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The Berezin Calculus

By

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Abstract

We present a canonical account of the Berezin integral and associated Berezin expectation over Hilbert spaces of arbitrary dimension. Our account is illustrated by an extensive discussion of Gaussians, by a Berezinian version of the kernel theorem for generalized functions, and by an extension of the Shale-Stinespring theorem on spin transformations.

Introduction

The calculus presented in these notes originated with F. A. Berezin in several papers and especially in the book [2]. One of its remarkable features is the way in which it enables a description of fermionic systems that is surprisingly close to a standard description of bosonic systems: thus, it facilitates a representation of intertwining operators in the fermionic Fock representation that is parallel to the integral kernel representation of intertwining operators in the bosonic Fock representation on Bargmann-Segal space; indeed, this appears to have been a primary reason for the initial development of the calculus. From a less exalted standpoint, the Berezin calculus may be regarded as a means of organizing and understanding the structure of exterior algebras and their relatives.

Our motivation for this work was a desire to appreciate the presentation in [2]. There, the author routinely identifies a Hilbert space with the space of square-integrable functions on some measure space and employs ideal or generalized elements as a matter of course. In these notes, we have attempted to formulate the theory in canonical form throughout: we work over abstract Hilbert spaces of arbitrary dimension, making no essential basis-dependent choices. We feel that reducing structural assumptions to a minimum in this way significantly clarifies the Berezin calculus. Of course, our debt to [2] is great; it is instructive to compare that presentation with the one offered here, setting up

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a dictionary for translation between the approaches.

It is perhaps worth drawing attention to a couple of elementary yet important points concerning our approach. Berezin refers to elements of a "Grassmann algebra with inner product" as "functionals of functions with anticommuting values". We take this reference almost literally: for V a complex Hilbert space, we set up alongside its exterior algebra $\wedge V$ the full antidual $\wedge V'$ comprising all antilinear functionals $\wedge V \to \mathbb{C}$; it is within this antidual that much of the formal calculus takes place. Our approach to an infinite-dimensional V is naturally via its set $\mathcal{F}(V)$ of finite-dimensional subspaces. To each $M \in \mathcal{F}(V)$ there corresponds the exterior algebra $\wedge M$: the direct limit of the algebras $\wedge M$ under inclusions is of course $\wedge V$; the inverse limit of the algebras $\wedge M$ under projections is canonically $\wedge V'$. These straightforward attitudes also appear to clarify the theory considerably.

"Finite dimensions" develops the Berezin calculus when the complex Hilbert space V is finite-dimensional. We study the Berezin integral and the Berezin expectation as linear functionals on the exterior algebra $\wedge V_{\mathbb{C}}$ of the complexification. We also study the Berezin partial integral and the Berezin conditional expectation relative to an orthogonal decomposition of V; for example, we include Berezinian versions of the Fubini theorem and differentiation under the integral. We note that the Berezin expectation yields a neat construction of the standard inner product on the exterior algebra over V.

"Infinite dimensions" passes to the case of an infinite-dimensional complex Hilbert space V. It is here that the antidual $\wedge V'$ makes it formal debut and is recognized as the inverse limit relative to projections of the exterior algebras $\wedge M$ for $M \in \mathcal{F}(V)$. We explicitly construct the complex Hilbert space completion $\wedge [V]$ of $\wedge V$ within $\wedge V'$; we define the Berezin integral and Berezin expectation on appropriate domains within $\wedge V'$. These notions are then related by a formula for the inner product on $\wedge [V]$ in terms of the Berezin expectation just as in the finite-dimensional case.

"Gaussian integrals" assembles the calculations of Berezin expectations for a variety of Gaussians, these being the exponentials of quadratic elements in either an exterior algebra or its antidual. In particular, we construct the Pfaffian pairing or relative Pfaffian between a pair of Hilbert-Schmidt antiskew operators on V in arbitrary dimension, for which see also [7] and [12]. These calculations are performed by various techniques and serve to illustrate the Berezin calculus developed in the preceding chapters.

"Integral kernels" centres upon a Berezinian version of the usual kernel theorem for generalized functions. The Berezinian theorem establishes a canonical isomorphism between $\wedge V'_{\mathbb{C}}$ and the space of all linear maps from $\wedge V$ to $\wedge V'$. In addition to analyzing the properties of this canonical isomorphism, we also calculate the Berezinian kernels of certain standard operators. For example, we study creators and annihilators, not only on $\wedge V$ but also on $\wedge [V]$ and indeed on $\wedge V'$.

"Spin transformations" draws upon the preceding theory to discuss the implementability of orthogonal transformations in the Fock representation. Recall that if π is the standard Fock representation of V on $\wedge [V]$ then the orthogonal transformation g of V admits a unitary operator $U: \wedge [V] \rightarrow \wedge [V]$ such that

$$v \in V \Rightarrow U\pi(v) = \pi(gv)U$$

precisely when its antilinear part $\frac{1}{2}(g + igi)$ is Hilbert-Schmidt: see [1] and [14]; see also [10] and [11]. Here, when $v \in V$ we regard $\pi(v)$ as acting not just on $\wedge [V]$ but actually on the triple $\wedge V \subset \wedge [V] \subset \wedge V'$. In this generalized setting, we obtain the interesting result that the orthogonal transformation g of V admits a (nonzero) generalized operator $U: \wedge V \rightarrow \wedge V'$ such that

$$v \in V \Rightarrow U\pi(v) = \pi(gv)U$$

precisely when its complex-linear part $\frac{1}{2}(g - igi)$ has finite-dimensional kernel; this extends the Shale-Stinespring theorem as far as one has a right to expect. After completing this work, we discovered that [9] contains a symplectic parallel to this discussion; of course, a closer symplectic parallel involves replacing $\wedge V$ and $\wedge V'$ by the symmetric algebra and its antidual, for which see [13].

"Remarks" addresses a number of matters arising from the body of the text: some of these pertain to alternative approaches or peripheral issues; others pertain to future directions or open problems. Even so, we do not aim at anything approaching completeness. In particular, the supersymmetric calculus combining bosons and fermions receives no mention; for this, see [5] and [6]. We refer to [3] and [15] for further discussion of the calculus in the context of quantum field theory, but remark that such discussions typically relate rigorously to finite dimensions only.

One final comment: the consistent conventions adopted in these notes were chosen after much deliberation. Though sufficient, they are by no means necessary; we encourage the reader to modify them if desired.

Finite Dimensions

Let V be a complex Hilbert space with J as its complex structure, with $\langle\cdot|\cdot\rangle$ as its complex inner product and with $(\cdot|\cdot)$ as the underlying real inner product, so that if $x, y \in V$ then

$$\langle x|y\rangle = (x|y) + i(Jx|y).$$

The complexification $V_{\mathbb{C}}=\mathbb{C}\otimes V$ possesses a canonical conjugation

$$\sigma : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}} : \nu\otimes z \mapsto \bar{\nu}\otimes z.$$

In addition, $V_{\mathbb{C}}$ carries a canonical complex inner product given by

$$\langle \lambda\otimes x | \mu\otimes y\rangle = \bar{\lambda}\mu\,(x|y)$$

and a canonical symmetric bilinear form given by

$$(\lambda\otimes x | \mu\otimes y) = \lambda\mu\,(x|y)$$

when λ, $\mu\in\mathbb{C}$ and $x, y\in V$. These structures are related by the fact that if $x, y\in V_{\mathbb{C}}$ then $\langle x|y\rangle = (\sigma x|y)$ and $(x|y) = \langle \sigma x|y\rangle$.

The complex-linear extension $J_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ induces a direct sum decomposition

$$V_{\mathbb{C}} = V^{+}\oplus V^{-}$$

in which

$$V^{\pm} = \{v \mp iJ_{\mathbb{C}}v : v\in V\}$$

is the $\pm i$ eigenspace. Note that V^{+} and V^{-} are both $(\cdot|\cdot)$-isotropic and $\langle\cdot|\cdot\rangle$-orthogonal; note also that V^{+} and V^{-} are interchanged by σ. The map

$$\gamma^{+} : V \rightarrow V^{+} : v \mapsto \frac{1}{\sqrt{2}}(v - iJ_{\mathbb{C}}v)$$

is a unitary isomorphism:

$$x, y\in V \Rightarrow \langle \gamma^{+}x | \gamma^{+}y\rangle = \langle x|y\rangle.$$

The map

$$\gamma^{-} : V \rightarrow V^{-} : v \mapsto \frac{1}{\sqrt{2}}(v + iJ_{\mathbb{C}}v)$$

is an antiunitary antiisomorphism:

$$x, y\in V \Rightarrow \langle \gamma^{-}x | \gamma^{-}y\rangle = \langle y|x\rangle.$$

Of course, the canonical conjugation σ interchanges these maps: $\gamma^{-} = \sigma\circ\gamma^{+}$ and $\gamma^{+} = \sigma\circ\gamma^{-}$.

Let $\wedge$ denote the functor associating to each complex vector space its exterior algebra; recall that this exterior algebra is graded by degree and that the vector 1 in its degree zero component $\mathbb{C}$ is called the vacuum. The canonical conjugation σ on $V_{\mathbb{C}}$ naturally extends to an antilinear antiautomorphism of $\wedge V_{\mathbb{C}}$ for which we use the same symbol: thus, $\sigma 1 = 1$ and if $v_1, \cdots, v_n \in V_{\mathbb{C}}$ then

$$\sigma(v_1\wedge\cdots\wedge v_n) = \sigma v_n\wedge\cdots\wedge\sigma v_1.$$

As $\sigma : \wedge V_{\mathbb{C}} \rightarrow \wedge V_{\mathbb{C}}$ is an adjunction, when $\phi\in\wedge V_{\mathbb{C}}$ we may write ϕ^{*} in place of $\sigma(\phi)$. By functoriality, the linear map $\gamma^+ : V \to V^+$ induces an injective linear homomorphism

$$\gamma^+ = \wedge \gamma^+ : \wedge V \to \wedge V^+ \subset \wedge V_{\mathbb{C}}$$

and the antilinear map $\gamma^- : V \to V^-$ induces an injective antilinear antihomomorphism

$$\gamma^- : \wedge V \to \wedge V^- \subset \wedge V_{\mathbb{C}}.$$

Matters are so arranged that the identities $\gamma^- = \sigma \circ \gamma^+$ and $\gamma^+ = \sigma \circ \gamma^-$ continue to hold at the exterior algebra level.

Among the other items of structure associated to the exterior algebra over a complex Hilbert space, we find it convenient to introduce annihilators at this juncture. Explicitly, if $v \in V$ then the annihilator $a(v) : \wedge V \to \wedge V$ is the linear antiderivation uniquely determined by the requirement that $a(v)1 = 0$ and that $a(v)w = \langle v|w \rangle$ whenever $w \in V$; in particular, if $v_0, \cdots, v_n \in V$ then

$$a(v)\,(v_0 \wedge \cdots \wedge v_n) = \sum_{k=0}^{n} (-1)^k \langle v|v_k \rangle v_0 \wedge \cdots \wedge \widehat{v_k} \wedge \cdots \wedge v_n$$

where a circumflex signifies omission as usual. Of course, the exterior algebra $\wedge V_{\mathbb{C}}$ is likewise acted upon by corresponding annihilators.

Now, let V be finite-dimensional: in fact, let V have complex dimension m so that $\wedge V$ has complex dimension 2^m. In this case, the exterior algebra $\wedge V$ possesses a unique minimal ideal: namely, the complex line $\wedge^m V$ comprising its degree m elements. For the complexification, the adjunction σ furnishes more structure; in particular, it distinguishes a specific element of the minimal ideal $\wedge^{2m} V_{\mathbb{C}} \subset \wedge V_{\mathbb{C}}$ as a consequence of the following.

(1.1) Theorem. *Each of the following conditions on $\zeta \in \wedge^2 V_{\mathbb{C}}$ is implied by the remaining pair:*

$$\begin{aligned}
&(0) \quad \zeta^* = \zeta \\
&(+) \quad v \in V^+ \Rightarrow a(v)\,\zeta = +v^* \\
&(-) \quad v \in V^- \Rightarrow a(v)\,\zeta = -v^*.
\end{aligned}$$

Moreover, there exists a unique $\zeta \in \wedge^2 V_{\mathbb{C}}$ satisfying these conditions.

Proof. A direct calculation establishes that if $x, y \in V_{\mathbb{C}}$ then $\langle y^*|x^* \rangle = \langle x|y \rangle$ whence if $v \in V_{\mathbb{C}}$ and $\zeta \in \wedge^2 V_{\mathbb{C}}$ then

$$(a(v)\,\zeta)^* + a(v^*)\,\zeta^* = 0.$$

From this, it follows that if $\zeta^* = \zeta$ then $a(v)\,\zeta = \pm v^*$ exactly when $a(v^*)\,\zeta = \mp v$. Thus, if (0) holds then $(+)$ and $(-)$ are equivalent. To complete the proof, let

$v_1,\cdots,v_n$ be a unitary basis for V and decompose the arbitrary $\zeta\in\wedge^2 V_{\mathbb{C}}$ as

$$\zeta=\sum_{i,j}A_{ij}\gamma^+(v_i)\gamma^-(v_j)+\frac{1}{2}\sum_{i,j}\{A^+_{ij}\gamma^+(v_i)\gamma^+(v_j)+A^-_{ij}\gamma^-(v_i)\gamma^-(v_j)\}.$$

Applying $(\pm)$ to $v=\gamma^\pm(v_k)$ for $k=1,\cdots,m$ yields $A=I$ and $A^\pm=0$. Thus $(+)$ and $(-)$ force ζ to be the selfadjoint

$$\zeta=\sum_{k=1}^m\gamma^+(v_k)\gamma^-(v_k).$$

□

The canonical element of $\wedge^2 V_{\mathbb{C}}$ singled out by this theorem is of such importance that we grant it a special symbol: we denote it by $\gamma=\gamma_V$ or by $\gamma^+\gamma^-$ as convenient, this notation being suggested by the fact that if $v_1,\cdots,v_m$ is a unitary basis for V then

$$\gamma_V=\sum_{k=1}^m\gamma^+(v_k)\gamma^-(v_k).$$

By passage to the m-fold exterior power, γ_V yields a canonical element of the minimal ideal in $\wedge V_{\mathbb{C}}$: explicitly, we define

$$\omega_V:=\frac{1}{m!}(-\gamma_V)^m\in\wedge^{2m}V_{\mathbb{C}}.$$

Note that if $v_1\cdots,v_m$ is a unitary basis for V then $\gamma^-(v_1)\gamma^+(v_1),\cdots,\gamma^-(v_m)\gamma^+(v_m)$ self annihilate and mutually commute, whence

$$\omega_V=\prod_{k=1}^m\gamma^-(v_k)\gamma^+(v_k).$$

We are now prepared to define the Berezin integral and to investigate its properties. Let

$$\tau_V:\wedge V_{\mathbb{C}}\to\wedge^{2m}V_{\mathbb{C}}$$

denote projection onto the (top) degree $2m$ homogeneous component. If $\phi\in\wedge V_{\mathbb{C}}$ then $\tau_V(\phi)$ is a scalar multiple of ω_V and we write

$$\tau_V(\phi)=\mathbb{I}_V[\phi]\,\omega_V$$

thus defining a complex linear functional

$$\mathbb{I}_V : \wedge V_{\mathbb{C}} \to \mathbb{C}$$

which we call the Berezin integral. We also write

$$\mathbb{I}_V[\phi] = \int_V \phi(\gamma^+, \gamma^-) d\gamma^+ d\gamma^-$$

as standard alternative notation for the Berezin integral of $\phi \in \wedge V_{\mathbb{C}}$.

For a variety of reasons, it is convenient to introduce a modification to the Berezin integral. We define the Berezin expectation as the complex linear functional

$$\mathbb{E}_V : \wedge V_{\mathbb{C}} \to \mathbb{C}$$

given by the rule that if $\phi \in \wedge V_{\mathbb{C}}$ then

$$\mathbb{E}_V[\phi] = \mathbb{I}_V[\phi e^{-\gamma_V}]$$

so that in the alternative notation

$$\mathbb{E}_V[\phi] = \int_V \phi(\gamma^+, \gamma^-) e^{-\gamma^+\gamma^-} d\gamma^+ d\gamma^-$$

where exponentiation takes place in the exterior algebra according to the usual power series expansion.

(1.2) Theorem. $\mathbb{E}_V[1] = 1$.

Proof. The exponential $e^{-\gamma_V} \in \wedge V_{\mathbb{C}}$ is defined by the usual expansion, thus

$$e^{-\gamma_V} = \sum_{k \geq 0} \frac{1}{k!} (-\gamma_V)^k = \sum_{k=0}^{m} \frac{1}{k!} (-\gamma_V)^k$$

and so

$$\tau_V(1e^{-\gamma_V}) = \frac{1}{m!}(-\gamma_V)^m = \omega_V.$$

□

(1.3) Theorem. *If T is a complex linear endomorphism of $V_{\mathbb{C}}$ extended functorially to a complex linear endomorphism of $\wedge V_{\mathbb{C}}$ then*

$$\phi \in \wedge V_{\mathbb{C}} \Rightarrow \mathbb{I}_V[T\phi] = (\text{Det } T)\mathbb{I}_V[\phi].$$

Proof. This property is simply the Berezin integral formulation of the familiar fact that the complex linear map T acts on the complex line $\wedge^{2m} V_{\mathbb{C}}$ as scalar multiplication by Det T. □

The preceding properties are most simply expressed separately in terms of the Berezin expectation and the Berezin integral respectively; the following properties are shared in form by both.

(1.4) Theorem. *The Berezin integral and the Berezin expectation are real in the sense that if* $\phi \in \wedge V_{\mathbb{C}}$ *then* $\mathbb{I}_V[\phi^*] = \overline{\mathbb{I}_V[\phi]}$ *and* $\mathbb{E}_V[\phi^*] = \overline{\mathbb{E}_V[\phi]}$.

Proof. Direct calculation reveals that τ_V commutes with adjunction: if $\phi \in \wedge V_{\mathbb{C}}$ then $\tau_V(\phi^*) = \tau_V(\phi)^*$. For the Berezin integral, simply note that ω_V is selfadjoint; for the Berezin expectation, note that γ_V itself is selfadjoint. □

In particular, $\mathbb{I}_V$ and $\mathbb{E}_V$ take real values on selfadjoint elements of $\wedge V_{\mathbb{C}}$.

(1.5) Theorem. *Let* $V = X \oplus Y$ *be a* $\langle \cdot | \cdot \rangle$*-orthogonal decomposition. If* $\xi \in \wedge X_{\mathbb{C}}$ *and* $\eta \in \wedge Y_{\mathbb{C}}$ *then the exterior product* $\xi\eta = \xi \wedge \eta$ *satisfies*

$$\mathbb{I}_V[\xi\eta] = \mathbb{I}_X[\xi]\,\mathbb{I}_Y[\eta]$$

$$\mathbb{E}_V[\xi\eta] = \mathbb{E}_X[\xi]\,\mathbb{E}_Y[\eta].$$

Proof. Decomposition into homogeneous components makes it clear that $\tau_V(\xi\eta) = \tau_X(\xi)\,\tau_Y(\eta)$. The asserted identities thus follow from the fact that $\gamma_V = \gamma_X + \gamma_Y$ which implies $\omega_V = \omega_X \omega_Y$ and $e^{-\gamma_V} = e^{-\gamma_X} e^{-\gamma_Y}$. □

The Berezin expectation has the further important property of being coherent as regards subspaces, in the following sense.

(1.6) Theorem. *If* $W \subset V$ *is a subspace and if* $\phi \in \wedge W_{\mathbb{C}}$ *then* $\mathbb{E}_V[\phi] = \mathbb{E}_W[\phi]$.

Proof. An immediate consequence of (1.2) and (1.5) upon considering the orthogonal decomposition $V = W \oplus W^{\perp}$. □

Now, the Berezin expectation on $\wedge V_{\mathbb{C}}$ enables us to introduce a complex inner product on $\wedge V$. Explicitly, for $\xi, \eta \in \wedge V$ we define

$$\langle \xi | \eta \rangle = \mathbb{E}_V[\gamma^-(\xi)\,\gamma^+(\eta)]$$

so that in the alternative notation

$$\langle \xi | \eta \rangle = \int_V \gamma^-(\xi)\,\gamma^+(\eta)\,e^{-\gamma^+\gamma^-}\,d\gamma^+ d\gamma^-.$$

We shall verify shortly that this formula does indeed define a complex inner product; before doing so, it is convenient to inspect a simple case and introduce more standard notation.

For the simple case, let $v\in V$ be a unit vector and $L\subset V$ its complex linear span. From

$$e^{\gamma^-(v)\gamma^+(v)}=1+\gamma^-(v)\gamma^+(v)$$

and (1.6) it follows that

$$\begin{aligned}\langle v|v\rangle &=\mathbb{E}_V[\gamma^-(v)\gamma^+(v)]=\mathbb{E}_L[\gamma^-(v)\gamma^+(v)]\\ &=\mathbb{I}_L[\gamma^-(v)\gamma^+(v)e^{\gamma^-(v)\gamma^+(v)}]\\ &=\mathbb{I}_L[\gamma^-(v)\gamma^+(v)]=1.\end{aligned}$$

In like manner, it may be seen that

$$\mathbb{E}_V[\gamma^-(v)]=\mathbb{E}_V[\gamma^+(v)]=0.$$

For the sake of brevity, write $\underline{m}$ for the set comprising $\emptyset$ together with all integer multiindices $C=(c_1,\cdots,c_t)$ such that $1\leq c_1<\cdots<c_t\leq m$. Let us agree that if $v_1,\cdots,v_m\in V$ then $v_\emptyset=1\in\wedge V$ and $v_C=v_{c_1}\cdots v_{c_t}\in\wedge V$ whenever $C=(c_1,\cdots,c_t)\in\underline{m}$. In terms of this notation, if $(v_c:1\leq c\leq m)$ is a basis for V then $(v_C:C\in\underline{m})$ is a basis for $\wedge V$.

(1.7) Theorem. *A complex inner product is defined on $\wedge V$ by the rule that if $\xi,\ \eta\in\wedge V$ then*

$$\langle\xi|\eta\rangle=\mathbb{E}_V[\gamma^-(\xi)\gamma^+(\eta)].$$

This inner product has the properly that if $(v_c:1\leq c\leq m)$ is a unitary basis for then $(v_C:C\in\underline{m})$ is a unitary basis for $\wedge V$.

Proof. The form $\langle\cdot|\cdot\rangle$ is manifestly sesquilinear; it is further Hermitian, in view of (1.4) and the fact that $\gamma^\mp=\sigma\circ\gamma^\pm$. If $C=(c_1,\cdots,c_t)\in\underline{m}$ then (1.5) and the simple case handled prior to the theorem imply that

$$\begin{aligned}\langle v_C|v_C\rangle &=\mathbb{E}_V[\gamma^-(v_{c_t})\cdots\gamma^-(v_{c_1})\gamma^+(v_{c_1})\cdots\gamma^+(v_{c_t})]\\ &=\prod_{k=1}^{t}\mathbb{E}_V[\gamma^-(v_{c_k})\gamma^+(v_{c_k})]=1.\end{aligned}$$

In order to complete the proof, it is enough to show that if $A,\ B\in\underline{m}$ and $\langle v_A|v_B\rangle$ is nonzero then $A=B$. With the vacuous interpretation when appropriate, let $A=(a_1,\cdots,a_r)$ with $A'=(a_1,\cdots,a_{r-1})$ and $B=(b_1,\cdots,b_s)$ with $B'=(b_1,\cdots,b_{s-1})$. From (1.5) and the simple case again, if $a_r<b_s$ then

$$\langle v_A|v_B\rangle=\mathbb{E}_V[\gamma^-(v_A)\gamma^+(v_{B'})]\mathbb{E}_V[\gamma^+(v_{b_s})]=0$$

and if $a_r>b_s$ then $\langle v_A|v_B\rangle=0$; accordingly, $a_r=b_s$ and

$$\begin{aligned}\langle v_A|v_B\rangle &=\mathbb{E}_V[\gamma^-(v_{a_r})\gamma^-(v_{A'})\gamma^+(v_{B'})\gamma^+(v_{b_s})]\\ &=(-1)^{r+s}\langle v_{A'}|v_{B'}\rangle.\end{aligned}$$

By reduction, it follows that $A=B$ and that $\langle v_A|v_B\rangle=1$ as required. □

This result shows that $\langle\cdot|\cdot\rangle$ on $\wedge V$ concides with the standard inner product, determined on decomposables by the rule that if $x_1, \cdots, x_r, y_1, \cdots y_s \in V$ then

$$\langle x_1 \cdots x_r | y_1 \cdots y_s \rangle = \delta_{rs} \mathrm{Det}[\langle x_a | y_b \rangle].$$

The foregoing construction in terms of the Berezin expectation has certain attractive features: for instance, $\langle \xi | \eta \rangle$ is at once defined when the vectors $\xi, \eta \in \wedge V$ are arbitrary and not merely decomposable.

As $\wedge V_{\mathbb{C}}$ is itself a complex Hilbert space of finite dimension, each of its complex linear functionals is given by evaluating its complex inner product against one of its vectors. The following result identifies the vector corresponding to the Berezin integral.

(1.8) Theorem. *If $\Phi \in \wedge V_{\mathbb{C}}$ then*

$$\mathbb{I}_V[\phi] = \langle \omega_V | \phi \rangle.$$

Proof. If $v_1, \cdots, v_m$ is a unitary basis for V then $(\gamma^-(v_k), \gamma^+(v_k): 1 \leq k \leq m)$ is a unitary basis for $V_{\mathbb{C}}$ so (1.7) implies in particular that

$$\prod_{k=1}^{m} \gamma^-(v_k) \gamma^+(v_k) = \omega_V$$

is a unit vector; consequently, if $\phi \in \wedge V_{\mathbb{C}}$ then

$$\tau_V(\phi) = \langle \omega_V | \phi \rangle \omega_V$$

whereupon the theorem follows at once from the definition of the Berezin integral. □

Henceforth, we fix a specific orthogonal decomposition $V = X \oplus Y$.

