h_0) = (1-\lambda)^{-\frac{1}{2}n} i^n \xi_{m,1} \hat{\varphi}_n, \qquad n \geq 0,$$

where, as in [6], we define

$$\xi_{n;1}=\sqrt{\frac{2^n}{n!}}\Gamma\left(\frac{1}{2}n+1\right), \qquad n\geq 0.$$

We deduce from this that

$$|\langle h_n, \Delta_{(\lambda,0)}[\varphi]h_0\rangle| \sim (1-\lambda)^{-\frac{1}{2}n}n^{-\frac{3}{4}}, \qquad n\to\infty.$$

Hence the sequence $(\langle h_n, \Delta_{(\lambda,0)}[\varphi]h_0\rangle)_{n\geq 0}$ does not belong to l^2 for any $0<\lambda<1$. Hence, although we can define $\Delta_{(\lambda,0)}[\varphi]$ as a sesquilinear form on $\Sigma(\mathbb{R})$, it is not possible to interpret this form as coming from a linear map from $\Sigma(\mathbb{R})$ to $L^2(\mathbb{R})$. In this sense we see that the quantization schemes that we have obtained for the series from Weyl to Wick ordering are much less satisfactory than those schemes we have obtained for the series from anti-Wick to Weyl ordering.

7. Smooth AW-Observables

We have seen that, while we can provide some form of quantization $\Delta_{(\lambda,0)}[\varphi]$ for any $-1\leq\lambda<1$, it is only in the case $-1\leq\lambda\leq 0$ that the resulting quantization takes the form of a unbounded linear operator on $L^2(\mathbb{R})$ (which is in fact bounded). For this section, therefore, we shall again restrict our attention to the case $-1\leq\lambda\leq 0$, and ask whether the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions is preserved under any of these quantizations of phase angle. When this is the case, the map $\Delta_{(\lambda,0)}[\varphi]$ will then provide us with a continuous linear map from the Fréchet space $\mathcal{S}(\mathbb{R})$ to itself.

Since $\mathcal{S}(\mathbb{R})$ is the natural domain for all polynomials in the operators Q and P, its elements have claim to be the set of wave functions which can actually be prepared. Moreover, the usual formulation of quantum mechanics can be recast in terms of operators which leave $\mathcal{S}(\mathbb{R})$ invariant [5]. Of course, not all of the standard operators considered in quantum mechanics have this property, but all operators can be deformed slightly in such a way that the resulting deformations do preserve $\mathcal{S}(\mathbb{R})$.

It is then desirable within this view of quantum mechanics that an operator should leave the space $\mathcal{S}(\mathbb{R})$ invariant, and consequently it is disappointing that the Weyl quantization $\Delta_{(0,0)}[\varphi]$ of the phase angle does not do so. However, since the quantizations $\Delta_{(\lambda,0)}[\varphi]$ are, in some sense, deformations of the Weyl quantization $\Delta_{(0,0)}[\varphi]$, it is worth considering whether any of these anti-Wick quantizations preserve $\mathcal{S}(\mathbb{R})$.

We start with the simplest case, namely that of $\Delta_{(-1,0)}[\varphi]$, and here the result is positive.

Proposition 7.1. *In the antiWick ordering case* $\lambda = -1$ *we have* $\boldsymbol{\Delta}_{(-1,0)}[\varphi] \in \mathcal{L}(\mathcal{S}(\mathbb{R}))$.

Proof. We are going to make use of the seminorms

$$p_k(f) = \|(N+1)^k f\|, \qquad f \in \mathcal{S}(\mathbb{R}),\ k \geq 0,$$

where N is the number operator, which define the usual Fréchet topology on $\mathcal{S}(\mathbb{R})$ (The intersection of the domains $\mathfrak{D}(N^k)$ for all integers $k \geq 0$ is equal to $\mathcal{S}(\mathbb{R})$.)
Direct calculation shows us that

$$\langle h_m, \boldsymbol{\Delta}_{(-1,0)}[\varphi] h_n \rangle = i^{m-n} \xi_{m+n;1} 2^{-\frac{1}{2}(m+n)} \sqrt{\binom{m+n}{m}} \widehat{\varphi}_{m-n}, \quad m, n \geq 0,$$

so, since there exists a constant $A > 0$ such that $\xi_{j,1} \leq A(j+1)^{\frac{1}{4}}$ for all $j \geq 0$, we see that

