

Generic and q -Rational Representation Theory

By

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Abstract

Part I of this paper develops various general concepts in generic representation and cohomology theories. Roughly speaking, we provide a general theory of orders in non-semisimple algebras applicable to problems in the representation theory of finite and algebraic groups, and we formalize the notion of a "generic" property in representation theory. Part II makes new contributions to the non-describing representation theory of finite general linear groups. First, we present an explicit Morita equivalence connecting $GL_n(q)$ with the theory of q -Schur algebras, extending a unipotent block equivalence of Takeuchi [T]. Second, we apply this Morita equivalence to study the cohomology groups $H^*(GL_n(q), L)$, when L is an irreducible module in non-describing characteristic. The generic theory of Part I then yields stability results for various groups $H^i(GL_n(q), L)$, reminiscent of our general theory [CPSK] with van der Kallen of generic cohomology in the describing characteristic case. (In turn, the stable value of such a cohomology group can be expressed in terms of the cohomology of the affine Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$.) The arguments entail new applications of the theory of tilting modules for q -Schur algebras. In particular, we obtain new complexes involving tilting modules associated to endomorphism algebras obtained from general finite Coxeter groups.

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This paper falls naturally into two parts. Part I (§§1-7, 13) introduces and develops some general concepts in generic representation theory. These ideas arose out of (and are applied to) issues in Part II (§§8-12), which deals with the non-describing characteristic representation and cohomology theories of the finite general linear groups $GL_n(q)$. We open here with a discussion of this latter topic.

Let G be a reductive group defined and split over a prime field \mathbb{F}_r . Assume that G has simply connected derived group G' . Fix $q=r^d$ and consider the split Chevalley group $G(q) = G(\mathbb{F}_q)$ of \mathbb{F}_q -rational points. Now let k be an algebraically closed field of positive characteristic p . The modular representation theory of the finite groups $G(q)$ over k naturally breaks into two cases—the describing characteristic theory in which $p=r$, and the *non-describing* characteristic theory in which $p \neq r$. In both cases, the central issues include the classification of the irreducible modules, the determination of their characters (or decomposition numbers), and other representation-theoretic properties (e.g., cohomology and submodule structure of natural modules):

• In the describing characteristic theory, Steinberg's pioneering work shows that the classification and characters of the irreducible modules are obtained (for all $q=p^d$) from the solution of the analogous problem for the ambient algebraic group. In that case, the characters are known generically, i.e., if p is sufficiently large (depending on the root type of G), thanks to [AJS], [KL2] and [KT].¹ Similarly, questions concerning the $\mathbb{F}_q G(q)$ -cohomology and submodule structure for natural modules can often be studied, and even explicitly answered for sufficiently large q or p , by passing to analogous problems for G [CPS1], [CPSK], [FP1,2].

• In the non-describing characteristic theory, however, even a natural parametrization of the irreducible modules remains problematic for some types. But, when $G=GL_n$, there is a parametrization of the irreducible modules, due to Dipper-James [DJ2], [D1]. Further, their work establishes that the Brauer characters of the irreducible $kG(q)$ -modules are determined by decomposition numbers of certain q^a -Schur algebras along with characteristic 0 character formulas due to Green [Gr]. In cohomology theory, the study of the groups $H^0(G(q), L)$ (L simple) has largely focused on the untwisted $L=k$ case, e.g., in famous work of Quillen and others (cf. [AM] for discussion and references).

Part II of this paper makes new contributions to the non-describing

¹However, no sufficient lower bound is known for p in any interesting cases.

characteristic theory in the case $G = GL_n$. In particular, we extend the work of Dipper and James described above to obtain a more intimate connection between the representation theory of $kGL_n(q)$ and that of q -Schur algebras. Namely, Theorem 9.17 presents a *Morita equivalence*:

$$(1) \quad kG(q)/J(q)_k \underset{\text{Morita}}{\sim} \bigoplus_{s \in \mathcal{C}_{ss'}} \bigotimes_{t=1}^{m(s)} S_{q^{e_t(s)}}(n_t(s), n_t(s))_k$$

between an algebra which is a direct sum of tensor products of certain q^a -Schur algebras and a quotient algebra $kG(q)/J(q)_k$ of $kG(q)$ by an ideal $J(q)_k \triangleleft kG(q)$. (See §9 for further explanation of notation.) Furthermore, the algebras $kG(q)$ and $kG(q)/J(q)_k$ have the same irreducible modules. A similar Morita equivalence has been obtained by Takeuchi [T] in the special case of unipoint blocks (in which case the right-hand side of (1) is replaced by a single q -Schur algebra).² Our Morita equivalence is derived from a second Morita equivalence:

$$(2) \quad \mathcal{O}G(q)/J(q) \underset{\text{Morita}}{\sim} \bigoplus_{s \in \mathcal{C}_{ss'}} \bigotimes_{t=1}^{m(s)} S_{q^{e_t(s)}}(n_t(s), n_t(s))_{\mathcal{O}},$$

over a discrete valuation ring \mathcal{O} with residue field k . (Here $J(q)$ is an ideal in $\mathcal{O}G(q)$ such that $J(q) \otimes_{\mathcal{O}} k$ is an ideal—namely, the ideal $J(q)_k$ of (1)—in $kG(q) = \mathcal{O}G(q) \otimes_{\mathcal{O}} k$.) Cast this way, the correspondence between irreducible $kGL_n(q)$ -modules and their decomposition numbers and the corresponding issues for certain q^a -Schur algebras becomes very conceptual—see (9.17(d)). The further connection between the representation theory of q -Schur algebras and the quantum general linear group $GL_{n,q}(k)$ explains at least part of the title of this paper.³ We emphasize that these results rely heavily on the work [DJ2] as well as that of Fong-Srinivasan [FS].

We wish to use the above Morita equivalence to study the cohomology groups $H^\bullet(GL_n(q), L)$, when L is an irreducible module in a non-describing characteristic p . When p divides $|G(q)|$, the algebra $kG(q)/J(q)_k$ has finite global dimension, while $kG(q)$ has infinite global dimension. Hence, the precise relation between the cohomology of $GL_n(q)$ and that of q -Schur algebras by means of the Morita equivalence (1) is quite subtle. Sections 10, 11, 12, are devoted to attacking this problem. To our knowledge, our results represent the first progress in understanding the cohomology of finite reductive groups with general irreducible *non-trivial* coefficients L in non-describing characteristic. (It would be interesting to pursue Dwyer’s general stability results [Dw] with respect to n with these coefficients in mind.)

²An announcement of our results here appears in the proceedings of the 1997 Newton Institute NATO conference [CPS8]. After our announcement was submitted, another generalization of Takeuchi’s result was given in a preprint of Dipper [D3]. His result involves a double centralizer property, but does not include our Morita equivalence.

³As we hope this introduction will make clear, the title also recalls our paper [CPSK], written with Wilberd van der Kallen. In that paper, somewhat similar topics were discussed in the describing characteristic case.

For applications to finite groups, e.g., to the study of maximal subgroups, the explicit calculation of 1-cohomology for quasi-simple groups plays an important role [AS]. In this case, our results are particularly strong and quite easy to obtain directly from the Morita theorem in §9. This work is presented in §10. For example, Theorem 10.1 establishes a direct connection between H^1 -calculations for $GL_n(q)$ and Ext^1 -calculations for q -Schur algebras. In Theorem 10.2, we use the work in Part I discussed below to prove, for a fixed n , and p sufficiently large (depending on n), that there are only finitely many possible values for $H^1(GL_n(q), L)$, when L is irreducible (or has bounded composition length), independent of q . The value is the same for all irreducible modules with the same parametrization, and all q of the same multiplicative order in $\overline{\mathbb{F}}_p$. The calculation of this p -stable value can then be translated, using the generic results of Part I, first to an Ext^1 -calculation for a q -Schur algebra over the field \mathbb{C} of complex numbers (taking q to be an appropriate root of unity), and then to an Ext^1 -calculation in the category of integrable modules for the quantum enveloping algebra $\tilde{U}_q(\mathfrak{gl}_n)$. Using [KL2], that calculation translates into an equivalent Ext^1 -calculation involving cohomology of the affine Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$ (see (10.3)). We expect that a precise answer can be obtained in the context of affine algebras,⁴ but we do not consider that problem (for either Ext^1 or the higher Ext^i discussed below) in this paper. Finally, Theorem 10.5 computes 1-cohomology for $SL_n(q)$ in terms of the cohomology of $GL_n(q)$ in most cases.

In §11 we begin to attack the higher cohomology groups of $GL_n(q)$ with nontrivial coefficients, especially irreducible modules. The idea, of course, is to exploit the above Morita equivalence and study instead questions for the q -Schur algebra. To that end, we develop an important projective resolution of the q -determinant representation of the q -Schur algebra $S_q(n, n)$. The result hinges ultimately on earlier work of Deodhar [De] on Hecke algebra complexes (reminiscent of the Solomon-Tits complex for the associated Tits building). But to bring these results to bear on our problem, we require the tilting theory work of [DPS3] for q -Schur algebras over the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials. In the process, we obtain another interesting complex, valid for endomorphism algebras associated to arbitrary finite Coxeter systems (W, S) ; see Theorem 11.10.

We apply the results of §§9, 11 in §12. For example, Theorem 12.4 establishes that the cohomology groups $H^i(GL_n(q), L)$ can be computed in a range, subject to arithmetic conditions on q and i , in terms of the cohomology of the q -Schur algebra $S_q(n, n)$. In order to achieve this goal, it is necessary to

⁴We have in mind formulas like those in [CPS4; §3]. For example, under the assumption of the Lusztig conjecture for the characters of the modular irreducible rational representations of a reductive group G , the groups $\text{Ext}_{\mathcal{O}}^i(L_1, L_2)$, for L_1, L_2 irreducible modules with regular highest weights in the Jantzen region, can be explicitly given in terms of Kazhdan-Lusztig polynomials [CPS4; (3.9.1)]. Similar remarks apply to the BGG category \mathcal{O} for a complex semisimple Lie algebra \mathfrak{g} and *all* irreducible modules; cf. [CPS4; (3.8.2)].

reinterpret the projective resolution of \det_q given in §11 as a complex of $GL_n(q)$ -modules, using the Morita equivalence of §9. This is done in the technical Lemma 12.1, and it leads to the arithmetic conditions on the cohomology calculations. As an application, Corollary 12.6 also presents a stability result for higher cohomology, similar to, but somewhat weaker than, the H^1 -stability result given in (10.2). As with the H^1 -result above, the stable value of a given $H^i(GL_n(q), L)$ can be expressed in terms of the cohomology of complex q -Schur algebras and quantum enveloping algebras, and then in terms of the cohomology of the affine Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$.

Generic cohomology and representation theory study behavior which stabilizes for large values of the parameters q, p , etc. For example, one can ask if, for sufficiently large p , there is some explicit formula for the Brauer characters of the irreducible $kGL_n(q)$ -modules, given in terms of ordinary characters in the same spirit as in the describing characteristic case (as proved by [AJS]). The answer is affirmative, and much easier in the non-describing case here—in fact, this observation has been made essentially in [GH; (10.2)]. We include a proof in §8 for completeness. As in the describing characteristic case, explicit formulas in terms of Kazhdan-Lusztig polynomials (and ordinary characters of $GL_n(q)$) can be given, using the work [KL2].

More generally, Part I of this paper undertakes an examination of generic representation and cohomology theory. Our development is new, though the methods are all quite easy. Roughly speaking, this work provides a general theory of orders in non-semisimple algebras, suitable for application to problems related to the representation theory of finite and algebraic groups. For example, let P be a property of finite dimensional algebras A over fields. Given a domain \mathcal{O} with quotient field K and an algebra A over \mathcal{O} , suppose that P holds for the K -algebra A_K . Then we call P *generic* with respect to \mathcal{O} provided there is a nonempty open subset $\Omega \subseteq \text{Spec } \mathcal{O}$ such that P holds for the residue algebras $A_{k(\mathfrak{p})}$ for all $\mathfrak{p} \in \Omega$. Section 1 studies this and related notions quite generally.

Sections 2,3 study the genericity of various cohomological properties for algebras over domains \mathcal{O} . For example, Theorem 2.1 states that, given finitely generated A -modules M, N , and an integer $m \geq 0$, there exists a nonempty open subset $\Omega \subseteq \text{Spec } \mathcal{O}$ such that, for $\mathfrak{p} \in \Omega$,

$$\dim \text{Ext}_{A_K}^m(M_K, N_K) = \dim \text{Ext}_{A_{k(\mathfrak{p})}}^m(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})}).$$

The stability results obtained in Part II (i.e., (8.6), (10.2) and (12.6)) are eventual applications of the above formula.

The paper contains a number of further general applications of our Ext-results; for example, we establish (in §3) that the property of being Koszul is generic, at least in the presence of a finite global dimension assumption.

Section 4 studies the genericity in the various senses of §1 of quasi-hereditary algebras. Given an algebra A such that A_K is quasi-hereditary, we observe that the Kazhdan-Lusztig polynomials associated to A_K also exhibit a generic property, so that the various parity conditions studied in [CPS3,4] are also generic phenomena. Finally, §§5-6 return to general considerations

involving Morita equivalence (in § 5) and derived equivalence (in § 6), while § 7 takes up the theory of quivers and relations. The appendix §13 introduces and briefly discusses the “constructible” topology. By using it, we are able to prove a new constructibility result for subsets of schemes arising in generic representation theory. Along the way, we recapture a classical theorem of Chevalley.

Part I: Generic Representation Theory

In the first seven sections (Part I) of this paper, \mathcal{O} will denote a commutative, Noetherian domain. (Unless we state otherwise, a “domain” will always be assumed to be commutative and Noetherian.) In the remaining sections, we shall place further restrictions on \mathcal{O} as needed. (In particular, in Part II, \mathcal{O} will be a suitable discrete valuation ring.) Let $X = \text{Spec } \mathcal{O}$. For $\mathfrak{p} \in X$, let $\mathcal{O}_{\mathfrak{p}}$ be the localization of \mathcal{O} at \mathfrak{p} and let $k(\mathfrak{p}) = \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ be the associated residue field. Often we write $K = k(0)$ (the quotient field of \mathcal{O}). For $0 \neq f \in \mathcal{O}$, let X_f be the basic open set consisting of those $\mathfrak{p} \in X$ satisfying $f \notin \mathfrak{p}$. If we write \mathcal{O}_f for the localization of \mathcal{O} at powers of f , then $X_f = \text{Spec } \mathcal{O}_f$.

Unless otherwise indicated, all \mathcal{O} -modules will be assumed to be finitely generated over \mathcal{O} (i.e., they are \mathcal{O} -finite). Similarly, \mathcal{O} -algebras A will be assumed to be \mathcal{O} -finite as modules. In particular, A -modules are all finitely generated as A -modules, unless we say otherwise. (We freely repeat these assumptions for the sake of clarity.) Let $A\text{-mod}$ (resp., $\text{mod-}A$) be the category of finitely generated left (resp., right) A -modules. If M is an \mathcal{O} -module and $\mathcal{O} \rightarrow R$ is a morphism of commutative rings, then $M_R = M \otimes_{\mathcal{O}} R$ is the R -module obtained by base change. When $0 \neq f \in \mathcal{O}$ (resp., $\mathfrak{p} \in X$), we contract the notation for the module $M_{\mathcal{O}_f}$ (resp., $M_{\mathcal{O}_{\mathfrak{p}}}$) to M_f (resp., $M_{\mathfrak{p}}$). A morphism $M \xrightarrow{g} N$ induces morphisms $M_f \xrightarrow{g_f} N_f$ and $M_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} N_{\mathfrak{p}}$ by base change.

§ 1. Representation Theory on an Open Set

We shall use continually, and often without comment, the following well known and elementary lemmas. We include proofs for the convenience of the reader.

Lemma 1.1. *Let V be a vector space over K and M an \mathcal{O} -submodule of V such that $KM = V$. Then the natural map*

$$\mu : M \otimes_{\mathcal{O}} K \rightarrow V : m \otimes t \mapsto tm$$

is an isomorphism of vector spaces. If W is any K -subspace of V , the restriction of μ to $(W \cap M)_K$ defines an isomorphism $(W \cap M)_K \cong W$.

Proof. Since M spans V , it contains a K -basis for V , hence a free \mathcal{O} -submodule N whose quotient M/N is an \mathcal{O} -torsion module. The exactness of localization determines a short exact sequence

$$0 \rightarrow N \otimes_{\mathcal{O}} K \rightarrow M \otimes_{\mathcal{O}} K \rightarrow M/N \otimes_{\mathcal{O}} K = 0 \rightarrow 0.$$

The first statement follows from this and the isomorphism $N \otimes_{\mathcal{O}} K \cong V$.

For the second statement, we may regard $(W \cap M)_K$ as a subspace of W . If $w \in W$, then w is a linear combination of elements of M , and for some $a \in \mathcal{O}$, $aw \in M$. Thus, $w \in (W \cap M)_K$, so $(W \cap M)_K = W$. \square

Lemma 1.2. ([Ma, p. 185; CE, Ch. VII]) *If M is a (finite) \mathcal{O} -module, then there exists a non-zero element $a \in \mathcal{O}$ such that M_a is a free \mathcal{O}_a -module. If M is an arbitrary torsion-free \mathcal{O} -module, then the canonical map $M \rightarrow M_K : m \mapsto m \otimes 1$ is an injection.*

Proof. If $T(M)$ denotes the torsion submodule of M , then there is a non-zero element $b \in \mathcal{O}$ which annihilates $T(M)$, hence $M_b \cong (M/T(M))_b$ is \mathcal{O}_b -torsion-free. The canonical map $M_b \rightarrow M_K$ becomes an isomorphism after base change by $-\otimes_{\mathcal{O}_b} K$. Let $\mathcal{B} = \{b_1, \dots, b_r\}$ be a subset of M_b which is mapped bijectively onto a K -basis \mathcal{B}_K of M_K . Then the \mathcal{O}_b -submodule $N = \mathcal{O}_b \mathcal{B}$ of M_b is free. Since $M_{bK} \cong M_K$, the quotient module M_b/N is a finite \mathcal{O}_b -torsion module, hence is annihilated by a non-zero element $a \in \mathcal{O}b$. Thus, $N_a \cong M_a$ is a free \mathcal{O}_a -module.

The final assertion of the lemma is clear from the definition of the localization of a module. \square

Corollary 1.3. *If M is a finite \mathcal{O} -module and $X \subset M_K$ is a finite subset of M_K , then there exists a non-zero $a \in \mathcal{O}$ such that, under the canonical map $M_a \rightarrow M_K$, M_a identifies with a free \mathcal{O}_a -submodule of M_K containing X .*

Proof. Choose $0 \neq b \in \mathcal{O}$ so that M_b is \mathcal{O}_b -free. Identify M_b with its image in M_K and consider the \mathcal{O}_b -submodule M'_b generated by M_b together with the finite set X . Then M'_b/M_b is a finite \mathcal{O}_b -torsion module. Thus, there exists $0 \neq a \in \mathcal{O}b$ such that $(M'_b/M_b)_a = 0$. This implies $X \subset M_a$. \square

Corollary 1.4. *Let A be an \mathcal{O} -algebra and M, N two A -modules which are \mathcal{O} -finite. If $g : M_K \rightarrow N_K$ is a homomorphism of A_K -modules, then there exists a non-zero $a \in \mathcal{O}$ and an A_a -module morphism $h : M_a \rightarrow N_a$ such that:*

- (a) M_a (resp., N_a) is an \mathcal{O}_a -free A_a -submodule of M_K (resp., N_K);
- (b) $h_K = g$.

If, in addition, $g = f_K$ for an A -module morphism $f : M \rightarrow N$, then

- (c) $h = f_a$.

Proof. Choose $0 \neq b \in \mathcal{O}$ so that M_b (resp., N_b) is an \mathcal{O}_b -free A_b -submodule of M_K (resp., N_K). If X is an \mathcal{O}_b -basis of M_b , then there exists $0 \neq a \in \mathcal{O}b$ such that N_a contains $g(X)$. Then M_a, N_a satisfy (a) and the restriction of g to M_b defines the required map $h : M_a \rightarrow N_a$ of part (b). If $g = f_K$ for a morphism $f : M \rightarrow N$, then the functoriality of localization implies that the restriction of g to M_a agrees with f_a proving (c). \square

Let P be a property of finite dimensional algebras over fields. Let A be a finite \mathcal{O} -algebra which is \mathcal{O} -torsion-free. There are at least two ways to say that “ P holds generically” for A :

(a) P holds for A_K ; or

(b) There exists a nonempty open subset $\Omega \subseteq X$ such that P holds for $A_{k(\mathfrak{p})}$ for each $\mathfrak{p} \in \Omega$.

Of course, (b) implies (a), and often (a) implies (b). When this latter implication holds for all \mathcal{O} -algebras A as above, the property P is called *generic with respect to \mathcal{O}* . If condition P is true for all \mathcal{O} in a class \mathcal{C} of domains, then call P *generic with respect to \mathcal{C}* . (Example: Any property P for algebras over fields is generic w.r.t. the class \mathcal{C} of discrete valuation rings.) If \mathcal{C} consists of all (commutative, Noetherian) domains, then P is *generic*.

If P is property of algebras over domains (and not just fields), there is yet a third way to say that P holds generically for A :

(c) There exists a nonempty open subset $\Omega \subseteq X = \text{Spec } \mathcal{O}$ such that P holds for the \mathcal{O}_f -algebra A_f for each non-zero $f \in \mathcal{O}$ satisfying $X_f \subseteq \Omega$. Equivalently, there is non-zero ideal $I \subseteq A$ such that P holds for A_f for each non-zero $f \in I$.

Property P is *integrally generic* for a domain \mathcal{O} if, whenever (a) above holds, then (c) holds with respect to some nonempty affine open subset Ω of X .

Similarly, we can define the notions of “integrally generic with respect to a class of domains,” and “integrally generic”.

There are many other variations possible when P is defined for algebras over domains (e.g., one could ask if P holds for the localization $A_{\mathfrak{p}}$, for all $\mathfrak{p} \in \Omega$, ...). However, we shall be content with the notions given, which seem the most useful.

To give a simple example, consider the property M_n : “ A is isomorphic to the $n \times n$ matrix algebra $M_n(\mathcal{O})$ over \mathcal{O} .” Then M_n makes sense for all domains \mathcal{O} . If $A_K \cong M_n(K)$, then for some non-zero $h \in \mathcal{O}$, the matrix units e_{ij} of A_K lie in A_h . Thus, we can regard $M_n(\mathcal{O}_h)$ as a submodule of A_h with torsion quotient, so $A_f \cong M_n(\mathcal{O}_f)$ for some non-zero $f \in \mathcal{O}_h$. It follows that M_n is an integrally generic property. If $A_f \cong M_n(\mathcal{O}_f)$, then $A_{k(\mathfrak{p})} \cong M_n(k(\mathfrak{p}))$ for any prime ideal \mathfrak{p} satisfying $f \notin \mathfrak{p}$. Hence, M_n is also a generic property. Similarly, the property that an algebra be split semisimple, i.e., that it is a direct product of matrix algebras, is both generic and integrally generic. However, we will see below in (1.7) that the property of being semisimple is *not* generic.

There are appropriate variations on the above theme for a property P which may hold for pairs (or triples, ...) of algebras, or for modules (or pairs of modules, ...) for an algebra (over a field or a domain). For example, let MORITA be the property on pairs (A, B) of algebras which asserts that A and B are Morita equivalent. Similarly, P might be a property holding for a morphism between modules, or it might be a property holding for a complex of modules, Rather than write down a plethora of formal definitions, the following remarks present some basic examples along this line.

Examples 1.5. (a) Given a morphism $M \xrightarrow{g} N$ of modules (for an algebra), the property that g is surjective (resp., an injection, an isomorphism) is denoted $\text{SUR}(g)$ (resp., $\text{INJ}(g)$, $\text{ISO}(g)$). Now assume g is a morphism of modules for an algebra A over \mathcal{O} .

Suppose first that $\text{SUR}(g_K)$ holds and N is \mathcal{O} -finite. By (1.4), we may choose $0 \neq b \in \mathcal{O}$ so that M_b (resp., N_b) is a free \mathcal{O}_b -submodule of M_K (resp., N_K). If $X \subset M_K$ is a finite set mapped bijectively onto an \mathcal{O}_b -basis of N_b by g_K , then, by (1.3), we may choose $0 \neq a \in \mathcal{O}b$ such that $X \subset M_a$. It follows that for $0 \neq c \in \mathcal{O}a$ the map $g_c: M_c \rightarrow N_c$ is surjective. In particular $\text{SUR}(g_c)$ is true for these elements c . Also $\text{SUR}(g_{k(\mathfrak{p})})$ holds for any $\mathfrak{p} \in X_a$. Thus, the property SUR of morphisms of A -modules with \mathcal{O} -finite codomains is both generic and integrally generic.

Second, suppose that $\text{INJ}(g_K)$ holds and both modules M, N are finite over \mathcal{O} . We choose $0 \neq a \in \mathcal{O}$ so that the canonical map $M_a \rightarrow M_K$ is an injection. The functoriality of localization implies $g_a: M_a \rightarrow N_a$ is also injective. We now choose $0 \neq b \in \mathcal{O}a$ so that $(N_a/g_a(M_a))_b$ is a free \mathcal{O}_b -module. Then each map $g_c, 0 \neq c \in \mathcal{O}b$, is a split injection of \mathcal{O}_c -modules. Consequently, for $\mathfrak{p} \in X_b$, $g_{\mathfrak{p}}: M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is also a split injection of $\mathcal{O}_{\mathfrak{p}}$ -modules. This implies that $g_{k(\mathfrak{p})}$ is an injective $A_{k(\mathfrak{p})}$ -homomorphism for such \mathfrak{p} . Therefore, INJ is both a generic and an integrally generic property for morphisms between A -modules which are \mathcal{O} -finite.

The statements for SUR and INJ , taken together, show that ISO is both generic and integrally generic for morphisms between A -modules which are \mathcal{O} -finite.

(b) The property SSUR (resp., SINJ) that a morphism of modules is a split surjection (resp., split injection) is both generic and integrally generic for A -modules finite over \mathcal{O} . Suppose that $M \xrightarrow{g} N$ is a morphism such that g_K is a split surjection with section map $s_K: N_K \rightarrow M_K$. By (1.4) we may choose $0 \neq a \in \mathcal{O}$ so that M_a (resp., N_a) is an \mathcal{O}_a -free A_a -submodule of M_K (resp., N_K). In addition, we may assume that the restriction h of s_K to N_a defines an A_a -homomorphism $h: N_a \rightarrow M_a$. Evidently h is a section for the map g_a . Further h_c is section for the map g_c for each $0 \neq c \in \mathcal{O}a$. For $\mathfrak{p} \in X_a$, the functoriality of base change by $k(\mathfrak{p})$ implies that $h_{k(\mathfrak{p})}$ is a section for $g_{k(\mathfrak{p})}$. This establishes our claim for SSUR ; a similar proof handles the property SINJ .