The complex inner product on $\wedge V_{\mathbb{C}}$ facilitates a significant generalization of the Berezin integral. We define the Berezin partial integral

$$\mathbb{I}_X : \wedge V_{\mathbb{C}} \to \wedge Y_{\mathbb{C}}$$

to be the linear map adjoint to

$$\wedge Y_{\mathbb{C}} \to \wedge V_{\mathbb{C}} : \eta \mapsto \omega_X \eta$$

so that if $\phi \in \wedge V_{\mathbb{C}}$ and $\eta \in \wedge Y_{\mathbb{C}}$ then

$$\langle \eta | \mathbb{I}_X[\phi] \rangle = \langle \omega_X \eta | \phi \rangle.$$

If we regard the Berezin integral $\mathbb{I}_V$ itself as integrating over V then $\mathbb{I}_X$ integrates only over X to leave a quantity depending on $Y = X^{\perp}$ alone.

Similarly, the Berezin expectation admits a significant generalization. We define the Berezin conditional expectation

$$\mathbb{E}_X : \wedge V_{\mathbb{C}} \to \wedge Y_{\mathbb{C}}$$

given by the rule that if $\phi \in \wedge V_{\mathbb{C}}$ then

$$\mathbb{E}_X[\phi] = \mathbb{I}_X[\phi e^{-\gamma_X}].$$

Our terminology is carefully chosen, for $\mathbb{E}_X$ is indeed a conditional expectation in the technical algebraic sense, as will be seen in due course.

The next result exhibits and resolves the ambiguity inherent in our notation: in the displayed equations, the original integral and expectation appear on the right while their generalizations appear on the left.

(1.9) Theorem. *If* $\xi \in \wedge X_{\mathbb{C}}$ *and* $\eta \in \wedge Y_{\mathbb{C}}$ *then* $\mathbb{I}_X[\xi\eta] = \mathbb{I}_X[\xi]\,\eta$ *and* $\mathbb{E}_X[\xi\eta] = \mathbb{E}_X[\xi]\,\eta$.

Proof. Let $\Gamma : \wedge V_{\mathbb{C}} \to \wedge V_{\mathbb{C}}$ be the standard parity automorphism, fixing the elements of even degree and acting as minus the identity on elements of odd degree. If also $\eta_0 \in \wedge Y_{\mathbb{C}}$ then from (1.5) it follows that if ξ is even then

$$\begin{aligned}\langle \omega_X \eta_0 | \xi\eta \rangle &= \mathbb{E}_{V_{\mathbb{C}}}[\gamma^-(\eta_0)\gamma^-(\omega_X)\gamma^+(\xi)\gamma^+(\eta)]\\ &= \mathbb{E}_{X_{\mathbb{C}}}[\gamma^-(\omega_X)\gamma^+(\xi)]\,\mathbb{E}_{Y_{\mathbb{C}}}[\gamma^-(\eta_0)\gamma^+(\eta)]\\ &= \langle \omega_X | \xi \rangle \langle \eta_0 | \eta \rangle\end{aligned}$$

while if ξ is odd then

$$\begin{aligned}\langle \omega_X \eta_0 | \xi\eta \rangle &= \mathbb{E}_{V_{\mathbb{C}}}[\gamma^-(\omega_X)\gamma^+(\xi)\gamma^-(\Gamma\eta_0)\gamma^+(\eta)]\\ &= \langle \omega_X | \xi \rangle \langle \Gamma\eta_0 | \eta \rangle = 0\\ &= \langle \omega_X | \xi \rangle \langle \eta_0 | \eta \rangle\end{aligned}$$

whence if ξ is arbitrary then

$$\langle \eta_0 | \mathbb{I}_X[\xi\eta] \rangle = \langle \eta_0 | \langle \omega_X | \xi \rangle \eta \rangle$$

and (1.8) applies. Finally, as γ_X is even so

$$\begin{aligned}\mathbb{E}_X[\xi\eta] &= \mathbb{I}_X[\xi e^{-\gamma_X}\eta]\\ &= \mathbb{I}_X[\xi e^{-\gamma_X}]\,\eta\\ &= \mathbb{E}_X[\xi]\,\eta.\end{aligned}$$

□

Some special cases are worth recording. Setting $\eta = 1$ we find that the notations $\mathbb{I}_X[\xi]$ and $\mathbb{E}_X[\xi]$ are unambiguous. Setting $\xi = 1$ we find that $\mathbb{I}_X[\eta] = 0$ and $\mathbb{E}_X[\eta] = \eta$.

(1.10) Theorem. *If* $\phi \in \wedge V_{\mathbb{C}}$ *and if* $\eta', \eta'' \in \wedge Y_{\mathbb{C}}$ *then*

$$\mathbb{I}_X[\eta'\phi\eta''] = \eta'\mathbb{I}_X[\phi]\eta''$$
$$\mathbb{E}_X[\eta'\phi\eta''] = \eta'\mathbb{E}_X[\phi]\eta''.$$

Proof. By linearity, it is enough to consider $\phi = \xi\eta$ with $\xi \in \wedge X_{\mathbb{C}}$ and $\eta \in \wedge Y_{\mathbb{C}}$. From (1.9) it follows that if ξ is even then

$$\begin{aligned}\mathbb{I}_X[\eta'\xi\eta\eta''] &= \mathbb{I}_X[\xi\eta'\eta\eta''] \\ &= \mathbb{I}_X[\xi]\eta'\eta\eta'' \\ &= \eta'\mathbb{I}_X[\xi]\eta\eta'' \\ &= \eta'\mathbb{I}_X[\xi\eta]\eta''\end{aligned}$$

while if ξ is odd then both $\mathbb{I}_X[\eta'\xi\eta\eta'']$ and $\eta'\mathbb{I}_X[\xi\eta]\eta''$ vanish. As $e^{-\gamma_X}$ is even we deduce that

$$\begin{aligned}\mathbb{E}_X[\eta'\xi\eta\eta''] &= \mathbb{I}_X[\eta'\xi\eta e^{-\gamma_X}\eta''] \\ &= \eta'\mathbb{I}_X[\xi\eta e^{-\gamma_X}]\eta'' \\ &= \eta'\mathbb{E}_X[\xi\eta]\eta''.\end{aligned}$$

□

It is now clear that $\mathbb{E}_X : \wedge V_{\mathbb{C}} \to \wedge Y_{\mathbb{C}}$ is indeed a conditional expectation: it is a linear map from the algebra $\wedge V_{\mathbb{C}}$ to its subalgebra $\wedge Y_{\mathbb{C}}$ restricting to $\wedge Y_{\mathbb{C}}$ as the identity and satisfying $\mathbb{E}_X[\eta'\phi\eta''] = \eta'\mathbb{E}_X[\phi]\eta''$ whenever $\phi \in \wedge V_{\mathbb{C}}$ and $\eta', \eta'' \in \wedge Y_{\mathbb{C}}$.

The Fubini theorem has a rather straightforward Berezinian analogue.

(1.11) Theorem. *If $\phi \in \wedge V_{\mathbb{C}}$ then*

$$\mathbb{I}_V[\phi] = \mathbb{I}_X[\mathbb{I}_Y[\phi]]$$
$$\mathbb{E}_V[\phi] = \mathbb{E}_X[\mathbb{E}_Y[\phi]].$$

Proof. By (1.8) and the factorization $\omega_V = \omega_X\omega_Y$ we have

$$\begin{aligned}\mathbb{I}_V[\phi] &= \langle\omega_V|\phi\rangle \\ &= \langle\omega_X\omega_Y|\phi\rangle \\ &= \langle\omega_Y|\mathbb{I}_X[\phi]\rangle \\ &= \mathbb{I}_Y[\mathbb{I}_X[\phi]]\end{aligned}$$

whence (1.9) yields

$$\begin{aligned}\mathbb{E}_V[\phi] &= \mathbb{I}_V[\phi e^{-\gamma_V}] \\ &= \mathbb{I}_Y[\mathbb{I}_X[\phi e^{-\gamma_X}e^{-\gamma_Y}]] \\ &= \mathbb{I}_Y[\mathbb{I}_X[\phi e^{-\gamma_X}]e^{-\gamma_Y}] \\ &= \mathbb{E}_Y[\mathbb{E}_X[\phi]].\end{aligned}$$

Symmetry concludes the argument. □

There is also an elementary Berezinian version of differentiation under the integral. Before we state it, observe that trilinearity and a routine check on basis vectors (for instance) show that if $v \in V_{\mathbb{C}}$ and ψ, $\phi \in \wedge V_{\mathbb{C}}$ then

$$\langle \psi | a(v)\phi \rangle = \langle v\psi | \phi \rangle.$$

(1.12) Theorem. *If $\phi \in \wedge V_{\mathbb{C}}$ and $y \in Y_{\mathbb{C}}$ then*

$$a(y)\mathbb{I}_X[\phi] = \mathbb{I}_X[a(y)\phi]$$
$$a(y)\mathbb{E}_X[\phi] = \mathbb{E}_X[a(y)\phi].$$

Proof. On account of the fact that ω_X is even, if also $\eta \in \wedge Y_{\mathbb{C}}$ then

$$\begin{aligned}\langle \eta | a(y)\mathbb{I}_X[\phi] \rangle &= \langle y\eta | \mathbb{I}_X[\phi] \rangle \\ &= \langle \omega_X y\eta | \phi \rangle \\ &= \langle y\omega_X\eta | \phi \rangle \\ &= \langle \omega_X\eta | a(y)\phi \rangle \\ &= \langle \eta | \mathbb{I}_X[a(y)\phi] \rangle.\end{aligned}$$

As X and Y are orthogonal so $a(y)e^{-\gamma_X}=0$ whence from above it follows that

$$\begin{aligned}a(y)\mathbb{E}_X[\phi] &= a(y)\mathbb{I}_X[\phi e^{-\gamma_X}] \\ &= \mathbb{I}_X[a(y)(\phi e^{-\gamma_X})] \\ &= \mathbb{I}_X[(a(y)\phi)e^{-\gamma_X}] \\ &= \mathbb{E}_X[a(y)\phi].\end{aligned}$$

□

Our next result calls for the introduction of some notation. Let $S : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ be any linear map and define a linear automorphism $T_S : V_{\mathbb{C}} \to V_{\mathbb{C}}$ by

$$x \in X_{\mathbb{C}},\ y \in Y_{\mathbb{C}} \Rightarrow T_S(x \oplus y) = x \oplus (Sx + y)$$

having adjoint $T_S^* : V_{\mathbb{C}} \to V_{\mathbb{C}}$ given by

$$x \in X_{\mathbb{C}},\ y \in Y_{\mathbb{C}} \Rightarrow T_S^*(x \oplus y) = (x + S^*y) \oplus y$$

so that $T_S | Y_{\mathbb{C}} = I$ and $T_S^* | X_{\mathbb{C}} = I$. By functoriality, T_S and T_S^* extend to automorphisms of $\wedge V_{\mathbb{C}}$ which we denote by the same symbols.

(1.13) Theorem. *If $S : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is a linear map and if $\phi \in \wedge V_{\mathbb{C}}$ then*

$$\mathbb{I}_X[T_S\phi] = \mathbb{I}_X[\phi].$$

Proof. Note that T_S^* acts identically on $\wedge X_{\mathbb{C}}$ and that T_S acts identically on $\wedge Y_{\mathbb{C}}$. Note also that if $P_Y : V_{\mathbb{C}} \to Y_{\mathbb{C}}$ is orthogonal projection and if $\eta \in \wedge Y_{\mathbb{C}}$ then $\omega_X T_S^* \eta = \omega_X P_Y T_S^* \eta$ since ω_X annihilates $\wedge X_{\mathbb{C}}$ under exterior multiplication.

Consequently, if $\eta \in \wedge Y_{\mathbb{C}}$ then

$$\begin{aligned}\langle \eta | \mathbb{I}_X [T_S \phi] \rangle &= \langle \omega_X \eta | T_S \phi \rangle \\ &= \langle T_S^* \omega_X T_S^* \eta | \phi \rangle \\ &= \langle \omega_X T_S^* \eta | \phi \rangle \\ &= \langle \omega_X P_Y T_S^* \eta | \phi \rangle \\ &= \langle P_Y T_S^* \eta | \mathbb{I}_X [\phi] \rangle \\ &= \langle T_S^* \eta | \mathbb{I}_X [\phi] \rangle \\ &= \langle \eta | T_S \mathbb{I}_X [\phi] \rangle \\ &= \langle \eta | \mathbb{I}_X [\phi] \rangle .\end{aligned}$$

□

We remark that this result implies a translation invariance of the Berezin integral.

The Berezin expectation furnishes an elegant proof for the following property of the canonical inner product.

(1.14) Theorem. *If $\xi', \xi'' \in \wedge X_{\mathbb{C}}$ and $\eta', \eta'' \in \wedge Y_{\mathbb{C}}$ then*

$$\langle \xi' \eta' | \xi'' \eta'' \rangle = \langle \xi' | \xi'' \rangle \langle \eta' | \eta'' \rangle .$$

Proof. Upon application of (1.10) and (1.11) we find that

$$\begin{aligned}\langle \xi' \eta' | \xi'' \eta'' \rangle &= \mathbb{E}_V [\gamma^- (\eta') \gamma^- (\xi') \gamma^+ (\xi'') \gamma^+ (\eta'')] \\ &= \mathbb{E}_Y [\mathbb{E}_X [\gamma^- (\eta') \gamma^- (\xi') \gamma^+ (\xi'') \gamma^+ (\eta'')]] \\ &= \mathbb{E}_Y [\gamma^- (\eta') \mathbb{E}_X [\gamma^- (\xi') \gamma^+ (\xi'')] \gamma^+ (\eta'')] \\ &= \mathbb{E}_Y [\gamma^- (\eta') \langle \xi' | \xi'' \rangle \gamma^+ (\eta'')] \\ &= \langle \xi' | \xi'' \rangle \mathbb{E}_Y [\gamma^- (\eta') \gamma^+ (\eta'')] \\ &= \langle \xi' | \xi'' \rangle \langle \eta' | \eta'' \rangle .\end{aligned}$$

□

Infinite Dimensions

Now let the complex Hilbert space V be of arbitrary dimension. We shall approach the Berezin calculus on V by way of its finite dimensional subspaces. Thus, let $\mathscr{F}(V)$ denote the set comprising all finite dimensional complex subspaces of V directed by inclusion; also, let $\mathscr{F}_M(V)$ denote the set comprising all elements of $\mathscr{F}(V)$ containing $M \in \mathscr{F}(V)$.

To each $M \in \mathscr{F}(V)$ we associate the orthogonal projection $P_M : V \to M$ of V on M along the orthocomplement $M^{\perp}$. Note that this is well defined whether or not V is complete: in fact, if $v_1, \cdots, v_m$ is a unitary basis for M then

$$v \in V \Rightarrow P_M v = \sum_{k=1}^{m} \langle v_k | v \rangle v_k.$$

The exterior algebra $\wedge V$ over V carries a canonical complex inner product. Indeed, if $\xi, \eta \in \wedge V$ then of course $\xi, \eta \in \wedge M$ for some $M \in \mathscr{F}(V)$ and we define

$$\langle \xi | \eta \rangle = \mathbb{E}_M [\gamma^-(\xi) \gamma^+(\eta)]$$

as for (1.7). The choice of M is immaterial, for if also $\xi, \eta \in \wedge N$ with $N \in \mathscr{F}(V)$ then both $\mathbb{E}_M[\gamma^-(\xi)\gamma^+(\eta)]$ and $\mathbb{E}_N[\gamma^-(\xi)\gamma^+(\eta)]$ equal $\mathbb{E}_{M+N}[\gamma^-(\xi)\gamma^+(\eta)]$ on account of (1.6). This inner product on $\wedge V$ continues to be determined by the rule that if $x_1, \cdots, x_r, y_1, \cdots, y_s \in V$ then

$$\langle x_1 \cdots x_r | y_1 \cdots y_s \rangle = \delta_{rs} \mathrm{Det} [\langle x_a | y_b \rangle].$$

(2.1) Theorem. *If $M \in \mathscr{F}(V)$ then the functorial extension $P_M : \wedge V \to \wedge M$ is precisely orthogonal projection of $\wedge V$ on $\wedge M$ along the orthocomplement $(\wedge M)^\perp$.*

Proof. By linearity, we need only consider decomposable vectors. If $v_1, \cdots, v_k \in V$ and if $w_1, \cdots, w_k \in M$ then

$$\begin{aligned} \langle w_1 \cdots w_k | v_1 \cdots v_k \rangle &= \mathrm{Det}[\langle w_a | v_b \rangle] \\ &= \mathrm{Det}[\langle w_a | P_M v_b \rangle] \\ &= \langle w_1 \cdots w_k | P_M v_1 \cdots P_M v_k \rangle \\ &= \langle w_1 \cdots w_k | P_M (v_1 \cdots v_k) \rangle \end{aligned}$$

whence $v_1 \cdots v_k - P_M(v_1 \cdots v_k)$ is orthogonal to $\wedge M$ as required. □

Fundamental to our approach is the full algebraic dual $\wedge V'$ comprising all antilinear functionals $\wedge V \to \mathbb{C}$. Of course, the complex inner product $\langle \cdot | \cdot \rangle$ on $\wedge V$ engenders a canonical linear embedding of $\wedge V$ in $\wedge V'$ which we shall usually view as an inclusion: namely, the map

$$\wedge V \to \wedge V' : \phi \mapsto \langle \cdot | \phi \rangle.$$

Let $M \in \mathscr{F}(V)$. If $\Phi \in \wedge V'$ then the restriction $\Phi | \wedge M$ is an antilinear functional on $\wedge M$ and so arises from a unique $\Phi_M \in \wedge M$ according to the rule

$$\phi \in \wedge M \Rightarrow \Phi(\phi) = \langle \phi | \Phi_M \rangle.$$

As our choice of notation suggests, the resulting linear map

$$P_M : \wedge V' \to \wedge M : \Phi \mapsto \Phi_M$$

extends the orthogonal projection from $\wedge V \subset \wedge V'$ onto $\wedge M$.

(2.2) Theorem. *If $M \in \mathscr{F}(V)$ then the restriction of the linear map P_M:*

$\wedge V' \to \wedge M$ to $\wedge V$ is precisely orthogonal projection of $\wedge V$ on $\wedge M$.

Proof. Direct calculation. Let $\phi \in \wedge V$ and write $\Phi := \langle \cdot | \phi \rangle \in \wedge V'$. If $\psi \in \wedge M$ then

$$\Phi(\psi) = \langle \psi | \phi \rangle = \langle \psi | P_M \phi \rangle$$

so that $\Phi_M = P_M \phi$ as required. □

Observe that if $N \in \mathscr{F}_M(V)$ and if $\Phi \in \wedge V'$ then $P_M \Phi_N = \Phi_M$: indeed, if also $\psi \in \wedge M$ then

$$\langle \psi | P_M \Phi_N \rangle = \langle \psi | \Phi_N \rangle = \Phi(\psi) = \langle \psi | \Phi_M \rangle .$$

The following converse to this observation provides a useful method for constructing elements of $\wedge V'$.

(2.3) Theorem. *If to each $M \in \mathscr{F}(V)$ is associated an element $\Phi_M \in \wedge M$ in such a way that $P_M \Phi_N = \Phi_M$ whenever $M, N \in \mathscr{F}(V)$ satisfy $M \subset N$ then there exists a unique $\Phi \in \wedge V'$ with $P_M \Phi = \Phi_M$ for each $M \in \mathscr{F}(V)$.*

Proof. If $\psi \in \wedge V$ then $\psi \in \wedge M$ for some $M \in \mathscr{F}(V)$ and we define $\Phi(\psi) := \langle \psi | \Phi_M \rangle$. The choice of M is immaterial, for if also $N \in \mathscr{F}(V)$ with $\psi \in \wedge N$ then each of $\langle \psi | \Phi_M \rangle$ and $\langle \psi | \Phi_N \rangle$ equals $\langle \psi | \Phi_{M+N} \rangle$ as $P_M \Phi_{M+N} = \Phi_M$ and $P_N \Phi_{M+N} = \Phi_N$. The uniqueness of Φ is plain. □

A little more concisely: the vector spaces $(\wedge M : M \in \mathscr{F}(V))$ constitute an inverse system when to each pair $M, N \in \mathscr{F}(V)$ with $M \subset N$ is associated the orthogonal projection $\wedge N \to \wedge M$; the inverse limit of this system is precisely $\wedge V'$. Similarly, $(\wedge M : M \in \mathscr{F}(V))$ is a direct system relative to the canonical inclusions, whose direct limit is exactly $\wedge V$.

As a consequence, the antidual $\wedge V'$ is more than just a vector space.

(2.4) Theorem. *The antidual $\wedge V'$ is naturally an anticommutative associative complex algebra according to the rule that if $\Phi, \Psi \in \wedge V'$ then*

$$M \in \mathscr{F}(V) \Rightarrow (\Phi \Psi)_M = \Phi_M \Psi_M.$$

Proof. From (2.1) it follows that if also $N \in \mathscr{F}_M(V)$ then

$$P_M(\Phi_N \Psi_N) = (P_M \Phi_N)(P_M \Psi_N) = \Phi_M \Psi_M$$

whence the consistency condition in (2.3) is satisfied. Verification that this product makes $\wedge V'$ into an anticommutative associative algebra is routine. □

Thus, $\wedge V'$ is the inverse limit for the inverse system $(\wedge M : M \in \mathscr{F}(V))$ of algebras; if $M \in \mathscr{F}(V)$ then the map $P_M : \wedge V' \to \wedge M$ is an algebra homomorphism. Of course, the canonical embedding $\wedge V \to \wedge V'$ is an algebra homomorphism: otherwise said, $\wedge V \subset \wedge V'$ is a subalgebra.

The foregoing considerations apply to the antidual of the exterior algebra over any complex Hilbert space. In particular, they apply to the antidual $\wedge V'_{\mathbb{C}}$ of $\wedge V_{\mathbb{C}}$. In this case, more can be said. Note the following elementary modification of (2.3): an element $\Phi \in \wedge V'_{\mathbb{C}}$ is determined by a consistent assignment of vectors $\Phi_{M_{\mathbb{C}}} \in \wedge M_{\mathbb{C}}$ as M runs over $\mathscr{F}(V)$. Note also that a check on decomposable vectors reveals the formula

$$\phi, \psi \in \wedge V_{\mathbb{C}} \Rightarrow \langle \phi^* | \psi^* \rangle = \langle \psi | \phi \rangle .$$

(2.5) Theorem. *The algebra $\wedge V'_{\mathbb{C}}$ carries a canonical adjunction defined by the rule that if $\Phi \in \wedge V'_{\mathbb{C}}$ then*

$$\Phi^* = \bar{\Phi} \circ \sigma$$

so that if also $M \in \mathscr{F}(V)$ then

$$(\Phi^*)_{M_{\mathbb{C}}} = (\Phi_{M_{\mathbb{C}}})^* .$$

Proof. The indicated rule plainly defines an adjunction on $\wedge V'_{\mathbb{C}}$. A little more explicitly, if $\psi \in \wedge V_{\mathbb{C}}$ then

$$\Phi^*(\psi) = \overline{\Phi(\psi^*)} .$$

In particular, if $\psi \in \wedge M_{\mathbb{C}}$ then

$$\begin{aligned} \Phi^*(\psi) &= \overline{\langle \psi^* | \Phi_{M_{\mathbb{C}}} \rangle} \\ &= \langle \psi | (\Phi_{M_{\mathbb{C}}})^* \rangle \end{aligned}$$

in view of the formula noted prior to the theorem. □

When convenient, we may denote this canonical adjunction by $\sigma: \wedge V'_{\mathbb{C}} \to \wedge V'_{\mathbb{C}}$. Of course, the canonical embedding of $\wedge V_{\mathbb{C}}$ in $\wedge V'_{\mathbb{C}}$ respects adjunctions.

Note that the functorial extension $\gamma^+ : \wedge V \to \wedge V^+ \subset \wedge V_{\mathbb{C}}$ of the unitary isomorphism $\gamma^+ : V \to V^+ : v \mapsto \frac{1}{\sqrt{2}}(v - iJv)$ is itself a unitary isomorphism.

(2.6) Theorem. *γ^+ extends further to an injective algebra homomorphism $\gamma^+ : \wedge V' \to \wedge V'_{\mathbb{C}}$ defined by the rule that if $\Phi \in \wedge V'$ and if $M \in \mathscr{F}(V)$ then $\gamma^+(\Phi)_{M_{\mathbb{C}}} = \gamma^+(\Phi_M)$.*

Proof. The indicated rule for defining $\gamma^+(\Phi)$ is consistent: if also $N \in \mathscr{F}_M(V)$ then $P_{M_{\mathbb{C}}}\gamma^+(\Phi_N) = \gamma^+(\Phi_M)$ on account of functoriality together with the identity $P_{M_{\mathbb{C}}} \circ \gamma^+ = \gamma^+ \circ P_M$ valid on V by direct calculation. □

Similarly, γ^- extends to an injective antilinear antihomomorphism $\gamma^-: \wedge V' \to \wedge V'_{\mathbb{C}}$ such that if $\Phi \in \wedge V'$ and $M \in \mathscr{F}(V)$ then $\gamma^-(\Phi)_{M_{\mathbb{C}}} = \gamma^-(\Phi_M)$. Naturally, the identities $\sigma \circ \gamma^{\pm} = \gamma^{\mp}$ continue to hold in this extended context.

Building upon this foundation, we proceed in a couple of directions when V is infinite dimensional: in the one we recognize that $\wedge V$ is incomplete, locating within $\wedge V'$ a model for its Hilbert space completion; in the other we develop the Berezin calculus, properly formulating the Berezin integral and the Berezin expectation.