$$\begin{aligned} |\langle h_m, \boldsymbol{\Delta}_{(-1,0)}[\varphi] h_n \rangle| &\leq A\sqrt{2}\, 2^{-\frac{1}{2}(m+n+1)} \sqrt{\binom{m+n+1}{m}} \sqrt{n+1}\, |\widehat{\varphi}_{m-n}| \\ &\leq A\sqrt{2}\, 2^{-\frac{1}{2}(m+n+1)} \sqrt{\frac{1}{2^{m+n+1}} \binom{m+n+1}{m}} \sqrt{n+1} \end{aligned}$$

for all $m, n \geq 0$. If we define the polynomials.

$$\zeta_k^+(x) = \prod_{j=1}^{k} (x+j), \qquad \zeta_k^-(x) = \prod_{j=0}^{k-1} (x-j), \qquad k \in \mathbb{N},$$

then we see that

$$\zeta_k^-(m) \binom{m+n+1}{m} = \frac{(m+n+1)!}{(m-k)!\,(n+1)!} = \binom{m+n+1}{m-k} \frac{\zeta_{k+1}^+(n)}{n+1},$$

$$m \geq k,\ n \geq 0,$$

and hence

$$\zeta_k^-(m) |\langle h_m, \boldsymbol{\Delta}_{-1,0}(\phi) h_n \rangle|^2 \leq 2A^2 \frac{1}{2^{m+n+1}} \binom{m+n+1}{m-k} \zeta_{k+1}^+(n) \leq 2A^2 \zeta_{k+1}^+(n)$$

for all $m \geq k$ and $n \geq 0$. Since $\zeta_k^-(m) = 0$ for all $0 \leq m \leq k-1$, it follows that

$$|\zeta_k^-(m)|\, |\langle h_m, \boldsymbol{\Delta}_{(-1,0)}[\varphi] h_n \rangle|^2 \leq 2A^2 \zeta_{k+1}^+(n), \qquad k, m, n \geq 0,$$

where we write $\zeta_0^-(x) = 1$. Since $\{\zeta_j^-(x) : j \geq 0\}$ is a basis for the space of all polynomials, for any $k \in \mathbb{N}$ we can find constants $\alpha_k(j)$ for $0 \leq j \leq 2k$ such that

$$(x+1)^{2k} = \sum_{j=0}^{2k} \alpha_k(j) \zeta_j^-(x).$$

Hence

$$(m+1)^{2k}|\langle h_m, \mathbf{\Delta}_{(-1,0)}[\varphi]h_n\rangle|^2 \leq 2A^2\sum_{j=0}^{2k}|\alpha_k(j)|\zeta_{j+1}^{+}(n)$$

$$\leq 2A^2\Big(\sum_{j=0}^{2k}|\alpha_k(j)|\Big)\zeta_{2k+1}^{+}(n)$$

$$\leq 2A^2\Big(\sum_{j=0}^{2k}|\alpha_k(j)|\Big)(n+2k+1)^{2k+1},$$

so we can certainly find a constant $A[k]>0$ such that

$$(m+1)^{2k}|\langle h_m, \mathbf{\Delta}_{(-1,0)}[\varphi]h_n\rangle|^2 \leq A[k]^2(n+1)^{2k+2}, \qquad m, n\geq 0.$$

Thus we deduce that $\mathbf{\Delta}_{(-1,0)}[\varphi]h_n\in\mathcal{S}(\mathbb{R})$ for all $n\geq 0$, with

$$p_k(\mathbf{\Delta}_{(-1,0)}[\varphi]h_n)^2 \leq \frac{1}{6}\pi^2 A[k+1]^2(n+1)^{2k+4}, \qquad k\geq 0.$$

Hence, if $f\in\mathcal{S}(\mathbb{R})$ then $\mathbf{\Delta}_{(-1,0)}[\varphi]f\in\mathcal{S}(\mathbb{R})$, with