(c) Suppose that A is an \mathcal{O} -algebra and M, N are A -modules. If $M_K \cong N_K$, then $M_{k(\mathfrak{p})}$ and $N_{k(\mathfrak{p})}$ have the same composition factors for all \mathfrak{p} belonging to some nonempty open subset Ω of X . In fact, by (1.4), there is an isomorphism $g: M_a \xrightarrow{\sim} N_a$ of \mathcal{O}_a -free A_a -modules for some non-zero $a \in \mathcal{O}$. Further, $g_{k(\mathfrak{p})}: M_{k(\mathfrak{p})} \rightarrow N_{k(\mathfrak{p})}$ is an isomorphism for all $\mathfrak{p} \in X_a$; *a fortiori* the two modules have the same composition factors.

(d) Let A be as in (c). Suppose that V is a finite dimensional A_K -module. Any finite A -module M which is \mathcal{O} -free and satisfies $M_K \cong V$ is called an A -lattice for V . Choose a finite A -submodule M of V such that $M_K \cong V$. By (1.2), there exists a non-zero $a \in \mathcal{O}$ such that M_a is \mathcal{O} -free. Thus, M_a is an A_a -lattice for V . Suppose $0 \neq b \in \mathcal{O}$ and we are given an A_b -lattice N for V .

Then (c) implies there exists a nonempty open set $\Omega \subseteq X_a \cap X_b$ such that for $\mathfrak{p} \in \Omega$ the $A_{k(\mathfrak{p})}$ -modules $M_{k(\mathfrak{p})}$ and $N_{k(\mathfrak{p})}$ are isomorphic (and hence have the same composition factors).

For the remainder of these examples, we assume that A is \mathcal{O} -finite and torsion-free.

(e) Let FREE (resp., PROJ, PGEN) denote the property that a (finitely generated) module for an algebra is free (resp., projective, a projective generator). Each of these properties is generic and integrally generic.

If F is an A -module such that F_K is a free A_K -module, then (1.3) implies there is a non-zero $a \in \mathcal{O}$ such that F_K is an A_K -submodule of F_K containing an A_K -basis $\mathcal{B}_K = \{b_1, \dots, b_r\}$ for F_a . The set \mathcal{B}_a generates a free A_a -submodule F' of F_a . The quotient F_a/F' is a finite \mathcal{O}_a -torsion module. Thus there is $0 \neq b \in \mathcal{O}_a$ such that $F_b = F'_b$ is a free A_b -submodule of F_K . Clearly, for $0 \neq c \in \mathcal{O}_b$, F_c remains free as an A_c -module. Further, for $\mathfrak{p} \in X_b$, $F_{k(\mathfrak{p})}$ is free as well.

Now suppose P is an A -module and P_K is a projective A_K -module. Then there is a free A -module F and a split surjection $F_K \xrightarrow{g} P_K$. By (1.4) and (1.3) there exists $0 \neq a \in \mathcal{O}$ such that F_a (resp., P_a) are A_a -submodules of F_K (resp., P_K), F_a is free as an A_a -module, and there is a map $h: F_a \rightarrow P_a$ such that $h_K = g$. Applying (b) to the \mathcal{O}_a -algebra A_a shows that h is generically and integrally generically a split surjection. Hence PROJ is a generic and integrally generic property.

Finally, since PROJ is generic and integrally generic, (b) above implies that the property PGEN that a module is a projective generator for a finite \mathcal{O} -algebra is both generic and integrally generic.

(f) Let $P_n^K \xrightarrow{d_{n-1}^K} \cdots \xrightarrow{d_0^K} P_0^K$ be a finite complex of finite dimensional A_K -modules. There exists a finite complex $P_n \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} P_0$ of A -modules which identifies with the original complex P_n^K after applying the base change $-\otimes_{\mathcal{O}} K$. If the P_i^K are A_K -projective, then there exists a non-zero $f \in \mathcal{O}$ such that each P_{if} is a projective A_f -module and free as a \mathcal{O}_f -module.

To see this, use induction on the length n of the complex. If $n=0$, let P_0 be the A -submodule generated by a finite set of A_K -generators of P_0^K . Applying induction to the complex $C_\bullet: P_n^K \xrightarrow{d_{n-1}^K} \cdots \xrightarrow{d_0^K} P_1^K$, we assume there exists an A -module subcomplex $P_n \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_1$ of which identifies with C_\bullet on base change to K . Let P_0 be the A -submodule of P_0^K generated by the image of P_1 in P_0^K , together with a K -basis for P_0^K . By (1.1), P_{0K} identifies with P_0^K . Finally, we define $d_0: P_1 \rightarrow P_0$ to be the restriction of d_0^K to P_1 .

The remaining statement follows by applying example (e) to each term in the complex P_\bullet .

(g) Let M_i , $0 \leq i \leq r$, be \mathcal{O} -torsion-free A -modules. Suppose there is given an exact complex

$$(1.5.1) \quad M_{rK} \xrightarrow{d_{r-1}^K} M_{r-1K} \xrightarrow{d_{r-2}^K} \cdots \xrightarrow{d_0^K} M_{0K}.$$

Then there is a basic open subset $X_f \subset X$ such that the restrictions d_i^f of d_i^K to M_{i+1f} define an exact complex

$$(1.5.2) \quad M_{rf} \xrightarrow{d_{r-1}^f} M_{r-1f} \xrightarrow{d_{r-2}^f} \cdots \xrightarrow{d_0^f} M_{0f}$$

of A_f -modules. Moreover, we may assume the sequence remains exact upon tensoring with any residue field $k(\mathfrak{p})$, $\mathfrak{p} \in X_f$.

To see this, we proceed as follows. By (1.2), we may assume, after identification, that each M_i is an A -submodule of M_{iK} . By (1.4), may choose $0 \neq a \in \mathcal{O}$ such that the restriction d_i^a of d_i^K to M_{i+1a} has image in M_{ia} . The exactness of (1.5.1), together with the fact that the M_i are all torsion-free, implies that the sequence of modules and homomorphisms

$$M_{ra} \xrightarrow{d_{r-1}^a} M_{r-1a} \xrightarrow{d_{r-2}^a} \cdots \xrightarrow{d_0^a} M_{0a}$$

is a complex.

Assume $r=2$ and let R_a denote the kernel of d_0^a . Because localization is an exact functor, we may identify R_K with the kernel of d_0^K . Since $d_1^K: M_{2K} \rightarrow R_K$ is surjective, and SUR is an integrally generic property, there is a non-zero multiple f of a such that $d_1^f: M_{2f} \rightarrow R_f$ is surjective. As with d_0^K , the exactness of localization implies R_f is the kernel of d_0^f . Thus the first statement is true for $r=2$. The general statement follows easily by induction. The second assertion is also clear, since by (1.2), we can assume that all the kernels and cokernels in (1.5.2) are free \mathcal{O}_f -modules.

We now study the generic properties of being separable and split (semisimple). For simplicity, we will assume that the algebra A is \mathcal{O} -torsion-free. Recall that A is *separable* provided that A is a projective $A^e = A \otimes A^{\text{op}}$ -module; equivalently, A is separable if the multiplication map $A \otimes A \xrightarrow{\pi} A$ is a split surjection of (A, A) -bimodules. We say that A is *split*, if it is isomorphic to a direct sum of full matrix algebras over \mathcal{O} . If A is split, then it is also separable: if $A = M_n(\mathcal{O})$, the map $A \xrightarrow{\sigma} A \otimes A$ given on matrix units by $e_{ij} \mapsto e_{i1} \otimes e_{1j}$ is a bimodule map splitting π above. We say that A is *nil-separable* (resp., *nil-split*) provided that there exists a nilpotent ideal N of A such that A/N is a separable (resp., split) algebra over \mathcal{O} . Let SPLIT (resp., SEP, NILSPLIT, and NILSEP) denote the property split (resp., separable, nil-split, and nil-separable) for algebra A over a domain \mathcal{O} . The ideal N for either NILSPLIT or NILSEP must necessarily be the Jacobson radical of A .

Lemma 1.6. *The properties SPLIT, SEP, NILSPLIT, and NILSEP are generic and integrally generic properties.*

Proof. We have already indicated earlier (above (1.5)) that SPLIT is

both a generic and an integrally generic property.

Suppose the algebra A is \mathcal{O} -finite and torsion-free. Assume that A_K is separable, so that the multiplication map $A_K \otimes A_K \xrightarrow{\pi} A_K$ is a split surjection as a (A_K, A_K) -bimodule morphism. By (1.5 (b)), the SSUR property of a morphism is both generic and integrally generic; whence, SEP is both generic and integrally generic.

Now assume that A_K is nil-split, so there exists a nilpotent ideal N_K of A_K such that A_K/N_K is a split semisimple algebra over K . The ideal $N = N_K \cap A$ of A satisfies $N \otimes_{\mathcal{O}K} K \cong N_K$ (so there is no ambiguity in notation) and $(A/N)_K \cong A_K/N_K$. Thus, there is a nonempty open set $\Omega \subseteq X$ such that $(A/N)_f \cong A_f/N_f$ is split if $X_f \subseteq \Omega$. It follows that NILSPLIT is an integrally generic property. We can also assume that $(A/N)_{k(\mathfrak{p})}$ is split $k(\mathfrak{p})$ -algebra for all $\mathfrak{p} \in \Omega$. By generic freeness, we can further assume, after possibly shrinking Ω , that $N_{\mathfrak{p}}$ is an $\mathcal{O}_{\mathfrak{p}}$ direct summand of $A_{\mathfrak{p}}$. Thus, for $\mathfrak{p} \in \Omega$, $N_{k(\mathfrak{p})}$ is a nilpotent ideal in $A_{k(\mathfrak{p})}$ satisfying $A_{k(\mathfrak{p})}/N_{k(\mathfrak{p})} \cong (A/N)_{k(\mathfrak{p})}$. This proves that NILSPLIT is generic.

A similar argument establishes that NILSEP is generic and integrally generic. \square

Example 1.7. The property that an algebra be semisimple (over a field) is not a generic property, although the properties that an algebra be split semisimple or separable are both generic. For example, let k be an algebraically closed field of positive characteristic p , and let $\mathcal{O} = k[t]$ be the polynomial algebra over k in an indeterminate t . Consider the \mathcal{O} -algebra $A = k[t^{1/p}]$. It is free of rank p over \mathcal{O} . Then A_K is a purely inseparable field extension of the function field $K = k(t)$, obtained by adjoining a p th root of t . However, let \mathfrak{p} be any non-zero prime ideal in \mathcal{O} . Then $\mathfrak{p} = (t - \lambda)$ for some $\lambda \in k$. The commutative $k(\mathfrak{p})$ -algebra $A_{k(\mathfrak{p})}$ is a free $k(\mathfrak{p})$ -module of rank p which is generated as a $k(\mathfrak{p})$ -algebra by the nilpotent element $t^{1/p} - \lambda^{1/p}$. Hence, $A_{k(\mathfrak{p})}$ is not semisimple, and “semisimple” is not generic.

We conclude this section by indicating some generic module-theoretic properties. We continue to assume that A is a fixed algebra which is finite and torsion-free over \mathcal{O} .

Lemma 1.8. *Assume that A_K is nil-separable. If M is an A -module such that M_K is a completely reducible A_K -module, then there exists a nonempty open subset $\Omega \subseteq X$ such that $M_{k(\mathfrak{p})}$ is a completely reducible $A_{k(\mathfrak{p})}$ -module for all $\mathfrak{p} \in \Omega$.*

Proof. The argument in (1.6) shows that there is an ideal N of A and a nonempty open subset Ω of X such that if $\mathfrak{p} \in \Omega$ then $N_{k(\mathfrak{p})}$ is the nilpotent radical of $A_{k(\mathfrak{p})}$ and $(A/N)_{k(\mathfrak{p})} \cong A_{k(\mathfrak{p})}/N_{k(\mathfrak{p})}$ is a separable $k(\mathfrak{p})$ -algebra. Since M_K is a completely reducible A_K -module, it follows that $N_K M_K = 0$. By shrinking Ω if necessary, we can assume that $M_{\mathfrak{p}}$ is $\mathcal{O}_{\mathfrak{p}}$ -torsion-free, hence is an $A_{\mathfrak{p}}$ -submodule of M_K . Thus, $N_{\mathfrak{p}} M_{\mathfrak{p}} = 0$. Then $N_{k(\mathfrak{p})} M_{k(\mathfrak{p})} = N_{\mathfrak{p}} M_{k(\mathfrak{p})} = 0$, so that $M_{k(\mathfrak{p})}$ is a completely reducible $A_{k(\mathfrak{p})}$ -module. \square

Lemma 1.9. *Assume that A_K is nil-split. Let $L_i^K, i=1, \dots, n$, be the distinct irreducible A_K -modules. There exists a non-zero $f \in \mathcal{O}$ such that each L_i^K has an A_f -lattice L_i^f and such that for $\mathfrak{p} \in X_f$, the set $\{L_{i\mathfrak{p}}^f, \dots, L_{n\mathfrak{p}}^f\}$ is a complete set of representatives for the isomorphism classes of irreducible $A_{k(\mathfrak{p})}$ -modules. Each $L_{i\mathfrak{p}}^f$ can be assumed to be absolutely irreducible.*

Proof. Let $\{a_j\}$ be a basis for A_K formed by first taking a basis $\{a_1, \dots, a_m\}$ for a Wedderburn complement S consisting of matrix units from each simple factor, and second taking a basis $\{a_{m+1}, \dots, a_n\}$ for $\text{rad}(A_K)$. For some non-zero $f \in \mathcal{O}$, A_f is a free \mathcal{O}_f -module with basis $\{a_i\}$. Let S' be the \mathcal{O}_f -subalgebra of A_f generated by $\{a_1, \dots, a_m\}$, so that S' is a direct sum of matrix algebras of the form $M_c(\mathcal{O}_f)$, $c \in \mathbb{Z}^+$. Also, $\{a_{m+1}, \dots, a_n\}$ is a basis for $A_f \cap \text{rad}(A_K)$. We have:

$$\begin{cases} A_f = S' \oplus (A_f \cap \text{rad}(A_K)), \\ S'_K \cong S, \text{ and} \\ (A_f \cap \text{rad}(A_K))_K \cong \text{rad}(A_K). \end{cases}$$

The lemma follows immediately from these observations. □

Lemma 1.10. *Assume that A_K is nil-split, and let $0 = M_0^K \subset M_1^K \subset \dots \subset M_n^K = M^K$ be a composition series of an A_K -module M^K . If $M \in \text{Ob}(A\text{-mod})$ is any \mathcal{O} -lattice for M^K , there is a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_n = M$ and a nonempty open subset $\Omega \subseteq X$ such that:*

- (a) $M_{iK} \cong M_i^K$ for all i ;
- (b) For $\mathfrak{p} \in \Omega$ and any extension field $E \supseteq k(\mathfrak{p})$, $0 = M_{0E} \subset M_{1E} \subset \dots \subset M_{nE} = M_E$ is a composition series of M_E .
- (c) For $0 < i \leq n$, set $L_i = M_i/M_{i-1}$. If $0 < i, j \leq n$, then $L_{iE} \cong L_{jE}$ if, and only if, $L_{iK} \cong L_{jK}$.

Proof. For each i , let $M_i = M_i^K \cap M$, so that $M_{iK} \cong M_i^K$. Choose a non-zero $f \in \mathcal{O}$ such that each

$$\tilde{M}_f \cong \bigoplus M_{if}/M_{i-1f}$$

is \mathcal{O}_f -free. If $L_i = M_i/M_{i-1}$, the L_{iK} are the composition factors of M_K . If $L_{iK} \cong L_{jK}$, we can assume by (1.5a) that $L_i \cong L_j$. The lemma is now clear from (1.9).

The following result will be used below and again in § 4.

Lemma 1.11. *Let M, N be \mathcal{O} -torsion-free A -modules. There exists a nonempty open set $\Omega \subseteq X = \text{Spec } \mathcal{O}$ such that if $\mathfrak{p} \in \Omega$,*

$$\text{Hom}_{A_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})}) \cong \text{Hom}_A(M, N)_{k(\mathfrak{p})}.$$

Also, if B is an \mathcal{O} -algebra such that $B_K \cong \text{End}_{A_K}(M_K)$, then we can assume that $B_{k(\mathfrak{p})} \cong \text{End}_{A_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})})$ for all $\mathfrak{p} \in \Omega$.

Proof. (Sketch) Let $0 \rightarrow L \rightarrow Q \rightarrow M \rightarrow 0$ be a short exact sequence in which Q is projective. Let X be the natural image of $\text{Hom}_A(Q, N)$ in $\text{Hom}_A(L, N)$. For any non-zero $f \in \mathcal{O}$, we have $\text{Hom}_A(M, N)_f \cong \text{Hom}_{A_f}(M_f, N_f)$. So by (1.2), after replacing \mathcal{O} by a suitable localization (of the form \mathcal{O}_f), we can assume that X is \mathcal{O} -free. Certainly, $\text{Hom}_A(Q, N)_k = \text{Hom}_{A_f}(Q_k, N_k)$ for $k = k(\mathfrak{p})$, $\mathfrak{p} \in \text{Spec } \mathcal{O}$, because Q is projective. Because the exact sequence $0 \rightarrow \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(Q, N) \rightarrow X \rightarrow 0$ is \mathcal{O} -split, $\text{Hom}_A(M, N)_k \subseteq \text{Hom}_{A_k}(M_k, N_k)$ for any such k . Similarly, localizing \mathcal{O} even further we can assume that $\text{Hom}_A(L, N)_k \subseteq \text{Hom}_{A_k}(L_k, N_k)$. (Let L play the role of M in the previous discussion.) Now an elementary diagram chase (using the snake lemma) proves the first assertion of the lemma. We leave the second easy assertion to the reader. \square

Lemma 1.12. *Let $M \in \text{Ob}(A\text{-mod})$ be \mathcal{O} -torsion-free with $M_K = \bigoplus_{i=1}^n M_i^K$ a decomposition of M_K into a direct sum of A_K -submodules. Put $M_i = M \cap M_i^K$. Then $M_{iK} \cong M_i^K$, and there exists a nonempty open subset Ω of X such that $M_f = \bigoplus_{i=1}^n M_{if}$ whenever $X_f \subseteq \Omega$ and $M_{k(\mathfrak{p})} = \bigoplus_{i=1}^n M_{k(\mathfrak{p})}^i$ whenever $\mathfrak{p} \in \Omega$. Moreover, if each M_i^K is absolutely indecomposable, we may assume that each $M_{ik(\mathfrak{p})}$ is absolutely indecomposable and that for each i, j , $M_{ik(\mathfrak{p})} \cong M_{jk(\mathfrak{p})}$ if and only if $M_{iK} \cong M_{jK}$.*

Proof. This result is clear provided we observe that if M is an \mathcal{O} -torsion-free A -module such that M_K is absolutely indecomposable, then there is a nonempty open subset Ω of X such that if $\mathfrak{p} \in \Omega$, $M_{k(\mathfrak{p})}$ is an absolutely indecomposable $A_{k(\mathfrak{p})}$ -module. But let $E = \text{End}_A(M)$. Then $E_K \cong \text{End}_{A_K}(M_K)$ is a nil-split algebra with radical quotient K . Thus, here is a nonempty open subset Ω such that if $\mathfrak{p} \in \Omega$ then $E_{k(\mathfrak{p})}$ is nil-split with radical quotient $k(\mathfrak{p})$ by (1.9). By (1.11), we can assume that $E_{k(\mathfrak{p})} \cong \text{End}_{A_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})})$. \square

§ 2. Ext Calculations: the Ungraded Case

In this section, we continue to assume that A is an algebra which is \mathcal{O} -finite and torsion-free for some domain \mathcal{O} . The main result below shows that for two fixed objects M, N in $A\text{-mod}$, the dimensions of the spaces $\text{Ext}_{A_{k(\mathfrak{p})}}^n(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})})$ are generically constant for n in a finite range. As an immediate consequence, the property FGLDIM that an algebra B over a field k has finite global dimension is a generic property.

Theorem 2.1. *Let $M, N \in \text{Ob}(A\text{-mod})$. If m is a non-negative integer, then there is a nonempty open subset $\Omega_m \subset X$, depending on m , such that for $\mathfrak{p} \in \Omega_m$, any extension field E of the residue field $k(\mathfrak{p})$, and $0 \leq n \leq m$*

$$\dim \text{Ext}_{A_E}^n(M_E, N_E) = \dim \text{Ext}_{A_K}^n(M_K, N_K).$$

Proof. Let $F^\bullet \rightarrow M$ denote a resolution of M by free A -modules of finite rank and let $d^i: F^{i+1} \rightarrow F^i$ denote the i^{th} differential of this resolution.

Let E be any commutative \mathcal{O} -algebra. Since the terms of F^\bullet are finite and

A -free, we have natural isomorphisms of complexes

$$(2.1.1) \quad \mathrm{Hom}_A(F^\bullet, N)_E \cong \mathrm{Hom}_A(F^\bullet, N_E) \cong \mathrm{Hom}_{A_E}(F_E^\bullet, N_E).$$

Let $C_{\bullet E}$ denote the truncated complex

$$(2.1.2) \quad 0 \rightarrow \mathrm{Hom}_{A_E}(F_E^0, N_E) \xrightarrow{d_E^{0*}} \mathrm{Hom}_{A_E}(F_E^1, N_E) \xrightarrow{d_E^{1*}} \cdots \\ \xrightarrow{d_E^{m-1*}} \mathrm{Hom}_{A_E}(F_E^m, N_E) \xrightarrow{d_E^{m*}} \mathrm{Im}(d_E^{m*}) \rightarrow 0.$$

When $E = \mathcal{O}$, we delete the subscripts. Thus, C_\bullet denotes the truncation of the original complex $\mathrm{Hom}_A(F^\bullet, N)$ constructed in parallel with (2.1.2).

By (1.2) we can choose $0 \neq f \in \mathcal{O}$ so that the terms of the resolution $F_f^\bullet \rightarrow M_f$, each term of the complex $C_{\bullet f}$, the kernels and images of the differentials of $C_{\bullet f}$, and the cohomology modules $H^\bullet(C_{\bullet f})$ are all free \mathcal{O}_f -modules.

If $\mathfrak{p} \in X_f = \Omega$ and E is an extension field of $k(\mathfrak{p})$, it follows that $F_E^\bullet \rightarrow M_E$ is a free resolution of the A_E -module M_E . Using the isomorphisms (2.1.1), it follows that the truncated complex $C_{\bullet E}$ defined in (2.1.2) is obtained from $C_{\bullet f}$ by base change from \mathcal{O}_f to E . Hence, for $0 \leq n \leq m$,

$$\dim \mathrm{Ext}_{A_E}^n(M_E, N_E) = \dim H^n(C_{\bullet E}) = \mathrm{rank}_{\mathcal{O}_f} H^n(C_{\bullet f}). \quad \square$$

The above result will be applied in Part II to q -Schur algebras and to the problem of calculating the cohomology of $GL_n(q)$ in non-describing characteristic. See (8.6), (10.2) and (12.6).

Corollary 2.2. *Assume that A_K is nil-split. The following statements hold:*

(a) *The property FGLDIM is generic.*

(b) *If A_K has finite global dimension, then there exists a nonempty open set $\Omega \subset X$ such that for each non-negative integer m and any two $M, N \in \mathrm{Ob}(A\text{-mod})$, the function $D_{M,N}^m: \Omega \rightarrow \mathbb{N}$ defined by setting*

$$D_{M,N}^m(\mathfrak{p}) = \dim \mathrm{Ext}_{A_{k(\mathfrak{p})}}^m(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})})$$

for $\mathfrak{p} \in \Omega$ is constant on Ω .

Proof. For (a), suppose A_K has finite global dimension d . If L_1^K, \dots, L_n^K are (up to isomorphism) the distinct irreducible A_K -modules, then, using (1.9), there is a non-zero element $g \in \mathcal{O}$ and, for each $1 \leq i \leq n$, an \mathcal{O}_g -lattice L_i in L_i^K , such that for $\mathfrak{p} \in X_g$, the set $\{L_{i k(\mathfrak{p})}\}_{1 \leq i \leq n}$ is a complete set of representatives for the isomorphism classes of irreducible $A_{k(\mathfrak{p})}$ -modules.

For $1 \leq i \leq n$, let $P_i^{\bullet K} \rightarrow L_i^K$ be a minimal projective resolution of L_i^K . Our assumption on the global dimension of A_K implies that this resolution has finite length at most d . By (1.5f), there exists a finite complex $P^\bullet \rightarrow L_i$ of A -modules which identifies with the original complex $P_i^{\bullet K} \rightarrow L_i^K$ after applying the base change $-\otimes_{\mathcal{O}} K$. Since the P_i^K are A_K -projective, there exists a non-zero multiple

f of g such that each term P_{if} in the localization P_{if}° of P_f° is a projective A_f -module (see (1.5e)). We may also assume, at the same time, that L_{if} , each P_{if}^j and each kernel and image in the sequence

$$(2.2.1) \quad P_{if}^\circ \rightarrow L_{if}$$

is a free \mathcal{O}_f -module of finite rank. Finally, by (1.5 (g)), we may assume that $P_{if}^\circ \rightarrow L_{if}$ is a projective resolution of L_{if} as an A_f -module. Since $P_{if}^\circ \rightarrow L_{if}$ can be viewed as a sequence of short exact sequences, split \mathcal{O}_f -modules, it follows for $\mathfrak{p} \in X_f$ that $P_{ik(\mathfrak{p})}^\circ \rightarrow L_{ik(\mathfrak{p})}$ is a projective resolution of the $A_{k(\mathfrak{p})}$ -module $L_{ik(\mathfrak{p})}$, of length at most d . Hence $A_{k(\mathfrak{p})}$ has global dimension at most d , proving (a).