We begin our discussion of the Hilbert space completion of $\wedge V$ by assigning to each $\Phi \in \wedge V'$ the (possibly infinite) number

$$\|\Phi\| = \sup\{\|\Phi_M\| : M \in \mathscr{F}(V)\}.$$

Note that the norms $(\|\Phi_M\| : M \in \mathscr{F}(V))$ constitute an increasing net: indeed, if $M, N \in \mathscr{F}(V)$ with $M \subset N$ then

$$\|\Phi_N\|^2 = \|\Phi_N - \Phi_M\|^2 + \|\Phi_M\|^2$$

by the Pythagorean identity, because $P_M : \wedge V \to \wedge N$ is orthogonal projection as in (2.1) and $P_M \Phi_N = \Phi_M$ as for (2.3). Consequently, if $M \in \mathscr{F}(V)$ is fixed then

$$\|\Phi\| = \sup\{\|\Phi_N\| : N \in \mathscr{F}_M(V)\}.$$

(2.7) Theorem. *If $\Phi \in \wedge V'$ then $\|\Phi\|$ is exactly its operator norm as an antilinear functional on $\wedge V$ in the sense that*

$$\|\Phi\| = \sup\{|\Phi(\phi)| : \phi \in \wedge V, \|\phi\| \leq 1\}.$$

Proof. On the one hand, if $\Phi_M \neq 0$ then $\phi = \Phi_M / \|\Phi_M\|$ is a unit vector and

$$\|\Phi_M\| = \langle \phi | \Phi_M \rangle = \Phi(\phi).$$

On the other hand, if the unit vector $\phi \in \wedge V$ lies in $\wedge M$ then

$$|\Phi(\phi)| = |\langle \phi | \Phi_M \rangle| \leq \|\Phi_M\|.$$

□

This prompts us to define

$$\wedge[V] = \{\Phi \in \wedge V' : \|\Phi\| < \infty\}$$

which we claim is a specific model for the Hilbert space completion of $\wedge V$ as an inner product space.

First of all, $\wedge[V]$ is plainly a complex vector space on which $\|\cdot\|$ is a norm. Further, the norm $\|\cdot\|$ actually arises from a complex inner product $\langle \cdot | \cdot \rangle$ on $\wedge[V]$. To see this, let $\Phi, \Psi \in \wedge[V]$: if $M \in \mathscr{F}(V)$ then the parallelogram law in $\wedge M$ yields

$$\|(\Phi - \Psi)_M\|^2 + \|(\Phi + \Psi)_M\|^2 = 2\{\|\Phi_M\|^2 + \|\Psi_M\|^2\}$$

whence taking the supremum as M runs over $\mathscr{F}(V)$ yields

$$\|\Phi-\Psi\|^2+\|\Phi+\Psi\|^2=2\{\|\Phi\|^2+\|\Psi\|^2\};$$

thus the parallelogram law holds in $\wedge[V]$ and so we may define

$$\langle\Phi|\Psi\rangle=\frac{1}{4}\sum_{p=0}^{3}i^p\|\Phi+i^p\Psi\|^2.$$

Note that the canonical embedding $\wedge V\to\wedge V'$ maps $\wedge V$ into $\wedge[V]$ isometrically. In fact, let $\phi\in\wedge V$ and $\Phi=\langle\cdot|\phi\rangle\in V'$: in light of (2.2) we see that if $M\in\mathscr{F}(V)$ then $\Phi_M=P_M\phi$ and $\|\Phi_M\|\leq\|\phi\|$ with equality precisely when $\phi\in\wedge M$. For convenience, we shall usually regard the canonical embedding $\wedge V\to\wedge[V]$ as an inclusion.

(2.8) Theorem. *If $M\in\mathscr{F}(V)$ then $P_M:\wedge V'\to\wedge M$ restricts to $\wedge[V]$ as orthogonal projection on $\wedge M$; in particular, if $\Phi\in\wedge[V]$ then*

$$\|\Phi\|^2=\|\Phi-\Phi_M\|^2+\|\Phi_M\|^2.$$

Proof. Let $\psi\in\wedge M$. If $N\in\mathscr{F}_M(V)$ then $P_N\psi=\psi$ to which $(\Phi-\Phi_M)_N=\Phi_N-P_M\Phi_N$ is orthogonal by (2.2) so

$$\|(\Phi-\Phi_M+\psi)_N\|^2=\|(\Phi-\Phi_M)_N\|^2+\|\psi\|^2$$

and the taking of suprema as N runs over $\mathscr{F}_M(V)$ yields

$$\|(\Phi-\Phi_M)+\psi\|^2=\|\Phi-\Phi_M\|^2+\|\psi\|^2.$$

That this is so for arbitrary $\psi\in\wedge M$ places $\Phi-\Phi_M$ in $(\wedge M)^{\perp}$. That this is so for the special case $\psi=\Phi_M$ concludes the proof. □

As a consequence, $\wedge V$ is dense in $\wedge[V]$: indeed, we have the following approximation theorem.

(2.9) Theorem. *If $\Phi\in\wedge[V]$ then the net $(\Phi_M: M\in\mathscr{F}(V))$ in $\wedge V$ converges to Φ in $\wedge[V]$.*

Proof. Let $\varepsilon>0$ and choose $M_\varepsilon\in\mathscr{F}(V)$ so that $\|\Phi_{M_\varepsilon}\|^2\geq\|\Phi\|^2-\varepsilon^2$. If $M\in\mathscr{F}_{M_\varepsilon}(V)$ then $\|\Phi_M\|^2\geq\|\Phi\|^2-\varepsilon^2$ so that $\|\Phi-\Phi_M\|\leq\varepsilon$ on account of (2.8). □

Incidentally, we now have an alternative construction for the complex inner product on $\wedge[V]$. Explicitly, if $\Phi,\Psi\in\wedge[V]$ then the nets $(\Phi_M: M\in\mathscr{F}(V))$ and $(\Psi_M: M\in\mathscr{F}(V))$ are Cauchy, whence so is the complex net $(\langle\Phi_M|\Psi_M\rangle: M\in\mathscr{F}(V))$ and we may define

$$\langle\Phi|\Psi\rangle=\lim_{M\in\mathscr{F}(V)}\langle\Phi_M|\Psi_M\rangle.$$

Not only is $\wedge [V]$ an inner product space in which $\wedge V$ is densely and isometrically embedded: $\wedge [V]$ is actually the Hilbert space completion of $\wedge V$.

(2.10) Theorem. *The complex inner product space $\wedge [V]$ is complete.*

Proof. Let $(\Phi^j : j \in \mathbb{N})$ be a Cauchy sequence in $\wedge [V]$. If $M \in \mathscr{F}(V)$ then the sequence $(\Phi^j_M : j \in \mathbb{N})$ is Cauchy, so we may define $\Phi_M \in \wedge M$ as its limit. If $N \in \mathscr{F}_M(V)$ and if $j \in \mathbb{N}$ then $P_M \Phi^j_N = \Phi^j_M$ so that $P_M \Phi_N = \Phi_M$ since P_M is continuous. By (2.3) we deduce the existence of a unique $\Phi \in \wedge V'$ such that if $M \in \mathscr{F}(V)$ then $\Phi_M = P_M \Phi$. Now, let $\varepsilon > 0$ and choose $j_\varepsilon \in \mathbb{N}$ so that if $p, q \geq j_\varepsilon$ then $\|\Phi^p - \Phi^q\| \leq \varepsilon$. If also $M \in \mathscr{F}(V)$ then $\|\Phi^p_M - \Phi^q_M\| \leq \varepsilon$ whence letting $p \to \infty$ and $q = j \geq j_\varepsilon$ yields $\|(\Phi - \Phi^j)_M\| \leq \varepsilon$. Taking the supremum as M runs over $\mathscr{F}(V)$ shows that if $j \geq j_\varepsilon$ then $\|\Phi - \Phi^j\| \leq \varepsilon$. This places Φ in $\wedge [V]$ as limit of the sequence $(\Phi^j : j \in \mathbb{N})$.

□

It is convenient here to record some remarks on grading. Recall that the exterior algebra is graded by degree: $\wedge V$ is the algebraic direct sum

$$\wedge V = \bigoplus_{n \geq 0} \wedge^n V$$

in which the homogeneous components $(\wedge^n V : n \geq 0)$ are orthogonal; when $n \geq 0$ we shall write $P^n : \wedge V \to \wedge^n V$ for orthogonal projection on the degree n component. Some of this structure passes to the antidual: When $n \geq 0$ we define $P^n : \wedge V' \to \wedge V'$ by declaring that

$$\Phi \in \wedge V',\ \psi \in \wedge V \Rightarrow (P^n \Phi)(\psi) = \Phi(P^n \psi).$$

Our notation is unambiguous, as these two meanings of P^n are respected by the canonical embedding $\wedge V \to \wedge V'$: if $\phi \in \wedge V$ and $\Phi = \langle \cdot | \phi \rangle$ then $P^n \Phi = \langle \cdot | P^n \phi \rangle$ for if also $\psi \in \wedge V$ then

$$(P^n \Phi)(\psi) = \Phi(P^n \psi) = \langle P^n \psi | \phi \rangle = \langle \psi | P^n \phi \rangle.$$

Let $\Phi \in \wedge V'$ and $M \in \mathscr{F}(V)$: if also $\psi \in \wedge M$ then $P^n \psi \in \wedge M$ so

$$(P^n \Phi)(\psi) = \Phi(P^n \psi) = \langle P^n \psi | \Phi_M \rangle = \langle \psi | P^n(\Phi_M) \rangle$$

whence

$$(P^n \Phi)_M = P^n(\Phi_M).$$

Accordingly, $P^n : \wedge V' \to \wedge V'$ maps $\wedge [V]$ to itself in continuous fashion.

(2.11) Theorem. *If $n \geq 0$ then $P^n : \wedge [V] \to \wedge [V]$ is orthogonal projection on the closure $\wedge^n [V] \subset \wedge [V]$ of $\wedge^n V \subset \wedge V$.*

Proof. As was noted prior to the theorem, $P^n|\wedge V$ agrees with the orthogonal projection $\wedge V \rightarrow \wedge^n V$. By continuity, $P^n: \wedge[V] \rightarrow \wedge[V]$ is thus a selfadjoint idempotent operator. On the one hand, its range contains $\wedge^n V$ and hence contains $\wedge^n[V]$. On the other hand, if $\Phi \in \wedge[V]$ and $P^n\Phi = \Phi$ then $P^n(\Phi_M) = (P^n\Phi)_M = \Phi_M$ so $\Phi_M \in \wedge^n V$ whenever $M \in \mathscr{F}(V)$ whence $\Phi \in \wedge^n[V]$ by (2.9). $\square$

Similar remarks apply to the parity automorphism Γ of $\wedge V$ which acts as $(-I)^n$ on degree n elements: it extends to an automorphism Γ of $\wedge[V]$ and indeed to an automorphism Γ of $\wedge V'$ defined by

$$\Phi \in \wedge V',\ \phi \in \wedge V \Rightarrow (\Gamma\Phi)(\phi) = \Phi(\Gamma\phi).$$

We now turn to the Berezin calculus on V.

We define the Berezin integral of $\Phi \in \wedge V'_{\mathbb{C}}$ by the formula

$$\mathbb{I}_V[\Phi] = \lim_{M \in \mathscr{F}(V)} \mathbb{I}_M[\Phi_{M_{\mathbb{C}}}]$$

provided that this limit exists; the complex vector space comprising all $\Phi \in \wedge V'_{\mathbb{C}}$ for which $\mathbb{I}_V[\Phi]$ exists will be denoted by $\mathscr{I}(V)$. The Berezin integral is thus a complex linear functional

$$\mathbb{I}_V: \mathscr{I}(V) \rightarrow \mathbb{C}$$

and when $\Phi \in \mathscr{I}(V)$ we may alternatively write

$$\mathbb{I}_V[\Phi] = \int_V \Phi(\gamma^+, \gamma^-)\, d\gamma^+ d\gamma^-.$$

We define the Berezin expectation of $\Phi \in \wedge V'_{\mathbb{C}}$ by the formula

$$\mathbb{E}_V[\Phi] = \lim_{M \in \mathscr{F}(V)} \mathbb{E}_M[\Phi_{M_{\mathbb{C}}}]$$

provided that this limit exists; the complex vector space comprising all $\Phi \in \wedge V'_{\mathbb{C}}$ for which $\mathbb{E}_V[\Phi]$ exists will be denoted by $\mathscr{E}(V)$. The Berezin expectation is thus a complex linear functional

$$\mathbb{E}_V: \mathscr{E}(V) \rightarrow \mathbb{C}$$

and when $\Phi \in \mathscr{E}(V)$ we may alternatively write

$$\mathbb{E}_V[\Phi] = \int_V \Phi(\gamma^+, \gamma^-)\, e^{-\gamma^+\gamma^-} d\gamma^+ d\gamma^-.$$

The alternative notation offered here is more than merely figurative. If $M, N \in \mathscr{F}(V)$ are such that $M \subset N$ then the explicit formula after (1.1) shows that $P_{M_{\mathbb{C}}}\gamma_N = \gamma_M$ whence (2.1) implies that $P_{M_{\mathbb{C}}} e^{-\gamma_N} = e^{-\gamma_M}$. Accordingly, the modified version of (2.3) for $\wedge V'_{\mathbb{C}}$ guarantees the existence of a unique $e^{-\gamma_V} \in \wedge V'_{\mathbb{C}}$ such that $P_{M_{\mathbb{C}}} e^{-\gamma_V} = e^{-\gamma_M}$ whenever $M \in \mathscr{F}(V)$. Via this element, the canonical multiplication in $\wedge V'_{\mathbb{C}}$ relates the Berezin integral and the Berezin expectation as follows.

(2.12) Theorem. *If $\Phi \in \wedge V'_{\mathbb{C}}$ then the conditions $\Phi \in \mathscr{E}(V)$ and $\Phi e^{-\gamma_V} \in \mathscr{I}(V)$ are equivalent; when they are satisfied,*

$$\mathbb{E}_V[\Phi] = \mathbb{I}_V[\Phi e^{-\gamma_V}].$$

Proof. A straightforward consequence of the definitions: simply pass to the limit as M runs over $\mathscr{F}(V)$ in

$$\mathbb{E}_M[\Phi_{M_{\mathbb{C}}}] = \mathbb{I}_M[\Phi_{M_{\mathbb{C}}} e^{-\gamma_M}] = \mathbb{I}_M[(\Phi e^{-\gamma_V})_{M_{\mathbb{C}}}].$$

□

The Berezin integral and Berezin expectation share certain properties; for example, both are real in the following sense.

(2.13) Theorem. *$\mathscr{I}(V)$ and $\mathscr{E}(V)$ are closed under adjunction, with $\mathbb{I}_V \circ \sigma = \overline{\mathbb{I}}_V$ and $\mathbb{E}_V \circ \sigma = \overline{\mathbb{E}}_V$.*

Proof. A direct consequence of (1.4) and the limit definitions, along with the fact that $\sigma(e^{-\gamma_V}) = e^{-\gamma_V}$. □

The Berezin integral overwhelms the subspace $\wedge V_{\mathbb{C}} \subset \wedge V'_{\mathbb{C}}$.

(2.14) Theorem. *$\wedge V_{\mathbb{C}} \subset \mathscr{I}(V)$ and $\mathbb{I}_V | \wedge V_{\mathbb{C}}$ is identically zero.*

Proof. Let $M \in \mathscr{F}(V)$. If $N \in \mathscr{F}(V)$ strictly contains M then $\mathbb{I}_N = \langle \omega_N | \cdot \rangle$ vanishes on $\wedge M_{\mathbb{C}}$ because ω_N is homogeneous of degree 2 dim N in which degree $\wedge M_{\mathbb{C}}$ is zero. Passing to the limit, $\mathbb{I}_V$ is defined and vanishes on $\wedge M_{\mathbb{C}}$. □

The Berezin expectation is more sensitive in this regard. From (1.2) and the limit definition, the vacuum $1 \in \wedge V_{\mathbb{C}}$ lies in $\mathscr{E}(V)$ and $\mathbb{E}_V[1] = 1$ so that in alternative notation

$$\int_V e^{-\gamma^+\gamma^-} d\gamma^+ d\gamma^- = 1.$$

More generally, the Berezin expectation is well defined on $\wedge V_{\mathbb{C}} \subset \wedge V'_{\mathbb{C}}$ and is there given by collating the linear functionals $\mathbb{E}_M$ for M in $\mathscr{F}(V)$.

(2.15) Theorem. *The domain $\mathscr{E}(V)$ of the Berezin expectation contains $\wedge V_{\mathbb{C}}$ and*

$$M \in \mathscr{F}(V) \Rightarrow \mathbb{E}_V | \wedge M_{\mathbb{C}} = \mathbb{E}_M.$$

Proof. Let $\phi \in \wedge M_{\mathbb{C}}$ and $\Phi = \langle \cdot | \phi \rangle \in \wedge V'_{\mathbb{C}}$. If $N \in \mathscr{F}_M(V)$ then (2.2) implies that $\Phi_{N_{\mathbb{C}}} = P_{N_{\mathbb{C}}}\phi = \phi$ and (1.6) implies that $\mathbb{E}_N[\Phi_{N_{\mathbb{C}}}] = \mathbb{E}_N[\phi] = \mathbb{E}_M[\phi]$. Thus the net $(\mathbb{E}_N[\Phi_{N_{\mathbb{C}}}] : N \in \mathscr{F}_M(V))$ has constant value $\mathbb{E}_M[\phi]$ and so $\Phi \in \mathscr{E}(V)$ with

$\mathbb{E}_V[\Phi] = \mathbb{E}_M[\phi]$ as required. $\square$

The Berezin expectation enables us to extend (1.7) to arbitrary dimensions as follows.

(2.16) Theorem. *If Φ, $\Psi \in \wedge [V]$ then the product $\gamma^-(\Phi)\gamma^+(\Psi)$ lies in $\mathscr{E}(V)$ and*

$$\mathbb{E}_V[\gamma^-(\Phi)\gamma^+(\Psi)] = \langle \Phi | \Psi \rangle.$$

Proof. If $M \in \mathscr{F}(V)$ then $[\gamma^-(\Phi)\gamma^+(\Psi)]_{M\mathbb{C}} = \gamma^-(\Phi_M)\gamma^+(\Psi_M)$ by definition, so (1.7) implies that

$$\mathbb{E}_M[(\gamma^-(\Phi)\gamma^+(\Psi))_{M\mathbb{C}}] = \langle \Phi_M | \Psi_M \rangle$$

whence the discussion of (2.9) implies that passage to the limit as M runs through $\mathscr{F}(V)$ establishes that $\mathbb{E}_V[\gamma^-(\Phi)\gamma^+(\Psi)]$ exists and equals $\langle \Phi | \Psi \rangle$. $\square$

In fact, if $\Phi \in \wedge V'$ and if $\gamma^-(\Phi)\gamma^+(\Phi) \in \mathscr{E}(V)$ then $\Phi \in \wedge [V]$: indeed, if $M \in \mathscr{F}(V)$ then

$$\mathbb{E}_M[(\gamma^-(\Phi)\gamma^+(\Phi))_{M\mathbb{C}}] = \|\Phi_M\|^2$$

as above, whence the increasing net $(\|\Phi_M\| : M \in \mathscr{F}(V))$ is convergent. Thus the Berezin expectation on $\mathscr{E}(V)$ yields an alternative description of the complex Hilbert space $\wedge [V]$.

Gaussian Integrals

In this section, we explicitly calculate Berezin expectations of various Gaussians and their relatives.

Let us begin by supposing that V is finite-dimensional. We claim that the second exterior power $\wedge^2 V$ may be canonically identified with the space $A^2 V$ comprising all (necessarily antilinear) maps $Z : V \to V$ that are antiskew in the sense

$$x, y \in V \Rightarrow \langle Zx | y \rangle + \langle Zy | x \rangle = 0.$$

(3.1) Theorem. *There is a canonical isomorphism $\wedge^2 V \leftrightarrow A^2 V$ under which $\zeta \in \wedge^2 V$ and $Z \in A^2 V$ correspond when either of the following equivalent conditions is satisfied:*

$$v \in V \Rightarrow Zv = a(v)\zeta$$
$$x, y \in V \Rightarrow \langle \zeta | xy \rangle = \langle Zx | y \rangle.$$

Proof. In the one direction, if $\zeta \in \wedge^2 V$ then the map

$$Z : V \to V : v \mapsto a(v)\zeta$$

has the property that if $x, y \in V$ then

$$\langle Zx|y\rangle = \langle a(x)\zeta|y\rangle = \langle \zeta|xy\rangle$$

and hence lies in A^2V. In the other direction, if $Z \in A^2V$ then the map

$$V\times V \to \mathbb{C}:\ (x, y) \mapsto \langle Zx|y\rangle$$

is alternating bilinear, inducing a linear functional on $\wedge^2 V$ given by inner product against some element ζ of $\wedge^2 V$. Verification that the assignments $\zeta \mapsto Z$ and $Z \mapsto \zeta$ are mutually inverse is straightforward. □

We remark that if $(v_1, \cdots, v_m)$ is a unitary basis for V then $(v_i v_j : 1 \leq i < j \leq m)$ is a unitary basis for $\wedge^2 V$ relative to which if $\zeta \in \wedge^2 V$ and $Z \in A^2 V$ correspond as in the theorem then

$$\zeta = \sum_{i<j} \langle v_j|Zv_i\rangle v_i v_j$$

or

$$\zeta = \frac{1}{2}\sum_{i,j} \langle v_j|Zv_i\rangle v_i v_j.$$

The following property of this canonical correspondence between $\wedge^2 V$ and A^2V will be important.

(3.2) Theorem. *Let the complex linear map $S: V \to V$ act functorially on $\wedge V$. If $\zeta \in \wedge^2 V$ and $Z \in A^2 V$ correspond then $S(\zeta) \in \wedge^2 V$ and $SZS^* \in A^2 V$ correspond.*

Proof. Direct calculation based on the definition of the correspondence. If $xy \in V$ then

$$\begin{aligned}\langle S(\zeta)|xy\rangle &= \langle \zeta|S^*(xy)\rangle \\ &= \langle \zeta|(S^*x)(S^*y)\rangle \\ &= \langle ZS^*x|S^*y\rangle \\ &= \langle SZS^*x|y\rangle.\end{aligned}$$

□

Now, if $Z \in A^2V$ then Z^2 is self-adjoint and indeed negative, for if $v \in V$ then $\langle v|Z^2v\rangle = -\|Zv\|^2$. Accordingly, Z^2 induces an orthogonal eigen-decomposition

$$V = V_0 \oplus \sum_{\lambda>0} V_\lambda$$

where $V_0 = \ker Z$ and where if $\lambda > 0$ then

$$V_\lambda=\{v\in V: Z^2v=-\lambda^2v\}$$

is of even complex dimension since Z is antilinear. This leads almost at once to the following useful decomposition theorem.

(3.3) Theorem. *If $\zeta\in\wedge^2V$ corresponds to $Z\in A^2V$ as usual, then there exist positive numbers $\lambda_1,\cdots,\lambda_l$ together with a unitary basis $(x_1, y_1,\cdots, x_l, y_l)$ for $(\ker Z)^\perp$ such that*

$$1\leq k\leq l\Rightarrow Zx_k=\lambda_k y_k,\ Zy_k=-\lambda_k x_k$$

and such that

$$\zeta=\sum_{k=1}^{l}\lambda_k x_k y_k.$$

Proof. Little remains to be done. In terms of the discussion prior to the theorem, for each nonzero V_λ with $\lambda>0$ we choose a unitary basis consisting of pairs $\left(x=v,\ y=\frac{1}{\lambda}Zv\right)$ and invoke the decomposition displayed after (3.1). □

In the notation established for the theorem, we record for later reference the formula

$$\mathrm{Det}(I-Z^2)=\prod_{k=1}^{l}(1+\lambda_k^2)^2.$$

We now regard an element of $\wedge^2V$ as a quadratic and form its exponential in $\wedge V$ according to the usual power series expansion; this series is a finite sum in the present situation, having no terms in degree greater than dim V. Thus, for $\zeta\in\wedge^2V$ we introduce the Gaussian

$$\exp(\zeta)=e^\zeta=\sum_{n\geq0}\frac{1}{n!}\zeta^n.$$

The first Berezin expectation calculated in this section yields the norm of such a Gaussian.