$$p_k(\mathbf{\Delta}_{(-1,0)}[\varphi]f) \leq \frac{\pi^2}{6}A[k+1]p_{k+3}(f)$$

for any $k\geq 0$, which establishes the required result. ■

In previous sections, it proved an important and useful technique to compare $\mathbf{\Delta}_{(\lambda,0)}$ with $\mathbf{\Delta}_{(0,0)}$. For example we showed (essentially) that $\mathbf{\Delta}_{(\lambda,0)}[T] = \mathbf{\Delta}_{(0,0)}[T * Q_\lambda]$, and proved that $\mathbf{\Delta}_{(\lambda,0)}[\varphi]$ was bounded by explicitly calculating the difference between $\mathbf{\Delta}_{(\lambda,0)}[\varphi]$ and $\mathbf{\Delta}_{(0,0)}[\varphi]$. However, this technique is of no use to us here, since we already know [6] that $\mathbf{\Delta}_{(0,0)}[\varphi]$ does not leave $\mathcal{S}(\mathbb{R})$ invariant. However, since we have just shown that $\mathbf{\Delta}_{(-1,0)}[\varphi]$ is an endomorphism of $\mathcal{S}(\mathbb{R})$, it will be interesting, and will prove useful, to compare $\mathbf{\Delta}_{(\lambda,0)}[\varphi]$ with $\mathbf{\Delta}_{(-1,0)}[\varphi]$ instead.

Lemma 7.1. *We have that*

$$\langle f, \mathbf{\Delta}_{(\lambda,0)}[\varphi]g\rangle =$$

$$\langle f, \mathbf{\Delta}_{(-1,0)}[\varphi]g\rangle + \frac{\pi}{2}\langle f, \Big[\mathrm{erf}\Big(\frac{Q}{\sqrt{|\lambda|}}\Big)-\mathrm{erf}(Q)\Big]g\rangle + \frac{i}{8}\int_{|\lambda|}^{1}\langle\mathscr{F}f, Y_{\mu,1}\mathscr{F}g\rangle\,d\mu, \qquad (7.1)$$

where the operator $Y_{\mu,n}$ is defined by the formula

$$[Y_{\mu,n}g](p) = \int_{\mathbb{R}}(p-q)^n e^{-\frac{1}{4}\mu(p-q)^2}\mathrm{erf}\Big(\frac{p+q}{2\sqrt{\mu}}\Big)g(q)\,dq \qquad (7.2)$$

for any $n\geq 0$, $\mu>0$ and $g\in\mathcal{S}(\mathbb{R})$.

The proof is a simple consequence of identities already given in this paper, and we shall omit it. We already know that the function $\boldsymbol{\Delta}_{(-1,0)}[\varphi]$ belongs to $\mathscr{L}(\mathscr{S}(\mathbb{R}))$, and it is clear, since the error function is smooth and all of its derivatives are bounded, that the map $\operatorname{erf}\left(\frac{Q}{\sqrt{|\lambda|}}\right) - \operatorname{erf}(Q)$ also belongs to $\mathscr{L}(\mathscr{S}(\mathbb{R}))$. Thus we only need concentrate on the last term.

Proposition 7.2. *The function $Y_{\mu,n}g$ belongs to $\mathscr{S}(\mathbb{R})$ for any $\mu>0$, $n\geq 0$ and $g\in\mathscr{S}(\mathbb{R})$.*

Proof. Since simple calculations show us that

$$p[Y_{\mu,n}g](p) = [Y_{\mu,n+1}g](p) + [Y_{\mu,n}Qg](p),$$
$$[Y_{\mu,n}g]'(p) = 2n[Y_{\mu,n-1}g](p) - [Y_{\mu,n+1}g](p) - [Y_{\mu,n}g'](p),$$

for any $\mu>0$, $n\geq 0$ and $g\in\mathscr{S}(\mathbb{R})$, it is clear that each such function $Y_{\mu,n}g$ is smooth, and it merely remains to show that each such function is bounded to deduce the result. Now since

$$|[Y_{\mu,n}g](p)| \leq \|g\|_\infty \int_{\mathbb{R}} |p-q|^n e^{-\frac{1}{4}\mu(p-q)^2} dp = \frac{2}{\mu^{\frac{1}{2}(n+1)}} \|g\|_\infty \int_0^\infty q^n e^{-\frac{1}{4}q^2} dq,$$

the result follows. ■

From this it is easy to show that the function

$$p \mapsto \int_\lambda^1 \mu^\alpha [Y_{\mu,n}g](p)\, d\mu$$

belongs to $\mathscr{S}(\mathbb{R})$ for any $\mu>0$, $\alpha\in\mathbb{R}$, $g\in\mathscr{S}(\mathbb{R})$, $n\geq 0$ and $0<\lambda\leq 1$, and so it follows that $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]g\in\mathscr{S}(\mathbb{R})$ for any $-1\leq\lambda<0$ and $g\in\mathscr{S}(\mathbb{R})$. Since $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ is self-adjoint as an element of $\mathscr{L}(L^2(\mathbb{R}))$, it is symmetric as an endomorphism of $\mathscr{S}(\mathbb{R})$, and so, by the Helliger-Toeplitz Theorem, it is a continuous endomorphism of $\mathscr{S}(\mathbb{R})$. We have proved the following result:

Theorem 7.1. *The operator $\boldsymbol{\Delta}_{(\lambda,0)}[\varphi]$ belongs to $\mathscr{L}(\mathscr{S}(\mathbb{R}))$ for any $-1\leq\lambda<0$.*

8. PQ-Quantization

Having considered quantization for the WAW-family, we turn now to the question of quantization for the PQ-family $\boldsymbol{\Delta}_{(0,\mu)}$ where $-1\leq\mu\leq 1$. Proceeding in the same manner as for WAW-quantization we can show that:

$$[\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)](p,q) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{g\left(q+(1+\mu)\frac{x}{2}\right)}\, f\left(q-(1-\mu)\frac{x}{2}\right) e^{ixp} dx. \quad (8.1)$$

Consequently $\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)$ belongs to $\mathscr{S}(\Pi)$ for all $f, g\in\mathscr{S}(\mathbb{R})$, and the map $(f, g)\mapsto\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)$ from $\mathscr{S}(\mathbb{R})\times\mathscr{S}(\mathbb{R})$ to $\mathscr{S}(\Pi)$ is jointly continuous and sesquilinear; hence the existence of PQ-quantization presents no difficulties, and we obtain the quantization scheme $\varDelta_{(0,\mu)}$: $\mathscr{S}'(\Pi)\to\mathscr{L}(\mathscr{S}(\mathbb{R}), \mathscr{S}'(\mathbb{R}))$ given by the formula

$$[\![\varDelta_{(0,\mu)}[T]f;\bar{g}]\!]=[\![T;\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)]\!],\qquad T\in\mathscr{S}'(\Pi),\ f, g\in\mathscr{S}(\mathbb{R}).\quad(8.2)$$

Having established existence, we consider the properties of the behaviour of these quantization schemes with respect to marginals and to the taking of adjoints. From the identities

$$\int_{\mathbb{R}}[\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)](p, q)\,dp=\overline{g(q)}f(q),\quad(8.3.\mathrm{a})$$

$$\int_{\mathbb{R}}[\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)](p, q)\,dq=\overline{(\mathscr{F}g)(p)}(\mathscr{F}f)(p),\quad(8.3.\mathrm{b})$$

for $f, g\in\mathscr{S}(\mathbb{R})$ we deduce that

$$\varDelta_{(0,\mu)}[1\otimes h]=h(Q),\quad(8.4.\mathrm{a})$$

$$\varDelta_{(0,\mu)}[h\otimes 1]=h(P),\quad(8.4.\mathrm{b})$$

for any $h\in\mathscr{S}'(\mathbb{R})$. This demonstrates that the PQ-quantizations have the correct marginals, as well as being well-defined in the $\mathscr{S}$-class based scheme. However they do not behave well with respect to the taking of adjoints and so are, in these various respects, complementary to the WAW-quantizations. To determine the behaviour of these quantizations with respect to taking adjoints, we note that

$$\overline{[\mathscr{G}_{(0,\mu)}(\bar{g}\otimes f)](p, q)}=[\mathscr{G}_{(0,-\mu)}(\bar{f}\otimes g)](p, q)\quad(8.5)$$

for $f, g\in\mathscr{S}(\mathbb{R})$, and hence it follows that

$$[\![\varDelta_{(0,\mu)}[T^*]f;\bar{g}]\!]=\overline{[\![\varDelta_{(0,-\mu)}[T]g;\bar{f}]\!]}\quad(8.6)$$

for $T\in\mathscr{S}'(\Pi)$ and $f, g\in\mathscr{S}(\mathbb{R})$. This result can be interpreted most clearly in the case where $\varDelta_{(0,\mu)}[T]$ is a bounded operator on $L^2(\mathbb{R})$, in which case we see that

$$\varDelta_{(0,\mu)}[T]^*=\varDelta_{(0,-\mu)}[T^*].\quad(8.7)$$

Except in very special cases, therefore, or in general in the case of Weyl quantization, the operator $\varDelta_{(0,\mu)}[T]$ will not be self-adjoint if $T=T^*$.