If $M \in \text{Ob}(A\text{-mod})$, we use induction on the length of M_K together with the Cartan-Eilenberg construction to obtain, for the element f above, a projective resolution $Q_f^\circ \rightarrow M_f$ (in the category of A_f -modules) which has length at most d . We now apply the argument of the theorem to this complex to establish (b). Note that the choice of f is independent of M . This proves (b). \square

§ 3. Ext Calculations: the Graded Case

In this section $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a \mathbb{Z} -graded \mathcal{O} -algebra, finite and torsion-free as an \mathcal{O} -module. We assume the structure map $\mathcal{O} \rightarrow A$ has image in the subalgebra A_0 of A consisting of terms of grade 0. Let $A\text{-grmod}$ denote the category of finitely generated graded left A -modules. If $M = \bigoplus_{i \in \mathbb{Z}} M_i \in \text{Ob}(A\text{-grmod})$, each M_i is naturally a finite \mathcal{O} -module. Given a non-zero $f \in \mathcal{O}$, $M_f = \bigoplus_i M_{if}$ is a graded module for $A_f = \bigoplus_i A_{if}$. Given M and $j \in \mathbb{Z}$, then $M(i)$ denotes the graded A -module whose n^{th} -grade is given by setting $M(i)_n = M_{n-i}$. The category $A\text{-grmod}$ has enough projectives. For $M, N \in \text{Ob}(A\text{-grmod})$, we have for any integer n ,

$$(3.1) \quad \text{Ext}_A^n(M, N) \cong \bigoplus_{i \in \mathbb{Z}} \text{Ext}_{A\text{-grmod}}^n(M, N(i)).$$

The main result below is to show that the Koszul property (for finite \mathcal{O} -algebras) is generic. To do this, we extend the results of §2 to the category $A\text{-grmod}$. For example, if $M^K \in \text{Ob}(A_K\text{-grmod})$ and $M' \subseteq M$ is an A -submodule such that $M'K = M^K$, define $M \in \text{Ob}(A\text{-grmod})$ by setting $M_j = M' \cap M_j^K$, $j \in \mathbb{Z}$. We have $M_{jK} \cong M_j^K$ for all j . Since M has only finitely many nonzero grades, there exists a non-zero $f \in \mathcal{O}$ such that the graded A_f -module M_f has \mathcal{O} -free grades M_{if} , by (1.2). It is now straightforward to extend the examples in (1.5) to the graded case. Thus, we obtain the following result whose proof we leave to the reader.

Theorem 3.2. *Let $M, N \in \text{Ob}(A\text{-grmod})$ be \mathcal{O} -torsion-free. For $m \geq 0$ and all n , $0 \leq n \leq m$, there is a nonempty open subset $\Omega \subset X$ depending on m such that for each $\mathfrak{p} \in \Omega$, each extension field E of the residue field $k(\mathfrak{p})$, we have:*

$$\dim \text{Ext}_{A_E\text{-grmod}}^n(M_E, N_E) = \dim \text{Ext}_{A_K\text{-grmod}}^n(M_K, N_K).$$

Now let B be a finite dimensional algebra over a field k . (Our insistence that B be finite dimensional is largely one of convenience here.) Assume that $B = \bigoplus_{n \geq 0} B_n$ is positively graded and that B_0 is a semisimple algebra. Each irreducible B -module L defines an irreducible graded B -module, still denoted L , by setting $L = L_0$. Recall that B is a *Koszul algebra* provided that for any two simple B -modules L, L' we have $\text{Ext}_{B\text{-grmod}}^n(L(i), L'(i')) \neq 0$ implies that $i' - i = n$ for all integers n, i, i' . There is an equivalent formulation which we will use in the proof below: B is Koszul provided that each irreducible B -module L (regarded as a graded B module concentrated in grade 0) has a projective resolution $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow L \rightarrow 0$ in the category $B\text{-grmod}$ in which the head of P_i is the graded term of P_i of grade i . For more discussion of the theory of Koszul algebras, we refer to [BGS], [CPS5], and [PS2].

Corollary 3.3. *Assume that A_K is a nil-algebra. If A_K is a Koszul algebra over K with finite global dimension, then there is an open subset $\Omega \subset X$ such that for each $\mathfrak{p} \in \Omega$, the $k(\mathfrak{p})$ -algebra $A_{k(\mathfrak{p})}$ is Koszul. Thus, the property FKOSZ that a finite dimensional graded algebra over a field is a Koszul algebra with finite global dimension is generic.*

Proof. Let $L_i^K, 1 \leq i \leq n$ denote the (up to isomorphism) distinct irreducible A_K -modules. Using (1.9), we can replace \mathcal{O} by a suitable localization (of the form \mathcal{O}_f for some non-zero $f \in \mathcal{O}$) so that each L_i^K has an A -lattice L_i and such that for $\mathfrak{p} \in X = \text{Spec } \mathcal{O}$, the $L_{ik(\mathfrak{p})}, i = 1, \dots, n$, are the distinct irreducible $A_{k(\mathfrak{p})}$ (which are absolutely irreducible).

Since A_K is Koszul and of finite global dimension, for each i, L_i^K has a graded projective resolution of the form

$$\cdots \rightarrow P_{ij}^K \rightarrow P_{ij-1}^K \rightarrow \cdots \rightarrow P_{i0}^K \rightarrow L_i^K \rightarrow 0$$

which has length $\leq d$ (for some positive integer d) such that each term P_{ij}^K has the form $P_{ij}^K \cong Q_{ij}^K(j)$, where Q_{ij}^K is a graded projective A_K -module generated by its grade 0 elements.

Adapting the proof of (2.2) to the graded setting yields a complex $P_{i\bullet} \rightarrow L_i$ in $A\text{-grmod}$ and $0 \neq f \in \mathcal{O}$ such that the complex $P_{i\bullet f} \rightarrow L_{if}$ is a projective resolution in $A_f\text{-grmod}$ whose terms P_{ijf} have the form $P_{ijf} \cong Q_{ijf}(j)$, where Q_{ijf} is a projective object in $A_f\text{-grmod}$ whose head is the term in grade 0. In addition, we can choose f so that the freeness conditions of (2.2.1) hold. Then $P_{ik(\mathfrak{p})} \rightarrow L_{ik(\mathfrak{p})}$ is a graded projective resolution of $L_{ik(\mathfrak{p})}$ of length at most d . The characterization of Koszul algebras mentioned above implies that $A_{k(\mathfrak{p})}$ is Koszul. \square

§ 4. Integral Quasi-Hereditary Algebras

We begin this section by recalling the definition of a split quasi-hereditary algebra over \mathcal{O} . That done, we will let QHA be the property that an algebra is a

split quasi-hereditary algebra. After showing QHA is both generic and integrally generic, we apply (2.2) to obtain a result on generic constancy of dimensions of all Ext groups. Finally we relate these results to the variations on the notion of abstract Kazhdan-Lusztig theory introduced in [CPS4, CPS5]. Here are the details.

Let A be an \mathcal{O} -algebra which is finite and projective as an \mathcal{O} -module. An ideal J in A is a split (resp., separable) heredity ideal the following properties hold:

- (1) A/J is \mathcal{O} -projective.
- (2) J is projective as a left A -module.
- (3) $J^2=J$.
- (4) The endomorphism algebra $\text{End}_A(A/J)$ satisfies the property SPLIT (resp., SEP).

By definition, A is a split (resp., separable) quasi-hereditary algebra over \mathcal{O} provided there exists a chain $0=J_0 \subset J_1 \subset \cdots \subset J_t=A$ of ideals such that each J_i/J_{i-1} is a split (resp., separable) heredity ideal A/J_{i-1} in the sense above. A detailed discussion of this notion is presented in [CPS3, § 3]. When $\mathcal{O}=K$ is an algebraically closed field, the split quasi-hereditary property coincides with the usual notion of a quasi-hereditary algebra over K [CPS1]. In this case, each ideal J_i is an idempotent ideal; observe that when $J=AeA$ for an idempotent e , then $\text{End}_A(A/J)^{\text{op}} \cong eAe$.

Let QHA denote the property of being a split quasi-hereditary algebra.

Theorem 4.1. *QHA is both a generic and an integrally generic property.*

Proof. Let A be an \mathcal{O} -algebra which is finite and torsion-free over \mathcal{O} . Assume that A_K is a split quasi-hereditary algebra as defined above. Then we can choose a defining sequence $0=J_0^K \subset J_1^K \subset \cdots \subset J_t^K=A_K$ of idempotent ideals satisfying the above properties. Let $J_i=A \cap J_i^K$, and consider the sequence $0=J_0 \subset J_1 \subset \cdots \subset J_t=A$ of ideals in A . For each i , consider the short exact sequence $0 \rightarrow J_i^2 \rightarrow J_i \rightarrow J_i/J_i^2 \rightarrow 0$. Then J_i/J_i^2 is a torsion \mathcal{O} -module since $J_i^{K^2}=J_i^K$. Therefore, there exists a non-zero $f \in \mathcal{O}$ such that each J_{if} is an idempotent ideal in A_f . By (1.2), we can further assume that A_f/J_{i-1f} is a free \mathcal{O}_f -module. Since PROJ is an integrally generic property (1.5e), we may further assume that J_{if}/J_{i-1f} is a projective right A_f -module. Finally, by (1.6), we can assume that $\text{End}_{A_f/J_{i-1f}}(J_{if}/J_{i-1f})$ satisfies SPLIT as does $\text{End}_{J_{ik(\mathfrak{p})}}(J_{i-1k(\mathfrak{p})}/J_{ik(\mathfrak{p})})$ for all $\mathfrak{p} \in X_f$. Thus, A_f is a split quasi-hereditary algebra over \mathcal{O}_f . Next, the local characterization of quasi-heredity [CPS3; (3.3)] implies that for $\mathfrak{p} \in X_f$, the algebra $A_{k(\mathfrak{p})}$ is a split quasi-hereditary algebra.

Theorem 4.2. *Let A be an \mathcal{O} -algebra which is finite and torsion-free over \mathcal{O} . Assume that A_K is a (split) quasi-hereditary algebra and let $f \in \mathcal{O}$ be as in the proof of Theorem (4.1). Then there is a basic open set $X_h \subseteq X_f$ such that for $\lambda, \mu \in \Lambda$, $\mathfrak{p} \in X_h$, we have:*

$$(4.2.1) \quad \dim \operatorname{Ext}_{A_K}^n(L(\lambda)_K, L(\mu)_K) = \dim \operatorname{Ext}_{A_E}^n(L(\lambda)_E, L(\mu)_E);$$

$$(4.2.2) \quad \dim \operatorname{Ext}_{A_K}^n(\Delta(\lambda)_K, L(\mu)_K) = \dim \operatorname{Ext}_{A_E}^n(\Delta(\lambda)_E, L(\mu)_E);$$

$$(4.2.3) \quad \dim \operatorname{Ext}_{A_K}^n(L(\lambda)_K, \nabla(\mu)_K) = \dim \operatorname{Ext}_{A_E}^n(L(\lambda)_E, \nabla(\mu)_E)$$

for all integers n and all field extensions E of $k(\mathfrak{p})$.

Proof. It is enough to prove these statements taking $E = k(\mathfrak{p})$. First, if Λ is a finite poset, and B is any quasi-hereditary algebra over a field k with poset Λ , then B has finite global dimension bounded above by $2c(\Lambda) - 2$, which $c(\Lambda)$ is the depth of Λ , i.e., the length of a maximal chain in Λ . See, for example, [DR]. It follows from (2.2a) that the global dimensions of the algebras $A_{k(\mathfrak{p})}$, $\mathfrak{p} \in X_f$, are uniformly bounded above. Therefore, the three statements (4.2.x) all follow immediately from (1.5e,g) and (2.2a). \square

Remark 4.3. In [CPS4, 5, 7] the authors studied various parity conditions for a quasi-hereditary algebra. Suppose that $B\text{-mod}$ is a highest weight category with finite poset Λ . Let $\ell: \Lambda \rightarrow \mathbb{Z}$ be a function. The left and right Kazhdan-Lusztig polynomials are defined, for $\lambda, \mu \in \Lambda$, as

$$\begin{cases} P_{\nu, \lambda}^L = t^{\ell(\lambda) - \ell(\mu)} \sum_n \dim \operatorname{Hom}_B^n(L(\lambda), \nabla(\nu)) t^{-n} \\ P_{\nu, \lambda}^R = t^{\ell(\lambda) - \ell(\mu)} \sum_n \dim \operatorname{Hom}_B^n(\Delta(\lambda), L(\nu)) t^{-n}. \end{cases}$$

When these (Laurent) polynomials contain only even powers of t , then $B\text{-mod}$ has a Kazhdan-Lusztig theory (with respect to ℓ). If $B\text{-mod}$ has a Kazhdan-Lusztig and the algebra B is Koszul, we say that $B\text{-mod}$ has a *graded* Kazhdan-Lusztig theory.⁵ In view of (4.2) and §4, we conclude that the property of having a Kazhdan-Lusztig theory (resp., a graded Kazhdan-Lusztig theory) is a generic property. Similar remarks apply to the strong Kazhdan-Lusztig theories (SKL) and (SKL') studied in [CPS7].

§ 5. Morita Theory

Let A and B be algebras over the domain \mathcal{O} (finite and torsion-free as \mathcal{O} -modules, as usual). Consider the property MORITA that A and B are Morita equivalent.

Theorem 5.1. *MORITA is generic and integrally generic.*

Proof. Suppose that the algebras A_K and B_K are Morita equivalent, so that there exists an (A_K, B_K) -bimodule M^K which, as a left A_K -module, is a projective generator for $A_K\text{-mod}$, and which satisfies $\operatorname{End}_{A_K}(M^K) \cong B_K$. Let M be the

⁵The original definition of a graded Kazhdan-Lusztig theory, discussed in [CPS4], was given in terms of a parity condition involving graded Ext-groups. It was later proved to be equivalent to the one given above [CPS5].

$A \otimes_{\mathcal{O}} B^{\text{op}}$ -submodule of M^K generated by a K -basis for M^K . Then $M_K \cong M^K$ as (A_K, B_K) -bimodules. By (1.5e), there is a nonempty open subset $\Omega \subseteq X = \text{Spec } \mathcal{O}$ such that if $X_f \subseteq \Omega$ (resp., $\mathfrak{p} \in \Omega$), then M_f (resp., $M_{k(\mathfrak{p})}$) is a projective generator for A_f (resp., $A_{k(\mathfrak{p})}$). By (1.11), we can also assume that $B_{k(\mathfrak{p})} \cong \text{End}_{A_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})})$ for all $\mathfrak{p} \in \Omega$. If $X_f \subseteq \Omega$, then the isomorphism $B_f \cong \text{End}_{A_f}(M_f)$ is automatic. We have shown that MORITA is both generic and integrally generic. \square

§ 6. Derived Category Constructions and Equivalences; Tilting Modules

Again, let \mathcal{O} be a Noetherian integral domain with quotient field K . Let A an \mathcal{O} -algebra, which is torsion-free and finite as an \mathcal{O} -module. Let $K^b(A)$ be the triangulated category whose objects are finite chain complexes $X = X^\circ: 0 \rightarrow X^m \rightarrow \dots \rightarrow X^n \rightarrow 0$, $m \leq n$, where each $X_i \in \text{Ob}(A\text{-mod})$, and whose morphisms are homotopy classes of chain maps. The distinguished triangles $X \rightarrow Y \rightarrow Z \rightarrow$ in $K^b(A)$ are those isomorphic to ones defined by taking Z to be the mapping cone of $X \rightarrow Y$. There are obvious localization functors $K^b(A) \rightarrow K^b(A_K)$, $K^b(A) \rightarrow K^b(A_f)$, $0 \neq f \in \mathcal{O}$, $K^b(A) \rightarrow K^b(A_{\mathfrak{p}})$, and specialization functors $K^b(A) \rightarrow K^b(A_{k(\mathfrak{p})})$, $\mathfrak{p} \in \text{Spec } \mathcal{O}$. If $u^K: X^K \rightarrow Y^K$ is a morphism in $K^b(A_K)$, there exists $X, Y \in \text{Ob}(K^b(A))$ such that $X_K \cong X^K$, $Y_K \cong Y^K$. Also, there exists a morphism $u: X_f \rightarrow Y_f$ in $K^b(A_f)$ for some non-zero $f \in \mathcal{O}$ such that $u_K = u^K$. Further, if u^K is a quasi-isomorphism, then we can assume that u is a quasi-isomorphism. There is also a nonempty open $\Omega \subseteq \text{Spec } \mathcal{O}_f$ such that $u_{k(\mathfrak{p})}$ is a quasi-isomorphism for all $\mathfrak{p} \in \Omega$. All these facts follows easily from the methods in §1 and will be used as needed below.

In this section, we will consider the derived category $D^b(A) = D^b(A\text{-mod})$. It is the localization of $K^b(A)$ by the multiplicative subset of quasi-isomorphisms. The distinguished triangles in $D^b(A)$ are those triangles isomorphic to the images in $D^b(A)$ of distinguished triangles in $K^b(A)$. For more details, see [W; §10].

As with $K^b(A)$, there are localization functors $D^b(A) \rightarrow D^b(A_K)$, $D^b(A) \rightarrow D^b(A_f)$ and $D^b(A) \rightarrow D^b(A_{\mathfrak{p}})$, taking X to X_K , X_f , and $X_{\mathfrak{p}}$ (in the notation above), respectively. (The categories $K^b(A)$ and $D^b(A)$ have the same objects.) We have that $X \otimes_{\mathcal{O}}^L \mathcal{O}_f \cong X_f$ and $X \otimes_{\mathcal{O}}^L A_{\mathfrak{p}} \cong X_{\mathfrak{p}}$. The exactness of localization also implies that $H^\circ(X_K) \cong H^\circ(X)_K$, $H^\circ(X_f) \cong H^\circ(X)_f$, and $H^\circ(X_{\mathfrak{p}}) \cong H^\circ(X)_{\mathfrak{p}}$ for any non-zero $f \in \mathcal{O}$ and any $\mathfrak{p} \in \text{Spec } \mathcal{O}$. Similarly, $\text{Hom}_{A_K}^\circ(X_K, Y_K) \cong \text{Hom}_A^\circ(X, Y)_K$, etc. for $X, Y \in D^b(A)$.

For $X \in \text{Ob}(K^b(A))$, $X_{k(\mathfrak{p})}$ is defined in $D^b(A)$ for any $\mathfrak{p} \in \text{Spec } \mathcal{O}$, though $X \mapsto X_{k(\mathfrak{p})}$ is *not* generally a functor! (The correct derived category functor is $X \mapsto X \otimes^L k(\mathfrak{p})$.) However, there exists a non-zero $f \in \mathcal{O}$ such that terms X^i of X , the cohomology groups $H^n(X)$, and the kernels and cokernels of the differentials $X^i \rightarrow X^{i+1}$ become free upon localization at f . Then $X_f \cong \bigoplus_n H^n(X)[-n]$. For $\mathfrak{p} \in \text{Spec } \mathcal{O}_f$, $X_{k(\mathfrak{p})} \cong X \otimes^L k(\mathfrak{p})$ and $H^n(X_{k(\mathfrak{p})}) \cong H^n(X)_{k(\mathfrak{p})}$. Similarly, given $X, Y \in D^b(A)$, there exists a nonempty, open $\Omega \subseteq \text{Spec } A$ such that $\text{Hom}_{A_{k(\mathfrak{p})}}^\circ(X_{k(\mathfrak{p})}, Y_{k(\mathfrak{p})})$

$$\cong \mathbf{RHom}_A^\bullet(X, Y)_{k(\mathfrak{p})}.$$

Recall that $X \in \text{Ob}(D^b(A))$ has finite projective dimension provided there exists an integer $N > 0$ such that $\text{Hom}_A^n(X, Y) = 0$ for all $Y \in \text{Ob}(A\text{-mod})$ and all $n > N$. In this case, there exists a finite complex P of finitely generated projective A -modules and a chain map $P \rightarrow X$ which is quasi-isomorphism. (For example, use the dual of [W; (10.7.2)].) Then for any $Y \in \text{Ob}(D^b(A))$, the derived complex $\mathbf{RHom}_A^\bullet(X, Y)$ is represented by $\text{Hom}_A^\bullet(P, Y)$, and so lies in $D^b(A)$ (rather than in just $D(A)$). Of course, $\mathbf{RHom}_A^\bullet(X, Y)_f \cong \mathbf{RHom}_{A_f}^\bullet(X_f, Y_f)$ by exactness of localization.

The following elementary result summarizes some further basic features of this localization/specialization process.

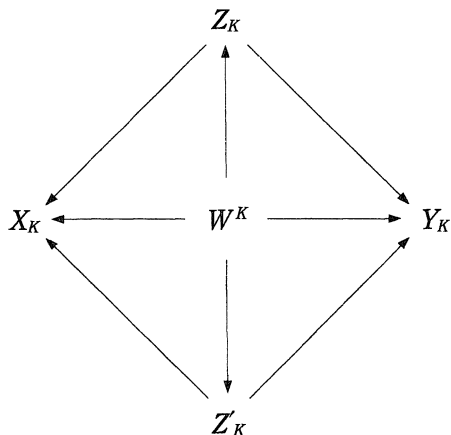
Proposition 6.1. (a) *The property that two morphisms $a, b : X \rightarrow Y$ in $D^b(A)$ be equal is both generic and integrally generic.*

(b) *Assume that $X \in \text{Ob}(D^b(A))$ has finite projective dimension. For any $Y \in \text{Ob}(D^b(A))$, there exists a nonempty open $\Omega \subseteq \text{Spec } \mathcal{O}$ so that*

$$\mathbf{RHom}_{A_{k(\mathfrak{p})}}^\bullet(X_{k(\mathfrak{p})}, Y_{k(\mathfrak{p})}) \cong \mathbf{RHom}_A^\bullet(X, Y) \otimes_{\mathcal{O}}^L k(\mathfrak{p}) \cong \mathbf{RHom}_A^\bullet(X, Y)_{k(\mathfrak{p})}$$

for all $\mathfrak{p} \in \Omega$.

Proof. To prove (a), consider two morphisms $a, b : X \rightarrow Y$ in $D^b(A)$ and assume that $a_K = b_K$. A morphism $a : X \rightarrow Y$ is represented by an equivalence class of diagrams $X \xleftarrow{s} Z \xrightarrow{g} Y$ (in $K^b(A)$) in which s is a quasi-isomorphism; see [W; § 10.3], for example. There are several ways to say that $a_K = b_K$ in $D^b(A_K)$. One way is to say that $a_K = b_K$ means precisely that there is a commutative diagram



in which $X_K \leftarrow Z_K \rightarrow Y_K$ defines a_K , $X_K \leftarrow Z'_K \rightarrow Y_K$ defines b_K and $W^K \rightarrow X_K$ is a quasi-isomorphism. We can replace \mathcal{O} by a localization \mathcal{O}_f to assume that $W^K = W_K$ for some $W \in D^b(A)$, that all the maps in the above diagram are defined over \mathcal{O} , and that the maps into X are quasi-isomorphisms (and remain so upon

specialization to any residue field $k(\mathfrak{p})$). Also, if two chain maps $c, d : S \rightarrow T$ are such that c_K and d_K are homotopic, then the chain homotopy can be defined over some \mathcal{O}_f . So, we can assume that $a = b$ in $D^b(A)$, replacing \mathcal{O} yet again by some \mathcal{O}_f . Also, by our construction, $a_{k(\mathfrak{p})} = b_{k(\mathfrak{p})}$ for any $\mathfrak{p} \in \text{Spec } \mathcal{O}$.

To prove (b), there is a chain map $P^K \rightarrow X_K$ which is a quasi-isomorphism, where P^K is a bounded complex of projective A_K -modules. Replacing \mathcal{O} by some localization \mathcal{O}_f , we can assume that $P^K \cong P_K$ for a bounded complex P of projective A -modules. We can also assume that there is a quasi-isomorphism $P \rightarrow X$ which induces a quasi-isomorphism $P_{k(\mathfrak{p})} \rightarrow X_{k(\mathfrak{p})}$ for all $\mathfrak{p} \in \text{Spec } \mathcal{O}$. Thus, after replacing $\text{Spec } \mathcal{O}$ by a smaller open subset in order to assume that all kernels and cokernels in $\text{Hom}_A^\circ(P, Y)$ are \mathcal{O} -free, we have

$$\begin{aligned} \mathbf{R}\text{Hom}_{A_{k(\mathfrak{p})}}^\circ(X_{k(\mathfrak{p})}, Y_{k(\mathfrak{p})}) &\cong \text{Hom}_{A_{k(\mathfrak{p})}}^\circ(P_{k(\mathfrak{p})}, Y_{k(\mathfrak{p})}) \\ &\cong \text{Hom}_A^\circ(P, Y)_{k(\mathfrak{p})} \\ &\cong \mathbf{R}\text{Hom}_A^\circ(X, Y)_{k(\mathfrak{p})} \\ &\cong \mathbf{R}\text{Hom}_A^\circ(X, Y) \otimes^L k(\mathfrak{p}). \end{aligned}$$

This completes the proof. \square

Lemma 6.2. *The property that a triangle $T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_1[1]$ in $D^b(A)$ be a distinguished triangle is both generic and integrally generic.*

Proof. A diagram $T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_1[1]$ in $D^b(A)$ defines a distinguished triangle if and only if it is isomorphic to the image in $D^b(A)$ of a distinguished triangle in $K^b(A)$. Thus, if $T_{1K} \rightarrow T_{2K} \rightarrow T_{3K} \rightarrow T_{1K}[1]$ defines a distinguished triangle in $D^b(A_K)$, there is a commutative diagram

$$(6.2.1) \quad \begin{array}{ccccccc} T_{1K} & \rightarrow & T_{2K} & \rightarrow & T_{3K} & \rightarrow & T_{1K}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_1^K & \rightarrow & X_2^K & \rightarrow & X_3^K & \rightarrow & X_1^K[1] \end{array}$$

in $D^b(A_K)$ in which the bottom row is the mapping cone sequence for a chain map $g^K : X_1^K \rightarrow X_2^K$ in $K^b(A_K)$ and the vertical maps are isomorphisms in $D^b(A_K)$. Now apply the remarks in the first paragraph of this section, together with (6.1a). \square

Let DERIVED be the property on pairs (A, B) of algebras (finitely generated and torsion-free over their common base ring) which holds if and only if the triangulated categories $D^b(A)$ and $D^b(B)$ are equivalent. Recall that $D^b(A)$ is equivalent to $D^b(B)$ if and only if there exists a bounded complex T of finitely generated projective A -modules such that A lies in the triangulated subcategory of $D^b(A)$ generated by the direct summands of T and, in addition,

$\text{Hom}_A^\bullet(T, T) \cong B$, i.e., $\text{Hom}_A^n(T, T) \cong \begin{cases} 0 & \text{if } n \neq 0 \\ B & \text{if } n = 0. \end{cases}$ See [R]. We will call such a T a *projective triangulated generator* relating $D^b(A)$ to $D^b(B)$. Now we can prove:

Theorem 6.3. *The property DERIVED is both generic and integrally generic.*

Proof. Suppose DERIVED holds for the pair (A_K, B_K) , with A, B algebras over \mathcal{O} as above. Let T^K be a projective triangulated generator relating $D^b(A_K)$ to $D^b(B_K)$. Using (1.5 (e, f)), we replace \mathcal{O} by some \mathcal{O}_f to assume that $T^K \cong T_K$, for a bounded complex T of finitely generated projective A_K -modules. Also, since A_K arises from T_K by a finite number of steps of forming distinguished triangles and direct summands, the same is true for A and T after a further appropriate localization of \mathcal{O} , as follows using (6.2). (The fact that “ M is a direct summand of N ” is both a generic and an integrally generic property is very easy to prove and is left to the reader.) Finally, using (6.1) and perhaps localizing \mathcal{O} further, we obtain that $B \cong \text{Hom}_A^\bullet(T, T)$ and $B_{k(p)} \cong \text{Hom}_{A_{k(p)}}^\bullet(T_{k(p)}, T_{k(p)})$. The theorem now follows. \square

§ 7. Generic Quiver Theory

As in the previous section, \mathcal{O} is a Noetherian domain with quotient field K , and A is an \mathcal{O} -algebra which is torsion-free and finite as an \mathcal{O} -module.