(3.4) Theorem. *If $\zeta\in\wedge^2V$ and $Z\in A^2V$ correspond as usual then*

$$\langle e^\zeta|e^\zeta\rangle=\mathrm{Det}^{\frac{1}{2}}(I-Z^2).$$

Proof. Diagonalize Z as in (3.3) so that $\zeta=\sum_{k=1}^{l}\zeta_k$ where if $1\leq k\leq l$ then $\zeta_k=\lambda_k x_k y_k$. As the even decomposables $\zeta_1,\cdots,\zeta_l$ mutually commute and self annihilate, so

$$\exp\left(\sum_{k=1}^{l}\zeta_k\right)=\prod_{k=1}^{l}\exp(\zeta_k)=\prod_{k=1}^{l}(1+\zeta_k).$$

Similarly, if for $1\leq k\leq l$ we denote by V_k the complex plane with (x_k, y_k) as unitary basis then

$$e^{-\gamma V_k}=(1+x_k^- x_k^+)(1+y_k^- y_k^+)$$

where $x_k^\pm=\gamma^\pm(x_k)$ and $y_k^\pm=\gamma^\pm(y_k)$ for convenience. Consequently, (1.5) implies by definition of $\langle\cdot|\cdot\rangle$ on $\wedge V$ that

$$\langle e^\zeta|e^\zeta\rangle=\prod_{k=1}^{l}\mathbb{E}_{V_k}[\gamma^-(e^{\zeta_k})\gamma^+(e^{\zeta_k})]$$

where if $1\leq k\leq l$ then the k-th factor may be calculated as follows: either indirectly via (1.7) as the inner product $\langle 1+\lambda_k x_k y_k|1+\lambda_k x_k y_k\rangle$ or directly as the Berezin integral over V_k of

$$(1+\lambda_k y_k^- x_k^-)(1+\lambda_k x_k^+ y_k^+)(1+x_k^- x_k^+)(1+y_k^- y_k^+);$$

either way, the k-th factor is $1+\lambda_k^2$. Finally, invocation of the formula recorded after (3.3) concludes the proof. □

Our next Berezin expectation calculates the inner product between a pair of Gaussians. As a preliminary, it is convenient to observe that the exponential

$$\exp: \wedge^2 V\to\wedge V: \zeta\mapsto e^\zeta$$

is holomorphic and indeed polynomial, whence

$$\wedge^2 V\times\wedge^2 V\to\mathbb{C}: (\xi,\eta)\mapsto\langle e^\xi|e^\eta\rangle$$

is antiholomorphic-holomorphic, as is

$$A^2V\times A^2V\to\mathbb{C}: (X,Y)\mapsto \mathrm{Det}(I-YX).$$

(3.5) Theorem. *If $\xi,\eta\in\wedge^2V$ correspond to $X, Y\in A^2V$ respectively, then*

$$\langle e^\xi|e^\eta\rangle^2=\mathrm{Det}(I-YX).$$

Proof. Both sides of the putative equality are antiholomorphic-holomorphic on $\wedge^2V\leftrightarrow A^2V$ and (3.4) asserts that they agree on the diagonal. All that remains is to apply the principle of analytic continuation. □

The fact that this construction singles out a preferred square root of a determinant prompts us to define the Pfaffian pairing

$$\mathrm{Pf}: A^2V\times A^2V\to\mathbb{C}$$

by the rule that if $X,Y\in A^2V$ correspond to $\xi,\eta\in\wedge^2V$ respectively then

$$\mathrm{Pf}(X,Y)=\langle e^\xi|e^\eta\rangle$$

whence

$$\mathrm{Pf}(X, Y)^2 = \mathrm{Det}(I - YX).$$

(3.6) Theorem. *Let $U: V \to V$ be a unitary operator. If $X, Y \in A^2V$ then*

$$\mathrm{Pf}(UXU^*, UYU^*) = \mathrm{Pf}(X, Y).$$

Proof. As the functorial extension of U to $\wedge V$ is unitary, so if $X, Y \in A^2V$ correspond to $\xi, \eta \in \wedge^2 V$ respectively then by (3.2) we deduce that

$$\begin{aligned}\mathrm{Pf}(UXU^*, UYU^*) &= \langle e^{U(\xi)} | e^{U(\eta)} \rangle \\ &= \langle U(e^{\xi}) | U(e^{\eta}) \rangle \\ &= \langle e^{\xi} | e^{\eta} \rangle \\ &= \mathrm{Pf}(X, Y).\end{aligned}$$

□

In addition to being unitarily invariant in this manner, the Pfaffian pairing is also plainly Hermitian in the sense that

$$X, Y \in A^2V \Rightarrow \overline{\mathrm{Pf}(X, Y)} = \mathrm{Pf}(Y, X).$$

We now introduce some convenient notation for dealing with quadratics. When $\zeta \in \wedge^2 V$ corresponds to $Z \in A^2V$ as usual, let us agree to write

$$\gamma^+(\zeta) = -\frac{1}{2}\gamma^+ Z \gamma^+$$

$$\gamma^-(\zeta) = +\frac{1}{2}\gamma^- Z \gamma^-$$

whence the relation $\sigma \circ \gamma^+ = \gamma^-$ implies

$$(\gamma^+ Z \gamma^+)^* = -\gamma^- Z \gamma^-.$$

(3.7) Theorem. *If $Z \in A^2V$ and if $(v_1, \cdots, v_m)$ is a unitary basis for V then*

$$\gamma^+ Z \gamma^+ = \sum_{k=1}^{m} \gamma^+(Zv_k)\gamma^+(v_k)$$

$$\gamma^- Z \gamma^- = \sum_{k=1}^{m} \gamma^-(Zv_k)\gamma^-(v_k).$$

Proof. If $Z_{ij} = \langle v_i | Zv_j \rangle$ when $1 \leq i, j \leq m$ then from the remark after (3.1) it follows that

$$\gamma^+ Z \gamma^+ = \sum_{i,j} Z_{ij} \gamma^+(v_i) \gamma^+(v_j)$$

$$= \sum_{j=1}^{m} \gamma^{+} \left(\sum_{i=1}^{m} Z_{ij} v_i\right) \gamma^{+}(v_j)$$

$$= \sum_{k=1}^{m} \gamma^{+}(Z v_k) \gamma^{+}(v_k).$$

This establishes the first formula; the second follows either similarly or by adjunction. □

As will shortly become clear, this result justifies the alternative notations

$$\gamma^{+} Z \gamma^{+} = [\gamma^{+} Z] \gamma^{+}$$
$$\gamma^{-} Z \gamma^{-} = [\gamma^{-} Z] \gamma^{-}$$

together with the identities

$$[\gamma^{+} Z] \gamma^{+} = -\gamma^{+} [\gamma^{+} Z] = \gamma^{+} [\gamma^{+} Z^{*}]$$
$$[\gamma^{-} Z] \gamma^{-} = -\gamma^{-} [\gamma^{-} Z] = \gamma^{-} [\gamma^{-} Z^{*}]$$

where if $W: V \to V$ is antilinear then its adjoint $W^{*}: V \to V$ is defined by

$$x, y \in V \Rightarrow \langle x | W^{*} y \rangle = \langle y | W x \rangle.$$

Let $S: V \to V$ be a complex linear map and define a complex linear map $\tilde{S}$: $V_{\mathbb{C}} \to V_{\mathbb{C}}$ by the rule that if $x, y \in V$ then

$$\tilde{S}[\gamma^{+}(x) + \gamma^{-}(y)] = \gamma^{-}(S^{*} x) - \gamma^{+}(S y).$$

By routine calculation, if $x_1, x_2, y_1, y_2 \in V$ then

$$\langle \tilde{S}[\gamma^{+} x_1 + \gamma^{-} y_1] | \gamma^{+} x_2 + \gamma^{-} y_2 \rangle = \langle S y_2 | x_1 \rangle - \langle S y_1 | x_2 \rangle$$

as a consequence of which $\tilde{S} \in A^2 V_{\mathbb{C}}$. Let us agree to denote by $[\gamma^{+} S] \gamma^{-} \in \wedge^2 V_{\mathbb{C}}$ its standard correspondent, so that if $x, y \in V_{\mathbb{C}}$ then

$$\langle [\gamma^{+} S] \gamma^{-} | x y \rangle = \langle \tilde{S} x | y \rangle.$$

The following result shows that our choice of notation for this correspondent is sensible.

(3.8) Theorem. *If $S: V \to V$ is complex linear and if $(v_1, \cdots, v_m)$ is a unitary basis for V then*

$$[\gamma^{+} S] \gamma^{-} = \sum_{k=1}^{m} \gamma^{+}(S v_k) \gamma^{-}(v_k)$$

$$= \sum_{k=1}^{m} \gamma^{+}(v_k) \gamma^{-}(S^{*} v_k).$$

Proof. Note that $(\gamma^- v_1, \gamma^- v_1, \cdots, \gamma^- v_m, \gamma^+ v_m)$ is a unitary basis for $V_{\mathbb{C}}$. Note also that if $i, j \in \{1, \cdots, m\}$ then both $\langle \tilde{S}\gamma^+ v_i | \gamma^+ v_j\rangle$ and $\langle \tilde{S}\gamma^- v_i | \gamma^- v_j\rangle$ vanish. Decomposing $[\gamma^+ S]\gamma^-$ along the lines of the formula after (3.1) therefore yields

$$\begin{aligned}[\gamma^+ S]\gamma^- &= \sum_{i,j} \langle \gamma^+ v_i \gamma^- v_j | [\gamma^+ S]\gamma^- \rangle \gamma^+ v_i \gamma^- v_j \\ &= \sum_{i,j} \langle \gamma^- v_j | \tilde{S}\gamma^+ v_i \rangle \gamma^+ v_i \gamma^- v_j \\ &= \sum_{i,j} \langle \gamma^- v_j | \gamma^- (S^* v_i) \rangle \gamma^+ v_i \gamma^- v_j \\ &= \sum_{i,j} \langle S^* v_i | v_j \rangle \gamma^+ v_i \gamma^- v_j\end{aligned}$$

so on the one hand

$$\begin{aligned}[\gamma^+ S]\gamma^- &= \sum_{i=1}^{m} \gamma^+ (v_i)\gamma^- (\sum_{j=1}^{m} \langle v_j | S^* v_i \rangle v_j) \\ &= \sum_{i=1}^{m} \gamma^+ (v_i)\gamma^- (S^* v_i)\end{aligned}$$

and on the other hand

$$\begin{aligned}[\gamma^+ S]\gamma^- &= \sum_{i=1}^{m} \gamma^+ (\sum_{j=1}^{m} \langle v_i | S v_j \rangle v_i)\gamma^- (v_j) \\ &= \sum_{j=1}^{m} \gamma^+ (S v_j)\gamma^- (v_j).\end{aligned}$$

□

This result also shows that it is sensible and valid to write

$$\gamma^+ [\gamma^- S] = [\gamma^+ S^*]\gamma^- = ([\gamma^+ S]\gamma^-)^*.$$

Of course, it is also justifiable to write

$$\gamma^- [\gamma^+ S] = -[\gamma^+ S]\gamma^-.$$

(3.9) Theorem. *If $S : V \to V$ is a complex linear map then*

$$\mathbb{I}_V[\exp(\gamma^- [\gamma^+ S])] = \operatorname{Det} S.$$

Proof. A direct proof in terms of a unitary basis for V and the associated matrix for S is of course possible, but an indirect proof is more elegant. As a special case, suppose that S is selfadjoint and diagonalize it by choosing for V a unitary basis of eigenvectors $(v_1, \cdots, v_m)$ with (real) eigenvalues $s_1, \cdots, s_m$. If for $1 \leq k \leq m$ we let V_k be the complex line spanned by v_k then from

$$\exp(\gamma^-[\gamma^+ S]) = \prod_{k=1}^{m} \exp(s_k \gamma^- v_k \gamma^+ v_k)$$

it follows by (1.5) that

$$\begin{aligned} \mathbb{I}_V[\exp(\gamma^-[\gamma^+ S])] &= \prod_{k=1}^{m} \mathbb{I}_{V_k}[1 + s_k \gamma^- v_k \gamma^+ v_k] \\ &= \prod_{k=1}^{m} s_k = \text{Det } S. \end{aligned}$$

The dependence of both $\mathbb{I}_V[\exp(\gamma^-[\gamma^+ S])]$ and Det S on S being holomorphic and indeed polynomial, the principle of analytic continuation applies to conclude the proof. □

Incidentally, it follows either by differentiation relative to t of the resulting formula

$$\mathbb{E}_V[\exp(t\gamma^-[\gamma^+ S])] = \text{Det}(I + tS)$$

or by expectation of the defining formula

$$\gamma^-[\gamma^+ S] = \sum_{k=1}^{m} \gamma^-(v_k)\gamma^+(Sv_k)$$

that

$$\mathbb{E}_V[\gamma^-[\gamma^+ S]] = \text{Tr } S.$$

Certain transformation laws for quadratics are important. For example, let $Z \in A^2 V$ correspond to $\zeta \in \wedge^2 V$ as usual: if the complex linear map $S: V \to V$ is extended functorially to $\wedge V$ as a linear homomorphism, then (3.2) informs us that SZS^* corresponds to $S \circ \zeta$; in our notation for quadratics, this becomes

$$(\gamma^+ S) Z (\gamma^+ S) = \gamma^+ (SZS^*) \gamma^+.$$

(3.10) Theorem. *If $Z \in A^2 V$ and if $W: V \to V$ is an antilinear map then*

$$(\gamma^- W) Z (\gamma^- W) = \gamma^- (WZW^*) \gamma^-.$$

Proof. Extend W functorially to $\wedge V$ as an antilinear antihomomorphism. An argument akin to that offered for (3.2) shows that if ζ corresponds to Z as usual then $W \cdot \zeta$ corresponds to $-WZW^*$. Now, if $(v_1,\cdots,v_m)$ is a unitary basis for V then the remark after (3.1) yields

$$\zeta=\frac{1}{2}\sum_{k=1}^{m} v_k(Zv_k)$$

whence

$$W\cdot\zeta=\frac{1}{2}\sum_{k=1}^{m}(WZv_k)(Wv_k)$$

and therefore

$$\begin{aligned}\gamma^-(WZW^*)\gamma^- &= -2\gamma^-(W\cdot\zeta)\\ &= \sum_{k=1}^{m}\gamma^-(WZv_k)\gamma^-(Wv_k)\\ &= [(\gamma^-W)Z](\gamma^-W).\end{aligned}$$

□

We now wish to perform more general Gaussian calculations, for which further preparation is required. Although a double direct sum would suffice, for aesthetic reasons we introduce the triple direct sum $V_{\mathbb{C}}\oplus V_{\mathbb{C}}\oplus V_{\mathbb{C}}$ in which we distinguish the subspaces

$$V_\alpha=V_{\mathbb{C}}\oplus 0\oplus 0 \qquad V_\beta=0\oplus 0\oplus V_{\mathbb{C}}$$
$$V_\gamma=0\oplus V_{\mathbb{C}}\oplus 0$$

to which V is mapped by the pair

$$\gamma^\pm : V\to V_\gamma : v\mapsto \frac{1}{\sqrt{2}}(0\oplus(v\mp iJv)\oplus 0)$$

together with $\alpha^\pm$ and $\beta^\pm$ defined likewise. We equip $V_{\mathbb{C}}\oplus V_{\mathbb{C}}\oplus V_{\mathbb{C}}$ with the standard inner product so that γ^+, α^+ and β^+ are isometric while γ^-, α^- and β^- are antiisometric. Further, we equip the exterior algebra $\wedge(V_{\mathbb{C}}\oplus V_{\mathbb{C}}\oplus V_{\mathbb{C}})$ with the adjunction σ interchanging the pairs $\gamma^\pm$, $\alpha^\pm$ and $\beta^\pm$ extended as usual.

For convenience, we shall denote the Berezin conditional expectation over the middle space in the triple direct sum $V_{\mathbb{C}}\oplus V_{\mathbb{C}}\oplus V_{\mathbb{C}}$ by

$$\mathbb{E}_\gamma : \wedge(V_\alpha\oplus V_\gamma\oplus V_\beta)\to\wedge(V_\alpha\oplus V_\beta).$$

Analogous to $\gamma^+\gamma^-=\gamma_V\in\wedge^2 V_{\mathbb{C}}$ is a quadratic $\alpha^+\beta^-\in\wedge^2(V_\alpha^+\oplus V_\beta^-)$ having the property that if $(v_1,\cdots,v_m)$ is a unitary basis for V then

$$\alpha^+\beta^- = \sum_{k=1}^{m} \alpha^+(v_k)\beta^-(v_k).$$

(3.11) Theorem. $\mathbb{E}_\gamma[\exp(\alpha^+\gamma^- + \gamma^+\beta^-)] = \exp(\alpha^+\beta^-)$.

Proof. We could argue by direct calculation, expanding in terms of a unitary basis. Instead, we shall offer an instructive illustration of (1.13) in action. Define a linear endomorphism T of $V_\mathbb{C} \oplus V_\mathbb{C} \oplus V_\mathbb{C}$ by

$$T|V_\alpha \oplus V_\beta = I$$

$$v \in V \Rightarrow \begin{cases} T(\gamma^+ v) = \gamma^+ v + \alpha^+ v \\ T(\gamma^- v) = \gamma^- v + \beta^- v \end{cases}$$

and extend it to $\wedge(V_\mathbb{C} \oplus V_\mathbb{C} \oplus V_\mathbb{C})$ functorially. From

$$\begin{gathered} T(\alpha^+\gamma^- + \gamma^+\beta^- - \gamma^+\gamma^-) = \\ \alpha^+(\gamma^- + \beta^-) + (\gamma^+ + \alpha^+)\beta^- - (\gamma^+ + \alpha^+)(\gamma^- + \beta^-) \\ = \alpha^+\beta^- - \gamma^+\gamma^- \end{gathered}$$

it follows by (1.9) and (1.13) that

$$\begin{aligned} \mathbb{E}_\gamma[\exp(\alpha^+\gamma^- + \gamma^+\beta^-)] &= \mathbb{I}_\gamma[\exp(\alpha^+\gamma^- + \gamma^+\beta^- - \gamma^+\gamma^-)] \\ &= \mathbb{I}_\gamma[T\exp(\alpha^+\gamma^- + \gamma^+\beta^- - \gamma^+\gamma^-)] \\ &= \mathbb{I}_\gamma[\exp(\alpha^+\beta^- - \gamma^+\gamma^-)] \\ &= \mathbb{E}_\gamma[\exp(\alpha^+\beta^-)] \\ &= \exp(\alpha^+\beta^-). \end{aligned}$$

□

Before embarking upon the next Gaussian calculation, it is convenient to make certain remarks about our notation for the algebra of quadratics. Thus, let $Z \in A^2V$ as usual, let S: $V \rightarrow V$ be complex linear and let W: $V \rightarrow V$ be antilinear. The discussion of (3.10) informs us that $(\alpha^+S)Z(\alpha^+S) = \alpha^+(SZS^*)\alpha^+$ and $(\beta^-W)Z(\beta^-W) = \beta^-(WZW^*)\beta^-$. Adjunction or similar arguments yield $(\alpha^+W)Z(\alpha^+W) = \alpha^+(WZW^*)\alpha^+$ and $(\beta^-S)Z(\beta^-S) = \beta^-(SZS^*)\beta^-$. The discussion of (3.8) implies that $[\alpha^+S]\beta^- = \alpha^+[\beta^-S^*]$; similarly, $[\alpha^+W]\beta^+ = \alpha^+[\beta^+W^*]$ and $[\alpha^-W]\beta^- = \alpha^-[\beta^-W^*]$. Further identities of this sort are readily derived and will be used freely in the sequel.

(3.12) Theorem. *If* $Z \in A^2V$ *then*

$$\begin{aligned} &\mathbb{E}_\gamma\left[\exp\left(\frac{1}{2}\gamma^-Z\gamma^- - \frac{1}{2}\gamma^+Z\gamma^+ + \alpha^+\gamma^- + \gamma^+\beta^-\right)\right] \\ &= \operatorname{Det}^{\frac{1}{2}}(I - Z^2)\exp\{[\alpha^+(I - Z^2)^{-1}]\beta^- + \end{aligned}$$

$$\frac{1}{2}\beta^-[Z(I-Z^2)^{-1}]\beta^- - \frac{1}{2}\alpha^+[(I-Z^2)^{-1}Z]\alpha^+\}.$$

Proof. We could proceed by a calculation broadly similar to but more elaborate than that for (3.4); in its place, we effectively complete the square in the exponent as another application of (1.13). Define a linear automorphism T of the sum $V_{\mathbb{C}}\oplus V_{\mathbb{C}}\oplus V_{\mathbb{C}}$ by

$$T|V_\alpha\oplus V_\beta = I$$

$$v\in V\Rightarrow\begin{cases} T(\gamma^+v) = \gamma^+v+\alpha^+(Sv)+\beta^-(Wv) \\ T(\gamma^-v) = \gamma^-v+\alpha^+(Wv)+\beta^-(Sv). \end{cases}$$

Of course, $T(\alpha^+\gamma^- + \gamma^+\beta^- - \gamma^+\gamma^-)$ equals

$$\alpha^+(\gamma^-+\alpha^+W+\beta^-S)+(\gamma^++\alpha^+S+\beta^-W)\beta^+$$
$$-(\gamma^++\alpha^+S+\beta^-W)(\gamma^-+\alpha^+W+\beta^-S).$$

The discussion prior to the theorem justifies the calculation of $T(\gamma^+Z\gamma^+)$ as

$$\begin{aligned}&[(\gamma^++\alpha^+S+\beta^-W)Z](\gamma^++\alpha^+S+\beta^-W)\\ &=\gamma^+Z\gamma^++\alpha^+(SZS^*)\alpha^++\beta^-(WZW^*)\beta^-\\ &\quad+2[\alpha^+SZ+\beta^-WZ]\gamma^++2[\alpha^+SZW^*]\beta^-\end{aligned}$$

together with a similar formula for $T(\gamma^-Z\gamma^-)$. Accordingly, when the expression

$$T\Big(\frac{1}{2}\gamma^-Z\gamma^- - \frac{1}{2}\gamma^+Z\gamma^+ + \alpha^+\gamma^- + \gamma^+\beta^- - \gamma^+\gamma^-\Big)$$

is expanded, the term independent of γ^- and linear in γ^+ is

$$\gamma^+[\alpha^+(SZ-W)+\beta^-[WZ+I-S)].$$

This term vanishes when S and W are chosen to satisfy $S=I+WZ$ and $W=SZ$ which forces $S=(I-Z^2)^{-1}$ and $W=(I-Z^2)^{-1}Z=Z(I-Z^2)^{-1}$. With this choice, routine calculations establish

$$\begin{aligned}&T\Big(\frac{1}{2}\gamma^-Z\gamma^- - \frac{1}{2}\gamma^+Z\gamma^+ + \alpha^+\gamma^- + \gamma^+\beta^- - \gamma^+\gamma^-\Big)\\ &=\frac{1}{2}\beta^-[Z(I-Z^2)^{-1}]\beta^- - \frac{1}{2}\alpha^+[(I-Z^2)^{-1}Z]\alpha^+\\ &\qquad+[\alpha^+(I-Z^2)^{-1}]\beta^- - \gamma^+\gamma^-.\end{aligned}$$

Invocation of (1.13) ends the proof. □

Note that if $Z\in A^2V$ corresponds to $\zeta\in\wedge^2V$ as usual then the theorem has calculated the Berezin conditional expectation

$$\mathbb{E}_\gamma[\exp(\gamma^-(\zeta)+\gamma^+(\zeta)+\alpha^+\gamma^-+\gamma^+\beta^-)]$$
$$=\int_\gamma \gamma^-(e^\zeta)\gamma^+(e^\zeta)e^{\alpha^+\gamma^-}e^{\gamma^+\beta^-}e^{-\gamma^+\gamma^-}d\gamma^+d\gamma^-.$$

(3.13) Theorem. *The subset of $A^2V\times A^2V$ comprising all (X, Y) for which the operator $I-YX$ is invertible is a connected open neighbourhood of the diagonal.*

Proof. That the indicated subset U of $A^2V\times A^2V$ is an open neighbourhood of the diagonal is clear. Let $(X, Y)\in U$ and let YX have spectrum $\{\lambda_1,\cdots,\lambda_l\}$ not containing unity. Choose $\alpha>0$ so that none of $\lambda_1e^{i\theta},\cdots,\lambda_le^{i\theta}$ is unity when $0\leq\theta\leq\alpha$ and none of $\lambda_1e^{i\alpha},\cdots,\lambda_le^{i\alpha}$ is real whence if $1\geq t\geq 0$ then $te^{i\alpha}YX$ does not have unity as an eigenvalue. Now (X, Y) is connected to $(X, 0)$ in U by following the arc $\{(X, e^{i\theta}Y): 0\leq\theta\leq\alpha\}$ and the line $\{(X, te^{i\alpha}Y): 1\geq t\geq 0\}$. Of course, $(X, 0)$ is connected to $(0, 0)$ in U along the line $\{(tX, 0): 1\geq t\geq 0\}$. □

Notice that the indicated subset of $A^2V\times A^2V$ is also symmetric about the diagonal, for if $X, Y\in A^2V$ then YX and XY are mutual adjoints.

(3.14) Theorem. *If $X, Y\in A^2V$ are such that $I-YX$ is invertible, then*

$$\mathbb{E}_\gamma\left[\exp\left(\frac{1}{2}\gamma^-X\gamma^--\frac{1}{2}\gamma^+Y\gamma^++\alpha^+\gamma^-+\gamma^+\beta^-\right)\right]$$
$$=\mathrm{Pf}(X, Y)\exp\{[\alpha^+(I-YX)^{-1}]\beta^-$$
$$+\frac{1}{2}\beta^-[X(I-YX)^{-1}]\beta^--\frac{1}{2}\alpha^+[(I-YX)^{-1}Y]\alpha^+\}.$$

Proof. Regarding the purported identity, each side is antiholomorphic (indeed, antipolynomial) in X and holomorphic (indeed, polynomial) in Y while both sides agree when $X=Y$ according to (3.12). The principle of analytic continuation from the diagonal applies thanks to (3.13). □

Note that if $X, Y\in A^2V$ correspond to $\xi,\eta\in\wedge^2V$ respectively then the theorem has calculated the Berezin conditional expectation

$$\mathbb{E}_\gamma[\exp(\gamma^-(\xi)+\gamma^+(\eta)+\alpha^+\gamma^-+\gamma^+\beta^-)]$$
$$=\int_\gamma \gamma^-(e^\xi)\gamma^+(e^\eta)e^{\alpha^+\gamma^-}e^{\gamma^+\beta^-}e^{-\gamma^+\gamma^-}d\gamma^+d\gamma^-.$$

Note further that the result may be reformulated: thus, $[\alpha^+(I-YX)^{-1}]\beta^-=\alpha^+[\beta^-(I-XY)^{-1}]$ as $I-YX$ and $I-XY$ are mutual adjoints; also, $X(I-YX)^{-1}=(I-XY)^{-1}X$ and $(I-YX)^{-1}Y=Y(I-XY)^{-1}$.

From these Gaussian integrals, others may be conveniently derived by differentiation under the expectation; for the derivation, the following facts will be useful.