Note that this result, while disappointing from the point of view of self-adjointness, has certain advantages, for it gives us a method of deducing a property for the quantization scheme $\varDelta_{(0,-\mu)}$ from a corresponding property for the scheme $\varDelta_{(0,\mu)}$. In general, then, it is only necessary to consider half the

PQ-family, say the range; $-1\leq\mu\leq 0$.

We turn now to the particular case of the phase operator. We start by showing that the operator $\boldsymbol{\Delta}_{(0,-1)}[\varphi]$ is bounded (and hence $\boldsymbol{\Delta}_{(0,1)}[\varphi]$ is bounded as well). It will be observed that even this result is by no means straightforward to prove. The analysis begins with the observation that

$$[\mathscr{G}_{(0,-1)}(\bar{f}\otimes g)](p,q)=\frac{1}{\sqrt{2\pi}}\overline{f(q)}(\mathscr{F}g)(p)e^{ipq}, \qquad f,g\in\mathscr{S}(\mathbb{R}). \quad (8.8)$$

and this in turn leads to the particularly simple expression

$$[\boldsymbol{\Delta}_{(0,-1)}[T]g](q)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}T(p,q)(\mathscr{F}g)(p)e^{ipq}dp, \quad T\in\mathscr{S}'(\Pi), g\in\mathscr{S}(\mathbb{R}). \quad (8.9)$$

We recall the Fresnel integrals

$$C(x)=\int_0^x\cos\left(\frac{\pi}{2}t^2\right)dt, \qquad S(x)=\int_0^x\sin\left(\frac{\pi}{2}t^2\right)dt, \quad (8.10)$$

noting that C and S are continuous odd functions, both of which tend to 1 as x tends to ∞, so there exists a constant $K>0$ such that

$$|C(x)+iS(x)|<K, \qquad x\in\mathbb{R}.$$

Some standard analysis enables us to establish the following result

Lemma 8.1. *If $\xi>0$, and if $h\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ is such that $\mathscr{F}^{-1}h\in L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}}\operatorname{sgn}(p)e^{\frac{ip^2}{2\xi}}(\mathscr{F}^{-1}h)(p)dp=-\sqrt{2\xi}\int_{\mathbb{R}}e^{-\frac{1}{2}\xi x^2}h(x)(C+iS)\left(x\sqrt{\frac{\xi}{\pi}}\right)dx. \quad (8.11)$$

This technical lemma turns out to be what we want to prove the boundedness of $\boldsymbol{\Delta}_{(0,-1)}[\varphi]$. To proceed further we shall have to employ geometric and group theoretic methods, as we did when proving similar results for Weyl quantization. By this we mean using the unitary rescaling map $E_\xi\in\mathscr{L}(L^2(\mathbb{R}))$ defined for any $\xi>0$ by

$$[E_\xi f](x)=\frac{1}{\sqrt{\xi}}f\left(\frac{x}{\xi}\right), \qquad f\in L^2(\mathbb{R}), \quad (8.12)$$

and the metaplectic unitary transformations $U_\xi\in\mathscr{L}(L^2(\mathbb{R}))$ for any real ξ, originally introduced in [6], where

$$[U_\xi f](x)=e^{-\frac{1}{2}i\xi x^2}f(x), \qquad f\in L^2(\mathbb{R}). \quad (8.13)$$

Then if $\xi>0$ and $f,g\in\mathscr{S}(\mathbb{R})$, we see that

$$\iint_{\mathbb{R}^2} \operatorname{sgn}(p+\xi q)\operatorname{sgn}(q)\,[\mathscr{G}_{(0,-1)}(\bar{g}\otimes f)]\,(p,q)\,dp\,dq$$

$$=-\sqrt{2}\int_{\mathbb{R}} e^{-\frac{1}{2}i\xi x^2}(C+iS)\left(x\sqrt{\frac{\xi}{\pi}}\right)(\mathscr{F}\operatorname{sgn}(Q)\,U_{\xi^{-1}}E_{\xi}\bar{g})\,(x)\,(\mathscr{F}U_{\xi^{-1}}\mathscr{F}g)\,(x)\,dx,$$

so that

$$\left|\iint_{\mathbb{R}^2} \operatorname{sgn}(p+\xi q)\operatorname{sgn}(q)\,[\mathscr{G}_{(0,-1)}(\bar{g}\otimes f)]\,(p,q)\,dp\,dq\right| \leq K\sqrt{2}\|f\|\cdot\|g\|.$$