We will assume throughout this section that the following holds.

Hypothesis 7.1a. *The algebra $A_K/\text{rad}(A_K)$ is separable. (Equivalently, the algebra A_K is nil-separable in the terminology of § 1.)*

If (7.1a) holds, the Wedderburn principal theorem implies that there is a sub-algebra A_0^K complementary to the radical N^K of A_K . Clearly, $N^K = NK \cong N_K$ where $N = A \cap N_K$. Similarly, $A_0^K = A_0K \cong A_{0K}$ where $A_0 = A \cap A_0^K$. We will usually require, in addition to (7.1a) that

Hypothesis 7.1b. *With the above notation, we have that $A = A_0 \oplus N$, and $N = M \oplus N^2$ for some (A_0, A_0) -bimodule M .*

In the presence of (7.1a), we can always replace \mathcal{O} by a localization \mathcal{O}_f in order to assume that (7.1b) does hold. Indeed, we could even assume that A_0 is separable, i.e., the algebra A is nil-separable. (In this case, standard homological arguments also show that A_0 is uniquely determined up to an inner automorphism of A . We do not know any way to canonically choose M inside A , though as a bimodule it is isomorphic to N/N^2 .)

We will now take up the question of generators and relations for A and its localizations and specializations. Observe that in (7.1b), A is generated as a ring, by A_0 and M . More precisely, since M is an (A_0, A_0) -bimodule, we can

form the tensor ring

$$T = T_{A_0}(M) = \bigoplus_n T^n,$$

where $T^n = M^{\otimes n} = \underbrace{M \otimes_{A_0} \cdots \otimes_{A_0} M}_{n \text{ times}}$ and the multiplication is defined in the obvious

way. Then there is a unique surjection $\pi : T \rightarrow A$ which respects the inclusions of $A_0 \oplus M$ into both T and A . We view M as a direct analogy of the “quiver module” used in the theory of finite dimensional algebras.

Now let R be any finitely generated (A_0, A_0) -sub-bimodule of T . We say R is a *module of relations for A* with respect to A_0 and M (or that the pair (M, R) gives generators and relations for A with respect to A_0) if $\text{Ker}(\pi)$ is the T -ideal generated by R . (Notice that R necessarily is contained in only finitely many grades of T , defined in terms of the number of tensors of M . We will usually take R to be contained in grades ≥ 2 , though the definition does not specifically require it.) Given a candidate finitely generated A_0 -sub-bimodule R , we will say the property $\text{QUIV}(A_0, M, R)$ holds if R is a module of relations for A . If, in addition, R is a direct sum of its homogeneous projections of various grades, we say $\text{HOMOG}(A_0, M, R)$ holds, and $\text{QUAD}(A_0, M, R)$ holds if R is entirely contained in grade 2. Also, if $\text{HOMOG}(A_0, M, R)$ (resp., $\text{QUAD}(A_0, M, R)$) holds for some A_0, M, R , we say that TIGHT (resp., QUADRATIC) holds for A .

Theorem 7.2. *Assume that A satisfies (7.1a, b) and that R is a finitely generated A_0 -submodule of $T = T_{A_0}(M)$. The properties*

$$\text{QUIV}(A_0, M, R), \text{HOMOG}(A_0, M, R), \text{QUAD}(A_0, M, R),$$

are all generic and integrally generic. Similarly, the properties TIGHT and QUADRATIC are generic and integrally generic for any algebra A satisfying (7.1a).

Proof. Suppose that $\text{QUIV}(A_{0K}, M_K, R_K)$ holds. Choose $n > 0$ so that $N_K^n = 0$; thus $N^n = 0$, and T_K^n must be contained in the T_K -ideal generated by R_K . The latter ideal is the K -submodule generated by all products $M^i R M^j$ with $i + j \geq 0$. Upon suitable localization, T^n will be contained in the ideal generated by R . In other words, we can assume that the T -ideal generated by R has as quotient an \mathcal{O} -algebra A' , finite as an \mathcal{O} -module, and $A'_K \cong A_K$. Localizing further, we may assume that $A'_f \cong A_f$ for some non-zero $f \in \mathcal{O}$. Now $\text{QUIV}(A_{0f}, M_f, R_f)$ follows, and certainly $\text{HOMOG}(A_{0f}, M_f, R_f)$, $\text{QUAD}(A_{0f}, M_f, R_f)$ follow from their counterparts over K just by requiring R_f to be torsion-free over \mathcal{O} . It follows easily that $\text{QUIV}(A_0, M, R)$, $\text{HOMOG}(A_0, M, R)$, and $\text{QUAD}(A_0, M, R)$ are generic and integrally generic, as are TIGHT and QUADRATIC . (Notice that if we do not have A_0, M, R to begin with, it is easy to obtain them, assuming $\text{QUIV}(A_0^K, M^K, R^K)$ for the algebra A^K . We can write $R^K = R_K$, where $R = T \cap R^K$ once A_0 and M have been defined (passing to a suitable localization, if

necessary). Observe that the intersection $T \cap R^K$ involves only finitely many terms.) \square

As a corollary of the proof, we have

Corollary 7.3. *Assume that A satisfies (7.1a, b). Then there is at least one module of relations R , in grades ≥ 2 , for A , with respect to some A_0 and M as above.*

Proof. This follows from the proof of the theorem, since it is well-known that $\text{QUIV}(A_0^K, M^K, R^K)$ may be satisfied using R^K concentrated in grades ≥ 2 . This property is inherited by $R = T \cap R^K$ and its further localizations in the above proof. It is also easy to argue more directly: By adjoining finitely many elements to R , we may assume that the ideal it generates contains T^n (where n is as in the above proof). Then, as above, the quotient by the ideal generated by R is a finitely generated by \mathcal{O} algebra which clearly has A as a homomorphic image (from (7.1b)). It is then an easy matter to add finitely many elements (of grades ≥ 2) to R so that this quotient is isomorphic to A . \square

Part II: q -Rational Representation Theory

In this part, we let $\mathcal{L} = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials in an indeterminate t . Let (W, S) be a finite Coxeter system, and consider the associated generic Hecke algebra $H = H(W, \mathcal{L})$ over \mathcal{L} . This algebra has basis $\{\tau_w\}_{w \in W}$ satisfying the relations

$$(II.1) \quad \tau_s \tau_w = \begin{cases} \tau_{sw}, & \text{if } sw > w \\ t\tau_{sw} + (t-1)\tau_w, & \text{otherwise} \end{cases} \quad (s \in S, w \in W).$$

For $\lambda \subseteq S$, $W_\lambda = \langle s \rangle_{s \in \lambda}$ denotes the corresponding parabolic subgroup of W , while $H_\lambda = \langle \tau_s \rangle_{s \in \lambda}$ is the corresponding parabolic subalgebra of H . Thus, $H_\lambda = H(W_\lambda, \mathcal{L})$. Also, put

$$(II.2) \quad x_\lambda = \sum_{w \in W_\lambda} \tau_w \quad \text{and} \quad y_\lambda = \sum_{w \in W_\lambda} (-t)^{-\ell(w)} \tau_w.$$

The left ideals Hx_λ and Hy_λ have natural interpretations as induced modules. Define linear characters on H by

$$(II.3) \quad \text{IND} : H \rightarrow \mathcal{L}, \tau_w \mapsto t^{\ell(w)}, \text{ and } \text{SGN} : H \rightarrow \mathcal{L}, \tau_w \mapsto (-1)^{\ell(w)}$$

for $w \in W$. Putting $\text{IND}_\lambda = \text{IND}|_{H_\lambda}$ and $\text{SGN}_\lambda = \text{SGN}|_{H_\lambda}$, we have

$$(II.4) \quad \tau_w x_\lambda = \text{IND}_\lambda(\tau_w) x_\lambda \text{ and } \tau_w y_\lambda = \text{SGN}_\lambda(\tau_w) y_\lambda, w \in W_\lambda,$$

so that $\mathcal{L}x_\lambda$ (resp., $\mathcal{L}y_\lambda$) is a module realization of IND_λ (resp., SGN_λ). It follows easily that

$$(II.5) \quad Hx_\lambda \cong \text{ind}_{H_\lambda}^H \text{IND}_\lambda \quad \text{and} \quad Hy_\lambda \cong \text{ind}_{H_\lambda}^H \text{SGN}_\lambda.$$

Similar comments hold for the right ideals $x_\lambda H$ and $y_\lambda H$.

There is a \mathcal{L} -automorphism $\Phi: H \rightarrow H$ given on generators by $\Phi(\tau_w) = (-t)^{\ell(w)} \tau_w^{-1}$. If M is a (right or left) H -module, let M^Φ denote the H -module obtained by twisting the action of H on M by Φ . For example, $(Hx_\lambda)^\Phi = Hy_\lambda$ for all $\lambda \subseteq S$ [DPS3; (1.4c)]. Also, $\text{IND}^\Phi = \text{SGN}$.

Let R be a commutative ring and $\mathcal{L} \rightarrow R$ be a ring homomorphism in which $t \mapsto q \in R$. The algebra $H(W, \mathcal{L})_R = H(W, \mathcal{L}) \otimes_{\mathcal{L}} R$ will be denoted by $H(W, R, q)$ and sometimes simply by just $H(W, q)_R$ or even $H(W, q)$. Although it should always be evident from context, we alert the reader that the “ q ” may have several different meanings in what follows: for example, it is often a power $q = r^d$ of a prime r , while if k is a field of characteristic $p \neq r$, then in the Hecke algebra $H(W, q)_k$, $q = q \cdot 1_k$ is a root of unity in the field k .

For $w \in W$, let τ_w denote the R -basis vector $\tau_w \otimes_{\mathcal{L}} 1 \in H(W, q)$. The relations (II.1) above provide a presentation for the R -algebra $H(W, q)$. The characters IND and SGN on $H(W, q)$ define linear characters $\text{IND}: H(W, q) \rightarrow R$ and $\text{SGN}: H(W, q) \rightarrow R$ by base-change, and the isomorphisms (II.5) hold over R . Finally, the automorphism Φ induces an automorphism on $H(W, q)$.

§ 8. The Geck-Gruber-Hiss Very Large Prime Result

In this section, let $(W, S) = (\mathfrak{S}_m, S)$, where \mathfrak{S}_m is the symmetric group of degree m and $S = \{(1, 2), (2, 3), \dots, (m-1, m)\}$ is the set of fundamental reflections. Then $H = H(\mathfrak{S}_m, \mathcal{L})$, $\mathcal{L} = \mathbb{Z}[t, t^{-1}]$.

Let V a free \mathcal{L} -module of rank $n > 0$. The Hecke algebra H acts on $V^{\otimes m}$ once an ordered basis $\{v_1, \dots, v_n\}$ for V has been fixed. If $J = (j_1, \dots, j_m)$ is a sequence of integers satisfying $1 \leq j_i \leq n$, for all i , then write $J\sigma = (j_{\sigma^{-1}(1)}, \dots, j_{\sigma^{-1}(m)})$ for $\sigma \in \mathfrak{S}_m$ and put $v_J = v_{j_1} \otimes \dots \otimes v_{j_m} \in V^{\otimes m}$. For $s = (i, i+1) \in S$, the formula

$$(8.1) \quad v_J \tau_s = \begin{cases} tv_{J_s}, & \text{if } j_i \leq j_{i+1} \\ v_{J_s} + (t-1)v_J, & \text{otherwise} \end{cases}$$

defines a right action of the generators τ_s , $s \in S$, of H on $V^{\otimes m}$. This action extends to define a right H -module structure on $V^{\otimes m}$. (See [DD; (3.1.5)].) As a right H -module, $V^{\otimes m}$ decomposes into a direct sum of various copies of the induced modules $x_\lambda H$, $\lambda \subseteq S$; see below. The endomorphism algebra

$$(8.2) \quad S_t(n, m) = \text{End}_H(V^{\otimes m})$$

is the t -Schur algebra of bidegree (n, m) over \mathcal{L} . For any commutative ring R and homomorphism $\mathcal{L} \rightarrow R$, $t \mapsto q$, we often write $S_q(n, m)$ or $S_t(n, m)_R$ for the algebra $S_t(n, m)_R = S_t(n, m) \otimes_{\mathcal{L}} R$ for the corresponding q -Schur algebra over R . The natural map $S_t(n, m) \rightarrow \text{End}_{H(\mathfrak{S}_m, R, q)}(V_R^{\otimes m})$ defines (by base-change) an isomorphism over R

$$(8.3) \quad S_q(n, m)_R \cong \text{End}_{H(\mathfrak{S}_m, R, q)}(V_R^{\otimes m}).$$

See [DPS1; (2.3.4), (2.3.5)], or the proof of [DJ1; (3.3)] using [DPS1; (1.1.1)].

The proof of (8.3) depends, in fact, on another description of the q -Schur algebras that will be useful in the sequel. Let $\Lambda(n, m)$ (resp., $\Lambda^+(n, m)$) be the set of compositions (resp., partitions) of m with n (resp., at most n non-zero) parts. Thus, $\lambda \in \Lambda(n, m)$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative integers such that $\lambda_1 + \dots + \lambda_n = m$. If $\lambda_1 \geq \lambda_2 \geq \dots$, then λ is a partition of m . Any $\lambda \in \Lambda(n, m)$ determines a subset $J(\lambda)$ of S as follows. Let $\mathcal{Y}(\lambda)$ be the Young diagram of shape λ . Filling in the boxes in the first row of $\mathcal{Y}(\lambda)$ consecutively with the integers $1, \dots, \lambda_1$, the second row with the integers $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$, etc. defines a standard tableau \mathbf{t}^λ of shape λ . Let $J(\lambda)$ be the subset of S consisting of those s which stabilize the rows of \mathbf{t}^λ . However, we usually write $W_\lambda, x_\lambda, y_\lambda$, etc. in place of $W_{J(\lambda)}, x_{J(\lambda)}, y_{J(\lambda)}$, etc. Then

$$(8.4) \quad \begin{aligned} V^{\otimes m} &\cong \bigoplus_{\lambda \in \Lambda(n, m)} x_\lambda H \cong \bigoplus_{\lambda \in \Lambda^+(n, m)} x_\lambda H^{\oplus k_\lambda} \\ &\cong \bigoplus_{\lambda \subseteq S} x_\lambda H^{\oplus r_\lambda} \end{aligned}$$

as right H -modules for some choice of non-negative integers $k_\lambda = k_\lambda(n, m)$ ($\lambda \in \Lambda^+(n, m)$) and r_λ ($\lambda \subseteq S$) [DD; (3.1.5)]. The second isomorphism above follows from the fact that W_λ and W_μ are conjugate in W , then $x_\lambda H \cong x_\mu H$. Now (8.3) follows as above. Sometimes it is useful to work with the algebra

$$(8.5) \quad A = \text{End}_H \left(\bigoplus_{\lambda \in \Lambda^+(n, m)} x_\lambda H \right).$$

By (8.4), A is Morita equivalent to $S_t(n, m)$.

If λ is a partition of m , write $\lambda \vdash m$. The poset structure \trianglelefteq on the set $\Lambda^+(m) = \{\lambda \mid \lambda \vdash m\}$ of partitions of m is defined by $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \trianglelefteq \mu = (\mu_1 \geq \mu_2 \geq \dots)$ if and only if $\lambda_1 \leq \mu_1, \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, \dots$. Thus, $\Lambda^+(n, m)$ is a coideal in $\Lambda^+(m)$. By [DPS2; (2.5)], there exist (left) $S_t(n, m)$ -modules $\Delta(\lambda)$, $\lambda \in \Lambda^+(n, m)$, such that for every field k and algebra homomorphism $\mathcal{Z} \rightarrow k, t \mapsto q$, $S_q(n, m)$ -mod is a highest weight category with poset $\Lambda^+(n, m)$, and standard objects $\Delta(\lambda)_k = \Delta_q(\lambda)$. In particular, the irreducible $S_q(n, m)$ -modules $L^k(\lambda) = L_q(\lambda)$ are indexed by the partitions $\lambda \in \Lambda^+(n, m)$.

We will need to know that the q -Schur algebras (or at least certain ones of them) are nil-split. Although this fact can be deduced using the theory of quantum enveloping algebras, it can also be established directly using the approach to q -Schur algebras given in [DPS2]. First, there exist (left) $S_t(n, m)$ -modules $\nabla(\lambda)$, $\lambda \in \Lambda^+(n, m)$, such that for any specialization $\mathcal{Z} \rightarrow k, t \mapsto q$, into a field k , the $\nabla(\lambda)_k, \lambda \in \Lambda^+(n, m)$, are the costandard objects in $S_q(n, m)$ -mod. (These modules can be defined first by taking \mathcal{Z} -linear duals of the standard modules for the category $\text{mod-}S_t(n, m)$ of right modules, using the right module version of [DPS2].) For k above, we have $\text{Ext}_{S_q(n, m)}^i(\Delta(\lambda)_k, \nabla(\lambda)_k) = 0$, for $i > 0$, by standard highest weight category theory. By [DPS3; (4.4)],

$$\text{Hom}_{S_t(n, m)}(\Delta(\lambda), \nabla(\lambda))_k \cong \text{Hom}_{S_q(n, m)}(\Delta(\lambda)_k, \nabla(\lambda)_k).$$

It is known that the $\mathbb{Q}(t)$ -algebra $H_{\mathbb{Q}(t)}$ is split semisimple, so that $S_t(n, m)_{\mathbb{Q}(t)}$

is a split semisimple algebra. In fact, it has irreducible modules $\Delta(\lambda)_{\mathbb{Q}(t)} \cong \nabla(\lambda)_{\mathbb{Q}(t)}$, $\lambda \in \Lambda^+(n, m)$. Thus, $\text{Hom}_{S_t(n, m)_{\mathbb{Q}(t)}}(\Delta(\lambda)_{\mathbb{Q}(t)}, \nabla(\lambda)_{\mathbb{Q}(t)}) \cong \mathbb{Q}(t)$. Since $\text{Hom}_{S_t(n, m)}(\Delta(\lambda), \nabla(\lambda))$ is a finitely generated projective module for \mathcal{Z} (by another commutative algebra argument using [DPS2; (0, 1)]), it follows that

$$\text{Hom}_{S_q(n, m)}(L^k(\lambda), L^k(\lambda)) \cong \text{Hom}_{S_q(n, m)}(\Delta(\lambda)_k, \nabla(\lambda)_k) \cong k,$$

so we conclude that $S_q(n, m)/\text{rad}(S_q(n, m))$ is a split semisimple algebra.

Now let ℓ be a positive integer, let $q = \zeta = \sqrt[\ell]{1} \in \mathbb{C}$ be a primitive ℓ th root of unity, and set $K = \mathbb{Q}(q)$. Consider the Dedekind domain $\mathcal{Z}(\ell)$ of algebraic integers in the number field $K = \mathbb{Q}(q)$. There is a homomorphism $\mathcal{Z} \rightarrow \mathcal{Z}(\ell)$ under which $t \mapsto q$. Applying (1.9) and the previous paragraph, with K the quotient field of $\mathcal{Z}(\ell)$, there exists a non-zero $f \in \mathcal{Z}(\ell)$ such that each $L^K(\lambda)$ has an $S_q(n, m)_f$ -lattice $L(\lambda)$ such that for all $\mathfrak{p} \in \Omega = \text{Spec } \mathcal{Z}(\ell)_f$, the $L^f(\lambda)_{k(\mathfrak{p})}$, $\lambda \in \Lambda^+(n, m)$, are the distinct irreducible $S_q(n, m)_{k(\mathfrak{p})}$ -modules. In the previous sentence, $S_q(n, m)$ denotes the $\mathcal{Z}(\ell)$ -algebra $S_t(n, m)_{\mathcal{Z}(\ell)}$. We can, in fact, assume that $L^f(\lambda)_{k(\mathfrak{p})} \cong L^{k(\mathfrak{p})}(\lambda)$.

We can now apply the results of §§ 1, 2 to obtain the following result. As we discuss in the remark following, an immediate consequence of the theorem is the Geck-Gruber-Hiss very large prime result.

Theorem 8.6. *With the above notation, we can also assume $\Omega \subseteq \text{Spec } \mathcal{Z}(\ell)$ has the property that for $\mathfrak{p} \in \Omega$ and $k = k(\mathfrak{p})$, we have:*

$$(8.6.1) \quad \begin{cases} [\Delta(\lambda)_K : L^K(\mu)] = [\Delta(\lambda)_k : L^k(\mu)]; \\ \dim \text{Ext}_{S_t(n, m)_k}^2(L^K(\lambda), L^K(\mu)) = \dim \text{Ext}_{S_t(n, m)_k}^2(L^k(\lambda), L^k(\mu)), \end{cases}$$

for all $\lambda, \mu \in \Lambda$, $n \in \mathbb{Z}^+$.

Proof. The first assertion on composition factors follows from (1.10) since we have already argued that $S_t(n, m)_K$ is nil-split. Finally, the second Ext-assertion follows from (2.1), together with the fact that the algebras $S_t(n, m)_k$ have finite global dimension uniformly bounded above by $2c(\Lambda^+(n, m)) - 2$, where $c(\Lambda^+(n, m))$ is the length of a maximal chain in $\Lambda^+(n, m)$ [DR]. \square

Remark 8.7. The first assertion in (8.6) proves “generically” an old conjecture of James [Ja; § 4]: Suppose that, modulo \mathfrak{p} , ℓ is equal to the smallest positive integer e with $1 + \zeta + \dots + \zeta^{e-1} = 0$.⁶ (Thus, $\ell = p$ if $\zeta = 1$ modulo p , and, otherwise, ℓ is the order of ζ modulo p . So we are requiring, simply, that ℓ be p whenever it happens to be divisible by p .) Then James conjectures that each $L^K(\lambda)$ remains irreducible upon reduction “modulo p ”, provided $\ell p > r$. Gruber and Hiss [GH; § 10] remark that this follows for (ℓ fixed and) p very large (that is, sufficiently large, with no explicit bound) from an argument of Geck,

⁶The formulation in [Ja] contains a misprint, omitting the “modulo \mathfrak{p} ”.

made originally in a Hecke algebra context. This conjecture is also a consequence of the above theorem, particularly of the first part, which we found before we were aware of the Geck-Gruber-Hiss remark. The result implies there are generic character formulas for modular characters of the finite general linear groups in “non-describing characteristics,” cf. [S].

There is an analogous large prime result in the “describing characteristic” case for each type of root system, by the much more difficult work of Andersen-Jantzen-Soergel [AJS]. It is an intriguing question as to whether or not an easier proof might be found of their result, better fitting the “generic” context of this paper.

§ 9. A Morita Equivalence for $GL_n(\mathfrak{q})$

Recall that k is a fixed algebraically closed field of positive characteristic p . Let (\mathcal{O}, K, k) be a p -modular system (i. e., \mathcal{O} is a discrete valuation ring with quotient field K and residue field k). The following simple lemma is key to our approach.

Lemma 9.1. *Let R be an \mathcal{O} -algebra which is free of finite rank over \mathcal{O} and has the property that R_K is a semisimple algebra over K . Consider an exact sequence $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ of right R -modules which are free of finite rank over \mathcal{O} . Suppose that the R_K -modules N_K and M_K have no composition factors in common and that P is a projective R -module. Then M is a projective R/J -module, where J is the annihilator of M in R .*

Proof. Because R_K is semisimple, and N_K and M_K have no common composition factors, we have that $N_K J_K = N_K$. Thus, N/NJ is a torsion \mathcal{O} -module. The surjective map $P \twoheadrightarrow M$ yields a surjective map $\pi: P/PJ \twoheadrightarrow M$. The right exactness of the functor $-\otimes_R R/J$ implies that $\text{Ker}(\pi)$ is a homomorphic image of $N \otimes_R R/J \cong N/NJ$, so $\text{Ker}(\pi)$ is also a torsion \mathcal{O} -module. But R/J is evidently \mathcal{O} -torsion-free, so, because P is a projective R -module, P/PJ is also \mathcal{O} -torsion-free. Since $\text{Ker}(\pi)$ is a submodule of P/PJ , it is also \mathcal{O} -torsion-free, so finally $\text{Ker}(\pi) = 0$. Thus, $M \cong P/PJ$ is a projective R/J -module, as required. \square

Now we can immediately obtain:

Theorem 9.2. *Assume the hypotheses of (9.1). If, in addition, every irreducible R/J -module lies in the head of the right R -module M , then the functor $F(-) = \text{Hom}_{R/J}(M, -)$ defines a Morita equivalence*

$$(9.2.1) \quad F: \text{mod-}R/J \xrightarrow[\text{Morita}]{\sim} \text{mod-End}_R(M)$$

of right module categories.

Proof. We regard M as a left $\text{End}_R(M)$ -module, so that in (9.2.1) $\text{Hom}_{R/J}(M, X)$ is a right $\text{End}_R(M)$ -module for any R/J -module X . The hypoth-

eses imply that every irreducible R/J -module is a homomorphic image of M . Thus, by (9.1), M is a projective generator for the algebra R/J and the conclusion follows. \square

For the rest of this section, we consider the group $G(q) = GL_n(q)$. It has order given by

$$(9.3) \quad |GL_n(q)| = q^{\binom{n}{2}} \cdot (q-1)^n \cdot \prod_{j=2}^n \frac{q^j - 1}{q - 1}.$$

Recall that $q = r^d$ is a power of a fixed prime r . In what follows, we assume that the quotient field K of \mathcal{O} has been taken large enough so that it is a splitting field for $G(q)$. We will verify that the hypotheses of (9.2) above can be achieved for $R = \mathcal{O}G(q)$.