(3.15) Theorem. *If $Z \in A^2V$ and if $v \in V$ then*

$$a(\alpha^+v)\exp\left(-\frac{1}{2}\alpha^+Z\alpha^+\right)=\alpha^+(Zv)\exp\left(-\frac{1}{2}\alpha^+Z\alpha^+\right)$$
$$a(\beta^-v)\exp\left(\frac{1}{2}\beta^-Z\beta^-\right)=-\beta^-(Zv)\exp\left(\frac{1}{2}\beta^-Z\beta^-\right).$$

Proof. Let $Z \in A^2V$ correspond to $\zeta \in \wedge^2V$. From (3.1) and the fact that $a(v)$ is an antiderivation, if $n \in \mathbb{N}$ then $a(v)\ \zeta^n = n(Zv)\ \zeta^{n-1}$ so that upon summation

$$a(v)\exp\zeta=(Zv)\exp\zeta.$$

As $\alpha^+\colon \wedge V \to \wedge V_\alpha^+$ is an isometric isomorphism, so

$$a(\alpha^+v)\exp\alpha^+(\zeta)=\alpha^+(Zv)\exp\alpha^+(\zeta).$$

Applying the antiisometric antiisomorphism $\beta^-\colon \wedge V \to \wedge V_\beta^-$ more explicitly, if $\phi := \exp\zeta$ and $\psi \in \wedge V$ is odd then

$$\begin{aligned}\langle a(\beta^-v)\beta^-\phi|\beta^-\psi\rangle &= \langle\beta^-\phi|\beta^-v\beta^-\psi\rangle\\ &=-\langle\beta^-\phi|\beta^-(v\psi)\rangle\\ &=-\langle v\psi|\phi\rangle\\ &=-\langle\psi|a(v)\phi\rangle\\ &=-\langle\beta^-(a(v)\phi)|\beta^-\psi\rangle\end{aligned}$$

thus parity considerations imply

$$a(\beta^-v)\beta^-\phi=-\beta^-(a(v)\phi)$$

and so

$$a(\beta^-v)\exp\beta^-(\zeta)=-\beta^-(Zv)\exp\beta^-(\zeta).$$

□

Of course, other identities follow similarly: for example, if $v \in V$ then

$$a(\alpha^+v)\exp(\alpha^+\beta^-)=(\beta^-v)\exp(\alpha^+\beta^-)$$
$$a(\beta^-v)\exp(\alpha^+\beta^-)=-(\alpha^+v)\exp(\alpha^+\beta^-).$$

(3.16) Theorem. *If $Z \in A^2V$ and if $v \in V$ then*

$$\mathbb{E}_\gamma\left[\exp\left(\frac{1}{2}\gamma^-Z\gamma^-+\alpha^+\gamma^-+\gamma^+\beta^-\right)\right]$$
$$=\exp\left(\frac{1}{2}\beta^-Z\beta^-+\alpha^+\beta^-\right)$$
$$\mathbb{E}_\gamma\left[(\gamma^+v)\exp\left(\frac{1}{2}\gamma^-Z\gamma^-+\alpha^+\gamma^-+\gamma^+\beta^-\right)\right]$$

$$= (\alpha^+(v) + \beta^-(Zv)) \exp\left(\frac{1}{2}\beta^- Z \beta^- + \alpha^+ \beta^-\right).$$

Proof. The first formula is a special case of (3.14) or may be established along the lines of (3.4). The second follows from the first by (1.12) in which the annihilator $a(\beta^- v)$ is applied. □

We remark that parallel arguments yield the identities

$$\mathbb{E}_\gamma\left[\exp\left(-\frac{1}{2}\gamma^+ Z \gamma^+ + \alpha^+ \gamma^- + \gamma^+ \beta^-\right)\right]$$
$$= \exp\left(-\frac{1}{2}\alpha^+ Z \alpha^+ + \alpha^+ \beta^-\right)$$
$$\mathbb{E}_\gamma\left[(\gamma^- v) \exp\left(-\frac{1}{2}\gamma^+ Z \gamma^+ + \alpha^+ \gamma^- + \gamma^+ \beta^-\right)\right]$$
$$= (\alpha^+(Zv) + \beta^-(v)) \exp\left(-\frac{1}{2}\alpha^+ Z \alpha^+ + \alpha^+ \beta^-\right).$$

For passage to infinite dimensions, we must modify (3.1) appropriately. By a quadratic we shall mean an element $\zeta \in \wedge V'$ that is homogeneous of degree two in the sense that $\zeta = P^2 \zeta$ with the notation introduced prior to (2.11). We shall say that the (necessarily antilinear) map $Z: V \to V'$ is antiskew when it satisfies the condition

$$x, y \in V \Rightarrow Zx(y) + Zy(x) = 0.$$

(3.17) Theorem. *A canonical isomorphism between the space of quadratics $\zeta \in \wedge V'$ and the space of antiskew maps $Z: V \to V'$ is determined by the rule*

$$x, y \in V \Rightarrow \zeta(xy) = Zx(y).$$

Proof. If ζ is given then the indicated rule plainly defines a map $Z: V \to V'$ which is antiskew because multiplication in $\wedge V'$ is anticommutative. If Z is given then the map

$$V \times V \to \mathbb{C}: (x, y) \mapsto Zx(y)$$

is alternating biantilinear, inducing an antilinear functional $\wedge^2 V \to \mathbb{C}$ which when precomposed with $P^2: \wedge V \to \wedge^2 V$ yields ζ. The correspondence $\zeta \leftrightarrow Z$ is evidently an isomorphism. □

We remark that if $M \in \mathcal{F}(V)$ then $\zeta_M = P_M(\zeta)$ corresponds to $Z_M = P_M Z|M$ under the isomorphism of (3.1): indeed, if $x, y \in M$ then

$$\langle xy | \zeta_M \rangle = \zeta(xy) = Zx(y) = \langle y | Z_M x \rangle.$$

Of course, an antiskew map $Z: V \to V$ induces an antiskew map $Z: V \to V'$ according to the rule

$$x, y \in V \Rightarrow Zx(y) = \langle y | Zx \rangle .$$

Plainly the antiskew maps $V \to V$ induce precisely those antiskew maps $V \to V'$ whose values are bounded antilinear functionals on V. Those antiskew maps $V \to V$ of Hilbert-Schmidt class are important and constitute the space $A^2[V]$.

(3.18) Theorem. *The canonical isomorphism between the quadratics in $\wedge V'$ and the antiskew maps $V \to V'$ induces an isomorphism $\wedge^2[V] \leftrightarrow A^2[V]$.*

Proof. Let the quadratic $\zeta \in \wedge V'$ and the antiskew $Z: V \to V'$ correspond. If $(v_1, \cdots, v_m)$ is a unitary basis for $M \in \mathscr{F}(V)$ then the remarks after (3.1) and (3.17) imply that

$$\zeta_M = \frac{1}{2} \sum_{i,j} \langle v_j | Z_M v_i \rangle v_i v_j$$

whence

$$\begin{aligned} 2\|\zeta_M\|^2 &= \sum_{i,j} | \langle v_j | Z_M v_i \rangle |^2 \\ &= \sum_i \|Z_M v_i\|^2 \\ &= \|Z_M\|^2_{HS} \end{aligned}$$

and therefore passage to the supremum as M runs over $\mathscr{F}(V)$ yields

$$\sqrt{2}\|\zeta\| = \|Z\|_{HS}$$

where $\| \cdot \|_{HS}$ signifies (complex) Hilbert-Schmidt norm. □

Now, if $\zeta \in \wedge V'$ is a quadratic then the associated Gaussian is its exponential

$$\exp\zeta = \sum_{n \geq 0} \frac{1}{n!} \zeta^n \in \wedge V'$$

where convergence is weak: in fact, if $\phi \in \wedge V$ then $P^n\phi = 0$ for all n sufficiently large, so the sum arising from the application of $\exp\zeta$ to ϕ is actually finite. Alternatively and equivalently, if $M \in \mathscr{F}(V)$ then $\exp(\zeta_M) \in \wedge M$ is defined and if also $N \in \mathscr{F}_M(V)$ then $P_M(\zeta_N) = \zeta_M$ so that $P_M(\exp\ \zeta_N) = \exp\ \zeta_M$; accordingly, (2.3) furnishes a unique $\exp\zeta \in \wedge V'$ such that $P_M(\exp\ \zeta) = \exp\zeta_M$ whenever $M \in \mathscr{F}(V)$.

(3.19) Theorem. *If $\zeta \in \wedge^2[V]$ corresponds to $Z \in A^2[V]$ as usual, then*

$$\langle e^\zeta | e^\zeta \rangle = \mathrm{Det}^{\frac{1}{2}}(I - Z^2).$$

Proof. As M runs through $\mathscr{F}(V)$ so $Z_M \to Z$ in Hilbert-Schmidt norm and $Z_M^2 \to Z^2$ in trace norm, whence $\mathrm{Det}(I-Z_M^2) \to \mathrm{Det}(I-Z^2)$ by continuity (or very definition) of the Fredholm determinant. As a consequence, it follows from (3.4) that the increasing net $(\|(e^\zeta)_M\|^4 : M \in \mathscr{F}(V))$ has limit

$$\lim \|e^{\zeta_M}\|^4 = \lim \mathrm{Det}(I-Z_M^2) = \mathrm{Det}(I-Z^2).$$

□

Conversely, if the quadratic $\zeta \in \wedge V'$ corresponds to the antiskew $Z: V \to V'$ then $e^\zeta \in \wedge [V]$ forces $Z \in A^2[V]$: if $M \in \mathscr{F}(V)$ and if $-Z_M^2 : M \to M$ has eigenvalues $\lambda_1^2, \cdots, \lambda_m^2$ repeated according to multiplicity, then the formula recorded after (3.3) yields

$$\sum_{j=1}^{m} \lambda_k^2 \leq \prod_{k=1}^{m} (1+\lambda_k^2) = \mathrm{Det}^{\frac{1}{2}}(I-Z_M^2) = \|e^{\zeta_M}\|^2$$

so

$$\mathrm{Tr}(Z_M^* Z_M) \leq \|e^\zeta\|^2.$$

(3.20) Theorem. *If $\xi, \eta \in \wedge^2[V]$ correspond to $X, Y \in A^2[V]$ respectively, then*

$$\langle e^\xi | e^\eta \rangle^2 = \mathrm{Det}(I-YX).$$

Proof. From (3.5) it follows that if $M \in \mathscr{F}(V)$ then

$$\langle e^{\xi_M} | e^{\eta_M} \rangle^2 = \mathrm{Det}(I-Y_M X_M).$$

Pass to the limit as M runs over $\mathscr{F}(V)$: the left member converges to $\langle e^\xi | e^\eta \rangle^2$ on account of the remark after (2.9); the right member converges to $\mathrm{Det}(I-YX)$ on account of trace norm continuity of the Fredholm determinant. □

Once again, we are justified in defining a Pfaffian pairing

$$\mathrm{Pf}: A^2[V] \times A^2[V] \to \mathbb{C}$$

by declaring that if $X, Y \in A^2[V]$ correspond to $\xi, \eta \in \wedge [V]$ as usual then

$$\mathrm{Pf}(X, Y) = \langle e^\xi | e^\eta \rangle.$$

Integral Kernels

Here we investigate an important canonical correspondence between the space $\wedge V'_{\mathbb{C}}$ and the space of all linear maps $\wedge V \to \wedge V'$. By its nature, this correspondence constitutes a kernel theorem; accordingly, when $U: \wedge V \to \wedge V'$ is linear we shall call the corresponding $u \in \wedge V'_{\mathbb{C}}$ its kernel.

Before proceeding, it is convenient to make a notational remark. To each $u \in \wedge V_{\mathbb{C}}$ we shall canonically associate $u(\gamma^+, \gamma^-) \in \wedge V_\gamma$ and $u(\alpha^+, \beta^-) \in \wedge (V^+_\alpha \oplus V^-_\beta)$ in the natural manner, handling elements of $\wedge V'_{\mathbb{C}}$ in similar fashion.

Once again, we begin by supposing that the complex Hilbert space V is finite dimensional.

To establish the kernel theorem in this context, let $(v_1, \cdots, v_m)$ be a unitary basis for V and $(v_C: C \in \underline{m})$ the corresponding unitary basis for $\wedge V$ as in (1.7). Let $U: \wedge V \to \wedge V$ be a linear map: if $\phi \in \wedge V$ then

$$\begin{aligned} U\phi &= U\Big(\sum_C \langle v_C | \phi \rangle v_C\Big) \\ &= \sum_C \mathbb{E}_\beta [\beta^-(v_C)\beta^+(\phi)]\, Uv_C \end{aligned}$$

whence (1.10) implies that

$$\begin{aligned} \alpha^+(U\phi) &= \sum_C \alpha^+(Uv_C)\, \mathbb{E}_\beta [\beta^-(v_C)\beta^+(\phi)] \\ &= \mathbb{E}_\beta [u(\alpha^+, \beta^-)\beta^+(\phi)] \end{aligned}$$

where

$$u = \sum_C \gamma^+(Uv_C)\, \gamma^-(v_C).$$

(4.1) Theorem. *There exists a canonical linear isomorphism*

$$\wedge V_{\mathbb{C}} \to \mathrm{End} \wedge V$$

associating to each $u \in \wedge V_{\mathrm{C}}$ the linear map $U: \wedge V \to \wedge V$ defined by the rule that if $\phi \in \wedge V$ then

$$\alpha^+(U\phi) = \mathbb{E}_\beta [u(\alpha^+, \beta^-)\beta^+(\phi)].$$

Proof. The assignment $u \mapsto U$ is well defined, for $\mathbb{E}_\beta [u(\alpha^+, \beta^-)\beta^+(\phi)]$ lies in $\wedge V^+_\alpha = \alpha^+(\wedge V)$ and therefore equals $\alpha^+(\psi)$ for a unique $\psi =: U\phi \in \wedge V$. Linearity of the assignment is plain; surjectivity is evident from the explicit formula displayed prior to the theorem, whence injectivity follows on account of the fact that $\wedge V_{\mathbb{C}}$ and End $\wedge V$ are equidimensional. □

An alternative explicit formula may be obtained by noting

$$\begin{aligned} U\phi &= \sum_C \langle v_C | U\phi \rangle v_C \\ &= \sum_C \langle U^* v_C | \phi \rangle v_C \end{aligned}$$

thus

$$\alpha^+(U\phi)=\sum_C \mathbb{E}_\beta[\beta^-(U^*v_C)\beta^+(\phi)]\alpha^+(v_C)$$
$$=\mathbb{E}_\beta\Big[\sum_C \alpha^+(v_C)\beta^-(U^*v_C)\beta^+(\phi)\Big]$$

and so

$$u=\sum_C \gamma^+(v_C)\gamma^-(U^*v_C).$$

Although we shall not employ it, the convention that if $\phi\in\wedge V$ then $\phi(\gamma^+):=\gamma^+(\phi)$ has its virtues: it enables us to cast the formula expressing $u\in\wedge V_{\mathbb{C}}$ as the kernel of $U\in\mathrm{End}\wedge V$ in the more familiar Gaussian form

$$U\phi(\alpha^+)=\int_\beta u(\alpha^+,\beta^-)\phi(\beta^+)e^{-\beta^+\beta^-}d\beta^+d\beta^-.$$

The canonical correspondence between $\wedge V_{\mathbb{C}}$ and End $\wedge V$ has a number of important properties. Thus, it respects the appropriate adjunctions: that defined by σ in $\wedge V_{\mathbb{C}}$ and that defined by the Hilbert space adjoint in End $\wedge V$.

(4.2) Theorem. *If $U\in\mathrm{End}\wedge V$ has kernel $u\in\wedge V_{\mathbb{C}}$ then U^* has kernel u^*.*

Proof. The explicit formula after (4.1) tells us that the kernel of U is

$$u=\sum_C \gamma^+(v_C)\gamma^-(U^*v_C)$$

whence application of σ yields

$$u^*=\sum_C \gamma^+(U^*v_C)\gamma^-(v_C)$$

which is the kernel of U^* by the explicit formula before (4.1). □

The kernel of a composite linear map is given by the standard formula.

(4.3) Theorem. *If $U_1, U_2\in\mathrm{End}\wedge V$ have kernels $u_1, u_2\in\wedge V_{\mathbb{C}}$ respectively then $U=U_1U_2$ has kernel u given by*

$$u(\alpha^+,\beta^-)=\mathbb{E}_\gamma[u_1(\alpha^+,\gamma^-)u_2(\gamma^+,\beta^-)].$$

Proof. Direct computation using (1.10) and (1.11): if $\phi\in\wedge V$ then

$$\alpha^+(U_1U_2\phi)=\mathbb{E}_\gamma[u_1(\alpha^+,\gamma^-)\gamma^+(U_2\phi)]$$
$$=\mathbb{E}_\gamma[u_1(\alpha^+,\gamma^-)\mathbb{E}_\beta[u_2(\gamma^+,\beta^-)\beta^+(\phi)]]$$
$$=\mathbb{E}_\gamma\mathbb{E}_\beta[u_1(\alpha^+,\gamma^-)u_2(\gamma^+,\beta^-)\beta^+(\phi)]$$

$$=\mathbb{E}_\beta\mathbb{E}_\gamma[u_1(\alpha^+\gamma^-)u_2(\gamma^+,\beta^-)\beta^+(\phi)]$$
$$=\mathbb{E}_\beta[\mathbb{E}_\gamma[u_1(\alpha^+,\gamma^-)u_2(\gamma^+,\beta^-)]\beta^+(\phi)]$$

so

$$u(\alpha^+,\beta^-)=\mathbb{E}_\gamma[u_1(\alpha^+,\gamma^-)u_2(\gamma^+,\beta^-)].$$

□

Recalling that the exterior algebra is graded by parity, the supertrace of a linear endomorphism of $\wedge V$ equals the Berezin expectation of its kernel in $\wedge V_{\mathbb{C}}$.

(4.4) Theorem. *If* $U\in\mathrm{End}\wedge V$ *has kernel* $u\in\wedge V_{\mathbb{C}}$ *then*

$$\mathbb{E}_V[u]=\mathrm{Tr}\Gamma U.$$

Proof. Of course, Γ is the parity automorphism of $\wedge V$. Let $C\in\underline{m}$: if v_C is even then

$$\langle v_C|Uv_C\rangle=\langle\Gamma v_C|Uv_C\rangle=\langle v_C|\Gamma Uv_C\rangle$$

and

$$\gamma^+(Uv_C)\gamma^-(v_C)=\gamma^-(v_C)\gamma^+(Uv_C)$$

while v_C is odd then

$$\gamma^+(Uv_C)\gamma^-(v_C)=\gamma^-(v_C)\gamma^+(\Gamma Uv_C).$$

Consequently, if $+/-$ indicates summation over even/odd then the explicit formula recorded before (4.1) yields

$$\begin{aligned}\mathbb{E}_V[u]&=\sum_C\mathbb{E}_V[\gamma^+(Uv_C)\gamma^-(v_C)]\\&=\sum_{C^+}\mathbb{E}_V[\gamma^-(v_C)\gamma^+(Uv_C)]+\sum_{C^-}\mathbb{E}_V[\gamma^-(v_C)\gamma^+(\Gamma Uv_C)]\\&=\sum_C\langle v_C|\Gamma Uv_C\rangle=Tr(\Gamma U).\end{aligned}$$

□

The kernel corresponding to the identity operator is fundamental.

(4.5) Theorem. *The kernel of* $I\in\mathrm{End}\wedge V$ *is precisely* $e^{\gamma^+\gamma^-}=e^{\gamma_V}\in\wedge V_{\mathbb{C}}$.

Proof. We offer two. For the first proof, denote the kernel corresponding to I by $u\in\wedge V_{\mathbb{C}}$ and let $(v_C\colon C\in\underline{m})$ be a standard unitary basis for $\wedge V$. The formula displayed before (4.1) yields

$$\begin{aligned} u &= \sum_C \gamma^+(v_C)\gamma^-(v_C) \\ &= \sum_C \gamma^+(v_{c_1})\cdots\gamma^+(v_{c_t})\gamma^-(v_{c_t})\cdots\gamma^-(v_{c_1}) \\ &= \sum_C \gamma^+(v_{c_1})\gamma^-(v_{c_1})\cdots\gamma^+(v_{c_t})\gamma^-(v_{c_t}) \\ &= \prod_{k=1}^m (1+\gamma^+(v_k)\gamma^-(v_k)) \\ &= \prod_{k=1}^m e^{\gamma^+(v_k)\gamma^-(v_k)} \\ &= e^{(\sum_{k=1}^m \gamma^+(v_k)\gamma^-(v_k))}. \end{aligned}$$

For the second proof, functorially extend to $\wedge(V_\alpha \oplus V_\beta)$ the linear endomorphism T of $V_\alpha \oplus V_\beta$ defined by $T|V_\alpha \oplus V_\beta^- = I$ and $T(\beta^+) = \beta^+ + \alpha^+$. Expansion shows that if $\phi \in \wedge V$ then $T(\beta^+\phi) - \alpha^+\phi$ lies in the ideal of $\wedge(V_\alpha \oplus V_\beta)$ generated by $\wedge V_\beta^+$ and so vanishes upon application of $\mathbb{E}_\beta$. Consequently, (1.13) yields

$$\begin{aligned} \mathbb{E}_\beta[e^{\alpha^+\beta^-}\beta^+(\phi)] &= \mathbb{I}_\beta[e^{\alpha^+\beta^- - \beta^+\beta^-}\beta^+(\phi)] \\ &= \mathbb{I}_\beta[e^{-\beta^+\beta^-}T\beta^+(\phi)] \\ &= \mathbb{E}_\beta[T\beta^+(\phi)] \\ &= \alpha^+(\phi). \end{aligned}$$

□

Let $v \in V$. We have already introduced the annihilator $a(v) \in \operatorname{End} \wedge V$ as the linear antiderivation determined by $a(v)1 = 0$ and the rule that if $w \in V$ then $a(v)w = \langle v|w\rangle$. We also introduce the creator $c(v) \in \operatorname{End} \wedge V$ as the linear operator of left multiplication by v so that if $\phi \in \wedge V$ then $c(v)\phi = v\phi$. Recall that $a(v)$ and $c(v)$ are mutually adjoint: if $\phi, \psi \in \wedge V$ then

$$\langle \phi | a(v)\psi\rangle = \langle c(v)\phi|\psi\rangle.$$

(4.6) Theorem. *If $v \in V$ then the creator $c(v) \in \operatorname{End} \wedge V$ has kernel $\gamma^+(v)e^{\gamma^+\gamma^-}$ and the annihilator $a(v) \in \operatorname{End} \wedge V$ has kernel $e^{\gamma^+\gamma^-}\gamma^-(v)$.*

Proof. For the creator, if $\phi \in \wedge V$ then (1.10) and (4.5) yield

$$\begin{aligned} \alpha^+(c(v)\phi) &= \alpha^+(v)\alpha^+(\phi) \\ &= \alpha^+(v)\mathbb{E}_\beta[e^{\alpha^+\beta^-}\beta^+(\phi)] \\ &= \mathbb{E}_\beta[\alpha^+(v)e^{\alpha^+\beta^-}\beta^+(\phi)]. \end{aligned}$$

For the annihilator, if $\phi \in \wedge V$ then (1.10) and (4.5) along with the remark after (3.15) yield

$$\begin{aligned}\alpha^+(a(v)\phi) &= a(\alpha^+ v)\alpha^+(\phi)\\ &= a(\alpha^+ v)\mathbb{E}_\beta[e^{\alpha^+\beta^-}\beta^+(\phi)]\\ &= \mathbb{E}_\beta[a(\alpha^+ v)e^{\alpha^+\beta^-}\beta^+(\phi)]\\ &= \mathbb{E}_\beta[e^{\alpha^+\beta^-}\beta^-(v)\beta^+(\phi)].\end{aligned}$$

□

Notice that the mutually adjoint nature of creators and annihilators is evident from their kernels in light of (4.2).

(4.7) Theorem. *If* S: $V \to V$ *is complex linear then its functorial extension* S: $\wedge V \to \wedge V$ *has kernel* $\exp([\gamma^+ S]\gamma^-)$.

Proof. Again let $(v_C: C \in \underline{m})$ be a standard unitary basis for $\wedge V$. Note that if $C = (c_1, \cdots, c_t) \in \underline{m}$ then

$$S(v_C) = (Sv_{c_1})\cdots(Sv_{c_t}).$$

Consequently, the formula prior to (4.1) reveals the kernel of $S: \wedge V \to \wedge V$ as

$$\begin{aligned}&\sum_C \gamma^+(Sv_C)\gamma^-(v_C)\\ &= \sum_C \gamma^+(Sv_{c_1})\cdots\gamma^+(Sv_{c_t})\gamma^-(v_{c_t})\cdots\gamma^-(v_{c_1})\\ &= \sum_C \gamma^+(Sv_{c_1})\gamma^-(v_{c_1})\cdots\gamma^+(Sv_{c_t})\gamma^-(v_{c_t})\\ &= \prod_{k=1}^m (1+\gamma^+(Sv_k)\gamma^-(v_k))\\ &= \exp\Big(\sum_{k=1}^m \gamma^+(Sv_k)\gamma^-(v_k)\Big).\end{aligned}$$

□

Of course, the identity operator I has kernel $\exp(\gamma^+\gamma^-)$ as a special case. As another special case, the parity automorphism Γ has kernel $\exp(-\gamma^+\gamma^-)$.

At this juncture, we pass on to a consideration of the case in which the complex Hilbert space V is infinite dimensional.

Naturally, our approach to the kernel theorem in case V is infinite dimensional proceeds via its finite dimensional subspaces. Before stating the theorem, it is convenient to introduce two more pieces of notation pertaining to a finite dimensional subspace $M\in\mathscr{F}(V)$. When $u\in\wedge V'_{\mathbb{C}}$ we simplify earlier notation, writing $u_{M_{\mathbb{C}}}$ as u_M so that

$$u_M=P_{M_{\mathbb{C}}}(u)\in\wedge M_{\mathbb{C}}.$$

When $U\colon \wedge V\rightarrow\wedge V'$ is complex linear we denote its compression to $\wedge M$ by

$$U_M=P_MU|\wedge M\in\mathrm{End}\wedge M.$$

(4.8) Theorem. *There exists a canonical linear isomorphism*

$$\wedge V'_{\mathbb{C}}\rightarrow\mathrm{Hom}(\wedge V,\ \wedge V')\colon u\mapsto U$$

uniquely determined by the requirement that if $M\in\mathscr{F}(V)$ then $u_M\in\wedge M_{\mathbb{C}}$ is the kernel of $U_M\in\mathrm{End}\wedge M$.