Taking the complex conjugate of this inequality, we also deduce that

$$\left|\iint_{\mathbb{R}^2} \operatorname{sgn}(p-\xi q)\operatorname{sgn}(q)\,[\mathscr{G}_{(0,-1)}(\bar{g}\otimes f)]\,(p,q)\,dp\,dq\right| \leq K\sqrt{2}\|f\|\cdot\|g\|.$$

In earlier work on Weyl quantization we needed to consider the distributions D_α for $0\leq\alpha\leq\pi$ which are symplectic distortions of sgn $\otimes$sgn, namely

$$D_\alpha(r\cos\beta,\, r\sin\beta)=\begin{cases} 1 & r>0,\ 0\leq\beta\leq\alpha \\ 1 & r>0,\ -\pi\leq\beta\leq-\pi+\alpha \\ -1 & r>0,\ \alpha<\beta<\pi \\ -1 & r>0,\ -\pi+\alpha<\beta<0 \\ 0 & r=0 \end{cases} \tag{8.14}$$

so that

$$D_{\frac{\pi}{2}-\arctan\xi}(p,q)=\operatorname{sgn}(p-\xi q)\operatorname{sgn}(q)$$

for any $\xi\in\mathbb{R}$. From the above we now deduce that

$$\|[\mathbf{\Delta}_{(0,-1)}[D_\alpha]f,\bar{g}]\| \leq K\sqrt{2}\|f\|\cdot\|g\|, \qquad f,g\in\mathscr{S}(\mathbb{R}),$$

for $0<\alpha<\frac{\pi}{2}$ and $\frac{\pi}{2}<\alpha<\pi$. Clearly

$$\mathbf{\Delta}_{(0,-1)}[D_0]=-I, \qquad \mathbf{\Delta}_{(0,-1)}[D_{\frac{\pi}{2}}]=\operatorname{sgn}(Q)\circ\operatorname{sgn}(P), \qquad \mathbf{\Delta}_{(0,-1)}[D_\pi]=I,$$

so we deduce that the family $\{\mathbf{\Delta}_{(0,-1)}[D_\alpha]: 0\leq\alpha\leq\pi\}$ is a uniformly bounded subset of $\mathscr{L}(L^2(\mathbb{R}))$, and since

$$\mathbf{\Delta}_{(0,-1)}[\varphi]=\frac{1}{2}\pi\operatorname{sgn}(Q)-\frac{1}{2}\int_0^\pi \mathbf{\Delta}_{(0,-1)}[D_\alpha]\,d\alpha,$$

it follows that the operator $\mathbf{\Delta}_{(0,-1)}[\varphi]$ is bounded.

Proving the general result that $\mathbf{\Delta}_{(0,\mu)}[\varphi]$ is bounded for any $-1<\mu<1$ is more complicated. We shall omit the details. It is possible, however, to extend the techniques used in [6] to establish the following identity:

$$[\![\boldsymbol{\Delta}_{(0,\mu)}[D_{\frac{1}{2}\pi-\arctan\xi}]f;\ \bar{g}\ ]\!]$$

$$=2[1-i\,\mathrm{sgn}(\mu\xi)]\int_{\mathbb{R}}\Big[C+i\,\mathrm{sgn}(\mu\xi)S\Big]\Big(\frac{y}{\sqrt{2|\mu\xi|\pi}}\Big)\overline{[\mathscr{F}\mathrm{sgn}(Q)U_{\xi}g](y)}\,[\mathscr{F}U_{\xi}f](y)\,dy$$

$$-\frac{2i}{\pi}\langle U_{\xi}g,\ \mathscr{K}_{E_{\mu\xi}}\Big(\frac{1-\mu}{1+\mu}\Big)U_{\xi}f\rangle \tag{8.15}$$

for any $0<|\mu|<1$ and $\xi\in\mathbb{R}\backslash\{0\}$, where the function $E_{\mu\xi}\in L^{\infty}(0,\infty)$ is given by the formula

$$E_{\mu\xi}(q)=e^{-\frac{1}{2}i\mu\xi q^{2}}. \tag{8.16}$$

From this it is now possible to show that the family of operators $\{\boldsymbol{\Delta}_{(0,\mu)}[D_{\alpha}]:0\leq\alpha\leq\pi\}$ is uniformly bounded for any $-1<\mu<1$, and hence it follows that $\boldsymbol{\Delta}_{(0,\mu)}[\varphi]$ is bounded for any $-1<\mu<1$. Indeed, we can deduce from this that the family of maps $\{\boldsymbol{\Delta}_{(0,\mu)}[\varphi]:-1\leq\mu\leq 1\}$ is also uniformly bounded.