Fix a set \mathcal{C} of representatives from the $G(q)$ -conjugacy classes. If b is a non-negative integer, let $\mathcal{C}_{b'}$ be the subset of $x \in \mathcal{C}$ having order relatively prime to b . If G_{ss} denotes the set of semisimple elements in the algebraic group G , put $G(q)_{ss} = G_{ss} \cap G(q)$ and $\mathcal{C}_{ss} = G(q)_{ss} \cap \mathcal{C}$. Let $\mathcal{C}_{ss, p'}$ be the set of p' -elements in \mathcal{C}_{ss} . Any $s \in G(q)_{ss}$ has centralizer $Z_{G(q)}(s)$ in $G(q)$ of the form:

$$(9.4) \quad Z_{G(q)}(s) \cong \prod_{i=1}^{m(s)} GL_{n_i(s)}(q^{a_i(s)}), \text{ where } \sum a_i(s) n_i(s) = n.$$

Let $\mathbf{n}(s) = (n_1(s), \dots, n_{m(s)}(s))$. Put

$$\Lambda^+(\mathbf{n}(s)) = \Lambda^+(Z_{G(q)}(s))$$

for the set of multi-partitions λ of $\mathbf{n}(s)$, i. e., $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m(s))})$, where $\lambda^{(1)} \vdash n_1(s), \dots, \lambda^{(m(s))} \vdash n_{m(s)}(s)$. For a fixed $s \in G(q)_{ss}$, we will make constant use of the following configuration of subgroups of $G(q)$:

$$\begin{array}{ccc}
 & G(q) = GL_n(q) & \\
 & \downarrow & \\
 & L_s(q) = \prod_{i=1}^{m(s)} GL_{a_i(s)n_i(s)}(q) & \\
 \swarrow & & \searrow \\
 H_s(q) = \prod_{i=1}^{m(s)} GL_{a_i(s)}(q)^{n_i(s)} & & Z_{G(q)}(s) = \prod_{i=1}^{m(s)} GL_{n_i(s)}(q^{a_i(s)}) \\
 \swarrow & & \searrow \\
 & T_s(q) = \prod_{i=1}^{m(s)} F_{q^{a_i(s)}}^{\times n_i(s)} &
 \end{array}$$

In this set-up, $N^{(m)} = N \times \cdots \times N$ (m times) for a subgroup N of $G(q)$. The inclusions are the natural ones; in particular, $T_s(q)$ is a maximal torus of $Z_{G(q)}(s)$.

If L is a Levi subgroup of GL_n defined over \mathbb{F}_q , let $R_{L(q)}^{G(q)}$ be the Deligne-Lusztig induction functor from the (complex) character group $X(L(q))$ of $L(q)$ to the character group $X(G(q))$ of $G(q)$. In case $L(q)$ is the Levi subgroup of a parabolic subgroup $P(q)$ of $G(q)$, then $R_{L(q)}^{G(q)}$ agrees with Harish-Chandra induction from $\text{mod-}\mathbb{C}L(q)$ to $\text{mod-}\mathbb{C}G(q)$, and so takes modules to modules. (More generally, for any commutative ring Z , define Harish-Chandra induction $R_{L(q)}^{G(q)} : \text{mod-Z}L(q) \rightarrow \text{mod-Z}G(q)$ as follows: let $P(q) = L_s(q) \rtimes V(q)$ be the standard parabolic subgroup associated to $L_s(q)$, and, for an $RL(q)$ -module Q , let Q' be the $RP(q)$ -module obtained from Q by inflating the action of $L_s(q)$ on Q through the natural homomorphism $P(q) \rightarrow L_s(q)$. Then $R_{L_s(q)}^{G(q)}Q = \text{ind}_{L_s(q)}^{G(q)}Q'$.)

The irreducible ordinary characters of $G(q)$ were first parameterized and determined by Green [Gr], while a recasting of those results in terms of Deligne-Lusztig theory⁷ is presented in [FS] : The distinct irreducible characters $\{\chi_{s,\eta}\}$ are indexed by pairs (s, η) , where $s \in \mathcal{C}_{ss}$ and η ranges over the irreducible unipotent characters of the centralizer $Z_{G(q)}(s)$ described in (9.4). In turn the set $A^+(\mathfrak{n}(s))$ indexes the irreducible unipotent characters of $Z_{G(q)}(s)$. For $\lambda \in A^+(\mathfrak{n}(s))$, let η_λ denote the corresponding unipotent character. For simplicity, write $\chi_{s,\lambda}$ (or $\chi_{s,\lambda,G(q)}$ in case $G(q)$ needs mentioning) in place of χ_{s,η_λ} . In the next paragraph, we elaborate on the construction of $\chi_{s,\lambda}$.

As in [FS; (1.16)], $s \in \mathcal{C}_{ss}$ can be associated with a linear K -valued character \widehat{s} on $Z_{G(q)}(s)$. For $\lambda \in A^+(\mathfrak{n}(s))$, $\varepsilon R_{Z_{G(q)}(s)}^{L_s(q)}\widehat{s}\eta_\lambda = \chi_{s,\lambda,L_s(q)}$ is an irreducible character of $L_s(q)$ for some choice $\varepsilon = \pm 1$. Further, $R_{L_s(q)}^{G(q)}\chi_{s,\lambda,L_s(q)} = \chi_{s,\lambda}$. If $s \in \mathcal{C}_{ss,p'}$ and $t \in Z_{G(q)}(s)$ is a p -element, then $Z_{G(q)}(st) \subseteq Z_{G(q)}(s)$, while $L_{st}(q)$ is a subgroup of $L_s(q)$. Therefore, the transitivity of Harish-Chandra induction and the above description of $\chi_{s,\lambda}$ imply that

$$(9.5a) \quad \chi_{st,\lambda} = R_{L_s(q)}^{G(q)}\chi_{st,\lambda,L_s(q)}, \quad s \in \mathcal{C}_{ss,p'}, \quad t \in Z_{G(q)}(s) \text{ a } p\text{-element.}$$

Since the unipotent characters of $Z_{G(q)}(s)$ are just the constituents of the trivial character induced from a Borel subgroup, the characters $\widehat{s}\eta_\lambda$ are precisely the irreducible constituents of $R_{T_s(q)}^{Z_{G(q)}(s)}(\widehat{s})$, where we have written \widehat{s} for $\widehat{s}|_{T_s(q)}$. Thus, by transitivity of Deligne-Lusztig induction, the characters $\chi_{s,\lambda}$ are just the characters appearing with nonzero coefficient (positive or negative) in $R_{T_s(q)}^{G(q)}(\widehat{s})$, i.e., in terms of the inner product on group characters, we have, for any irreducible character ξ on $G(q)$.

$$(9.5b) \quad (\xi, R_{T_s(q)}^{G(q)}(\widehat{s})) \neq 0 \Leftrightarrow \xi = \chi_{s,\lambda}, \text{ some } \lambda \in A^+(\mathfrak{n}(s)).$$

⁷It would be possible to avoid Deligne-Lusztig theory, and just use Green's results, by taking the approach in [DJ2; § 7] to the results in [FS]. However, some aspects of the Deligne-Lusztig formalism are conceptually simplifying.

Let $f(X) \in \overline{\mathbb{F}}_q[X]$ be an irreducible monic polynomial of positive degree d and distinct from X . For any root $\zeta \in \overline{\mathbb{F}}_q$ of $f(X) = 0$, [DJ2; p.29] defines an irreducible cuspidal representation $C_K(\zeta)$ of $KGL_d(q)$.⁸ The representation $C_K(\zeta)$ depends on a choice of an embedding $\overline{\mathbb{F}}_q^{\times d} \hookrightarrow \mathbb{C}^{\times}$. They also define an $\mathcal{O}GL_d(q)$ -lattice $C_{\theta}(\zeta)$ of $C_K(\zeta)$. By [DF; (4.4)], the character of the $GL_d(q)$ -module $C_K(\zeta)$ equals $\pm R_{T^{\times}(q)}^{GL_d(q)}(\zeta)$ where ζ is a (regular) linear (K -valued) character on a torus $T^{\times}(q) \cong \overline{\mathbb{F}}_q^{\times d}$ of $GL_d(q)$.

Now fix $s \in \mathcal{C}_{ss}$. We can assume that

$$s = \text{diag}(\underbrace{s_1, \dots, s_1}_{n_1 \text{ times}}, \dots, \underbrace{s_m, \dots, s_m}_{n_m \text{ times}}),$$

where the semisimple matrices s_1, \dots, s_m have distinct irreducible characteristic polynomials $f_1(X), \dots, f_m(X) \in \overline{\mathbb{F}}_q[X]$. For each i , let $\zeta_i \in \overline{\mathbb{F}}_q$ be a fixed root of $f_i(X) = 0$ and form the representation

$$(9.6) \quad C_K(s) = \bigotimes_{i=1}^{m(s)} C_K(\zeta_i)^{\otimes n_i(s)}$$

of the Levi subgroup $H_s(q)$ of $L_s(q)$. It has an $\mathcal{O}H_s(q)$ -lattice

$$(9.7) \quad C_{\theta}(s) = \bigotimes_{i=1}^{m(s)} C_{\theta}(\zeta_i)^{\otimes n_i(s)}.$$

Since Deligne-Lusztig induction behaves well with respect to group products, the $H_s(q)$ -module $C_K(s)$ has character $\pm R_{T_s^{\times}(q)}^{H_s(q)}(\mu)$, where $\mu = \bigotimes_{i=1}^{m(s)} \zeta_i^{\otimes n_i(s)}$.

As already mentioned above, [FS; (1.16)] associates a linear character \widehat{s} of $T_s(q)$ to s . Tracing through the construction used in [FS; (1.16)], we find that \widehat{s} can be defined as a product of characters $\mu = \bigotimes_{i=1}^{m(s)} \widehat{s}_i^{n_i(s)}$. Each character \widehat{s}_i is linear and regular on $\overline{\mathbb{F}}_q^{\times}$, obtained as follows: All finite fields in sight are identified in a fixed way with a subfield of $\overline{\mathbb{F}}_q$, and a generator is chosen for the multiplicative group of each such subfield. Thus, all finite fields under consideration may be regarded as having a fixed generator. The linear character \widehat{s}_i is obtained by sending the generator of $\overline{\mathbb{F}}_q^{\times}$ to $s_i \in \overline{\mathbb{F}}_q^{\times}$ and following this map by a chosen fixed injection $\overline{\mathbb{F}}_q^{\times} \subset \overline{\mathbb{Q}}_{p_1}^{\times} \cong \mathbb{C}^{\times}$ (for some prime p_1 , denoted ℓ in [FS]). We now *redefine* $C_K(s)$ to be compatible with [FS]'s choice of \widehat{s} . That is, $\zeta_i = \widehat{s}_i$ for each i , and thus we take $\mu = \widehat{s}$ above. None of the properties quoted from [DJ2] depend on their specific indexing of the cuspidal representations $C_K(\zeta)$.⁹

Recall again that Harish-Chandra induction $R_{H_s^{\times}(q)}^{L_s^{\times}(q)}$ agrees with the Deligne-Lusztig induction functor on characters.

⁸In the notation of [DJ2], the root ζ is denoted s .

⁹Also, the meaning of the original $C_K(\zeta)$ is changed in this new notation only by a new embedding $\overline{\mathbb{F}}_q^{\times} \hookrightarrow \mathbb{C}^{\times}$, which is a matter of choice in any case.

Lemma 9.8. For $s \in \mathcal{C}_{ss}$, let $M_{s, L_s(q), R} = R_{H_s(q)}^{L_s(q)} C_R(s)$, where $R = K$ or \mathcal{O} . Then

$$(9.8.1) \quad \text{End}_{RL_s(q)}(M_{s, L_s(q), R}) \cong \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, R, q^{a_i(s)}).$$

Furthermore, $M_{s, L_s(q), K}$ has character of the form

$$(9.8.2) \quad \text{ch } M_{s, L_s(q), K} = \sum_{\lambda \vdash \mathbf{n}(s)} c_\lambda \chi_{s, \lambda, L_s(q)}$$

with all multiplicities $c_\lambda \neq 0$.

Proof. Let $m = m(s)$. In case $m = 1$, (9.8.1) is noted below [DJ2; (2.17)]. The case $m > 1$ follows formally from that fact. The second assertion (9.8.2) follows from (9.5b). \square

For a partition $\lambda \vdash m$ and a commutative \mathcal{L} -algebra R (with $t \mapsto q$), let

$$y_\lambda = \sum_{w \in W_\lambda} (-q)^{-\ell(w)} \tau_w \in H(\mathfrak{S}_m, R, q)$$

as in (II.2) in the Preamble to Part II. Any $\lambda \vdash \mathbf{n}(s)$ defines an element

$$y_\lambda = y_{\lambda^{(1)}} \otimes \cdots \otimes y_{\lambda^{(m(s))}} \in \bigotimes_{i=1}^{m(s)} H(\mathfrak{S}_{n_i(s)}, R, q^{a_i(s)}).$$

If N is a submodule of an \mathcal{O} -module M , let \sqrt{N} be the smallest \mathcal{O} -submodule of M containing N such that M/\sqrt{N} is \mathcal{O} -torsion-free. With the notation of (8.4), let $m_\lambda = \prod k_{\lambda^{(i)}}(n_i(s), n_i(s))$ and form the $\mathcal{O}L_s(q)$ -module

$$(9.9) \quad \widehat{M}_{s, L_s(q), \mathcal{O}} = \bigoplus_{\lambda \vdash \mathbf{n}(s)} \sqrt{y_\lambda \widehat{M}_{s, L_s(q), \mathcal{O}}}^{\oplus m_\lambda}.$$

For $R = K$ or k , put $\widehat{M}_{s, L_s(q), R} = \widehat{M}_{s, L_s(q), \mathcal{O}} \otimes_{\mathcal{O}} R$. If $\lambda = ((1^{n_1(s)}), \dots, (1^{n_{m(s)}(s)}))$, then $y_\lambda = 1$, so $\sqrt{y_\lambda \widehat{M}_{s, L_s(q), \mathcal{O}}} = \widehat{M}_{s, L_s(q), \mathcal{O}}$. Hence,

$$(9.10) \quad \text{ch } \widehat{M}_{s, L_s(q), K} = \sum_{\lambda \vdash \mathbf{n}(s)} c'_\lambda \chi_{s, \lambda, L_s(q)}, \text{ where all } c'_\lambda \neq 0$$

by (9.8.2).

Lemma 9.11. For $s \in \mathcal{C}_{ss}$ and $\widehat{M}_{s, L_s(q), \mathcal{O}}$ as in (9.9), we have an isomorphism:

$$\text{End}_{\mathcal{O}L_s(q)}(\widehat{M}_{s, L_s(q), \mathcal{O}}) \cong \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}.$$

Proof. Use (8.4) and the proof of [DJ2; (2.24vi)] for each factor. \square

The main goal of [FS] is to classify the p -blocks of $G(q)$. These are indexed by pairs (s, λ) , with $s \in \mathcal{C}_{ss, p'} = \mathcal{C}_{ss} \cap \mathcal{C}_{p'}$ and λ an e -core of a

multi-partition in $\Lambda^+(Z_{G(q)}(s))$. (Here $e = (e_1, \dots, e_{m(s)})$, where e_i is defined in terms of the order of s_i . We do not require the precise definition.) We have the following result, which follows immediately from remarks above and [FS; (7A)] (see also [DJ2; § 7]):

Lemma 9.12. *For distinct elements $s, s' \in \mathcal{C}_{ss,p'}$, the characters $\chi_{s,\lambda}$, $\chi_{s',\lambda'}$ belong to distinct blocks of $KG(q)$ for any choice of multi-partitions $\lambda \vdash \mathbf{n}(s)$ and $\lambda' \vdash \mathbf{n}(s')$. Thus, the characters in $R_{L_s(q)}^{G(q)} \widehat{M}_{s,L_s(q),\emptyset}$ and $R_{L_{s'}(q)}^{G(q)} \widehat{M}_{s',L_{s'}(q)}$ belong to distinct blocks.*

For $s \in \mathcal{C}_{ss,p'}$, define $B_{s,G(q)}$ (resp., $B_{s,L_s(q)}$) to be the sum of the blocks of $\mathcal{O}G(q)$ (resp., $\mathcal{O}L_s(q)$) which contain a character of the form $\chi_{st,\lambda}$ (resp., $\chi_{st,\lambda,L_s(q)}$), where $t \in Z_{G(q)}(s)$ is a p -element.

Lemma 9.13. *Harish-Chandra induction defines a full embedding*

$$(9.13.1) \quad R_{L_s(q)}^{G(q)} : \text{mod-}B_{s,L_s(q)}^{\text{torsion-free}} \hookrightarrow \text{mod-}B_{s,G(q)}^{\text{torsion-free}}$$

from the category of \mathcal{O} -torsion-free $B_{s,L_s(q)}$ -modules to the category of \mathcal{O} -torsion-free $B_{s,G(q)}$ -modules. The functor induces a character isometry, takes projective indecomposable modules to projective indecomposable modules, preserving distinct isomorphism types.

Proof. The fact that Harish-Chandra induction $R_{L_s(q)}^{G(q)}$ from characters in $B_{s,L_s(q)}$ to characters in $B_{s,G(q)}$ is an isometry follows from (9.5a). Now let M, N be \mathcal{O} -torsion-free $B_{s,L_s(q)}$ -modules. The Mackey decomposition theorem implies that

$$\text{Hom}_{\mathcal{O}G(q)}(R_{L_s(q)}^{G(q)}M, R_{L_s(q)}^{G(q)}N) \cong \text{Hom}_{\mathcal{O}L_s(q)}(M, N) \otimes X,$$

where X is a torsion-free \mathcal{O} -module. Since $R_{L_s(q)}^{G(q)}$ is an isometry, we have $X=0$. Thus,

$$\text{Hom}_{\mathcal{O}G(q)}(R_{L_s(q)}^{G(q)}M, R_{L_s(q)}^{G(q)}N) \cong \text{Hom}_{\mathcal{O}L_s(q)}(M, N),$$

so that the functor in (9.13.1) is a full embedding. Obviously, this functor takes indecomposable modules to indecomposable modules, and it preserves distinct isomorphism types. It remains to check that if Q is a projective $\mathcal{O}L_s(q)$ -module, then $R_{L_s(q)}^{G(q)}Q$ is a projective $\mathcal{O}G(q)$ -module. Let $P(q) = L_s(q) \ltimes V(q)$ be the standard parabolic subgroup associated to $L_s(q)$, and let Q' be the $\mathcal{O}P(q)$ -module obtained from Q by inflating the action of $L_s(q)$ on Q through the natural homomorphism $P(q) \rightarrow L_s(q)$. Since $V(q)$ has order prime to p , $|V(q)|$ is a unit in \mathcal{O} , and a standard argument shows that Q' is a projective $\mathcal{O}P(q)$ -module. Thus, $R_{L_s(q)}^{G(q)}Q = \text{ind}_{P(q)}^{G(q)}Q'$ is a projective $\mathcal{O}G(q)$ -module, as required. \square

Corollary 9.14. For $s \in \mathcal{C}_{ss,p'}$, define $\widehat{M}_{s,G(q),\emptyset} = R_{L_s(q)}^{G(q)} \widehat{M}_{s,L_s(q),\emptyset}$. Then

$$\text{End}_{\mathcal{O}G}(\widehat{M}_{s,G(q),\emptyset}) \cong \bigotimes_i^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\emptyset}.$$

Proof. This follows from (9.11) and (9.13). \square

Lemma 9.15. For $s \in \mathcal{C}_{ss,p'}$, the $\mathcal{O}G(q)$ -module $\widehat{M}_{s,G(q),\emptyset}$ satisfies the hypothesis of (9.1).

Proof. By [DJ2; (3.7)], the $\mathcal{O}L_s(q)$ -modules $\widehat{M}_{s,L_s(q),\emptyset}$ satisfies the hypothesis of (9.1), so there is an exact sequence

$$(9.15.1) \quad 0 \rightarrow \widehat{N}_{s,L_s(q),\emptyset} \rightarrow \widehat{P}_{s,L_s(q),\emptyset} \rightarrow \widehat{M}_{s,L_s(q),\emptyset} \rightarrow 0$$

in which the irreducible characters in $\widehat{N}_{s,L_s(q),\emptyset}$ all have the form $\chi_{st,\lambda,L_s(q)}$, where $1 \neq t \in Z_{G(q)}(s)$ is a p -element. But $R_{L_s(q)}^{G(q)} \chi_{st,\lambda,L_s(q)} = \chi_{st,\lambda}$ and $R_{L_s(q)}^{G(q)} \chi_{s,\lambda,L_s(q)} = \chi_{s,\lambda}$ by (9.5a). Thus, we can apply Harish-Chandra induction to (9.15.1) to obtain the desired conclusion. \square

For each $s \in \mathcal{C}_{ss,p'}$, let $J_s(q)$ be the annihilator in $B_{s,G(q)}$ of $\widehat{M}_{s,G(q),\emptyset}$. Define $J(q) = \sum_{s \in \mathcal{C}_{ss,p'}} J_s(q)$. Since $\mathcal{O}G(q) = \bigoplus_{s \in \mathcal{C}_{ss,p'}} B_{s,G(q)}$, we have:

$$(9.16) \quad \mathcal{O}G(q)/J(q) \cong \bigoplus_{s \in \mathcal{C}_{ss,p'}} B_{s,G(q)}/J_s(q).$$

Now we can establish the following fundamental result.

Theorem 9.17. Let k be an algebraically closed field of characteristic $p > 0$ and let $G(q) = GL_n(q)$, where p does not divide q . The algebras $\mathcal{O}G(q)$ and $\mathcal{O}G(q)/J(q)$ (defined in (9.16)) have the same irreducible modules over k . There is a Morita equivalence

$$(9.17.1) \quad F : \text{mod-}\mathcal{O}G(q)/J(q) \xrightarrow{\text{Morita}} \text{mod-} \bigoplus_{s \in \mathcal{C}_{ss,p'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_{\emptyset},$$

defined by $F(-) = \text{Hom}_{\mathcal{O}G(q)/J(q)}(\bigoplus_{s \in \mathcal{C}_{ss,p'}} \widehat{M}_{s,G(q),\emptyset}, -)$. The functor F induces a Morita equivalence

$$(9.17.2) \quad \bar{F} : \text{mod-}kG(q)/J(q)_k \xrightarrow{\text{Morita}} \text{mod-} \bigoplus_{s \in \mathcal{C}_{ss,p'}} \bigotimes_{i=1}^{m(s)} S_{q^{a_i(s)}}(n_i(s), n_i(s))_k$$

by putting $\bar{F}(-) = \text{Hom}_{kG(q)/J(q)_k}(\bigoplus_{s \in \mathcal{C}_{ss,p'}} \widehat{M}_{s,G(q),k}, -)$.

Proof. For $s \in \mathcal{C}_{ss,p'}$, $\widehat{M}_{s,G(q),\emptyset}$ is a projective $B_{s,G(q)}/J_s(q)$ -module by (9.15) and (9.1). Hence, it is a projective $\mathcal{O}G(q)/J(q)$ -module. It follows that

$$\widehat{M} = \bigoplus_{s \in \mathcal{C}_{ss,p'}} \widehat{M}_{s,G(q),\emptyset}$$

is a projective $\mathcal{O}G(q)/J(q)$ -module.

By [DJ2; p.356], the irreducible $kL_s(q)$ -modules appearing in the head of $\widehat{M}_{s, L_s(q), \emptyset}$ are indexed by the set of multi-partitions $\lambda \vdash \mathfrak{n}(s)$. Hence, using (9.13), we see that the head of $\widehat{M}_{s, G(q), \emptyset} = R_{L_s(q)}^{G(q)} \widehat{M}_{s, L_s(q), \emptyset}$ has as many non-isomorphic irreducible modules as there are multi-partitions $\lambda \vdash \mathfrak{n}(s)$. On the other hand, (9.12) implies that for distinct $s, s' \in \mathcal{C}_{ss, p'}$, the modules $\widehat{M}_{s, G(q), \emptyset}$ and $\widehat{M}_{s', G(q), \emptyset}$ belong to distinct blocks. Since the number of irreducible $kG(q)$ -modules equals the number of p' -conjugacy classes in $G(q)$, namely, the cardinality of the set $\{(s, \lambda) \mid s \in \mathcal{C}_{ss, p'}, \lambda \vdash \mathfrak{n}(s)\}$, we see that $\widehat{M}_{s, G(q), \emptyset}$ has all the irreducible $kG(q)$ -modules of $B_{s, G(q)}$ in its head. Thus, $\mathcal{O}G(q)/J(q)$ and $\mathcal{O}G(q)$ have the same irreducible modules over k .

We can apply (9.2) and (9.14) to conclude that the \mathcal{O} -algebras $B_{s, G(q)}/J_s(q)$ and

$$\begin{aligned} \text{End}_{B_{s, G(q)}}(\widehat{M}_{s, G(q), \emptyset}) &\cong \text{End}_{\mathcal{O}G(q)}(\widehat{M}_{s, G(q), \emptyset}) \\ &\cong \bigotimes_i^{m(s)} S_{q^{s(i)}}(n_i(s), n_i(s))_{\emptyset} \end{aligned}$$

are Morita equivalent by the functor $F_s(-) = \text{Hom}_{\mathcal{O}G(q)}(\widehat{M}_{s, G(q), \emptyset}, -)$. Therefore, $\bar{F}(-) = \text{Hom}_{\mathcal{O}G(q)}(\widehat{M}, -)$ defines a Morita equivalence as required in (9.17.1). Finally, base change defines the Morita equivalence \bar{F} in (9.17.2). \square

As explained in § 8, the irreducible modules for $S_q(n, m) = S_t(n, m)_k$ are naturally indexed by the poset $A^+(n, m)$ of partitions of m into at most n parts. For $\lambda \in A^+(n, m)$, we let $L_q(\lambda)$ or sometimes $L^k(\lambda)$ denote the corresponding irreducible $S_q(n, m)$ -module. Thus, we can use (9.17) to obtain a natural indexing of the irreducible $kG(q)$ -modules. Explicitly, given $s \in \mathcal{C}_{ss, p'}$ and $\lambda \vdash \mathfrak{n}(s)$, let $D(s, \lambda) \in \text{Irr}(kG(q))$ correspond, under the Morita equivalence \bar{F} to the irreducible module $\bigotimes_{i=1}^{m(s)} L_{q^{s(i)}}(\lambda^{(i)}) \in \text{Irr}(\bigotimes_{i=1}^{m(s)} S_{q^{s(i)}}(n_i(s), n_i(s)))$.

Remarks 9.18. (a) The category $\text{mod-}S_q(n, m)_k$ can be regarded as fully embedded in the module category for various versions of the quantum general linear group or of a quantum enveloping algebra (see (e) below). For example, let $GL_{n, q}(k)$ be the Manin quantum general linear group over k .¹⁰ As explained in [PW; § 10], $\text{mod-}S_q(n, m)_k$ (or $S_q(n, m)_k\text{-mod}$) identifies with the full subcategory of rational $GL_{n, q}(k)$ -modules which are homogeneous of degree m . We can assume that $L_q(\lambda)$, when regarded as a $GL_{n, q}(k)$ -module, is the irreducible module of highest weight λ (in the usual Lie-theoretic sense).