Proof. Let $u\in\wedge V'_{\mathbb{C}}$. When $M\in\mathscr{F}(V)$ let $u_M:=P_{M_{\mathbb{C}}}(u)$ act as kernel for $\widetilde{U}_M\in\mathrm{End}\wedge M$ according to (4.1). Define $U\colon\wedge V\rightarrow\wedge V'$ by the rule that if $\phi,\ \psi\in\wedge V$ then

$$U\phi(\psi)=\langle\psi|\widetilde{U}_M\phi\rangle$$

for any $M\in\mathscr{F}(V)$ such that $\wedge M\ni\phi,\ \psi$. Once this definition is sound, it is plain that $U\colon\wedge V\rightarrow\wedge V'$ is linear and that if $M\in\mathscr{F}(V)$ then $P_MU|\wedge M=\widetilde{U}_M$ has u_M as its kernel. To establish soundness, let $N\in\mathscr{F}_M(V)$. By linearity, we may assume that $u_N(\alpha^+,\beta^-)=\alpha^+(\xi)\beta^-(\eta)$ for $\xi,\ \eta\in\wedge N$. Under this assumption, as $\phi,\ \psi\in\wedge M$ so (1.10) yields

$$\begin{aligned}\langle\psi|\widetilde{U}_N\phi\rangle&=\mathbb{E}_\alpha[\alpha^-(\psi)\alpha^+(\widetilde{U}_N\phi)]\\&=\mathbb{E}_\alpha\mathbb{E}_\beta[\alpha^-(\psi)u_N(\alpha^+,\beta^-)\beta^+(\phi)]\\&=\mathbb{E}_\alpha\mathbb{E}_\beta[\alpha^-(\psi)\alpha^+(\xi)\beta^-(\eta)\beta^+(\phi)]\\&=\langle\psi|\xi\rangle\langle\eta|\phi\rangle\\&=\langle\psi|P_M\xi\rangle\langle P_M\eta|\phi\rangle\\&=\mathbb{E}_\alpha\mathbb{E}_\beta[\alpha^-(\psi)P_{M_{\mathbb{C}}}u_N(\alpha^+,\beta^-)\beta^+(\phi)]\\&=\mathbb{E}_\alpha\mathbb{E}_\beta[\alpha^-(\psi)u_M(\alpha^+,\beta^-)\beta^+(\phi)]\\&=\langle\psi|\widetilde{U}_M\phi\rangle.\end{aligned}$$

Let $U\in\mathrm{Hom}(\wedge V,\ \wedge V')$. When $M\in\mathscr{F}(V)$ let $U_M:=P_MU|\wedge M$ have kernel $\widetilde{u}_M\in\wedge M_{\mathbb{C}}$ according to (4.1). We claim that if $N\in\mathscr{F}_M(V)$ then $P_{M_{\mathbb{C}}}(\widetilde{u}_N)=\widetilde{u}_M$. Once this is verified, the modification of (2.3) furnishes a $u\in\wedge V'_{\mathbb{C}}$ such that if $M\in\mathscr{F}(V)$ then $P_{M_{\mathbb{C}}}(u)=\widetilde{u}_M$ is the kernel of U_M. For the verification, let $(v_c\colon 1\leq c\leq m)$ be a unitary basis for M engendering $(v_C\colon C\in\underline{m})$ for $\wedge M$; extend to a unitary basis $(v_d\colon 1\leq d\leq n)$ for N engendering $(v_D\colon D\in\underline{n})$ for $\wedge N$. Of course, $\underline{m}\subset\underline{n}$ and if $D\in\underline{n}$ then

$$P_M(v_D)=\begin{cases}v_D & D\in\underline{m}\\0 & D\notin\underline{m}.\end{cases}$$

As $P_M U_N | \wedge M = U_M$ so the formula displayed before (4.1) yields

$$\begin{aligned} P_{M_{\mathbb{C}}}(\tilde{u}_N) &= P_{M_{\mathbb{C}}} \sum_D \gamma^+(U_N v_D)\gamma^-(v_D) \\ &= \sum_D \gamma^+(P_M U_N v_D)\gamma^-(P_M v_D) \\ &= \sum_C \gamma^+(U_M v_C)\gamma^-(v_C) \\ &= \tilde{u}_M. \end{aligned}$$

□

As before, the correspondence between linear operators and their kernels respects the appropriate adjunctions. The adjunction on $\wedge V'_{\mathbb{C}}$ is defined by σ as in (2.5) so that if $u \in \wedge V'_{\mathbb{C}}$ and $M \in \mathscr{F}(V)$ then in simplified notation

$$(u^*)_M = (u_M)^*.$$

If $U \in \operatorname{Hom}(\wedge V, \wedge V')$ then its adjoint $U^* \in \operatorname{Hom}(\wedge V, \wedge V')$ is defined by the rule that if $\phi, \psi \in \wedge V$ then

$$U^*\phi(\psi) = \overline{U\psi(\phi)}$$

so that if $M \in \mathscr{F}(V)$ and $\phi, \psi \in \wedge M$ then

$$\begin{aligned} \langle \psi | (U^*)_M \phi \rangle &= \langle \psi | P_M U^* \phi \rangle \\ &= U^*\phi(\psi) \\ &= \overline{U\psi(\phi)} \\ &= \langle P_M U\psi | \phi \rangle \\ &= \langle U_M \psi | \phi \rangle \end{aligned}$$

whence

$$(U^*)_M = (U_M)^*.$$

(4.9) Theorem. *If $U \in \operatorname{Hom}(\wedge V, \wedge V')$ has kernel $u \in \wedge V'_{\mathbb{C}}$ then U^* has kernel u^*.*

Proof. If $M \in \mathscr{F}(V)$ then $(U_M)^*$ has kernel $(u_M)^*$ by (4.2) whence $(U^*)_M$ has kernel $(u^*)_M$ by the remarks preceding the theorem. The kernel theorem (4.8) itself completes the proof. □

Of course, elements of $\wedge V_{\mathbb{C}} \subset \wedge V'_{\mathbb{C}}$ act as kernels for linear operators $\wedge V \to \wedge V'$ in a rather special class.

(4.10) Theorem. *Let $M \in \mathscr{F}(V)$. The linear operator $U : \wedge V \to \wedge V'$*

satisfies the conditions Ran $U \subset \wedge M$ *and* $(\wedge M)^{\perp} \subset$ Ker U *precisely when its kernel* u *lies in* $\wedge M_{\mathbb{C}}$.

Proof. Assume the stated conditions on U. As for the kernel theorem, let $N \in \mathscr{F}_M(V)$ and choose a standard unitary basis $(v_D: D \in \underline{n})$ for $\wedge N$ so that $(v_C: C \in \underline{m})$ is standard for $\wedge M$. Note that if $D \in \underline{n} - \underline{m}$ then $U_N v_D = 0$ since $(\wedge M)^{\perp} \subset$ Ker U and that if $C \in \underline{m}$ then $U_N v_C = U v_C$ since Ran $U \subset \wedge M$. Accordingly, the formula preceding (4.1) yields the kernel of U_N as

$$\begin{aligned} u_N &= \sum_D \gamma^+(U_N v_D)\gamma^-(v_D) \\ &= \sum_C \gamma^+(U_N v_C)\gamma^-(v_C) \\ &= \sum_C \gamma^+(U v_C)\gamma^-(v_C). \end{aligned}$$

This forces

$$u = \sum_C \gamma^+(U v_C)\gamma^-(v_C) \in \wedge M_{\mathbb{C}}.$$

Assume $u \in \wedge M_{\mathbb{C}}$. Linearity entitles us to assume further that $u = \gamma^+(\xi)\gamma^-(\eta)$ for $\xi, \eta \in \wedge M$. As for the kernel theorem, if $\phi \in \wedge V$ and if $N \in \mathscr{F}_M(V)$ is such that $\wedge N \ni \phi$ then

$$\begin{aligned} \alpha^+(P_N U\phi) &= \mathbb{E}_\beta[u_N(\alpha^+, \beta^-)\beta^+(\phi)] \\ &= \mathbb{E}_\beta[\alpha^+(\xi)\beta^-(\eta)\beta^+(\phi)] \\ &= \langle \eta | \phi \rangle \alpha^+(\xi). \end{aligned}$$

Thus

$$\phi \in \wedge V \Rightarrow U\phi = \langle \eta | \phi \rangle \xi \in \wedge M$$

and so

$$\phi \in (\wedge M)^{\perp} \Rightarrow U\phi = 0.$$

□

Further to the proof, we remark that if $\Phi^+, \Phi^- \in \wedge V'$ then $u = \gamma^+(\Phi^+)\gamma^-(\Phi^-) \in \wedge V'_{\mathbb{C}}$ is the kernel of $U: \wedge V \rightarrow \wedge V'$ given by

$$\phi \in \wedge V \Rightarrow U\phi = \overline{\Phi^-(\phi)}\,\Phi^+.$$

Again, $\exp(\gamma_V)$ is the kernel corresponding to the identity operator on $\wedge V$ or more precisely the canonical inclusion $\wedge V \rightarrow \wedge V'$.

(4.11) Theorem. *The identity operator* $I \in \mathrm{End} \wedge V \subset \mathrm{Hom}(\wedge V, \wedge V')$ *has* *kernel* $\exp(\gamma_V) \in \wedge V'_{\mathbb{C}}$.

Proof. Let $u = \exp(\gamma_V)$ correspond to $U \in \mathrm{Hom}(\wedge V, \wedge V')$. If $M \in \mathscr{F}(V)$ then $U_M \in \mathrm{End} \wedge M$ has kernel $P_{M_{\mathbb{C}}}(u) = P_{M_{\mathbb{C}}}(\exp \gamma_V) = \exp \gamma_M$ as before (2.12) so that U_M is the identity by (4.5). This forces $U = I$. □

Let $v \in V$. We have already defined the creator $c(v) \in \mathrm{End} \wedge V$ and the annihilator $a(v) \in \mathrm{End} \wedge V$. We have also noted that these are mutually adjoint: if $\phi, \psi \in \wedge V$ then

$$\langle c(v)\phi | \psi \rangle = \langle \phi | a(v)\psi \rangle.$$

We may reformulate these operators, defining the creator $c(v) \in \mathrm{Hom}(\wedge V, \wedge V')$ by

$$\phi, \psi \in \wedge V \Rightarrow [c(v)\phi](\psi) = \langle \psi | c(v)\phi \rangle$$

and the annihilator $a(v) \in \mathrm{Hom}(\wedge V, \wedge V')$ by

$$\phi, \psi \in \wedge V \Rightarrow [a(v)\phi](\psi) = \langle \psi | a(v)\phi \rangle.$$

Naturally, the operators $c(v)$ and $a(v)$ are mutually adjoint in the sense appropriate to $\mathrm{Hom}(\wedge V, \wedge V')$.

(4.12) Theorem. *If $v \in V$ then $c(v) \in \mathrm{Hom}(\wedge V, \wedge V')$ has kernel $\gamma^+(v)e^{\gamma^+\gamma^-}$ and $a(v) \in \mathrm{Hom}(\wedge V, \wedge V')$ has kernel $e^{\gamma^+\gamma^-}\gamma^-(v)$.*

Proof. If $M \in \mathscr{F}(V)$ then $c(v)_M = P_{MC}(v) | \wedge M = c(v_M) \in \mathrm{End} \wedge M$ has kernel $\gamma^+(v_M)e^{\gamma_M} = P_{M_{\mathbb{C}}}(\gamma^+(v)e^{\gamma_V})$ according to (4.6) whence $c(v)$ has kernel $\gamma^+(v)e^{\gamma_V}$ as claimed. The proof may be completed either by a parallel argument or by adjunction. □

In fact, these operators may be extended further in canonical fashion: $c(v) \in \mathrm{End} \wedge V'$ is defined by

$$\Phi \in \wedge V', \psi \in \wedge V \Rightarrow [c(v)\Phi](\psi) = \Phi(a(v)\psi);$$

$a(v) \in \mathrm{End} \wedge V'$ is defined by

$$\Phi \in \wedge V', \psi \in \wedge V \Rightarrow [a(v)\Phi](\psi) = \Phi(c(v)\psi).$$

These extensions are continuous when $\wedge V'$ has the weak topology according to which the net $(\Phi_j : j \in \mathscr{J})$ in $\wedge V'$ converges to zero precisely when the complex net $(\Phi_j(\psi) : j \in \mathscr{J})$ converges to zero for all $\psi \in \wedge V$: indeed, if $\Phi_j \to 0$ in $\wedge V'$ then

$$\psi \in \wedge V \Rightarrow [c(v)\Phi_j](\psi) = \Phi_j(a(v)\psi) \to 0$$

so $c(v)\Phi_j \to 0$ in $\wedge V'$ while $a(v)\Phi_j \to 0$ in like manner. These extensions to $\wedge V'$ are uniquely determined by their effect on $\wedge V$: indeed, it is plain that each $\Phi \in$

$\wedge V'$ is the weak limit of the net $(\Phi_M \colon M \in \mathscr{F}(V))$.

(4.13) Theorem. *If $v \in V$ and $\Phi \in \wedge V'$ then*

$$[c(v)\Phi]_M = c(v_M)\Phi_M$$
$$[a(v)\Phi]_M = P_M(a(v)\Phi_{M_v})$$

where M_v is the complex linear span of v and the arbitrary $M \in \mathscr{F}(V)$.

Proof. Observe that if $\phi \in \wedge M$ then of course $c(v)\phi \in \wedge M_v$ so

$$\begin{aligned}\langle \phi | [a(v)\Phi]_M \rangle &= [a(v)\Phi](\phi) \\ &= \Phi(c(v)\phi) \\ &= \langle c(v)\phi | \Phi_{M_v} \rangle \\ &= \langle \phi | a(v)\Phi_{M_v} \rangle \\ &= \langle \phi | P_M(a(v)\Phi_{M_v}) \rangle\end{aligned}$$

as required for the annihilator. The creator succumbs to similar treatment, simplified by the fact that if $\phi \in \wedge M$ then $a(v)\phi \in \wedge M$. □

Recall that the anticommutator $\{S, T\}$ between linear operators S and T on any vector space is defined by

$$\{S, T\} = ST + TS.$$

In these terms, creators and annihilators satisfy the canonical anticommutation relations in End$\wedge V'$ as follows.

(4.14) Theorem. *If $x, y \in V$ then*

$$\{a(x), a(y)\} = 0$$
$$\{a(x), c(y)\} = \langle x|y \rangle I$$
$$\{c(x), c(y)\} = 0.$$

Proof. These familiar identities hold automatically in End $\wedge V'$ once they hold in End $\wedge V$: here, the last holds by anticommutativity of exterior multiplication whence the first holds by adjunction; for the central identity, if $\phi \in \wedge V$ then

$$\begin{aligned}a(x)c(y)\phi &= a(x)[y\phi] \\ &= [a(x)y]\phi - y[a(x)\phi] \\ &= \langle x|y \rangle \phi - c(y)a(x)\phi.\end{aligned}$$

□

Incidentally, the scalar multiples of the vacuum $1 \in \wedge V \subset \wedge V'$ are precisely the vectors annihilated by every annihilator: if $\Phi \in \wedge V'$ and if $v \in V$ implies $a(v)\Phi = 0$ then

$$\phi \in \wedge V \Rightarrow \Phi[v\phi] = [a(v)\Phi](\phi) = 0$$

thus Φ vanishes on all elements of $\wedge V$ having positive homogeneous degree and so Φ is a scalar multiple of $\langle \cdot | 1 \rangle$.

(4.15) Theorem. *If $v \in V$ then $c(v)$ and $a(v)$ define mutually adjoint bounded linear operators on $\wedge [V]$ such that if $\Phi \in \wedge [V]$ then*

$$\| c(v)\Phi \|^2 + \| a(v)\Phi \|^2 = \| v \|^2 \| \Phi \|^2.$$

Proof. By virtue of the canonical anticommutation relations, if $\phi \in \wedge V$ then

$$\begin{aligned} \| c(v)\phi \|^2 + \| a(v)\phi \|^2 &= \langle c(v)\phi | c(v)\phi \rangle + \langle a(v)\phi | a(v)\phi \rangle \\ &= \langle \phi | a(v)c(v)\phi \rangle + \langle \phi | c(v)a(v)\phi \rangle \\ &= \langle \phi | \{a(v), c(v)\}\phi \rangle \\ &= \|v\|^2 \|\phi\|^2 \end{aligned}$$

since $c(v)$ and $a(v)$ are mutually adjoint on $\wedge V$. Now, if $\Phi \in \wedge [V]$ and if $M \in \mathscr{F}(V)$ is chosen to contain v then (4.13) yields $(c(v)\Phi)_M = c(v)\Phi_M$ and $(a(v)\Phi)_M = a(v)\Phi_M$ so that

$$\| (c(v)\Phi)_M \|^2 + \| (a(v)\Phi)_M \|^2 = \|v\|^2 \|\Phi_M\|^2$$

whence passage to the supremum over M establishes the claimed equality. In particular, $c(v) \in \operatorname{End} \wedge V'$ and $a(v) \in \operatorname{End} \wedge V'$ restrict to $\wedge [V]$ as bounded linear operators; their mutually adjoint nature on $\wedge V$ extends to $\wedge [V]$ by continuity. □

Spin Transformations

As an extended illustration of the various concepts introduced thus far, we now consider in some detail the implementation of orthogonal transformations in the Fock representation. Throughout, the underlying complex Hilbert space V has arbitrary dimension, unless otherwise specified.

The orthogonal group $O(V)$ comprises all real-linear automorphisms g of V that are orthogonal transformations in the sense

$$x, y \in V \Rightarrow (gx | gy) = (x | y).$$

It contains as subgroups both the group $U(V)$ comprising all unitary transformations of V and the larger group containing also the antiunitaries.

Each $g \in O(V)$ decomposes uniquely as a sum $g = C_g + A_g$ in which C_g is complex-linear and A_g antilinear: explicitly,

$$C_g = \frac{1}{2}(g - JgJ)$$

$$A_g = \frac{1}{2}(g + JgJ).$$

(5.1) Theorem. *If $g \in O(V)$ then $C_g^* = C_{g^{-1}}$ and $A_g^* = A_{g^{-1}}$ where adjunction is relative to the real inner product $(\cdot|\cdot)$.*

Proof. As J is skew-adjoint, this is a direct consequence of the explicit formulae for C_g and A_g. Note that for complex-linear endomorphisms, adjunction relative to $\langle\cdot|\cdot\rangle$ and $(\cdot|\cdot)$ have the same effect. □

Taking into account complex-linearity and antilinearity as appropriate, it follows that if also $x, y \in V$ then

$$\langle C_g x|y\rangle = \langle x|C_{g^{-1}}y\rangle$$
$$\langle A_g x|y\rangle = \langle A_{g^{-1}}y|x\rangle.$$

(5.2) Theorem. *If $g \in O(V)$ then*

$$C_{g^{-1}}C_g + A_{g^{-1}}A_g = I$$
$$A_{g^{-1}}C_g + C_{g^{-1}}A_g = O.$$

Proof. This applies to any real-linear automorphism g of V and follows upon taking complex-linear and antilinear parts in the expansion of

$$I = g^{-1}g = (C_{g^{-1}} + A_{g^{-1}})(C_g + A_g).$$

□

We remark that a particular consequence of (5.1) and (5.2) is the fact that if $g \in O(V)$ and if $v \in V$ then

$$\|C_g v\|^2 + \|A_g v\|^2 = \|v\|^2.$$

(5.3) Theorem. *If $g \in O(V)$ then A_g restricts to an antiunitary isomorphism* Ker $C_g \rightarrow$ Ker $C_{g^{-1}}$.

Proof. The second identity in (5.2) implies that A_g maps Ker C_g to Ker $C_{g^{-1}}$ and the first identity in (5.2) implies that $A_{g^{-1}}A_g|\text{Ker } C_g = I$; apply (5.1) and symmetry. □

The following factorization of orthogonal transformations is useful.

(5.4) Theorem. *If $g \in O(V)$ then there exists $h \in O(V)$ such that gh^{-1} is unitary and C_h is self adjoint.*

Proof. We offer a sketch. Let f_g be the partially isometric factor in the polar decomposition $C_g = f_g|C_g|$ and invoke (5.3) to choose a partial isometry e on V having Ker C_g as initial space and Ker $C_{g^{-1}}$ as final space. Now $k := e + f_g : V \rightarrow V$

is unitary and $k|C_g|=C_g$ since if $v\in V$ then $|C_g|v\in(\operatorname{Ker}C_g)^{\perp}$ and so $k|C_g|v=f_g|C_g|v=C_gv$. Further, the complex-linear part of $h=k^{-1}g=|C_g|+k^{-1}A_g$ is $C_h=|C_g|$ which is actually positive. □

Alternatively and in the notation of the proof, $g=gk^{-1}k$ where $k: V\to V$ is unitary and $gk^{-1}\in O(V)$ has self-adjoint complex linear part $k|C_g|k^{-1}$.

We define the Fock representation of V in terms of creators and annihilators as usual. The prescription

$$v\in V\Rightarrow\pi(v)=c(v)+a(v)$$

defines a real-linear map from V to both $\operatorname{End}(\wedge V)$ and $\operatorname{End}(\wedge V')$; extending the traditional terminology, we shall refer to each of these maps as the Fock representation of V. By (4.15) we see that the same prescription defines a real-linear map π from V to the algebra $B(\wedge[V])$ comprising all bounded linear operators on $\wedge[V]$; this is the traditional Fock representation of V. Note that the creators and annihilators may be recovered from the Fock representation: indeed, if $v\in V$ then

$$c(v)=\frac{1}{2}\{\pi(v)-i\pi(Jv)\}$$
$$a(v)=\frac{1}{2}\{\pi(v)+i\pi(Jv)\}.$$

(5.5) Theorem. *The Fock representation π of V is irreducible in the sense that $T\in\operatorname{Hom}(\wedge V,\wedge V')$ is a scalar operator if it satisfies*

$$v\in V\Rightarrow T\pi(v)=\pi(v)T.$$

Proof. Let T map $1\in\wedge V$ to $\Phi\in\wedge V'$. If $v\in V$ then

$$a(v)\Phi=a(v)T1=Ta(v)1=0$$

whence the remark after (4.14) yields a scalar $\lambda\in\mathbb{C}$ such that $T1=\Phi=\lambda 1$. It follows that if $\phi\in\wedge V$ then $T\phi=\lambda\phi$: indeed, T is linear and if $v_1,\cdots,v_n\in V$ then

$$\begin{aligned}T(v_1\cdots v_n)&=Tc(v_1)\cdots c(v_n)1\\&=c(v_1)\cdots c(v_n)T1\\&=\lambda v_1\cdots v_n.\end{aligned}$$

□

Of course, our proof of this result is modelled on a standard proof that the traditional Fock representation is irreducible.

Now, we shall refer to the nonzero $U\in\operatorname{Hom}(\wedge V,\wedge V')$ as a generalized Fock implementer for the orthogonal transformation $g\in O(V)$ precisely when

$$v\in V\Rightarrow U\pi(v)=\pi(gv)U$$

where $\pi: V\to\operatorname{End}\wedge V$ on the left and $\pi: V\to\operatorname{End}\wedge V'$ on the right. This generalizes the usual notion of a unitary Fock implementer, in which U: $\wedge[V] \rightarrow \wedge[V]$ is unitary and $\pi: V \rightarrow B(\wedge[V])$ is the traditional Fock representation. Recall that in this familiar context, $g \in O(V)$ admits a unitary Fock implementer if and only if its antilinear part A_g is of Hilbert-Schmidt class. As we shall see, Fock implementers in our generalized sense exist almost universally: in fact, $g \in O(V)$ admits a generalized Fock implementer if and only if Ker C_g is finite-dimensional.

Our analysis is facilitated by the introduction of a little more notation: when $g \in O(V)$ and $v \in V$ we define

$$\begin{aligned} c_g(v) &= c(C_g v) + a(A_g v) \\ a_g(v) &= a(C_g v) + c(A_g v). \end{aligned}$$

(5.6) Theorem. *If $g \in O(V)$ and if $x, y \in V$ then*

$$\begin{aligned} \{a_g(x), a_g(y)\} &= 0 \\ \{a_g(x), c_g(y)\} &= \langle x|y\rangle I \\ \{c_g(x), c_g(y)\} &= 0. \end{aligned}$$

Proof. These versions of the canonical anticommutation relations follow easily from (4.14) together with (5.1) and (5.2): taking the central identity for example,

$$\begin{aligned} \{a_g(x), c_g(y)\} &= \langle C_g x|C_g y\rangle + \langle A_g y|A_g x\rangle \\ &= \langle x|C_{g^{-1}}C_g y + A_{g^{-1}}A_g y\rangle \\ &= \langle x|y\rangle. \end{aligned}$$

□

Note further from (5.2) with $g \in O(V)$ replaced by g^{-1} that if $v \in V$ then

$$\begin{aligned} c_g(C_{g^{-1}}v) + a_g(A_{g^{-1}}v) &= c(v) \\ a_g(C_{g^{-1}}v) + c_g(A_{g^{-1}}v) &= a(v). \end{aligned}$$

These transformed creators and annihilators are significant in several respects. Thus, it is obvious from the definitions that if $g \in O(V)$ then

$$v \in V \Rightarrow \pi(gv) = c_g(v) + a_g(v).$$

Also, the nonzero $U \in \mathrm{Hom}(\wedge V, \wedge V')$ is a generalized Fock implementer for $g \in O(V)$ precisely when

$$v \in V \Rightarrow \begin{cases} U\, c(v) = c_g(v)\, U \\ U\, a(v) = a_g(v)\, U. \end{cases}$$

(5.7) Theorem. *If $U \in \mathrm{Hom}(\wedge V, \wedge V')$ is a generalized Fock implementer* *for* $g \in O(V)$ *then* U^* *is a generalized Fock implementer for* g^{-1}.

Proof. If $v \in V$ then the note after (5.6) implies that

$$\begin{aligned} a(v)U &= [a_g(C_{g^{-1}}v) + c_g(A_{g^{-1}}v)]U \\ &= U[a(C_{g^{-1}}v) + c(A_{g^{-1}}v)] \\ &= Ua_{g^{-1}}(v) \end{aligned}$$

whence if also ϕ, $\psi \in \wedge V$ then

$$\begin{aligned} (U^*c(v)\phi)(\psi) &= \overline{U\psi(c(v)\phi)} \\ &= \overline{a(v)U\psi(\phi)} \\ &= \overline{Ua_{g^{-1}}(v)\psi(\phi)} \\ &= U^*\phi(a_{g^{-1}}(v)\psi) \\ &= (c_{g^{-1}}(v)U^*\phi)(\psi) \end{aligned}$$

and therefore

$$U^*c(v) = c_{g^{-1}}(v)U^*.$$

A parallel argument ends the proof. □

We shall refer to the nonzero $\Phi \in \wedge V'$ as a displaced Fock vacuum for the orthogonal transformation $g \in O(V)$ precisely when

$$v \in V \Rightarrow a_g(v)\Phi = 0.$$

(5.8) Theorem. *If* $g \in O(V)$ *then the rule* $\Phi = U1$ *sets up a bijective correspondence between its displaced Fock vacua* $\Phi \in \wedge V'$ *and its generalized Fock implementers* $U \in \mathrm{Hom}(\wedge V, \wedge V')$.