However, as has already been indicated, the adjoint of $\boldsymbol{\Delta}_{(0,-1)}[\varphi]$ is equal to $\boldsymbol{\Delta}_{(0,1)}[\varphi]$, and so one does not expect $\boldsymbol{\Delta}_{(0,-1)}[\varphi]$ to be self-adjoint. We can prove this by calculating the expectation value of this operator in the state h_0. For

$$[\boldsymbol{\Delta}_{(0,-1)}[\varphi]h_0](q)=\frac{\pi}{2}\mathrm{sgn}(q)h_0(q)-\frac{i}{2}h_0(q)\int_0^{\infty}e^{-\frac{1}{2}x^2}x^{-1}[1-e^{-2x|q|}]\,dx,$$

so that

$$\langle h_0,\ \boldsymbol{\Delta}_{(0,-1)}[\varphi]h_0\rangle=-\frac{i}{2}\int_0^{\infty}e^{-\frac{1}{2}x^2}x^{-1}[1-e^{x^2}\mathrm{erfc}(x)]\,dx.$$

Now since

$$e^{x^2}\mathrm{erfc}(x)<1$$

for all $x>0$, it is clear that $\langle h_0,\ \boldsymbol{\Delta}_{(0,-1)}[\varphi]h_0\rangle$ is not real, and hence $\boldsymbol{\Delta}_{(0,-1)}[\varphi]$ is not self-adjoint, as required. Finally, we note that this sort of quantization does not behave well in respect of radial functions. For example, we see that

$$[\boldsymbol{\Delta}_{(0,-1)}[(h_0)_{\mathrm{rad}}]h_0](q)=\frac{1}{\sqrt{2\pi}}e^{-\frac{3}{4}q^2},$$

and so $\boldsymbol{\Delta}_{(0,-1)}[(h_0)_{\mathrm{rad}}]h_0$ is not a scalar multiple of h_0, and hence $\boldsymbol{\Delta}_{(0,-1)}[(h_0)_{\mathrm{rad}}]$ is not diagonal with respect to the Hermite-Gaussian functions.

9. Conclusion

Thus we have seen that the Weyl quantization enjoys a special position amongst the various quantization schemes that we have considered, in that it is the only one of the various schemes which enjoys all of the following properties:

- $\mathbf{\Delta}_{(0,0)}$ provides the correct marginal distributions,
- $\mathbf{\Delta}_{(0,0)}[T]$ is self-adjoint whenever T is real,
- $\mathbf{\Delta}_{(0,0)}[f_{\mathrm{rad}}]$ is diagonal with respect to the Hermite-Gaussians for all suitable functions f,
- $\mathbf{\Delta}_{(0,0)}[\varphi]$ is a bounded operator on $L^2(\mathbb{R})$.

After the Weyl quantization scheme, the AW-quantization are the most well-behaved, satisfying all of the above properties except for the first one. However the AW-quantization have the added property that $\mathbf{\Delta}_{(\lambda,0)}[\varphi]$ is also a continuous endomorphism of $\mathcal{S}(\mathbb{R})$ for $-1 \leq \lambda < 0$. Although we still prefer to use Weyl quantization, particularly in the light of its central role in the laser model as discussed by Alli and Sewell [1], in view of general view that quantum mechanics should be expressed in terms of observables which are endomorphisms of $\mathcal{S}(\mathbb{R})$. the fact that $\mathbf{\Delta}_{(0,0)}[\varphi]$ can be approximated, at least weakly, by AW-quantizations of φ is of interest in itself.

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Errata

Correction to Vol. 35, No. 1:

Daniel A. DUBIN, Mark A. HENNINGS and Thomas B. SMITH, "Existence theorems for ordered variants of Weyl quantization", pp. 1-29.

page 1; the first line of the footnote should be replaced by

Communicated by T. Kawai, October 28, 1997. Revised April 7, 1998.

We sincerely apologize for the misprint.

Editors