(b) The algebra $S_q(n, n)_k$ has a unique one-dimensional representation

¹⁰Alternatively, one could use the quantum general linear group $GL'_{n, q}(k)$ studied in [DD]. Both quantum general linear groups lead to the same q -Schur algebra [DPW]. The action of H on $V^{\otimes m}$ used in [PW; (11.3c)] can be replaced by the equivalent action described in (8.1) above, so that the quantum group $GL_{n, q}(k)$ can be used, rather than $GL'_{n, q}(k)$.

$L_q(\lambda)$, obtained by taking $\lambda = (1^n)$. This fact follows immediately from the connection described above with the representation theory of $GL_{n,q}(k)$ and the fact that the weights in any irreducible $GL_{n,q}(k)$ -module are stable under the Weyl group \mathfrak{S}_n [PW; (8.8.2)]. In fact, $L_q((1^n))$ corresponds precisely to the one-dimensional module on $GL_{n,q}(k)$ defined by the quantum determinant \det_q . For this reason, we denote the $S_q(n, n)_k$ -module $L_q((1^n))$ by simply \det_q . The quantum determinant is defined for the algebra $S_t(n, n)$ over \mathcal{X} and is the unique group-like element \det_t in the coalgebra $A_t(n, n) = \text{Hom}_{\mathcal{X}}(S_t(n, n), \mathcal{X})$.

We give another description of \det_t . If $\emptyset \neq \lambda \subseteq S$, then $\text{Hom}_H(x_\lambda H, \text{SGN}) = 0$. Let V be a free \mathcal{X} -module of rank n (with fixed ordered basis). It follows from (8.4) with $m = n$ that there is an isomorphism

$$\text{Hom}_H(V^{\otimes n}, \text{SGN}) \cong \text{Hom}_H(x_\emptyset H, \text{SGN}) \cong \mathcal{X}$$

of \mathcal{X} -modules because in the decomposition (8.4) $H \cong x_\emptyset H$ appears with multiplicity 1. Thus,

$$(9.18.1) \quad \det_t \cong \text{Hom}_H(V^{\otimes m}, \text{SGN}).$$

(c) Consider the trivial module k for $kG(q)$. In the notation above, it has the form $D(1, \lambda)$ for some partition $\lambda \vdash n$. We claim that $\lambda = (1^n)$. To see this, it suffices from (b) to show, in the language of (9.17.2), that

$$\dim \bar{F}(k) = \dim \text{Hom}_{kG(q)}(\widehat{M}_{1,G(q),k}, k)$$

is equal to 1. This is easily done by reducing to a corresponding problem over K . Here are the details. We have, taking $s=1$ in (9.9),

$$\widehat{M}_{1,G(q),k} = \left(\bigoplus_{\lambda \vdash n} \sqrt{y_\lambda \widehat{M}_{1,G(q),\emptyset}^{\oplus m_\lambda}} \right)_k,$$

where $M_{1,G(q),\emptyset} = \text{ind}_B^{G(q)} \mathcal{O}$. Since $(\sqrt{y_\lambda \widehat{M}_{1,G(q),\emptyset}^{\oplus m_\lambda}})_k \cong \text{ind}_B^{G(q)} k$ for $\lambda = (1^n)$, the equality $\dim \text{Hom}_{kG(q)}(\text{ind}_B^{G(q)} k, k) = 1$ shows that it suffices to check that

$$(9.18.2) \quad \text{Hom}_{kG(q)} \left(\left(\sqrt{y_\lambda \text{ind}_B^{G(q)} \mathcal{O}} \right)_k, k \right) = 0 \text{ for } \lambda \neq (1^n).$$

Let $M = \text{ind}_B^{G(q)} \mathcal{O}$. Because $\sqrt{y_\lambda M}$ is a projective $\mathcal{O}G(q)/J(q)$ -module (as a direct summand of the projective $\mathcal{O}G(q)/J(q)$ -module $\widehat{M}_{1,G(q),\emptyset}$) and because \mathcal{O} is an $\mathcal{O}G(q)/J(q)$ -module, we have

$$\begin{aligned} \text{Hom}_{\mathcal{O}G(q)}(\sqrt{y_\lambda M}, \mathcal{O}) &\cong \text{Hom}_{\mathcal{O}G(q)/J(q)}(\sqrt{y_\lambda M}, \mathcal{O})_k \\ &\cong \text{Hom}_{kG(q)/J(q)_k}((\sqrt{y_\lambda M})_k, k) \\ &\cong \text{Hom}_{kG(q)}((\sqrt{y_\lambda M})_k, k). \end{aligned}$$

Therefore, it suffices to show that $\text{Hom}_{\mathcal{O}G(q)}(\sqrt{y_\lambda M}, \mathcal{O}) = 0$ for $\lambda \neq (1^n)$, or equivalently, we must prove that

$$\text{Hom}_{KG(q)}((\sqrt{y_\lambda M})_K, K) \cong \text{Hom}_{KG(q)}(y_\lambda \text{ind}_B^{G(q)} K, K) = 0$$

for $\lambda \neq (1^n)$. If $p_n(q) = \sum_{w \in \mathfrak{S}_n} q^{\ell(w)}$, then $(\sum_{w \in \mathfrak{S}_n} \tau_w)^2 = p_n(q) \sum_{w \in \mathfrak{S}_n} \tau_w$. Thus,

$$e = p_n(q)^{-1} \sum_{w \in \mathfrak{S}_n} \tau_w \in \text{End}_{kG(q)}(\text{ind}_B^{G(q)} K) \cong H(\mathfrak{S}_n, q)_K,$$

is the idempotent projection from $\text{ind}_B^{G(q)} K$ onto its unique constituent isomorphic to the trivial module. Obviously $ey_\lambda = 0$ for $\lambda \neq (1^n)$. This proves our claim.

It also follows that the trivial $\mathcal{O}G(q)$ -module \mathcal{O} (which is annihilated by $J(q)$) corresponds under the Morita equivalence (9.17.1) to the quantum determinant representation \det_q for $S_q(n, n)_\mathcal{O}$.

(d) The Grothendieck group of the algebra on the right hand side of (9.17.2) has a \mathbb{Z} -basis consisting of the reductions “mod p ” (i.e., tensoring with k) of (lattices of) irreducible modules over K of the algebra on the right hand side of (9.17.1). Therefore, the same result holds (with the same formulas) for the algebras on the left hand side of (9.17.2) and (9.17.1). In this way, one obtains formulas in terms of ordinary characters for all the irreducible Brauer characters for $kG(q)$. We regard the problem of determining these Brauer character formulas as more important than the dual problem of determining the decomposition numbers of ordinary characters. Theoretically, the latter problem can be solved from the former through character values and linear algebra. In this way, the statement of Theorem 9.17 generalizes the original decomposition number results of Dipper-James [DJ2] (though we use their results in deriving (9.17)). Finally, it follows from the results in [DJ2] and [D1], comparing decomposition numbers for $kG(q)$ with those for q^a -Schur algebras, that the indexing above for $\text{Irr}(kG(q))$ is compatible with that taken by Dipper and James. However, we will not need to use this result in the sequel.

(e) Let $\tilde{U}_q(\mathfrak{gl}_n)$ be the divided power $\mathbb{Z}[q, q^{-1}]$ -form of the quantized enveloping algebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of $n \times n$ matrices over \mathbb{C} . For any field k and any specialization $\mathbb{Z}[q, q^{-1}] \rightarrow k$, the corresponding q -Schur algebra $S_q(n, m)_k$ is a homomorphic image of $\tilde{U}_q(\mathfrak{gl}_n(\mathbb{C}))$ [Du]. Therefore, we could also index the irreducible $kG(q)$ -modules by using the quantum enveloping algebra $\tilde{U}_q(\mathfrak{gl}_n(\mathbb{C}))_k$. This would lead to the same indexing by highest weights as above. More importantly, it provides, through the results of §2, together with the work [KL2] a connection between the representation theory of $G(q)$ and that of the affine Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$.

§ 10. H^1 -Cohomology in Non-Describing Characteristic

In this section, we show how the Morita equivalence (9.17) can be effectively applied to study $H^1(GL_n(q), V)$, when V is an irreducible $kGL_n(q)$ -module and k is an algebraically closed field of characteristic p not dividing q . Such H^1 -cohomology is important for the theory of maximal subgroups of finite groups. As explained in [AS], the two basic ingredients needed to understand all maximal subgroups of a given finite group G , modulo easier or smaller problems, are:

- (1) knowledge of the maximal subgroups of quasi-simple groups; and

(2) determination of the cohomology groups $H^1(G, V)$ for quasi-simple groups G and irreducible G -modules V .

We will show immediately below, that when p is prime to $q(q-1)$, the answer to (2) above for $G = GL_n(q)$ is given completely in terms of a corresponding Ext^1 -calculation for q -Schur algebras $S_q(n, n)_k$. Using the generic results of §§ 2, 8, together with [KL2], one can expect, if p is large, that the structure of these Ext^1 -groups can be understood eventually in terms of the Kazhdan-Lusztig polynomials related to the theory of affine Lie algebras. Here we allow q to vary subject to our hypothesis on p .

We will use the description of the irreducible $kG(q)$ -modules $D(s, \lambda)$ given above (9.18). Recall that s belongs to the set $\mathcal{C}_{ss, p'}$ of representatives of the set of semisimple p' -conjugacy classes of $GL_n(q)$, for a prime p' not dividing q , while λ is a certain multipartition. We now prove the following result, which includes the H^1 -calculations alluded to above.

Theorem 10.1. *Let $G(q) = GL_n(q)$. Assume that the characteristic p of the field k is relatively prime to $q(q-1)$. For any $kG(q)/J(q)_k$ -module V , we have*

$$(10.1.1) \quad H^1(G(q), V) \cong \text{Ext}_{S_q(n, n)_k}^1(\det_q, \bar{F}(V)),$$

where $\bar{F}(V)$ is the image of V under the Morita equivalence \bar{F} defined in (9.17.2). Also, there is an injection

$$(10.1.2) \quad \text{Ext}_{S_q(n, n)_k}^2(\det_q, \bar{F}(V)) \hookrightarrow \text{Ext}_{G(q)}^2(k, V) \cong H^2(G(q), V).$$

In particular, for any $s \in \mathcal{C}_{ss, p'}$ and any multi-partition $\lambda \vdash \mathbf{n}(s)$, we have

$$(10.1.3) \quad H^1(G(q), D(s, \lambda)) \cong \begin{cases} \text{Ext}_{S_q(n, n)_k}^1(\det_q, L^k(\lambda)), & s=1 \\ 0, & s \neq 1. \end{cases}$$

Proof. Let k denote the trivial $kG(q)$ -module and form the short exact sequence

$$(10.1.4) \quad 0 \rightarrow U \rightarrow \text{ind}_{B(q)}^{G(q)} k \rightarrow k \rightarrow 0$$

of $kG(q)$ -modules (where k denotes the trivial module). Since $k|_{B(q)}^{G(q)}$ is a $kG(q)$ -direct summand of the module $\widehat{M}_{1, G(q), k}$ defined in (9.9), the definition of $J(q)_k$ implies that (10.1.4) is actually an exact sequence of $kG(q)/J(q)_k$ -modules in which $\text{ind}_{B(q)}^{G(q)} k$ is a projective $kG(q)/J(q)_k$ -module. By hypothesis, p is relatively prime to $q(q-1)$, so that k is a projective $kB(q)$ -module, and hence $\text{ind}_{B(q)}^{G(q)} k$ is also a projective $kG(q)$ -module. Hence, writing $M = k|_{B(q)}^{G(q)}$, the long exact sequence of cohomology gives a commutative diagram

$$\begin{array}{ccccccc} \text{Hom}_{kG(q)/J(q)_k}(M, V) & \rightarrow & \text{Hom}_{kG(q)/J(q)_k}(U, V) & \rightarrow & \text{Ext}_{kG(q)/J(q)_k}^1(k, V) & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Hom}_{G(q)}(M, V) & \rightarrow & \text{Hom}_{G(q)}(U, V) & \rightarrow & \text{Ext}_{G(q)}^1(k, V) & \rightarrow & 0 \end{array}$$

in which the rows are exact and the two left vertical arrows are isomorphisms. This gives

$$H^1(G(q), V) \cong \text{Ext}_{kG(q)}^1(k, V) \cong \text{Ext}_{kG(q)/J(q)_k}^1(k, V).$$

Now (9.17) implies that

$$\text{Ext}_{kG(q)/J(q)_k}^1(k, V) \cong \text{Ext}_{\mathbb{F}_q(n,n)_k}^1(\det_q, \bar{F}(V)),$$

using the fact proved in (9.18(c)) that $\bar{F}(k) = \bar{F}(D(1, (1^n))) \cong \det_q$. This proves (10.1.1).

Next, by general elementary principles, we have

$$\text{Ext}_{kG(q)/J(q)_k}^1(U, V) \subseteq \text{Ext}_{kG(q)}^1(U, V).$$

By dimension shifting,

$$\text{Ext}_{kG(q)/J(q)_k}^1(U, V) \cong \text{Ext}_{kG(q)/J(q)_k}^2(k, V).$$

Another application of (9.17) yields (10.1.3).

Finally, (10.1.3) is a special case of (10.1.1), taking $V = D(s, \lambda)$ and observing that if $s \neq 1$, then $D(s, \lambda)$ and $D(1, (1^n))$ belong to different blocks. \square

Now we prove the following stability result for H^1 mentioned in the introduction.

Theorem 10.2. (*H^1 -stability*) *Let n be a fixed positive integer and k an algebraically closed field of characteristic p . Let q be a prime power not divisible by p , and consider the group $GL_n(q)$. For $s \in \mathcal{C}_{ss, p'}$ and $\lambda \vdash \mathfrak{n}(s)$, let $D(s, \lambda)$ denote the associated irreducible $kGL_n(q)$ -module. Then:*

- (a) *If $1 \neq s \in \mathcal{C}_{ss, p'}$, then $H^1(GL_n(q), D(s, \lambda)) = 0$.*
- (b) *There exists an integer $N(n)$, depending on n , such that if $p > N(n)$, then $\dim H^1(GL_n(q), D(1, \lambda))$ depends only on λ and the order ℓ of q modulo p .*

Proof. First, (a) is clear, since the modules $D(s, \lambda)$ with $s \neq 1$ are not even in the principal p -block.

To prove (b), we may fix the order ℓ of q modulo p to a fixed value $\leq n$, since these are the only cases for which p divides $|GL_n(q)|$. Also, we can assume that $N(n) > n$.

Case 1. $\ell > 1$. In other words, p does not divide $q - 1$, so that (10.1) implies that

$$(10.2.1) \quad \dim H^1(G(q), D(1, \lambda)) = \dim \text{Ext}_{\mathbb{F}_q(n,n)_k}^1(\det_q, L^k(\lambda)).$$

Consider the ring $\mathcal{Z}(\ell)$ of algebraic integers in $K = \mathbb{Q}(\sqrt[\ell]{1})$ as in (8.6). By (8.6), there exists a non-empty open subset Ω of $\text{Spec } \mathcal{Z}(\ell)$ such that if $\mathfrak{p} \in \Omega$, then (1) each $L^k(\lambda)$, $\lambda \in \Lambda^+(n, m)$, is obtained by “reduction mod \mathfrak{p} ” from a lattice for $L^k(\lambda)$, and (2) there is an equality

$$(10.2.2) \quad \dim \operatorname{Ext}_{S_q(n, n)_k}^1(\det_q, L^{k(p)}(\lambda)) = \dim \operatorname{Ext}_{S_q(n, n)_K}^1(\det_q, L^K(\lambda)).$$

Thus,

$$(10.2.3) \quad \begin{aligned} \dim \operatorname{Ext}_{S_q(n, n)_k}^1(\det_q, L^k(\lambda)) &= \dim \operatorname{Ext}_{S_q(n, n)_{k(p)}}^1(\det_q, L^{k(p)}(\lambda)) \\ &= \dim \operatorname{Ext}_{S_q(n, n)_K}^1(\det_q, L^K(\lambda)), \end{aligned}$$

provided that $k(\mathfrak{p}) \subseteq k$, which means precisely that \mathfrak{p} lies over (p) , where now p is the characteristic of k .

Since $\mathcal{X}(\ell)$ as above is finite over \mathbb{Z} , for any prime integer $p \in \mathbb{Z}$, there exist only finitely many prime ideals $\mathfrak{p} \in \operatorname{Spec} \mathcal{X}(\ell)$ satisfying $\mathfrak{p} \cap \mathbb{Z} = (p)$. Thus, we can assume that $N(n)$ is sufficiently large that if $p > N(n)$ and $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}(\ell)$ lies over (p) , then $\mathfrak{p} \in \Omega$. The theorem is proved in this case.

Case 2. $\ell = 1$. For any subset $\lambda \subseteq S$, let $P_\lambda(q)$ the parabolic subgroup of $G(q)$ whose Levi factor $L_\lambda(q)$ has simple roots λ . Since $p > n$, the dimension of $H^1(P_\lambda(q), k) \cong \operatorname{Hom}(P_\lambda(q)/P_\lambda(q)', k)$ equals the rank of the center of the Levi factor L_λ , and so depends only on λ .

Now let \mathcal{O}, K, k be as in § 9. In particular, K is a splitting field for $G(q)$, so that $KG(q)/J(q)_K$ is a split semisimple algebra. Since $kG(q)/J(q)_k$ is also a split semisimple algebra, we conclude that any $D(1, \lambda)$ lifts to an irreducible ordinary unipotent representation. Also, [CR; (68.24)] implies that the ordinary irreducible characters are \mathbb{Z} -linear combinations of the characters of the transitive permutation modules $\mathbb{C} |_{P_\lambda(q)}^{G(q)}$, $\lambda \subseteq S$. Hence, in the Grothendieck group of $kG(q)/J(q)_k$, any $D(1, \lambda)$ is a \mathbb{Z} -linear combination of various $k |_{P_\lambda(q)}^{G(q)}$. So, by the Shapiro–Eckmann lemma, it suffices to prove that the dimensions of the $H^1(G(q), k |_{P_\lambda(q)}^{G(q)}) \cong H^1(P_\lambda(q), k)$ for $\lambda \subseteq S$ stabilize if p is a sufficiently large prime (and p still divides $q - 1$). But for $p > n$, this is clear from the previous paragraph. \square

Remarks 10.3. (a) The cohomology of the q -Schur algebras $S_q(n, m)_k$ can be expressed in terms of the cohomology of the quantum enveloping algebra $\tilde{U}_q(\mathfrak{gl}_n(k))$ because of the natural surjection $\tilde{U}_q(\mathfrak{gl}_n(k)) \rightarrow S_q(n, m)_k$ of algebras. Then we have:

$$(10.3.1) \quad \operatorname{Ext}_{S_q(n, m)_k}^\bullet(V, W) \cong \operatorname{Ext}_{\tilde{U}_q(\mathfrak{gl}_n(k))}^\bullet(V, W),$$

for any $S_q(n, m)_k$ -modules V, W . In (10.3.1), on the right hand side, V, W are regarded as $\tilde{U}_q(\mathfrak{gl}_n(k))$ -modules by inflation, and the $\operatorname{Ext}^\bullet$ -group is computed in the category of integrable modules. For a proof of this fact, see [DS]. Variations on (10.3.1), involving the various quantum general linear groups, are also established in [Do2] and [PW].

(b) Using the generic result (8.6) as in Case 1 of the proof of (10.2), the dimension of $\operatorname{Ext}_{S_q(n, n)_k}^n(\det_q, L^k(\mu))$ can be equated, when p is sufficiently large, with $\operatorname{Ext}_{S_q(n, n)_K}^n(\det_q, L^K(\mu))$ over a field K of characteristic 0 in which q is

regarded as a suitable root of unity. In turn, using the category equivalence given in [KL2], this latter dimension can be expressed as the dimension of an Ext^n -group for the affine algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$. It seems likely, as with the case of the category \mathcal{O} for a complex semisimple Lie algebra (see [CPS4; (3.8.1)]), that these dimensions can be given in terms of values of certain Kazhdan-Lusztig polynomials.

We next briefly consider the relationship between the H^1 -cohomology of $GL_n(q)$ and that of $SL_n(q)$. By Clifford theory, the restriction from $GL_n(q)$ to $SL_n(q)$ of an irreducible module is always completely reducible, isomorphic to a sum of conjugates of a single irreducible $SL_n(q)$ -module. In fact, all unipotent ordinary characters $\chi_{1,\lambda}$ restrict *irreducibly* to $SL_n(q)$. The $\chi_{1,\lambda}$ are precisely the irreducible constituents of the permutation module $K|_{B(q)}^{GL_n(q)}$. If $B'(q) = B(q) \cap SL_n(q)$, then the isomorphism $K|_{B(q)}^{GL_n(q)}|_{SL_n(q)} \cong K|_{B'(q)}^{SL_n(q)}$ induces an isomorphism

$$\text{End}_{GL_n(q)}(K|_{B(q)}^{GL_n(q)}) \cong \text{End}_{SL_n(q)}(K|_{B'(q)}^{SL_n(q)})$$

of endomorphism algebras. There are two ways to see this. One may observe that both sides have dimension equal to

$$|B(q) \backslash GL_n(q) / B(q)| = |B'(q) \backslash SL_n(q) / B'(q)|.$$

Alternatively, both endomorphism algebras have identical descriptions in terms of double cosets. Using this isomorphism, it is immediate that each $\chi_{1,\lambda}$ restricts irreducibly to $SL_n(q)$.

It is false, in general, that the modular irreducible unipotent modules $D(1, \lambda)$ restrict irreducibly to $SL_n(q)$. However, the exceptions occur for only relatively small primes. *We claim that, if p does not divide $(n, q-1)$, then any $D(1, \lambda)|_{SL_n(q)}$ is irreducible.* To see this, let $Z \cong \mathbb{F}_q^\times$ be the center of $GL_n(q)$. Because the elements of Z act by scalar multiplication on $D(1, \lambda)$, it suffices to check that each restriction $D(1, \lambda)|_{ZSL_n(q)}$ is irreducible.

Suppose that χ_1 is an ordinary irreducible character of $GL_n(q)$ which has a common $SL_n(q)$ -constituent with $\chi_2 = \chi_{1,\lambda}$. An easy argument shows that $\chi_1 = \widehat{s} \chi_2$ for a linear character \widehat{s} on $GL_n(q)$. This character \widehat{s} corresponds to a semisimple element $s \in Z$ as in [FS; p. 116].

We may assume \widehat{s} is trivial on $ZSL_n(q)$, hence that \widehat{s} , as a linear character on $GL_n(q) / ZSL_n(q)$, has order dividing $(n, q-1)$. Now, by our hypothesis on p , \widehat{s} has order prime to p . Since the correspondence of [FS; (1.16)] is an isomorphism of abelian groups, s also has order prime to p . It follows that $\chi_1 = \chi_{s,\lambda}$. Thus, by [FS; (7A)], χ_1 and χ_2 are in distinct p -blocks unless $s = 1$.

Now let $(\mathcal{O}, K, \mathfrak{k})$ be the p -modular system used in (9.1). Because we have shown above that the $\chi_{1,\lambda}$ restrict irreducibly $SL_n(q)$, we have

$$(*) \quad \text{End}_{\mathcal{O}GL_n(q)}(\widehat{M}_{1,G(q),\mathcal{O}}) \cong \text{End}_{\mathcal{O}SL_n(q)}(\widehat{M}_{1,G(q),\mathcal{O}}),$$

since both sides become isomorphic over K . This isomorphism implies that any indecomposable $GL_n(q)$ -summand X of $\widehat{M}_{1,G(q),\mathcal{O}}$ remains indecomposable upon restriction to $SL_n(q)$. By the previous paragraph, the hypothesis of (9.1) is

satisfied for $M = \widehat{M}_{1,G(q),\mathcal{O}}|_{SL_n(q)}$ and P the projective cover of M for $\mathcal{O}GL_n(q)$, but viewed as an $\mathcal{O}SL_n(q)$ -module. Thus, X is projective for a suitable quotient algebra A of $\mathcal{O}SL_n(q)$. The results of [CPS6; (1.5.6)] imply that X_k is indecomposable for A_k ; thus, a standard argument shows that X has a simple head. Since the restrictions of irreducible $GL_n(q)$ -modules to $SL_n(q)$ are completely reducible, the $GL_n(q)$ -head of X must remain irreducible upon restriction of $SL_n(q)$. But every $D(1, \lambda)$ appears in the head of some indecomposable summand X of $\widehat{M}_{1,G(q),\mathcal{O}}$. This completes the proof of our claim.

Clearly, if p does not divide $(n, q-1)$, (*) implies that the $D(1, \lambda)|_{SL_n(q)}$ are distinct for distinct λ . We will denote the restriction $D(1, \lambda)|_{SL_n(q)}$ again by $D(1, \lambda)$. As is well-known, the restriction of a p -block for a finite group G to a normal subgroup N is a union of conjugate blocks. The following definition, which agrees with more general terminology for unipotent blocks for reductive groups [GeH], is justified by this fact.

Definition 10.4. Assume that p does not divide both n and $q-1$, and, also does not divide q . The modular representations of $SL_n(q)$ having only composition factors of the form $D(1, \lambda)$ will be called the *unipotent* modular representations of $SL_n(q)$. They are the irreducible modular representations in a union of p -blocks for $SL_n(q)$ —these blocks will be called the *unipotent* blocks for $SL_n(q)$.

One corollary of our discussion above is that, for such unipotent p -blocks for $SL_n(q)$, there is a Morita equivalence like that given in (9.17) for $SL_n(q)$. In fact, the corresponding factor algebras for the group algebras over $GL_n(q)$ and $SL_n(q)$ are even isomorphic. Moreover, the previous two theorems essentially hold in the SL_n case. In more detail, we have

Theorem 10.5. Assume that p does not divide both n and $q-1$, and, also, does not divide q . Let L be an irreducible module for $SL_n(q)$ over the algebraically closed field k of characteristic p . If L is not unipotent, then $H^*(SL_n(q), L) = 0$. If $L = D(1, \lambda)$ is unipotent, then:

$$H^1(SL_n(q), L) \cong H^1(GL_n(q), D(1, \lambda)).$$

Finally, if p divides $q-1$, $\dim H^1(SL_n(q), L)$ depends only on λ when $p \geq n$.

The proof is very similar to the GL_n -proofs of the previous two theorems, with minor adjustments in the case where p divides $q-1$. Further details are left to the reader. We note that, as a corollary, the stability Theorem 10.2 also holds for SL_n .