Proof. If U is a generalized Fock implementer and if $v \in V$ then

$$a_g(v)U1 = Ua(v)1 = 0$$

so that $U1$ is a displaced Fock vacuum. Conversely, if $\Phi \in \wedge V'$ is a displaced Fock vacuum then the canonical anticommutation relations of (5.6) permit us to define a generalized Fock implementer $U \in \mathrm{Hom}(\wedge V, \wedge V')$ by $U1 = \Phi$ and the requirement that if $v_0, \cdots, v_n \in V$ then

$$U(v_0 \cdots v_n) = c_g(v_0) \cdots c_g(v_n)\Phi.$$

The bijective nature of the correspondence $\Phi \leftrightarrow U$ is plain. □

Our advertised necessary condition for generalized Fock implementability is rather easily established. We make use of the following observation: let $\Phi \in \wedge V'$ be nonzero and let $X \subset V$ be a complex subspace; if

$$x \in X \Rightarrow c(x)\Phi = 0$$

then X is finite-dimensional. For a contradiction, let X be infinite-dimensional: if the vectors $v_1,\cdots,v_n$ lie in V then we may choose a unit vector $x \in X$ orthogonal to each so that

$$\begin{aligned} 0 &= c(x)\,\Phi(xv_1\cdots v_n) \\ &= \Phi(a(x)\,(xv_1\cdots v_n)) \\ &= \Phi(v_1\cdots v_n) \end{aligned}$$

while of course it is also true that

$$0 = c(x)\,\Phi(x) = \Phi(a(x)x) = \Phi(1).$$

(5.9) Theorem. *If $g \in O(V)$ admits a generalized Fock implementer then* Ker C_g *is finite dimensional.*

Proof. Let $U: \wedge V \to \wedge V'$ be a generalized Fock implementer with $\Phi \doteq U1$ the corresponding displaced Fock vacuum. The note following (5.6) shows that if $x \in \mathrm{Ker} C_{g^{-1}}$ then

$$\begin{aligned} c(x)\,\Phi &= [c_g(C_{g^{-1}}x) + a_g(A_{g^{-1}}x)]\,U1 \\ &= Ua(A_{g^{-1}}x)\,1 = 0 \end{aligned}$$

whence the observation made prior to the proof reveals that Ker $C_{g^{-1}}$ is finite-dimensional. All that remains is to call upon (5.3). □

We approach the valid converse to this theorem by a combination of special cases.

(5.10) Theorem. *Each unitary $g \in O(V)$ has its functorial extension $\wedge g$: $\wedge V \to \wedge V$ as generalized Fock implementer.*

Proof. That $\wedge g$ is a generalized Fock implementer is readily verified: if $v \in V$ and $\phi \in \wedge V$ then

$$\begin{aligned} \wedge g\,(c(v)\,\phi) &= \wedge g\,(v\phi) = \wedge g\,(v) \wedge g\,(\phi) \\ &= gv \wedge g\,(\phi) = c(gv) \wedge g\,(\phi) \end{aligned}$$

whence if also $\psi \in \wedge V$ then

$$\begin{aligned} \langle \psi | \wedge (g)\, a(v)\,\phi \rangle &= \langle \wedge (g^{-1})\,\psi |\, a(v)\,\phi \rangle \\ &= \langle c(v) \wedge (g^{-1})\,\psi | \phi \rangle \\ &= \langle \wedge (g^{-1})\, c(gv)\,\psi | \phi \rangle \\ &= \langle c(gv)\,\psi | \wedge (g)\,\phi \rangle \\ &= \langle \psi | a(gv) \wedge (g)\,\phi \rangle. \end{aligned}$$

□

Note that in this case, the indicated implementer is actually a unitary operator on $\wedge V$ and so extends to $\wedge [V]$ as a unitary Fock implementer in the traditional sense.

(5.11) Theorem. *Let V be finite dimensional with $(v_1, \cdots, v_m)$ as unitary basis. Each antiunitary $g \in O(V)$ has a generalized Fock implementer U with kernel*

$$u = \prod_{k=1}^{m} \{\gamma^+(gv_k) - (-1)^m \gamma^-(v_k)\}.$$

Proof. We merely indicate how u may be determined; verification that u serves as the kernel of a generalized Fock implementer is routine. First, the displaced Fock vacuum $U1 \in \wedge V$ has the property that if $1 \leq k \leq m$ then $c(v_k)U1 = Ua(g^{-1}v_k)1 = 0$ and is therefore in $\wedge^m V$; by scaling, suppose $U1 = w_1 \cdots w_m$ where if $1 \leq k \leq m$ then $w_k = gv_k$. Now, if $C = (c_1, \cdots, c_t) \in \underline{m}$ then

$$\begin{aligned} Uv_C &= Uc(v_{c_1}) \cdots c(v_{c_t})1 \\ &= a(w_{c_1}) \cdots a(w_{c_t})(w_1 \cdots w_m) \end{aligned}$$

whence if $z^{\pm} = \gamma^{\pm}(z)$ when $z \in V$ then

$$\begin{aligned} \gamma^+(Uv_C)\gamma^-(v_C) &= [a(w^+_{c_1}) \cdots a(w^+_{c_t})(w^+_1 \cdots w^+_m)] v^-_{c_t} \cdots v^-_{c_1} \\ &= (-1)^{t(m-t)} v^-_{c_t} \cdots v^-_{c_1} [a(w^+_{c_1}) \cdots a(w^+_{c_t})(w^+_1 \cdots w^+_m)] \end{aligned}$$

since the bracketed term has degree $m-t$. Here, if $1 \leq i, j \leq m$ then the canonical anticommutation relations of (4.14) imply that $c(v^-_i)a(w^+_i)$ and $c(v^-_j)$ commute. Consequently,

$$\begin{aligned} \gamma^+(Uv_C)\gamma^-(v_C) &= (-1)^{mt+t} c(v^-_{c_1})a(w^+_{c_1}) \cdots c(v^-_{c_t})a(w^+_{c_t})(w^+_1 \cdots w^+_m) \\ &= z_1 \cdots z_m \end{aligned}$$

where if $1 \leq k \leq m$ then

$$z_k = \begin{cases} w^+_k & k \notin \{c_1, \cdots, c_t\} \\ (-1)^{m+1} v^-_k & k \in \{c_1, \cdots, c_t\}. \end{cases}$$

Substitution into the formula preceding (4.1) completes the argument. □

Note that in this case too, the indicated implementer is unitary. Note also that finite dimensionality of V is essential here: an orthogonal transformation is antiunitary precisely when its complex-linear part is identically zero.

(5.12) Theorem. *Each $g \in O(V)$ for which C_g is invertible has a generalized Fock implementer U whose kernel is the Gaussian*

$$u = \exp\left(\frac{1}{2}\gamma^- X \gamma^- - \frac{1}{2}\gamma^+ Y \gamma^+ + \gamma^+[\gamma^- S]\right)$$

where $S=C_g^{-1}$, $X=C_g^{-1}A_g$ *and* $Y=-A_gC_g^{-1}$.

Proof. The indicated kernel $u\in\wedge V'_{\mathbb{C}}$ certainly defines a linear operator $U\in \mathrm{Hom}(\wedge V, \wedge V')$ according to (4.8). What must be established is that if $v\in V$ then $Ua(v)=a_g(v)U$ and $Uc(v)=c_g(v)U$. Let us show in detail that if also ϕ, $\psi \in\wedge V$ then

$$(Ua(v)\phi)(\psi)=(a_g(v)U\phi)(\psi).$$

When $M\in\mathscr{F}(V)$ we shall indicate compression to M by a subscript, so that $S_M=P_MS|M$ and

$$\begin{aligned} u_M &= P_{M_{\mathbb{C}}}(u) \\ &= \exp\Big(\frac{1}{2}\gamma^- X_M\gamma^- - \frac{1}{2}\gamma^+ Y_M\gamma^+ + \gamma^+[\gamma^- S_M]\Big). \end{aligned}$$

Choose M to contain, v, C_gv and A_gv in such a way that ϕ, $\psi\in\wedge M$. By definition, if $u_M\in\wedge M_{\mathbb{C}}$ corresponds to $U_M\in\mathrm{End}\wedge M$ as in (4.1) then

$$(Ua(v)\phi)(\psi)=\langle\psi|U_M a(v)\phi\rangle$$

and

$$\begin{aligned} (a_g(v)U\phi)(\psi) &= U\phi(c_g(v)\psi) \\ &= \langle c_g(v)\psi|U_M\phi\rangle \\ &= \langle\psi|a_g(v)U_M\phi\rangle. \end{aligned}$$

Accordingly, it is enough to see that the endomorphisms $U_M a(v)$ and $a_g(v)U_M$ of $\wedge M$ coincide. For convenience, suppress the subscripts g and M: (4.3) and (4.6) along with (3.16) tell us that $U_M a(v)$ has kernel given at (α^+, β^-) by

$$\mathbb{E}_\gamma[u_M(\alpha^+, \gamma^-)e^{\gamma^+\beta^-}\beta^-(v)]=\beta^-(v)\exp\zeta$$

while $(c(Av)+a(Cv))U_M$ has kernel given at (α^+, β^-) by

$$\begin{aligned} &\mathbb{E}_\gamma[(\alpha^+(Av)+\gamma^-(Cv))e^{\alpha^+\gamma^-}u_M(\gamma^+, \beta^-)] \\ &\quad = (\alpha^+(Av)+\alpha^+(YCv)+\beta^-(SCv))\exp\zeta \end{aligned}$$

where

$$\zeta:=\frac{1}{2}\beta^- X\beta^- - \frac{1}{2}\alpha^+ Y\alpha^+ + \alpha^+[\beta^- S].$$

Reinstating the subscripts, it is now enough to see that $S_MC_gv=v$ and $Y_MC_gv=-A_gv$. These identities hold because M contains v, C_gv and A_gv: explicitly,

$$S_MC_gv=P_MSC_gv=P_MC_g^{-1}C_gv=v$$

and

$$Y_MC_gv=P_MYC_gv=-P_MA_gv=-A_gv.$$

In summary, we have established that $Ua(v) = a_g(v)\,U$ as required. The argument for creators is similar and will be omitted; we merely remark that it involves the identities $X=SA_g$ and $S^*=C_g+YA_g$. □

In all cases considered thus far, the generalized Fock implementers are unique up to scalar multiples; equivalently, the displaced Fock vacua are similarly unique by (5.8). Let $g\in O(V)$ have displaced Fock vacuum $\Phi\in\wedge V'$ with decomposition

$$\Phi=\sum_{n\geq 0}\Phi_n$$

into homogeneous components, where if $n\geq 0$ then $\Phi_n=P^n\Phi$ as in the discussion leading to (2.11); here, if $\psi\in\wedge V$ then $P^n\psi=0$ when n is large, whence the sum $\sum_{n\geq 0}\Phi_n(\psi)$ is actually finite. Taking homogeneous components in the displaced vacuum condition

$$v\in V\Rightarrow a_g(v)\,\Phi=0$$

reveals that if $v\in V$ then

$$n=0:\qquad a(C_g v)\,\Phi_1=0$$
$$n>0:\quad a(C_g v)\,\Phi_{n+1}+c(A_g v)\,\Phi_{n-1}=0.$$

Now let C_g be invertible as in (5.12): it follows that if $w\in V$ then

$$n=0:\qquad a(w)\,\Phi_1=0$$
$$n>0:\quad a(w)\,\Phi_{n+1}+c(A_g C_g^{-1}w)\,\Phi_{n-1}=0.$$

The $n=0$ equation forces Φ_1 to vanish while the $n=1$ equation forces the canonical correspondent of the quadratic $\Phi_2/\Phi_0\in\wedge V'$ to be the antiskew map $-A_g C_g^{-1}\colon V\to V$; the remaining equations then force Φ to be a scalar multiple of $\exp(\Phi_2/\Phi_0)$. The situation of (5.10) is a special case, while (5.11) may be handled similarly.

(5.13) Theorem. *Each $g\in O(V)$ for which* Ker C_g *is finite dimensional and the restriction $C_g\colon(\text{Ker } C_g)^\perp\to(\text{Ker } C_{g^{-1}})^\perp$ is invertible has a generalized Fock implementer.*

Proof. As following (5.4) we may factorize g as $g=hk$ with $C_h=C_{h^{-1}}$ self-adjoint and k unitary. The factorization $C_h=C_g k^{-1}$ implies that the restriction C_h: $(\text{Ker } C_h)^\perp\to(\text{Ker } C_h)^\perp$ is invertible, whence the orthogonal transformation $h\,|\,(\text{Ker } C_h)^\perp$ of $(\text{Ker } C_h)^\perp$ admits a generalized Fock implementer by (5.12). The antiunitary transformation $h\,|\text{Ker } C_h$ of Ker C_h admits a generalized Fock implementer by (5.11). It follows that $h\in O(V)$ admits a generalized Fock implementer $U_h: \wedge V \to \wedge V'$: as corresponding displaced Fock vacuum, take the obvious product of the respective vacua for $h|(\mathrm{Ker}\, C_h)^{\perp}$ and $h|\mathrm{Ker}\, C_h$. The unitary transformation k of V admits $U_k = \wedge k$: $\wedge V \to \wedge V$ as generalized Fock implementer by (5.10). Finally, $U_h \circ U_k$ is plainly a generalized Fock implementer for g. □

We remark that these slightly elaborate hypotheses on $g \in O(V)$ are automatically satisfied under the more convenient assumption that its antilinear part A_g be compact. Of course, the antiunitary nature of A_g: Ker $C_g \to$ Ker $C_{g^{-1}}$ in (5.3) forces finite dimensionality upon Ker C_g. The restriction C_g: $(\mathrm{Ker}\, C_g)^{\perp} \to (\mathrm{Ker}\, C_{g^{-1}})^{\perp}$ is certainly injective with dense range, so the open mapping theorem completes the proof once it is seen that $C_g|(\mathrm{Ker}\, C_g)^{\perp}$ is bounded below. For a contradiction, let $(v_n: n \geq 0)$ be a sequence of unit vectors in $(\mathrm{Ker}\, C_g)^{\perp}$ such that $C_g v_n \to 0$. Passing to a subsequence, we may suppose that $(A_g v_n: n \geq 0)$ converges, say to $w \in V$. The remark after (5.2) yields

$$1 = \|v_n\|^2 = \|C_g v_n\|^2 + \|A_g v_n\|^2$$

whence passage to the limit reveals w as a unit vector. On the one hand (5.2) implies that

$$C_{g^{-1}} w = \lim C_{g^{-1}} A_g v_n = -\lim A_{g^{-1}} C_g v_n = 0$$

whence $w \in \mathrm{Ker}\, C_{g^{-1}}$. On the other hand, (5.1) and (5.2) imply that

$$A_g \cdot (\mathrm{Ker}\, C_g)^{\perp} = A_g \cdot \overline{\mathrm{Ran}\, C_{g^{-1}}} \subset \overline{\mathrm{Ran} C_g} = (\mathrm{Ker}\, C_{g^{-1}})^{\perp}$$

whence $w \in (\mathrm{Ker}\, C_{g^{-1}})^{\perp}$. The unit vector w therefore vanishes, a contradiction.

We are now prepared to consider the converse of (5.9) in general. Suppose first that $h \in O(V)$ has complex-linear part C_h that is both self-adjoint and injective, so that C_h has dense range since $\overline{\mathrm{Ran}\, C_h} = (\mathrm{Ker}\, C_h^*)^{\perp}$ always. Let $W \subset V$ be any complex subspace with the property that $V = W \oplus \mathrm{Ran}\, C_h$. The linear map Z: $V \to V'$ defined by the rule that if $v, v_0 \in V$ and $w, w_0 \in W$ then

$$Z(w_0 + C_h v_0)(w + C_h v) = \langle C_h v_0 | A_h v \rangle + \langle w_0 | A_h v \rangle - \langle w | A_h v_0 \rangle$$

is readily verified to be antiskew and to satisfy the condition

$$v \in V \Rightarrow Z C_h v = -A_h v.$$

Accordingly, the corresponding quadratic $\zeta \in \wedge V'$ has exponential $\Phi = \exp \zeta$ satisfying the displaced Fock vacuum condition

$$v \in V \Rightarrow a_g(v)\, \Phi = 0.$$

Thus the Gaussian Φ is a displaced Fock vacuum for h and so h admits a generalized Fock implementer by (5.8).

(5.14) Theorem. *Each* $g \in O(V)$ *for which* $\operatorname{Ker} C_g$ *is finite dimensional admits a generalized Fock implementer.*

Proof. Essentially as for (5.13): having factorized g as the product hk with C_h self-adjoint and k unitary, the only modification required is to observe that the orthogonal transformation $h|(\operatorname{Ker} C_h)^\perp$ of $(\operatorname{Ker} C_h)^\perp$ admits a generalized Fock implementer as in the discussion prior to the statement of the theorem. □

It is interesting to note that we encounter a new phenomenon here. In the more restricted context of (5.13) the generalized Fock implementers of an orthogonal transformation are unique up to scalar multiples; see the remarks immediately preceding (5.13). In the present quite general context, such need no longer be the case.

(5.15) Theorem. *Each* $g \in O(V)$ *for which* $\operatorname{Ker} C_g$ *is finite dimensional and* $C_g: (\operatorname{Ker} C_g)^\perp \to (\operatorname{Ker} C_{g^{-1}})^\perp$ *is not invertible admits independent generalized Fock implementers.*

Proof. After factorization and reduction, we may suppose that $h=g$ has the property that C_h is self-adjoint and injective but not surjective. Choose any complement W to $\operatorname{Ran} C_h$ in V and define an antiskew map $Z: V \to V'$ by declaring that if $v, v_0 \in V$ and $w, w_0 \in W$ then

$$Z(w_0+C_h v_0)(w+C_h v) = z(w_0)(w) + \langle C_h v_0 | A_h v\rangle + \langle w_0 | A_h v\rangle - \langle w | A_h v_0\rangle$$

where $z: W \to W'$ is any antiskew map. It is readily verified that the quadratic $\zeta \in \wedge V'$ corresponding to Z exponentiates to a (Gaussian) displaced Fock vacuum for h. □

It is perhaps worth pointing out that this theorem is not vacuous. Let V be the Hilbert space direct sum

$$V = \bigoplus_{n \geq 0} V_n$$

where if $n \geq 0$ then V_n has a preferred quaternionic structure K so that $K \in O(V_n)$ is antiunitary and $K^2 = -I$. Let

$$h = \bigoplus_{n \geq 0} h_n$$

where if $n \geq 0$ then $h_n = c_n I + s_n K$ with $c_n = \cos(\theta_n)$ and $s_n = \sin(\theta_n)$ for some $\theta_n \in \mathbb{R}$. Plainly, $h \in O(V)$ and $C_h = \bigoplus_{n \geq 0} c_n I$ is self-adjoint. If each c_n is nonzero then of course C_h is injective; if also $c_n \to 0$ then C_h is not surjective, for its spectrum contains 0.

A discussion of unitary Fock implementation in the traditional sense is appropriate. As recalled earlier, $g \in O(V)$ admits a unitary Fock implementer if and only if A_g is of Hilbert-Schmidt class. We are content to discuss here only the harder direction of this equivalence.

(5.16) Theorem. *Each $g \in O(V)$ for which A_g is Hilbert-Schmidt admits a unitary Fock implementer.*

Proof. As A_g is compact, so (5.13) and the subsequent remark furnish a generalized Fock implementer unique up to scalar multiples. The proof of (5.13) makes it plain that by factorization and reduction we may suppose C_g to be invertible. Direct computation shows that the generalized Fock implementer U of (5.12) has as displaced Fock vacuum the Gaussian $U1 = \exp \eta$ where the quadratic $\eta \in \wedge^2 [V]$ corresponds canonically to the Hilbert-Schmidt antiskew map $-A_g C_g^{-1} \in A^2[V]$ so that (3.19) places $U1$ in $\wedge [V]$. Normalize U so that $\Phi = U1$ is a unit vector: recall from (5.8) that if $v_1, \cdots, v_n \in V$ then

$$U(v_1 \cdots v_n) = c_g(v_1) \cdots c_g(v_n) \Phi$$

and note from (4.15) that U actually maps $\wedge V$ to $\wedge [V]$. The canonical anticommutation relations of (5.6) show that if $x_1, \cdots, x_r, y_1, \cdots, y_s \in V$ then

$$\begin{aligned} &\langle U(x_1 \cdots x_r) | U(y_1 \cdots y_s) \rangle \\ &= \langle c_g(x_1) \cdots c_g(x_r) \Phi | c_g(y_1) \cdots c_g(y_s) \Phi \rangle \\ &= \langle \Phi | a_g(x_r) \cdots a_g(x_1) c_g(y_1) \cdots c_g(y_s) \Phi \rangle \\ &= \langle x_1 \cdots x_r | y_1 \cdots y_s \rangle \end{aligned}$$

since Φ is a unit displaced Fock vacuum. Thus $U: \wedge V \to \wedge [V]$ is an isometry and so it extends to an isometric Fock implementer $U: \wedge [V] \to \wedge [V]$. The (Hilbert space) adjoint $U^*: \wedge [V] \to \wedge [V]$ implements g^{-1} so that UU^* is a scalar operator by irreducibility of the Fock representation as after (5.5). Now $U^* \Phi = U^* U1 = 1$ as U is isometric, whence $UU^* \Phi = U1 = \Phi$; thus $UU^* = I$ and so U is unitary. □

As a matter of definition, the restricted orthogonal group $O_{\text{res}}(V)$ comprises precisely all $g \in O(V)$ for which A_g (equivalently, the commutator $[g, J] = gJ - Jg$) is Hilbert-Schmidt. Thus, unitary Fock implementers constitute a central circle extension of the restricted orthogonal group. We remark that placing the operator topology on complex-linear parts and the Hilbert-Schmidt topology on antilinear parts makes $O_{\text{res}}(V)$ a topological group; its identity component $SO_{\text{res}}(V)$ comprises precisely all $g \in O_{\text{res}}(V)$ for which C_g has even-dimensional kernel.

Remarks

We close by offering a variety of remarks, many related to alternative approaches and future directions; for convenience they are grouped by section, though there is some overlap.

Finite dimensions

Our account of the Berezin calculus is based upon the complexification of an m-dimensional complex Hilbert space V. This context is natural, because the complexification $V_{\mathbb{C}}$ admits a canonical quadratic γ_V from which to fashion a canonical volume form $\omega_V = (-\gamma_V)^m/m! \in \wedge^{2m} V_{\mathbb{C}}$ for the calculus. More generally, we may erect a Berezin calculus on V itself: for example, we may define a Berezin integral on $\wedge V$ by evaluating the inner product against a unit vector in $\wedge^m V$ to mimic (1.8). The subsequent development may be left to the imagination.

Certain natural Berezinian counterparts to results in the standard integral calculus have been omitted from our account. Thus, a version of the rule for integration by parts is valid. When $v \in V_{\mathbb{C}}$ it is convenient to write $\tilde{a}(v) = \Gamma a(v) = -a(v)\Gamma$ where Γ is the parity automorphism as usual. With this notation, the rule asserts that if also $\phi, \psi \in \wedge V_{\mathbb{C}}$ then

$$\mathbb{I}_V[(\tilde{a}(v)\phi)\psi] = \mathbb{I}_V[\phi(a(v)\psi)].$$

In fact, as $\omega_V \in \wedge^{2m} V_{\mathbb{C}}$ so $v\omega_V = 0$ and therefore

$$\begin{aligned} 0 &= \langle v\omega_V | (\Gamma\phi)\psi\rangle = \langle \omega_V | a(v)((\Gamma\phi)\psi)\rangle \\ &= \mathbb{I}_V[(a(v)\Gamma\phi)\psi + \phi(a(v)\psi)] \end{aligned}$$

whence rearrangement yields the rule. Also, our change of variables formula (1.2) applies only to linear transformations; nonlinear transformations have their own change of variables formulae involving Jacobians.

Incidentally, it is perhaps worth recording the counterpart to (1.8) for the Berezin expectation: if $\phi \in \wedge V_{\mathbb{C}}$ then

$$\mathbb{E}_V[\phi] = \langle e^{-\gamma_V} | \phi\rangle.$$

Infinite dimensions

It is worth remarking that if V is infinite dimensional then the special Gaussian $e^{-\gamma_V} \in \wedge V'_{\mathbb{C}}$ introduced for (2.12) does not lie in $\wedge[V_{\mathbb{C}}]$. In fact, the quadratic γ_V corresponds to the antiskew map $Z_V : V_{\mathbb{C}} \to V_{\mathbb{C}}$ defined by $Z_V(\gamma^+) = \gamma^-$ and $Z_V(\gamma^-) = -\gamma^+$ so that $Z_V^2 = -I$. Now, if $e^{-\gamma_V} \in \wedge[V_{\mathbb{C}}]$ then from the remark after (3.19) it follows that Z_V is Hilbert-Schmidt which forces V to be finite dimensional. We can be a little more precise than this: (3.4) implies that if $M \in \mathscr{F}(V)$ has complex dimension m then $\| e^{-\gamma_M} \| = \sqrt{2}^m$.