Remark 10.6. Let $G(q) = GL_n(q)$, as above, but assume that $q \equiv 1 \pmod p$, so that the hypotheses of (10.1) fail. If $p > n$, then p does not divide the order of the Weyl group GL_n . In this case, if V is an arbitrary $kG(q)$ -module, it is possible to use the methods of [CPS1; §6] to obtain information concerning $H^*(G(q), V)$, in general, and $H^1(G(q), V)$, in particular, for any $kG(q)$ -module

V . Namely, let T (resp., B) be the maximal torus (resp., Borel subgroup) of GL_n consisting of diagonal (resp., upper triangular) matrices. By (9.3), p does not divide $[G(q) : B(q)]$. As shown in [CPS1; §6], there is a right action of the Hecke algebra $H_k = H(\mathfrak{S}_n, k, 1)$ on $H^n(B(q), V)$ given by

$$(10.6.1) \quad \mu\tau_w = \mu^w|_{B(q) \cap B(q)}|^{B(q)}, \quad w \in \mathfrak{S}_n, \mu \in H^n(B(q), V).$$

Here $|^{B(q)}$ is corestriction. But observe that $H_k \cong k\mathfrak{S}_n$, the group algebra of \mathfrak{S}_n . Since $T(q)$ contains a Sylow p -subgroup of $G(q)$, the restriction map $H^\circ(G(q), V) \rightarrow H^\circ(B(q), V)$ is injective, and the main result in [CPS1; §6] establishes that the image of $H^\circ(G(q), V)$ in $H^\circ(B(q), V)$ identifies with the space of fixed points, i. e.,

$$(10.6.2) \quad H^\circ(G(q), V) \cong H^\circ(B(q), V)^{\mathfrak{S}_n}.$$

The equality (10.6.2) provides a more conceptual reformulation of the classical Cartan-Eilenberg stability theorem for cohomology, cf. [CE; p. 258]. Since $B(q) = T(q) \times U(q)$, where U is the unipotent radical of B , and p does not divide $|U(q)|$, an elementary spectral sequence argument shows that there is a natural identification $H^\circ(B(q), V) \cong H^\circ(T(q), V^{U(q)})$. Thus, if we transfer the action of \mathfrak{S}_n from $H^\circ(B(q), V)$ to an action on $H^\circ(T(q), V^{U(q)})$, (10.6.2) can be rewritten in the form

$$(10.6.3) \quad H^\circ(G(q), V) \cong H^\circ(T(q), V^{U(q)})^{\mathfrak{S}_n}.$$

Because the set S of simple reflections $(i, i+1)$ generates \mathfrak{S}_n , an element $\mu \in H^\circ(T(q), V^{U(q)})$ is \mathfrak{S}_n -fixed if it is fixed by every simple reflection $s = (i, i+1)$. This holds if the image of μ in $H^\circ(T(q), V^{U(q) \cap U(q)^s})$ is fixed by the usual action of s . This condition, for all s , is necessary as well as sufficient for μ to correspond to an element of $H^\circ(G(q), V)$. See [CPS1; (6.1)] for further discussion.

§ 11. Resolutions

In the previous section, we used the Morita equivalence proved in (9.17) to relate the H^1 -cohomology of $GL_n(q)$ at modules in non-describing characteristic to Ext^1 -calculations for q -Schur algebras. By using results from Part I, we were led to the H^1 -stability Theorem 10.2. We wish to extend this work to include the higher cohomology groups $H^i(GL_n(q), V)$, $i > 1$. Section 12 below presents our results in that direction. The present section lays the foundation for that work. Our first main result, given in Theorem 11.10, develops an interesting complex of “tilting modules” for endomorphism algebras A associated to general finite Coxeter systems (W, S) .¹¹ The complex itself comes about by “dualizing” the Deodhar complex [De] for generic Hecke algebras. The proof of

¹¹A study of the homological properties of such endomorphism algebras A forms a central theme in previous papers [CPS6], [DPS1], [DPS2], [DPS3].

(11.10) makes essential use of certain aspects of the theory of Kazhdan-Lusztig cells; these facts are reviewed below. Next, we specialize to the case in which $W = \mathfrak{S}_m$ is a symmetric group. Our final result, Theorem 11.15, gives a very specific projective resolution of the quantum determinant representation \det_t for the t -Schur algebra $S_t(m, m)$ over $\mathcal{L} = \mathbb{Z}[t, t^{-1}]$, discussed in (9.18b). In order to pass from the complex in (11.10) to that in (11.15), we must use the theory of tilting modules for $S_t(m, m)$ developed in [DPS3].

We will use the notation introduced in the preamble to Part II. Thus, let (W, S) be a finite Coxeter system, and let H be the Hecke algebra over the ring $\mathcal{L} = \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials with basis $\{\tau_w\}_{w \in W}$ satisfying the relations (II.1). Write $m = |S|$ for the rank of W . For $\lambda \subseteq S$, let $x_\lambda \in H$ be the element defined in display (II.2).

For an integer $i \geq 0$, form the left H -module

$$(11.1) \quad N_i = \bigoplus_{\lambda \subseteq S, |\lambda|=i} Hx_\lambda.$$

Observe that $N_0 = H$, while $N_m = \mathcal{L}$ with H acting through the index map $\text{IND} : H \rightarrow \mathcal{L}$, $\tau_w \mapsto t^{\ell(w)}$. If $\lambda \subseteq \mu \subseteq S$, there is a unique H -module mapping $Hx_\lambda \xrightarrow{\phi_{\mu,\lambda}} Hx_\mu$ satisfying $\phi_{\mu,\lambda}(x_\lambda) = x_\mu$.

Fix a linear ordering $<$ on the set S . If $\lambda \subset \mu$ and $\mu \setminus \lambda = \{s'\}$ for some $s' \in S$, define

$$(11.2) \quad \varepsilon(\mu, \lambda) = (-1)^{|\{s \in \lambda \mid s < s'\}|}.$$

Then define the H -module homomorphism $d_i : N_i \rightarrow N_{i+1}$ by setting for any $\lambda \subseteq S$ satisfying $|\lambda|=i$:

$$(11.3) \quad d_i|_{Hx_\lambda} = \sum_{\mu \supset \lambda, |\mu|=i+1} \varepsilon(\mu, \lambda) \phi_{\mu,\lambda}.$$

Suppose that $h \in H$ satisfies $d_0(h) = 0$. Then $(1 + \tau_s)h = 0$ for all $s \in S$. By a direct calculation, any $h \in H$ satisfying $\tau_s h = -h$ for all $s \in S$ is a scalar multiple of $y_S = \sum_w (-t)^{-\ell(w)} \tau_w$. Hence, $\text{Ker}(d_0)$ identifies with the sign character $\text{SGN} : H \rightarrow \mathcal{L}$ defined in (II.3). This suggests part of the following basic result.

Lemma 11.4. (Deodhar [De]) *The above definitions define a complex*

$$(11.4.1) \quad (N_\bullet, d_\bullet) : 0 \rightarrow N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} \dots \xrightarrow{d_{m-1}} N_m \rightarrow 0$$

of left H -modules which has cohomology groups satisfying

$$(11.4.2) \quad H^i(N_\bullet) = \begin{cases} \text{SGN}, & i=0 \\ 0, & i>0. \end{cases}$$

Actually, [De] proves this lemma for the complex $N'_\bullet = N_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}[t^{1/2}, t^{-1/2}]$ of left modules for the Hecke algebra $H_0 = H \otimes_{\mathbb{Z}} \mathbb{Z}[t^{1/2}, t^{-1/2}]$. However, since $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ is a free \mathcal{L} -module, the validity of the lemma for N'_\bullet implies its

truth for N_{\bullet} .¹²

There is a \mathcal{L} -automorphism $\Phi: H \rightarrow H$ given on generators by $\Phi(\tau_w) = (-t)^{\ell(w)} \tau_w^{-1}$. If M is a (right or left) H -module, let M^Φ denote the H -module obtained by twisting the action of H on M by Φ . For example, $(Hx_\lambda)^\Phi = Hy_\lambda$ for all $\lambda \subseteq S$ [DPS2; (1.4c)]. Since $\text{SGN}^\Phi \cong \text{IND}$, the complex N_i^Φ has terms $N_i^\Phi = \bigoplus_{\lambda \subseteq S, |\lambda|=i} Hy_\lambda$ and cohomology which vanishes except in degree 0, where it identifies with IND .

Recall the $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ -bases

$$\{C_w\}_{w \in W} \text{ and } \{C'_w\}_{w \in W}$$

for H_0 defined in [KL1]. Explicitly,

$$C_w = (-t^{1/2})^{\ell(w)} \sum_{y \leq w} (-t)^{-\ell(y)} \bar{P}_{y,w} \tau_y \text{ and } C'_w = t^{-\ell(w)/2} \sum_{y \leq w} P_{y,w} \tau_y$$

where $P_{y,w} \in \mathcal{L}$ is the Kazhdan-Lusztig polynomial corresponding to the pair $(y, w) \in W \times W$ and $\bar{P}_{y,w}$ denotes the image of $P_{y,w}$ under the automorphism of \mathcal{L} satisfying $t \mapsto t^{-1}$.

We put $C_w^+ = t^{\ell(w)/2} C'_w$ and $C_w^- = (-t^{1/2})^{-\ell(w)} C_w$ for $w \in W$.¹³ Then $\{C_w^\varepsilon\}_{w \in W}$, $\varepsilon = \pm$, forms a \mathcal{L} -basis for H which satisfies the following multiplicative rules. For $w \in W$, $s \in S$, we have:

$$(11.4.3a) \quad \tau_s C_w^- = \begin{cases} -C_w^-, & sw < w \\ tC_w^- - tC_{sw}^- - t \sum_{z < w, sz < z} \mu(z, w) t^{\frac{\ell(z) - \ell(w) + 1}{2}} C_z^-, & sw > w. \end{cases}$$

$$(11.4.3b) \quad \tau_s C_w^+ = \begin{cases} tC_w^+, & sw < w \\ -C_w^+ + C_{sw}^+ + \sum_{z < w, sz < z} \mu(z, w) t^{\frac{\ell(w) - \ell(z) - 1}{2}} C_z^+, & sw > w. \end{cases}$$

In both cases, $z < w$ indicates that $P_{z,w}$ has degree $(\ell(w) - \ell(z) - 1)/2$. When $z < w$, $\mu(z, w)$ denotes the coefficient of $t^{(\ell(w) - \ell(z) - 1)/2}$ in $P_{z,w}$. There are completely analogous formulas to both (11.4.3a, b) for computing $C_w^- \tau_s$ and $C_w^+ \tau_s$. The following lemma was essentially observed in [M2].

Lemma 11.5. *For $\lambda \subseteq S$, let w_λ be the long word in the parabolic subgroup W_λ . Let ${}^\lambda W$ be the set of distinguished left coset representatives of W_λ in W , i.e., $w \in {}^\lambda W$ if and only if $ws > w$ for all $s \in \lambda$. Then:*

$$(a) \quad x_\lambda = C_{w_\lambda}^+ \text{ and } y_\lambda = C_{w_\lambda}^-.$$

¹²The paper [M1] by Mathas takes note of Deodhar's paper, but could be read as suggesting that Deodhar obtains an exact complex, namely one resolving SGN , only for infinite Coxeter groups. That is not the case. We wish to make it clear that the exact complex is due to Deodhar for both the finite and infinite Coxeter groups.

¹³Here we follow [DPS1], except that C_w^- may differ by a sign from the definition given in [DPS1].

- (b) For $\lambda \subseteq S$, write $C_{\bar{\lambda}} = C_{w_{\lambda}}^-$ and $C_{\lambda}^{\dagger} = C_{w_{\lambda}}^+$. The set $\{C_{\bar{x}}C_{\lambda}^{\dagger}\}_{x \in {}^{\lambda}W}$ forms a \mathcal{L} -basis for Hx_{λ} and the set $\{C_x^{\dagger}C_{\bar{\lambda}}\}_{x \in {}^{\lambda}W}$ forms a \mathcal{L} -basis for Hy_{λ} .
- (c) If $x \notin {}^{\lambda}W$, then $C_{\bar{x}}C_{\lambda}^{\dagger} = 0 = C_x^{\dagger}C_{\bar{\lambda}}$.

Proof. (a) follows from the definition of $C_{w_{\lambda}}$ and $C'_{w_{\lambda}}$ given earlier, together with the fact that $P_{y, w_{\lambda}} = 1$ for any $y \leq w_{\lambda}$ [KL1; (2.6vi)]. Next, we prove (c). Suppose that $x \notin {}^{\lambda}W$, so that there exists $s \in \lambda$ with $xs < x$. By (11.4.3a, b) (and their right-hand versions), we obtain

$$-C_{\bar{x}}C_{\lambda}^{\dagger} = C_{\bar{x}\tau_s}C_{\lambda}^{\dagger} = tC_{\bar{x}}C_{\lambda}^{\dagger},$$

so $C_{\bar{x}}C_{\lambda}^{\dagger} = 0$, as required. This proves the first equality in (c), while the second is similar. Now since $\{C_{\bar{x}}\}_{x \in W}$ is a \mathcal{L} -basis for H and Hx_{λ} is \mathcal{L} -free of rank $|{}^{\lambda}W|$, the first assertion in (b) follows. The second assertion in (b) is similar. \square

Given a \mathcal{L} -module M , write $M^* = \text{Hom}_{\mathcal{L}}(M, \mathcal{L})$ for the corresponding \mathcal{L} -linear dual. Then \mathcal{L} -duality interchanges left and right H -modules, e. g., if M is a left H -module M^* is a right H -module. It is well-known that $Hx_{\lambda} \cong (x_{\lambda}H)^*$ and $Hy_{\lambda} \cong (y_{\lambda}H)^*$ for any $\lambda \subseteq S$. Also, $\text{SGN}^* \cong \text{SGN}$ and $\text{IND}^* \cong \text{IND}$, if we let SGN and IND denote the left/right H -modules defined by the sign and index homomorphisms. Since the \mathcal{L} -modules N_i and N_i^{ϕ} are \mathcal{L} -free, we can dualize the complexes $(N_{\bullet}, d_{\bullet})$ and $(N_{\bullet}^{\phi}, d_{\bullet}^{\phi})$ to obtain complexes $(N_{\bullet}^*, d_{\bullet}^*)$ and $(N_{\bullet}^{\phi*}, d_{\bullet}^{\phi*})$ which have sole non-vanishing homology groups SGN and IND , respectively, in degree 0.

For each $\lambda \subseteq S$, let r_{λ} be some fixed non-negative integer, and define

$$(11.6) \quad T = \bigoplus_{\lambda \subseteq S} x_{\lambda}H^{\oplus r_{\lambda}}, \text{ and } A = \text{End}_H(T).$$

We will assume that, given any $\mu \subseteq S$, the parabolic subgroup W_{μ} is W -conjugate to some W_{λ} in which $r_{\lambda} > 0$.¹⁴ We view T as a (A, H) -bimodule. Let $(-)^{\diamond}$ denote either of the two contravariant functors

$$(11.7) \quad \begin{cases} (-)^{\diamond} = \text{Hom}_H(-, T) : \text{mod-}H \rightarrow A\text{-mod} \\ (-)^{\diamond} = \text{Hom}_A(-, T) : A\text{-mod} \rightarrow \text{mod-}H. \end{cases}$$

We will consider the complex

$$(11.8) \quad (X_{\bullet}, \partial_{\bullet}) = (N_{\bullet}^{\phi*}, d_{\bullet}^{\phi*}).$$

Thus, its term in degree i is given as

$$(11.9) \quad X_i = \text{Hom}_H\left(\bigoplus_{\lambda \subseteq S, |\lambda|=i} y_{\lambda}H, T\right),$$

¹⁴Recall that, given subsets $\lambda, \mu \subseteq S$, the right H -modules $x_{\lambda}H$ and $x_{\mu}H$ are isomorphic provided that the parabolic subgroups W_{λ} and W_{μ} are W -conjugate. See [DJ1; (4.3)]. Thus, our assumption guarantees that the algebra A defined in (11.6) is Morita equivalent to the algebra $\text{End}_H(\bigoplus_{\lambda \subseteq S} x_{\lambda}H)$.

while the differential $X_i \xrightarrow{\partial_i} X_{i+1}$ is the map $f \mapsto f \circ d_i^*$, $f \in \text{Hom}_H(N_i^{\phi^*}, T)$. Then $(X_\bullet, \partial_\bullet)$ is a complex of left A -modules.

The next theorem determines the cohomology of the complex (11.8) of left A -modules.

Theorem 11.10. *Let (W, S) be an arbitrary finite Coxeter system, and consider the corresponding complex $(X_\bullet, \partial_\bullet)$ defined in (11.8) - (11.9) above. It has cohomology groups as follows:*

$$(11.10.1) \quad H^i(X_\bullet) \cong \begin{cases} \text{IND}^\diamond = \text{Hom}_H(\text{IND}, T), & i=0 \\ 0, & i>0. \end{cases}$$

Proof. If M, N are right H -modules which are \mathcal{L} -free, we have $\text{Hom}_H(M, N) \cong \text{Hom}_H(N^*, M^*)$. Also, $M^{**} \cong M$. Thus, we can rewrite (11.9) as:

$$\begin{aligned} X_i &= \text{Hom}_H(N_i^{\phi^*}, T) = \text{Hom}_H(T^{*\phi}, N_i) \\ &= \bigoplus_{\mu \subseteq S} \text{Hom}_H(Hy_\mu, N_i)^{\oplus r_\mu}. \end{aligned}$$

It is enough to show that, for a fixed $\mu \subseteq S$, the cohomology of the complex

$$0 \rightarrow \text{Hom}_H(Hy_\mu, N_0) \rightarrow \text{Hom}_H(Hy_\mu, N_1) \rightarrow \cdots \rightarrow \text{Hom}_H(Hy_\mu, N_m) \rightarrow 0,$$

obtained by applying the functor $\text{Hom}_H(Hy_\mu, -)$ to the complex N_\bullet , vanishes in positive degree, and equals $\text{Hom}_H(Hy_\mu, \text{SGN})$ in degree 0. The assertion about its cohomology in degree 0 follows from the left exactness of $\text{Hom}_H(Hy_\mu, -)$, together with (11.4). So fix $i > 0$ and let $K_i = \text{Ker } d_i$. By the long exact sequence of cohomology, it suffices to check that the natural map

$$(11.10.2) \quad \text{Ext}_H^1(Hy_\mu, K_i) \rightarrow \text{Ext}_H^1(Hy_\mu, \bigoplus_{\lambda \subseteq S, |\lambda|=i} Hx_\lambda) = \text{Ext}_H^1(Hy_\mu, N_i)$$

is an *injection*. Since $H \otimes \mathbb{Q}(t)$ is a semisimple algebra, the \mathcal{L} -module $\text{Ext}_H^1(Hy_\mu, K_i)$ is torsion. Let $0 \neq \pi \in \mathcal{L}$ be in the annihilator of $\text{Ext}_H^1(Hy_\mu, K_i)$. If M is a \mathcal{L} -module, we will denote $M/\pi M$ by \bar{M} below. Since the N_i are \mathcal{L} -torsion free, and since K_i is a submodule of N_i , we have exact sequences $0 \rightarrow K_i \xrightarrow{\pi} K_i \rightarrow \bar{K}_i \rightarrow 0$ and $0 \rightarrow N_i \xrightarrow{\pi} N_i \rightarrow \bar{N}_i \rightarrow 0$. It follows from (11.4) and the isomorphisms $N_i/K_i \cong K_{i+1}$ that multiplication by π defines an injection $N_i/K_i \rightarrow N_i/K_i$. Hence, by the snake lemma, \bar{K}_i is a submodule of \bar{N}_i . Also, we obtain a commutative diagram

$$(11.10.3) \quad \begin{array}{ccccc} \text{Hom}_H(Hy_\mu, K_i) & \xrightarrow{a} & \text{Hom}_H(Hy_\mu, \bar{K}_i) & \rightarrow & \text{Ext}_H^1(Hy_\mu, K_i) & \longrightarrow 0 \\ \downarrow c & & \downarrow b & & \downarrow a & \\ \text{Hom}_H(Hy_\mu, N_i) & \xrightarrow{e} & \text{Hom}_H(Hy_\mu, \bar{N}_i) & \rightarrow & \text{Ext}_H^1(Hy_\mu, N_i) & \end{array}$$

with exact rows. In particular, the top row of (11.10.3) is exact because the element π annihilates the torsion module $\text{Ext}_H^1(Hy_\mu, K_i)$.

Now any H -module morphism $\xi: Hy_\mu \rightarrow N_i = \bigoplus_\lambda Hx_\lambda$ is completely determined by the image $\xi(y_\mu)$. In turn, $\xi(y_\mu)$ can be any vector $\delta \in N_i = \bigoplus_\lambda Hx_\lambda$ which satisfies the condition

$$(11.10.4) \quad \tau_s \delta = -\delta \text{ for all } s \in \mu.$$

By (11.5b), N_i has a basis consisting of the products $C_{\bar{x}} C_\lambda^+$, $|\lambda|=i$, $x \in {}^\lambda W$. We claim that (11.10.4) holds if and only if δ is a \mathcal{L} -linear combination of terms $C_{\bar{x}} C_\lambda^+$, for $|\lambda|=i$, $x \in {}^\lambda W$, and μ contained in the left-set $\mathcal{L}(x) = \{s \in S \mid sx < x\}$ of x . In fact, such a linear combination δ does satisfy (11.10.4) by (11.4.3a). Conversely, suppose δ satisfies (11.10.4). We can assume that, when expressed as a linear combination $\delta = \sum a_{x,\lambda} C_{\bar{x}} C_\lambda^+$, where $|\lambda|=i$ and $x \in {}^\lambda W$, the coefficient $a_{x,\lambda} = 0$ if $\mu \not\subseteq \mathcal{L}(x)$. Among the x, λ with $a_{x,\lambda} \neq 0$, choose x', λ so that x' has maximal length. For $s \in \mu$ and $s \notin \mathcal{L}(x')$, (11.4.3a) and (11.5) imply, using $\tau_s \delta = -\delta$, that $ta_{x',\lambda} = -a_{x',\lambda}$, a contradiction. This establishes our claim.

Let $\bar{\xi}: Hy_\mu \rightarrow \bar{N}_i$ be the image of ξ under the map $e: \text{Hom}_H(Hy_\mu, N_i) \rightarrow \text{Hom}_H(Hy_\mu, \bar{N}_i)$ above. The previous paragraph establishes that $\bar{\xi}(y_\mu)$ lies in the \mathcal{L} -span of the images under the maps $N_i \rightarrow \bar{N}_i$ of the $C_{\bar{x}} C_\lambda^+$ for $|\lambda|=i$ and $\mu \subseteq \mathcal{L}(x)$. Assume that $\bar{\xi}(y_\mu) \in \bar{K}_i$. For a given $x \in {}^\lambda W$, let $K_i(x)$ be the \mathcal{L} -submodule of K_i consisting of all terms which are expressible as a \mathcal{L} -linear combination of terms $C_{\bar{x}} C_\lambda^+$, $|\lambda|=i$. Then $K_i = \bigoplus_x K_i(x)$ by (11.5) and the definition of the complex N_\bullet . Also, $\bar{K}_i = \bigoplus_x \bar{K}_i(x)$. Again using (11.5), together with the above discussion of $\bar{\xi}(y_\mu)$, this implies that $\bar{\xi}(y_\mu) \in \bar{K}_i$ lifts to an element $\omega \in K_i$ satisfying $\tau_s \omega = -\omega$ for all $s \in \mu$. Therefore, $\omega = f(y_\mu)$ for an H -module morphism $f: Hy_\mu \rightarrow K_i$, and we note that $c(f)$ and ξ have the same image in $\text{Hom}_H(Hy_\mu, \bar{N}_i)$. Next, observe that the map b above is an injection. Finally, an elementary application of the snake lemma to (11.10.3), or a direct diagram chase, shows that the map a is an injection, i.e., the map (11.10.2) is an injection, as required. \square

For the remainder of this section, we assume that $(W, S) = (\mathfrak{S}_m, S)$, where $S = \{(1, 2), \dots, (m-1, m)\}$. In particular, this means, contrary to the notation above, that W has rank $m-1$. We choose the integers r_λ so that $T \cong V^{\otimes m}$ in (8.4) with $n=m$. Thus, in the notation of (11.6), the endomorphism algebra A identifies with the t -Schur algebra $S_t(m, m)$ defined in (8.2).

Define

$$(11.11) \quad Y = \text{Hom}_H\left(\bigoplus_{\lambda \in S} y_\lambda H^{\oplus r_\lambda}, T\right), \text{ and } E = \text{End}_A(Y).$$

View Y as a left E -module and consider the contravariant functor

$$(11.12) \quad G = \text{Hom}_A(-, Y) : A\text{-mod} \rightarrow E\text{-mod}$$

We want to apply the functors G to the complex (11.8) of left A -modules. In order to establish the next result concerning the behavior of the complex $G(X_\bullet)$, we need to make use of the theory of tilting modules for the quasi-hereditary algebras A_k for fields k . The original reference for tilting module theory for quasi-hereditary algebras is [Ri]. Donkin [Do1] has obtained many further results in the context of algebraic groups, especially for the groups of type GL_n . However, [DPS3; §4] contains a brief summary of everything that is required. In addition, our proofs use some of the main results in [DPS3] which deal with tilting module theory for q -Schur algebras over \mathcal{L} .

Lemma 11.13. *The following sequence of left A -modules is exact:*

$$(11.13.1) \quad 0 \rightarrow \text{IND}^\diamond \rightarrow X_0 \rightarrow \cdots \rightarrow X_{m-1} \rightarrow 0.$$

The functor G carries this sequence to an exact sequence

$$(11.13.2) \quad 0 \rightarrow G(X_{m-1}) \rightarrow \cdots \rightarrow G(X_0) \rightarrow G(\text{IND}^\diamond) \rightarrow 0$$

of left E -modules.

Proof. The exactness of (11.13.1) is the conclusion of Theorem 11.10. Let k be a field which is a \mathcal{L} -algebra. By [DPS3; (7.4a)], the A_k -module $(y_\lambda H)_k^\diamond = \text{Hom}_H(y_\lambda H, T)_k$ is a tilting module in the highest weight category $A_k\text{-mod}$ for any subset $\lambda \subseteq S$.¹⁵ This means that $(y_\lambda H)_k^\diamond$ has two filtrations, one having sections which are standard modules $\Delta_k(\tau)$ and the other having sections which are costandard modules $\nabla_k(\tau)$. By [CPS2], we therefore have:

$$(11.13.3) \quad \text{Ext}_{A_k}^i((y_\lambda H)_k^\diamond, Y_k) = 0, \quad \forall i > 0, \forall \lambda \subseteq S.$$

Suppose B is an arbitrary \mathcal{L} -algebra which is finitely generated and projective as a \mathcal{L} -module, and let M, N be B -modules which are finitely generated and projective over \mathcal{L} . In a proof which comes down to an elementary commutative algebra argument, [DPS3; (4.4)] shows that if $\text{Ext}_B^i(M, N) \neq 0$ for some positive integer i , then there exists a field k which is a \mathcal{L} -algebra such that $\text{Ext}_{A_k}^i(M_k, N_k) \neq 0$. By (11.13.3), taking $B=A$, we conclude that¹⁶

$$(11.13.4) \quad \text{Ext}_A^i((y_\lambda H)^\diamond, Y) = 0 \quad \forall i > 0.$$

For any $j=0, \dots, m-1$, X_j is a direct sum of various $\text{Hom}_H(y_\lambda H, T) = (y_\lambda H)^\diamond$, so (11.13.4) implies that $\text{Ext}_A^i(X_j, Y) = 0$ for all $i > 0$. Thus, working from right to left in (11.13.1), we obtain that $\text{Ext}_A^i(K, Y) = 0, i > 0$, for any kernel K of a differential ∂_j in (11.8). The exactness of (11.13.2) is immediate from this

¹⁵The argument in [DPS3] applies to subsets λ determined by partitions of m , but [DPS3; (1.4d)] establishes that the same assertion holds when the subset λ is determined by a composition of m , as explained above.

¹⁶It is well-known that A is a finitely generated projective \mathcal{L} -module; in fact, the projectivity of endomorphism algebras $\text{End}_B(Z)$ for algebras B over regular rings of Krull dim. at most 2 holds quite generally under mild hypotheses—see [DPS2; § 1].

fact. □

Since each X_i is a direct sum of various $(y_\lambda H)^\diamond$ which are, in turn, direct summands of Y , it follows that $G(X_i)$ is a projective left E -module. In the following result, we identify $G(X_i)$ in another way.

Lemma 11.14. *There is an isomorphism $E \xrightarrow{\sim} A^{\text{op}}$ such that the projective left E -module $G(X_i)$, when viewed as a right A -module by means of this isomorphism, becomes isomorphic to $\text{Hom}_H(T, \bigoplus_{|\lambda|=i} x_\lambda H^{\oplus r_\lambda})$. Also, the left E -module $G(\text{IND}^\diamond)$ identifies with the q -determinant representation det_t of A over \mathcal{L} .*

Proof. The isomorphism $E \cong A^{\text{op}}$ is proved in [DPS3; (2.5)]. In [DPS3; (2.6.1)], it proved that, under this isomorphism, $G(X_i)$ identifies with the left A^{op} -module

$$\text{Hom}_H(T^\phi, \bigoplus_{|\lambda|=i} y_\lambda H^{\oplus r_\lambda}) = \text{Hom}_H(T, \bigoplus_{|\lambda|=i} x_\lambda H^{\oplus r_\lambda}).$$

The identification of $G(X_i)$ as the right A -module described in the statement of the theorem then follows immediately. Next, we can apply [DPS3; (2.6.1)], taking \tilde{N} there to be IND in the present notation, to obtain that $G(\text{IND}^\diamond)$ identifies with $\text{Hom}_H(T^\phi, \text{IND})$ as a right A -module. The functor $(-)^\phi$ is defined below (11.4.2). But using (9.18.1),

$$\text{Hom}_H(T^\phi, \text{IND}) \cong \text{Hom}_H(T, \text{SGN}) \cong \text{det}_t.$$

Therefore, $G(\text{IND}^\diamond) \cong \text{det}_t$, as A -modules. □

Putting everything together, we obtain the following important result:

Theorem 11.15. *For the finite Coxeter system (\mathfrak{S}_m, S) , let $A = \text{End}_H(T)$, where $T = \bigoplus_{\lambda \in S} x_\lambda H^{\oplus r_\lambda}$ and the positive integers r_λ are chosen so that $A \cong S_t(m, m)$. There exists a finite resolution*

$$(11.15.1) \quad 0 \rightarrow Q(m-1) \rightarrow \dots \rightarrow Q(0) \rightarrow \text{det}_t \rightarrow 0$$

of det_t by projective right A -modules $Q(i)$, where

$$(11.15.2) \quad Q(i) = \text{Hom}_H(T, \bigoplus_{|\lambda|=i} x_\lambda H^{\oplus r_\lambda}).$$

For any commutative \mathcal{L} -algebra \mathcal{O} in which $t \mapsto q$, the complex $Q(\bullet)_\mathcal{O}$ defines a projective $A_\mathcal{O}$ -resolution of det_q over \mathcal{O} .

Proof. This is immediate from (11.12) and (11.13). Observe that, because all modules in (11.15.1) are \mathcal{L} -projective, it remains exact after tensoring with \mathcal{O} . □

Remarks 11.16. (a) The above Theorem (11.15) has been proved only when $W \cong \mathfrak{S}_m$. It would be interesting to extend (11.15) to include all finite

Coxeter systems. However, at present, the proof of the essential (11.13) uses highest weight category theory.

(b) Let V be a vector space of dimension m over a field k . For an integer j , let $\Lambda^j(V)$ (resp., $S^j(V)$) be the j th exterior (resp., symmetric) power of V . For $W = \mathfrak{S}_m$, consider the Schur algebra $S(m, m) = \text{End}_{\mathfrak{S}_m}(V^{\otimes m})$. One can obtain a Koszul resolution of the $S(m, m)$ -module \det with terms in degree n given by $\Lambda^{m-j-1}(V) \otimes S^{j+1}(V)$. Applying the standard duality, we obtain a resolution P_\bullet of \det . Over $\mathbb{Q}(t)$, P_0 identifies with the adjoint representation of GL_m . In general, P_\bullet , or, more precisely, its q -version, is distinct from our resolution (11.15.1). We have not seriously investigated the connection of (11.15.1) with other commonly used complexes for GL_m , e.g., the so-called Akin-Buchbaum resolution [AB].

(c) In the next section, we will see the utility of using the resolution (11.15) *in precisely the form given*. Thus, in (12.1), we will apply the Morita equivalence (9.17) to (11.15) to obtain yet another resolution involving the finite groups $GL_n(q)$ in non-describing characteristic. This resolution looks very much like the Solomon-Tits complex associated to the building of type A_{m-1} .

§ 12. Higher Cohomology in Non-Describing Characteristic

In this section, we apply (9.17) and (11.15) to obtain results on the higher cohomology of $G(q) = GL_n(q)$ in non-describing characteristic. To begin, we must interpret the projective $S_q(n, n)_k$ -modules given in (10.13.1) as $GL_n(q)$ -modules. Let \mathcal{O} be a commutative \mathcal{X} -algebra which is a discrete valuation ring and such that its quotient field K is a splitting field for $G(q)$. Assume the residue field k of \mathcal{O} has characteristic p which is relatively prime to q . As in (11.15), for each $\lambda \subseteq S = \{(1, 2), \dots, (n-1, n)\}$, we fix a positive integer r_λ so that, with $H = H(\mathfrak{S}_n, Z)$, we have $\text{End}_H(\bigoplus_{\lambda \subseteq S} x_\lambda H^{\oplus r_\lambda}) \cong S_t(n, n)$. Now we can state:

Lemma 12.1. *There is a resolution of $\mathcal{O}G(q)$ -modules*

$$(12.1.1) \quad 0 \rightarrow M(n-1) \rightarrow \dots \rightarrow M(1) \rightarrow M(0) \rightarrow \mathcal{O} \rightarrow 0$$

where

$$(12.1.2) \quad M(i) = \bigoplus_{\lambda \subseteq S, |\lambda|=i} \sqrt{y_\lambda} \bar{M}^{\oplus r_\lambda} \text{ with } M = \mathcal{O} \Big|_{B(q)}^{G(q)}$$

is a projective $\mathcal{O}G(q)/J(q)$ -module.

Proof. We will use the complex of right $S_t(m, m)$ -modules given in (11.15.1). Tensoring with \mathcal{O} , gives a complex of $S_q(m, m)_{\mathcal{O}}$ -modules which provide a projective resolution of \det_q . It suffices to show that this complex corresponds under the Morita equivalence (9.17) to a complex of the form indicated in (12.1.1).

First, by (9.18a), the q -determinant module \det_q for $S_q(n, n)_\theta$ corresponds, under the Morita equivalence (9.17.1), to the trivial module \mathcal{O} . Next, we need to identify $Q(i) = \text{Hom}_H(T, \bigoplus_{|\lambda|=i} x_\lambda H)$ with an $\mathcal{O}G(q)/J(q)$ -module under the Morita equivalence (9.17.1). By (9.17), this Morita equivalence is obtained by applying the functor $\text{Hom}_{\mathcal{O}G(q)/J(q)}(\widehat{M}, -)$, where $\widehat{M} = \widehat{M}_{1,G(q),\theta}$ as defined in (9.14), and using the identification $\text{End}_{\mathcal{O}G(q)}(\widehat{M}) \cong S_q(n, n)_\theta$ given in (9.11) (following [DJ2]). Letting $M = \mathcal{O}|_{\mathcal{B}(q)}^{\mathcal{G}(q)}$, the identification arises through a natural isomorphism

$$\text{Hom}_{\mathcal{O}G(q)}(\sqrt{y_\mu M}, \sqrt{y_\lambda M}) \cong \text{Hom}_H(x_\mu H_\theta, x_\lambda H_\theta);$$

see [DJ2; (2.24)]. Thus, $\text{Hom}_{\mathcal{O}G(q)}(\widehat{M}, \sqrt{y_\lambda M})$ identifies with $\text{Hom}_H(T, x_\lambda H)$, as required. \square

For $\lambda \subseteq S$, define $f_\lambda(q) \in \mathcal{O}$ by the condition that $y_\lambda^2 = f_\lambda(q) y_\lambda$. Since $\tau_w y_\lambda = \text{SGN}(\tau_w) y_\lambda = (-1)^{\ell(w)} y_\lambda$ for $w \in W_\lambda$, it follows that $f_\lambda(q) = \sum_{w \in W_\lambda} q^{-\ell(w)}$. When identified with a set of simple roots in the root system of type A_{n-1} , the set λ has a disjoint decomposition $\lambda = \cup_{i=1}^a \lambda_i$, where each λ_i corresponds to a connected subset of the set of simple roots. Then

$$(12.2) \quad f_\lambda(q) = \prod_{i=1}^a f_{|\lambda_i|}(q), \quad \text{where } f_{|\lambda_i|}(q) = \prod_{i=1}^{|\lambda_i|} \frac{q^{-i}-1}{q^{-i}-1}.$$

Now we have

Lemma 12.3. *Fix a non-negative integer $m \leq n$ and assume that $q-1$ and $f_\lambda(q)$ is invertible in \mathcal{O} for all $\lambda \subseteq S$ satisfying $|\lambda|=m$. Then the $\mathcal{O}G(q)$ -module $M(i)$ in (12.1.2) is projective. In particular, if p does not divide $\prod_{j=1}^m (q^j-1)$, then $M(j)$ is a projective $\mathcal{O}G(q)$ -module for all $j \leq m$.*

Proof. Because p does not divide $q-1$ (and is distinct from q), \mathcal{O} is a projective $\mathcal{O}B(q)$ -module and so $M = \mathcal{O}|_{\mathcal{B}(q)}^{\mathcal{G}(q)}$ is a projective $\mathcal{O}G(q)$ -module. The condition that $f_\lambda(q)$ be invertible for all λ satisfying $|\lambda|=m$ implies that $e_\lambda = \frac{1}{f_\lambda(q)} y_\lambda$ is an idempotent in H_θ , so that

$$\sqrt{y_\lambda M} = y_\lambda M = e_\lambda M$$

is also a projective $\mathcal{O}G(q)$ -module. By (12.1.2), this proves the first assertion of the lemma. The final assertion follows immediately from formula (12.2). \square

Now we are ready to prove the following important result which generalizes (10.1).

Theorem 12.4. *Consider the group $G(q) = GL_n(q)$. Assume that p does not divide $q \prod_{j=1}^{m+1} (q^j-1)$ for some integer $m \geq 0$. Let V be an arbitrary $kG(q)/J(q)_k$ -*

module, e. g., any $V = D(s, \lambda)$, $s \in \mathcal{C}_{ss, p'}$ and $\lambda \vdash \mathbf{n}(s)$. Then we have an isomorphism

$$(12.4.1) \quad H^i(G(q), V) \cong \text{Ext}_{S_q(\mathbf{n}, \mathbf{n})_k}^i(\det_q, \bar{F}(V)), \quad 0 \leq i \leq m+1$$

and an injection

$$(12.4.2) \quad \text{Ext}_{S_q(\mathbf{n}, \mathbf{n})_k}^{m+2}(\det_q, \bar{F}(V)) \hookrightarrow H^{m+2}(G(q), V).$$

Proof. By (12.3), the $\mathcal{O}G(q)$ -modules $M(0), \dots, M(m)$ are projective. (Of course, they are automatically projective as $\mathcal{O}G(q)/J(q)_k$ -modules.) Since the acyclic complex (12.1.1) consists of projective \mathcal{O} -modules, it remains exact after applying the functor $-\otimes_{\mathcal{O}} k$. Thus, we obtain an acyclic complex

$$(12.4.3) \quad 0 \rightarrow \bar{M}(m+1) \rightarrow \dots \rightarrow \bar{M}(0) \rightarrow k \rightarrow 0$$

in which, for $j \leq m$, the modules $\bar{M}(j) = M(j)_k$ are projective for both $kG(q)$ and $kG(q)/J(q)_k$, while $\bar{M}(m+1) = \text{Im}(M(m+1)_k \rightarrow M(m)_k)$. The theorem now follows by a dimension shifting argument extending that given in the proof of (10.1). More precisely, consider the double complex $C_{\bullet \bullet}$ obtained by applying the functor $\text{Hom}_R(-, V)$ to a Cartan-Eilenberg resolution of the complex M_{\bullet} in (12.4.3), where $R = kG(q)$ or $kG(q)/J(q)_k$. Filtering $C_{\bullet \bullet}$ by columns $C_{\bullet \circ}$ leads to the well-known spectral sequence

$$(12.4.4) \quad E_{iR}^s = \text{Ext}_R^i(\bar{M}(s), V) \Rightarrow \text{Ext}_R^{s+t}(k, V).$$

For $s \leq m$ and $t > 0$, we have $E_{iR}^s = 0$, so (12.4.1) follows from the equality $\text{Hom}_{kG(q)}(-, -) = \text{Hom}_{kG(q)/J(q)_k}(-, -)$ of bifunctors on $\text{mod-}kG(q)/J(q)_k$, together with (9.17).

Also, we have

$$\begin{aligned} \text{Ext}_{kG(q)/J(q)_k}^{m+2}(k, V) &\cong \text{Ext}_{kG(q)/J(q)_k}^1(\bar{M}(m+1), V) \\ &\hookrightarrow \text{Ext}_{kG(q)}^1(k, V) \cong \text{Ext}_{kG(q)}^{m+2}(k, V). \end{aligned}$$

In the above diagram, the isomorphisms are provided by the shapes of the spectral sequences E_{iR}^s . Now (12.4.2) follows from this, together with (9.17). \square

Remark 12.5. Although we have stated the above result for $kG(q)$, the argument also establishes a similar result comparing the cohomology of $\mathcal{O}G(q)$ with that of $S_q(\mathbf{n}, \mathbf{n})_{\mathcal{O}}$. More precisely, let V be a $\mathcal{O}G(q)/J(q)$ -module. Then, under the arithmetic hypothesis of (12.4), we have

$$(12.5.1) \quad H^i(G(q), V) \cong \text{Ext}_{S_q(\mathbf{n}, \mathbf{n})_{\mathcal{O}}}^i(\det_q, F(V)), \quad 0 \leq i \leq m+1$$

and an injection

$$(12.5.2) \quad \text{Ext}_{S_q(\mathbf{n}, \mathbf{n})_{\mathcal{O}}}^{m+2}(\det_q, F(V)) \hookrightarrow H^{m+2}(G(q), V),$$

where F is the Morita equivalence in (9.17.1).

Combining (12.4) and (8.6), we obtain the following stability result,

somewhat analogous to H^1 -stability proved in (10.2).

Corollary 12.6. *Let ℓ, n be fixed positive integers. There exists an integer $N=N(n, \ell)$ such that if k is an algebraically closed field of characteristic $p > N$, $\lambda \vdash n$, and $0 \leq i \leq \ell$ and q has order ℓ modulo p , then $H^i(GL_n(q), D(1, \lambda))$ depends only on λ and i .*

Using (8.6), which is based on the results of Part I, and the remarks in (10.3b), the stable value of $H^i(GL_n(q), D(1, \lambda))$ in (12.6) above can be expressed in terms of the dimension of a certain Ext^i -group for the affine Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$.

We conclude this section with the following result concerning SL_n -cohomology, extending (10.5). We leave the proof to the interested reader.

Theorem 12.7. *Assume that p does not divide both n and $q-1$. Also, assume that p does not divide q . Let L be an irreducible unipotent module for $SL_n(q)$ over k , and let $D(1, \lambda)$ be the unique irreducible module of $GL_n(q)$ which restricts to it. Suppose p also does not divide $\prod_{i=1}^m (q^i - 1)$ for some non-negative integer m . Then*

$$H^m(SL_n(q), L) \cong H^m(GL_n(q), D(1, \lambda)).$$

§ 13. Appendix: The Constructible Topology

Consider a property P for algebras A over domains \mathcal{O} , or their module morphisms g . For example, in Part I, we defined P to be a generic property for \mathcal{O} -finite and \mathcal{O} -torsion free algebras provided that: given an algebra A over \mathcal{O} such that P holds for $A_K (K = k(0))$, then P holds for $A_{k(\mathfrak{p})}$ for \mathfrak{p} belonging to a nonempty open subset Ω of $X = \text{Spec } \mathcal{O}$. In certain cases, it is possible to say more about the set of those points $\mathfrak{p} \in X$ at which P holds for $A_{k(\mathfrak{p})}$. This discussion makes use of the *constructible topology* on X . We introduce this concept and give a brief exposition, though the results are not required elsewhere in this paper.

To consider a specific example, let A be an \mathcal{O} -algebra, which is finitely generated and torsion-free as an \mathcal{O} -module. Consider a morphism $M \xrightarrow{g} N$ of A -modules as in (1.5a). The set

$$Y = \{\mathfrak{p} \in X \mid \text{Sur}(g_{k(\mathfrak{p})}) \text{ holds}\}$$

is not necessarily open in X , but instead satisfies — at least — the following property (see also (13.6) below). If Y is a subset of a topological space, we let \bar{Y} denote its closure.

Property GZ. If $\mathfrak{p} \in Y$, there is a Zariski open neighborhood W of \mathfrak{p} in $X = \text{Spec } \mathcal{O}$ such that $\overline{\{\mathfrak{p}\}} \cap W \subset \{\mathfrak{p}\} \cap Y$.

This property follows easily from Nakayama's lemma, together with an

elementary localization argument. If $\mathfrak{p} \notin Y$, then $g_{k(\mathfrak{p})}$ is not surjective, and in order to track further information, it is useful to recast the surjectivity property in terms of the dimension of the image of $g_{k(\mathfrak{p})}$. Thus, it makes sense to consider, for each non-negative integer n , the set Y_n of points $\mathfrak{p} \in X$ such that the image of $g_{k(\mathfrak{p})}$ has $k(\mathfrak{p})$ -dimension n . The sets Y_n satisfy (GZ) as well.

Since similar considerations apply to other properties of A -modules and maps, we define a new topology based on the property (GZ). In fact, we do this for an arbitrary nonempty topological space.

Let \mathcal{T} be a topology on a nonempty set X . Define \mathcal{T}^g to be the collection of subsets $V \subseteq X$ with the following property:

Property C. If $x \in V$, then there is a $W \in \mathcal{T}$ containing x such that $\overline{\{x\}} \cap W \subseteq \overline{\{x\}} \cap V$. In other words, for each $x \in V$, $\overline{\{x\}} \cap V$ is a neighborhood of x in $\overline{\{x\}}$ (given its subspace topology).

We verify directly that $\mathcal{T} \subset \mathcal{T}^g$ and that \mathcal{T}^g is a topology on X . We call \mathcal{T}^g the \mathcal{T} -constructible topology on X . As we note in (13.2) below, \mathcal{T}^g has a basis consisting of the \mathcal{T} -locally closed sets. By analogy with the Zariski topology, we say that a subset of X is \mathcal{T} -constructible if it is the union of a finite number of \mathcal{T} -locally closed subsets.

The proofs of the following elementary statements about the \mathcal{T} -constructible topology will be left to the reader.

Lemma 13.1. *Let \mathcal{T} be a topology on a nonempty set X . The \mathcal{T} -constructible topology \mathcal{T}^g contains both the \mathcal{T} -open and the \mathcal{T} -closed subsets of \mathcal{T} as members. In particular, if $Y \subseteq X$ is either \mathcal{T} -open or closed, then Y is both \mathcal{T}^g -open and \mathcal{T}^g -closed. For example, if $x \in X$, then $\overline{\{x\}}$ is both open and closed in the \mathcal{T} -constructible topology.*

Lemma 13.2. *Each of the two sets*

$$\begin{aligned} \mathcal{B} &= \{\overline{\{x\}} \cap W \mid W \in \mathcal{T}\} \\ \mathcal{B}' &= \{C \cap W \mid C \text{ is } \mathcal{T}\text{-closed and } W \text{ is } \mathcal{T}\text{-open}\} \end{aligned}$$

is a basis for the \mathcal{T} -constructible topology on X .

We describe the effect of passing to the constructible topology in several common situations. The proof is immediate from (13.2).

Lemma 13.3. *The following statements hold:*

- (a) *If $\mathcal{T} = \{X, \emptyset\}$ is the minimal topology on X , then $\mathcal{T} = \mathcal{T}^g$.*
- (b) *If (X, \mathcal{T}) is a T_0 -space, then (X, \mathcal{T}^g) is Hausdorff.*
- (c) *If (X, \mathcal{T}) is a T_1 -space, then \mathcal{T}^g is the discrete topology on X .*

The Zariski topology on a scheme satisfies the T_0 -separation axiom, so that

the following result holds:

Corollary 13.4. *Let \mathcal{T} be the Zariski topology on a scheme X . Then the \mathcal{T} -constructible topology is Hausdorff, and $(\mathcal{T}^a)^a$ is the discrete topology on X .*

Example 13.5. *For the affine scheme $X = \text{Spec } \mathbb{Z}$, the constructible topology coincides with the one-point compactification of the discrete topological space whose elements are the positive prime integers.*

In the context of Noetherian schemes, we next wish to indicate a technique for parlaying the Property GZ, which is a kind of infinite constructibility, into the finite version of constructibility.

Let X be a Noetherian scheme or, more generally, any Zariski space in the sense of [H; Ex. (3.17)]. Then the following property holds for its Zariski topology:

Property Z. If Y is any nonempty closed subset of X , there is a point $y \in Y$ whose closure $\overline{\{y\}}$ contains a nonempty open subset of Y .

Property Z, together with the usual descending chain condition on closed sets, allows for iteration and the construction of many filtrations $X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$ of X by closed subsets X_i , with points $x_i \in X_i$ ($i = 0, \dots, m - 1$), such that $X_i \setminus X_{i+1} \subseteq \overline{\{x_i\}}$. For example, consider our original example above involving the dimensions of images of maps $g_{k(\mathfrak{p})}$ for $\mathfrak{p} \in \text{Spec } \mathcal{O}$ and $g: M \rightarrow N$. By Property Z, we can choose $x \in X$ whose closure $\overline{\{x\}}$ contains a nonempty Zariski open subset. Now applying Property GZ to this point x , we find that there is a closed subset X_1 with $x \notin X_1$ and with $\dim \text{Im } g_{k(\mathfrak{p})}$ constant on $X \setminus X_1$. Continuing in this way yields a filtration $X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$ by closed subsets such that the function $\mathfrak{p} \mapsto \dim \text{Im } g_{k(\mathfrak{p})}$ is constant on each stratum $X_i \setminus X_{i+1}$. Since these strata are all locally closed, we have proved the following prototypic result:

Corollary 13.6. *Let A be an \mathcal{O} -algebra, which is finitely generated and torsion-free as an \mathcal{O} -module. Consider a morphism $M \xrightarrow{g} N$ of A -modules as in (1.5a), and fix any positive integer n . Then the set*

$$Y_n = \{\mathfrak{p} \in X = \text{Spec } \mathcal{O} \mid \dim \text{Im } g_{k(\mathfrak{p})} = n\}$$

is constructible (i.e., it is a finite union of locally closed subsets).

To conclude, it is worthwhile to review the classical constructibility result of Chevalley. Although our proof is based on that given in [H; Ex (3.19)], our present point of view makes it more conceptual.

Proposition 13.7. (Chevalley) *Let $f: X \rightarrow Y$ be a morphism of finite type of Noetherian schemes. Then the image of f is constructible.*

Proof. Induct on $\dim X + \dim Y$. Because f has finite type, we can reduce, as suggested in [H; Ex. (3.19(a))], to the case $f : X = \text{Spec } B \rightarrow Y = \text{Spec } A$ in which case f is induced by an inclusion $A \rightarrow B$ of domains. In addition, B can be taken to be a finitely generated A -algebra, so we can assume $B = A[\xi]$.

Now we claim that $\text{Im } f$ is open in the constructible topology on Y . Let $y = \mathfrak{q} \in \text{Im } f$. If ξ is transcendental over A , then $X \cong Y \times \mathbb{A}^1$ and f is projection onto Y , so $\text{Im } f = Y$. Otherwise, there exists a nonzero $a \in A$ such that the localization $B' = B_a$ is integral over $A' = A_a$. If $a \notin \mathfrak{q}$, then y belongs to the (Zariski) open subscheme $\text{Spec } A' \subseteq \text{Im } f$ of Y . Otherwise, f induces a morphism $g : \text{Spec } B/(a) \rightarrow \text{Spec } A/(a)$ in which $y \in \text{Im } g$. By induction on $\dim A$, $\text{Im } g$ is open in the constructible topology on $\text{Spec } A/(a)$. Since $\text{Spec } A/(a)$ is Zariski closed in $\text{Spec } A$, our claim is established.

Finally, Property Z yields an $x \in X$ so that $\overline{\{x\}}$ contains a nonempty Zariski open subset of X . By (13.2) and the claim, $f(x) \in V \subseteq \text{Im } f$ for a Zariski locally closed subset V of Y . Now x belongs to the Zariski locally closed subset $f^{-1}(V)$, so $f^{-1}(V)$ contains a nonempty Zariski open subset W of X . By induction, $f(X \setminus W)$ is constructible in Y . Since $\text{Im } f = V \cup f(X \setminus W)$, $\text{Im } f$ is also constructible. \square

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