Our account of the Berezin calculus in infinite dimensions has not mentioned the notions of Berezin partial integral and Berezin conditional expectation: on the one hand, we have not required the development of such notions; on the other hand, such notions are a little complicated in their details. To indicate what is involved, let $V=X\oplus Y$ be an orthogonal decomposition into closed complex subspaces. Notice that an arbitrary finite dimensional subspace of V need not decompose as the direct sum of subspaces in X and Y. Accordingly, it is natural to replace $\mathscr{F}(V)$ by its subset $\mathscr{F}(X, Y)$ comprising all $L=M\oplus N$ with $M\in\mathscr{F}(X)$ and $N\in\mathscr{F}(Y)$. In special cases, it might be appropriate to consider an alternative replacement: for example, we may replace $\mathscr{F}(V_{\mathbb{C}})$ by its subset comprising all $M_{\mathbb{C}}$ as M runs through $\mathscr{F}(V)$. With this understanding, several versions of the Berezin conditional expectation may be considered: we may consider the space comprising those $\Phi\in\wedge V'_{\mathbb{C}}$ for which the net $(\mathbb{E}_M[\Phi_{L_{\mathbb{C}}}]: L=M\oplus N\in\mathscr{F}(X, Y))$ converges in $\wedge Y'_{\mathbb{C}}$ relative to its weak topology; more specially, we may consider the space comprising those $\Phi\in\wedge V'_{\mathbb{C}}$ for which the same net converges in $\wedge[V_{\mathbb{C}}]$. For example, if $\xi\in\mathscr{E}(X)$ and $\Phi=\xi\eta$ then the indicated net converges to $\mathbb{E}_X[\xi]\eta$ in $\wedge Y'_{\mathbb{C}}$ when $\eta\in\wedge Y'_{\mathbb{C}}$ and to $\mathbb{E}_X[\xi]\eta$ in $\wedge[Y_{\mathbb{C}}]$ when $\eta\in\wedge[Y_{\mathbb{C}}]$. Moreover, if $\xi\in\wedge X'_{\mathbb{C}}$ and $\eta\in\wedge Y'_{\mathbb{C}}$ are such that the indicated net converges in either sense, then $\xi\in\mathscr{E}(X)$ and convergence is to $\mathbb{E}_X[\xi]\eta$.

Gaussian integrals

The basic Gaussian expectation (3.4) was calculated by diagonalization. Instead, it may be calculated by the application of (1.13) and (1.3) to (1.2). We omit the details of the argument, mentioning only the following points: when $W\in A^2V$ we consider the functorial extension to $\wedge V_{\mathbb{C}}$ of $T\in\mathrm{End}\ V_{\mathbb{C}}$ defined by $T(\gamma^+)=\gamma^+-\gamma^-W$ and $T(\gamma^-)=\gamma^--\gamma^+W$; from $T\sigma=\sigma T$ there follows the equality $\mathrm{Det}\ T=\mathrm{Det}(I-W^2)$ between complex determinants.

When V is finite dimensional, it is possible to consider still more general Gaussian expectations than those considered here. For example, we may consider $\mathbb{E}_V[\exp\ \zeta]$ where $\zeta\in A^2(V_{\mathbb{C}})$ is arbitrary; equivalently, we may consider

$$\mathbb{E}_V\Big[\exp\Big(\frac{1}{2}\gamma^-X\gamma^--\frac{1}{2}\gamma^+Y\gamma^++[\gamma^+S]\gamma^-\Big)\Big]$$

when, $X, Y\in A^2V$ are such that $I-YX\in\mathrm{End}\ V$ is a singular operator. Further cases. For another example, we may consider

$$\mathbb{E}_\gamma\Big[\exp\Big(\frac{1}{2}\gamma^-X\gamma^--\frac{1}{2}\gamma^+Y\gamma^++\alpha^+\gamma^-+\gamma^+\beta^-\Big)\Big]$$

when, $X, Y\in A^2V$ are such that $I-YX\in\mathrm{End}\ V$ is a singular operator. Further examples may be derived by repeated differentiation under the expectation as for (3.16).

We remark that formulae of a different nature may be derived from those of Gaussian type. For instance, if $\xi, \eta \in \wedge^2 V$ correspond to X, $Y \in A^2 V$ respectively then

$$\langle \xi | \eta \rangle = -\frac{1}{2}\operatorname{Tr}(YX).$$

Explicitly, if s, $t \in \mathbb{R}$ then

$$\langle e^{s\xi} | e^{t\eta} \rangle^2 = \operatorname{Det}(I - tYsX)$$

from which the asserted formula follows upon differentiation by s and t at zero. Alternatively, the asserted formula may be established by diagonalization and analytic continuation.

Integral kernels

The isomorphism in (4.1) is actually isometric when End $\wedge V$ has the Hilbert-Schmidt norm. In fact, if $U \in \operatorname{End} \wedge V$ has kernel $u \in \wedge V_{\mathbb{C}}$ and if $(v_C : C \in \underline{m})$ is a standard unitary basis for $\wedge V$ then the formula displayed prior to (4.1) yields

$$u = \sum_{A,B} \langle v_A | U v_B \rangle \gamma^+(v_A) \gamma^-(v_B)$$

so that

$$\|u\|^2 = \sum_{A,B} |\langle v_A | U v_B \rangle|^2 = \sum_C \|U v_C\|^2 = \|U\|_{HS}^2.$$

Certain natural questions present themselves in connexion with the kernel theorem (4.8). Thus, we may ask how various properties of an operator $U \in \operatorname{Hom}(\wedge V, \wedge V')$ are reflected in its kernel $u \in \wedge V'_{\mathbb{C}}$. One result in this direction is (4.10): if $M \in \mathscr{F}(V)$ then U maps to $\wedge M$ and vanishes on $(\wedge M)^{\perp}$ precisely when $u \in \wedge M_{\mathbb{C}}$. It is not difficult to establish that U maps to $\wedge [V]$ and extends to a Hilbert-Schmidt operator $\wedge [V] \to \wedge [V]$ precisely when $u \in \wedge [V_{\mathbb{C}}]$: indeed, if $M \in \mathscr{F}(V)$ then $U_M = P_M U | \wedge M$ and $u_M = P_{M_{\mathbb{C}}}(u)$ satisfy $\|U_M\|_{HS} = \|u_M\|$ as noted above; now pass to the supremum as M runs over $\mathscr{F}(V)$. In a similar vein, (4.4) suggests a relationship between trace class operators on $\wedge [V]$ and kernels in $\mathscr{E}(V)$.

The kernel theorem admits a reformulation in terms of sesquilinear forms on $\wedge V$ rather than linear operators $\wedge V \to \wedge V'$. Thus, a canonical isomorphism between $\operatorname{Hom}(\wedge V, \wedge V')$ and the space of sesquilinear forms on $\wedge V$ is set up by the rule $U \leftrightarrow \widehat{u}$ according to which

$$\phi, \psi \in \wedge V \Rightarrow U\phi(\psi) = \widehat{u}(\psi, \phi).$$

In these terms, (4.8) asserts the existence of a canonical isomorphism between

$\wedge V'_{\mathbb{C}}$ and the space of sesquilinear forms on $\wedge V$ in which $u \leftrightarrow \widehat{u}$ when

$$\phi, \psi \in \wedge V \Rightarrow \widehat{u}(\psi, \phi) = u(\psi^+ \phi^-)$$

where the application of $\gamma^{\pm}$ is indicated by a superscript $\pm$ for convenience.

For example, if $g \in O(V)$ then its generalized Fock implementers $U : \wedge V \to \wedge V'$ correspond to sesquilinear forms $\widehat{u}$ on $\wedge V$ having the property that if $v \in V$ and if $\phi, \psi \in \wedge V$ then

$$\widehat{u}(\psi, \pi(v)\phi) = \widehat{u}(\pi(gv)\psi, \phi).$$

Spin transformations

The triple $\wedge V \subset \wedge [V] \subset \wedge V'$ actually carries another canonical Fock representation $\widetilde{\pi}$. Let $v \in V$: the (right) creator $\widetilde{c}(v) : \wedge V \to \wedge V : \phi \mapsto \phi v$ is right multiplication by v; the (right) annihilator $\widetilde{a}(v) \in \operatorname{End} \wedge V$ is determined by $\widetilde{a}(v)1 = 0$ together with the requirements that if $w \in V$ then $a(v)w = \langle v|w\rangle$ and if $\phi, \psi \in \wedge V$ then

$$\widetilde{a}(v)(\phi\psi) = \phi(\widetilde{a}(v)\psi) + (\widetilde{a}(v)\phi)\Gamma\psi.$$

In fact, it is readily verified that $\widetilde{c}(v) = c(v)\Gamma$ and $\widetilde{a}(v) = \Gamma a(v) = -a(v)\Gamma$. As for the original creators and annihilators, these right versions extend not only to $\wedge [V]$ but also to $\wedge V'$ and satisfy the canonical anticommutation relations: if $x, y \in V$ then

$$\{\widetilde{a}(x), \widetilde{a}(y)\} = 0$$
$$\{\widetilde{a}(x), \widetilde{c}(y)\} = \langle x|y\rangle I$$
$$\{\widetilde{c}(x), \widetilde{c}(y)\} = 0.$$

It may be checked that if $x, y \in V$ then $c(x)$ commutes with $\widetilde{c}(y)$ and $a(x)$ commutes with $\widetilde{a}(y)$ while

$$[a(x), \widetilde{c}(y)] = \langle x|y\rangle \Gamma.$$

In terms of these operators, the alternative Fock representation $\widetilde{\pi}$ of V is defined by the prescription

$$v \in V \Rightarrow \widetilde{\pi}(v) = \widetilde{c}(v) + \widetilde{a}(v).$$

Our approach to generalized Fock implementers was essentially just to exhibit them. It is perhaps more instructive to derive equations for their corresponding kernels; such may be formulated in terms of creators and annihilators on left and right. In fact, let $g \in O(V)$ have generalized Fock implementer $U \in \operatorname{Hom}(\wedge V, \wedge V')$ with kernel $u \in \wedge V'_{\mathbb{C}}$. It transpires that if $v \in V$ then $Ua(v) = a_g(v)U$ corresponds to

$$\widetilde{a}(\gamma^- v)u = a(\gamma^+ A_g v)u + c(\gamma^+ C_g v)u$$

while $Uc(v)=c_g(v)U$ corresponds to

$$\tilde{c}(\gamma^- v)u=c(\gamma^+ A_g v)u+a(\gamma^+ C_g v)u.$$

These may be combined into the single condition

$$v\in V\Rightarrow\tilde{\pi}(\gamma^- v)u=\pi(\gamma^+ gv)u$$

which expresses the kernel u corresponding to the generalized Fock implementer U for g as a pure spinor.

For example, suppose we seek a generalized Fock implementer for g with kernel in the standard Gaussian form

$$u=\exp\left(\frac{1}{2}\gamma^- X\gamma^- -\frac{1}{2}\gamma^+ Y\gamma^+ +\gamma^+[\gamma^- S]\right).$$

In this case, the $\tilde{a}$ equations force

$$SC_g=I,\quad YC_g+A_g=0$$

while the $\tilde{c}$ equations force

$$SA_g=X,\quad YA_g+C_g=S^*.$$

In particular, if C_g is invertible then $S=C_g^{-1}$, $X=C_g^{-1}A_g$ and $Y=-A_gC_g^{-1}$ as in (5.12).

It is of interest to note that if V is infinite dimensional then the elements of $O(V)$ admitting generalized Fock implementers do not constitute a subgroup. That this might perhaps be the case is suggested by the general lack of a composition in $\mathrm{Hom}(\wedge V, \wedge V')$; that it is actually so may be seen by the following simple example. Let K be a quaternionic structure on V and note by direct calculation that the map

$$\mathbb{R}\to O(V):\ \theta\mapsto g_\theta=(\cos\theta)I+(\sin\theta)K$$

is a group homomorphism. If 2θ is not an odd multiple of π then g_θ admits generalized Fock implementers: indeed, $C_{g_\theta}=(\cos\theta)I$ is invertible and (5.12) is applicable. If 2θ is an odd multiple of π then $C_{g_\theta}=0$ so that Ker $C_{g_\theta}=V$ and (5.9) precludes the existence of generalized Fock implementers.

Coherent states

Thus far, coherent states have been conspicuous by their absence; here we partially repair this omission in finite dimensions.

To begin, on $\wedge(V_\alpha\oplus V_\gamma\oplus V_\beta)$ we define a $\wedge(V_\alpha\oplus V_\beta)$-valued inner product $\langle\cdot|\cdot\rangle$ by the rule that if $\Phi,\ \Psi\in\wedge(V_\alpha\oplus V_\gamma\oplus V_\beta)$ then

$$\langle\Phi|\Psi\rangle=\mathbb{E}_\gamma[\Phi^*\Psi].$$

Straightforward arguments establish that if Φ, $\Psi \in \wedge(V_\alpha \oplus V_\gamma \oplus V_\beta)$ then

$$\langle \Phi | \Psi \rangle^* = \langle \Psi | \Phi \rangle$$

and that if also $\Theta \in \wedge(V_\alpha \oplus V_\beta)$ then

$$\langle \Phi | \Psi \Theta \rangle = \langle \Phi | \Psi \rangle \Theta.$$

Note in particular that (1.7) yields

$$\phi, \psi \in \wedge V \Rightarrow (\gamma^+ \phi | \gamma^+ \psi) = \langle \phi | \psi \rangle.$$

Now, we claim that the vectors $e^{\gamma^+ \alpha^-} \in \wedge(V_\alpha \oplus V_\gamma)$ and $e^{\gamma^+ \beta^-} \in \wedge(V_\gamma \oplus V_\beta)$ play the rôle of coherent states.

First of all, from (4.5) it follows that if $\phi \in \wedge V$ then

$$\langle e^{\gamma^+ \alpha^-} | \gamma^+ \phi \rangle = \mathbb{E}_\gamma [e^{\alpha^+ \gamma^-} \gamma^+ \phi] = \alpha^+(\phi).$$

The vector $e^{\gamma^+ \alpha^-}$ is an "eigenvector" for annihilators, but with coefficients in the algebra $\wedge(V_\alpha \oplus V_\gamma)$: the remark after (3.15) shows that if $x, y \in V$ then

$$a(\gamma^+ x + \alpha^- y) e^{\gamma^+ \alpha^-} = (\alpha^- x - \gamma^+ y) e^{\gamma^+ \alpha^-}.$$

A resolution of the identity for coherent states follows from (1.7): if ϕ, $\psi \in \wedge V$ then

$$\begin{aligned}\langle \phi | \psi \rangle &= \mathbb{E}_\alpha [\alpha^+(\phi)^* \alpha^+(\psi)] \\ &= \mathbb{E}_\alpha [\langle \gamma^+ \phi | e^{\gamma^+ \alpha^-} \rangle \langle e^{\gamma^+ \alpha^-} | \gamma^+ \psi \rangle] \\ &= \int \langle \gamma^+ \phi | e^{\gamma^+ \alpha^-} \rangle \langle e^{\gamma^+ \alpha^-} | \gamma^+ \psi \rangle e^{-\alpha^+ \alpha^-} d\alpha^+ d\alpha^-.\end{aligned}$$

A coherent state formula for the integral kernel $u \in \wedge V_{\mathbb{C}}$ of a complex-linear map $U: \wedge V \to \wedge V$ may be derived as follows. Let $\gamma^+(U)$ be the induced endomorphism of $\wedge(V_\alpha \oplus V_\gamma \oplus V_\beta)$ acting canonically on $\wedge(V_\gamma^+)$. Let $(v_1, \cdots, v_m)$ be a unitary basis for V and $(v_C: C \in \underline{m})$ the corresponding unitary basis for $\wedge V$. The formula displayed prior to (4.1) tells us that $u(\alpha^+, \beta^-) \in \wedge(V_\alpha^+ \oplus V_{\bar{\beta}})$ is given by

$$\begin{aligned}u(\alpha^+, \beta^-) &= \sum_C \alpha^+(Uv_C) \beta^-(v_C) \\ &= \sum_C \langle e^{\gamma^+ \alpha^-} | \gamma^+(Uv_C) \rangle \beta^-(v_C) \\ &= \langle e^{\gamma^+ \alpha^-} | \sum_C \gamma^+(Uv_C) \beta^-(v_C) \rangle \\ &= \langle e^{\gamma^+ \alpha^-} | \gamma^+(U) \sum_C \gamma^+(v_C) \beta^-(v_C) \rangle\end{aligned}$$

or

$$u(\alpha^+, \beta^-) = \langle e^{\gamma^+\alpha^-} | \gamma^+(U) e^{\gamma^+\beta^-} \rangle.$$

In particular, we remark that the overlap between the coherent states $e^{\gamma^+\alpha^-}$ and $e^{\gamma^+\alpha^-}$ is given by

$$\langle e^{\gamma^+\alpha^-} | e^{\gamma^+\beta^-} \rangle = e^{\alpha^+\beta^-}.$$

Further information on fermionic coherent states, especially as they relate to fermionic path integrals, may be found in [8].

Number operator

Along with creators and annihilators, the number operator in fermionic Fock space is of fundamental importance. Define a linear operator $\mathcal{N} : \wedge V \to \wedge V$ by the rule

$$\mathcal{N} = \sum_{n \geq 0} n P^n$$

so that $\mathcal{N}$ multiplies each homogeneous summand by its degree. Extend $\mathcal{N}$ to the antidual $\wedge V'$ by antiduality, so that if $\Phi \in \wedge V'$ and $\phi \in \wedge V$ then

$$[\mathcal{N}\Phi](\phi) = \Phi(\mathcal{N}\phi).$$

The number operator in fermionic Fock space $\wedge [V]$ is the restriction of this $\mathcal{N}$ to the natural domain

$$\mathbb{D}[\mathcal{N}] = \{\Phi \in \wedge [V] : \mathcal{N}\Phi \in \wedge [V]\}.$$

It is readily verified that the number operator $\mathcal{N}$ in $\wedge [V]$ is selfadjoint and has $(\wedge^n [V] : n \in \mathbb{N})$ as spectral subspaces.

In order to express the number operator in terms of creators and annihilators, we proceed as follows. Let $M \in \mathcal{F}(V)$ have unitary basis $(v_1, \cdots, v_n)$ and independently define $\mathcal{N}_M$ (in $\mathrm{End} \wedge V$ or $\mathrm{End} \wedge V'$ and as a positive bounded linear operator on $\wedge [V]$) by

$$\mathcal{N}_M = \sum_{k=1}^{m} c(v_k) a(v_k)$$

so that if $\Phi \in \wedge [V]$ then

$$\langle \Phi | \mathcal{N}_M \Phi \rangle = \sum_{k=1}^{m} \| a(v_k) \Phi \|^2.$$

Thus, if $\Phi \in \wedge [V]$ then the real net $(\langle \Phi | \mathcal{N}_M \Phi \rangle : M \in \mathcal{F}(V))$ is increasing;

without much difficulty, it may be shown that if $\Phi \in \mathbb{D}[\mathcal{N}]$ then

$$\langle \Phi | \mathcal{N} \Phi \rangle = \sup_M \langle \Phi | \mathcal{N}_M \Phi \rangle = \lim_M \langle \Phi | \mathcal{N}_M \Phi \rangle .$$

As an application, let us calculate the expected value of the number operator in the state $\Phi = U1$ corresponding to the Fock vacuum displaced by a unitary Fock implementer U for the restricted orthogonal transformation $g \in O_{\text{res}}(V)$. Note first that if $v \in V$ is a unit vector then as $a(v)U = Ua_{g^{-1}}(v)$ and $c(v)U = Uc_{g^{-1}}(v)$ so

$$\begin{aligned}\langle \Phi | c(v)a(v)\Phi \rangle &= \langle U1 | Uc_{g^{-1}}(v)a_{g^{-1}}(v)1 \rangle \\ &= \langle 1 | c_{g^{-1}}(v)a_{g^{-1}}(v)1 \rangle \\ &= \langle a_{g^{-1}}(v)1 | a_{g^{-1}}(v)1 \rangle \\ &= \langle A_{g^{-1}}v | A_{g^{-1}}v \rangle \\ &= \langle v | A_g A_{g^{-1}} v \rangle .\end{aligned}$$

Summation as v runs over a unitary basis for $M \in \mathscr{F}(V)$ now yields

$$\langle \Phi | \mathcal{N}_M \Phi \rangle = \operatorname{Tr}(A_g A_{g^{-1}})_M$$

where the subscript M indicates compression to M. Lastly, passage to the supremum as M runs over $\mathscr{F}(V)$ establishes the formula

$$\langle \Phi | \mathcal{N} \Phi \rangle = \operatorname{Tr}(A_g A_{g^{-1}}).$$

Of course, this result lends further meaning to the Hilbert-Schmidt condition on A_g for $g \in O_{\text{res}}(V)$: the expected number of particles in the corresponding displaced vacuum state is finite.

Spin cocycle

As was discussed earlier, the Fock representation of V on $\wedge[V]$ engenders a central circle extension of the restricted orthogonal group $O_{\text{res}}(V)$. Equivalently, it engenders a projective unitary representation of $O_{\text{res}}(V)$: if $g, h \in O_{\text{res}}(V)$ then

$$U(g)U(h) = \Delta(g, h)U(gh)$$

where U associates to each restricted orthogonal transformation a unitary Fock implementer and where $\Delta: O_{\text{res}}(V) \times O_{\text{res}}(V) \to \mathbb{T}$ is the corresponding cocycle. By defining U appropriately it is possible to make the spin cocycle Δ quite explicit, at least over a substantial portion of $O_{\text{res}}(V)$.

To be precise, let $\mathcal{O}_{\text{res}}(V)$ denote the set comprising all $g \in O_{\text{res}}(V)$ for which C_g is invertible. Recall that $O_{\text{res}}(V)$ has the topology determined by operator norm on complex-linear parts and Hilbert-Schmidt norm on antilinear parts; its identity component $SO_{\text{res}}(V)$ comprises those $g \in O_{\text{res}}(V)$ for which $\ker C_g$ is even-dimensional. Of course, $\mathcal{O}_{\text{res}}(V)$ is an open subset of $O_{\text{res}}(V)$; in fact, $\mathscr{O}_{\text{res}}(V)$ is dense in $SO_{\text{res}}(V)$. To see this last point, note by (5.4) that we may assume C_g self-adjoint, in which case g, C_g and A_g all preserve both Ker C_g and its orthocomplement, so we may restrict attention to Ker C_g where A_g is antiunitary by (5.3). Accordingly, let $W=\text{Ker}\, C_g$ be of even complex dimension and let $A \in O(W)$ be antiunitary: if $K \in O(W)$ is a quaternionic (that is, antilinear complex) structure then

$$0 \leq t \leq 1 \Rightarrow g_t = \sqrt{1-t^2}AK + tA$$

defines a continuous curve in $SO(W)$ such that C_{g_t} is invertible when $0 \leq t < 1$ and $g_1 = A$.

Now, if $g \in \mathscr{O}_{\text{res}}(V)$ then the proof of (5.16) shows that its Fock implementers send the Fock vacuum 1 to scalar multiples of the Gaussian $\exp(\zeta_g)$ where $\zeta_g \in \wedge^2[V]$ corresponds to $Z_g = -A_g C_g^{-1} \in A^2[V]$. Choose the implementer $U_g : \wedge V \to \wedge V'$ for which $U_g 1 = \exp(\zeta_g)$ so that $\langle 1 | U_g 1 \rangle = 1$ and write $\overline{U}_g : \wedge[V] \to \wedge[V]$ for its continuous extension. Of course, $\|U_g 1\|^2 = \text{Det}^{\frac{1}{2}}(I - Z_g^2)$ so that $\text{Det}^{\frac{1}{4}}(I - Z_g^2)^{-1}\overline{U}_g$ is unitary.

With this normalization, the spin cocycle may be computed over $\mathscr{O}_{\text{res}}(V)$ as follows: if $g, h \in \mathscr{O}_{\text{res}}(V)$ then

$$\overline{U}_g \overline{U}_h = \delta(g, h)\, \overline{U}_{gh}$$

where

$$\delta(g, h) = \text{Det}^{\frac{1}{2}}(I - Z_h Z_{g^{-1}}).$$

Indeed, routine calculations show that $\overline{U}_g \overline{U}_h$ satisfies

$$v \in V \Rightarrow \begin{cases} \overline{U}_g \overline{U}_h c(v) = c_{gh}(v)\, \overline{U}_g \overline{U}_h \\ \overline{U}_g \overline{U}_h a(v) = a_{gh}(v)\, \overline{U}_g \overline{U}_h \end{cases}$$

so that $\overline{U}_g \overline{U}_h = \delta(g, h)\, \overline{U}_{gh}$ where $\delta(g, h) \in \mathbb{C}$ is given by

$$\begin{aligned} \delta(g, h) &= \langle 1 | \delta(g, h) \overline{U_{gh}} 1 \rangle \\ &= \langle 1 | \overline{U}_g \overline{U}_h 1 \rangle \\ &= \langle \overline{U}_{g^{-1}} 1 | \overline{U}_h 1 \rangle \\ &= \langle \exp(\zeta_{g^{-1}}) | \exp(\zeta_h) \rangle \\ &= \text{Det}^{\frac{1}{2}}(I - Z_h Z_{g^{-1}}). \end{aligned}$$

Pure spinors

In the spin representation of a finite-dimensional complex vector space with a nonsingular symmetric bilinear form, a special place is occupied by the spinors annihilated by a maximal isotropic subspace, such spinors being called pure for the subspace. Here we offer a tantalisingly partial account of the infinite-dimensional version: thus, when $F \subset V_{\mathbb{C}}$ is maximally isotropic relative to the complex-bilinear extension of $(\cdot|\cdot)$ to $V_{\mathbb{C}}$ we shall consider the space

$$(\wedge V')^F = \{\Phi \in \wedge V' : v \in F \Rightarrow \pi(v)\Phi = 0\}.$$

First, let $F \subset V_{\mathbb{C}}$ be a maximal isotropic subspace: if $x, y \in F$ then $(x|y) = 0$ and if $z \in V_{\mathbb{C}}$ satisfies $(z|F) = 0$ then $z \in F$. As $v \in V$ implies that $(iv|iv) = -\|v\|^2$ so $F \cap iV = 0$. Thus

$$F = \{w + iLw : w \in W\}$$

for some real subspace $W \subset V$ and some real linear $L: W \to V$. As $F \subset V_{\mathbb{C}}$ is a complex subspace, so L leaves W invariant and satisfies $LL = -I$. The isotropic nature of F implies that if $x, y \in W$ then $(Lx|Ly) = (x|y)$ and $(Lx|y) + (x|Ly) = 0$. Maximality of F forces W to be closed. Thus, $W \subset V$ is a closed real subspace on which $L \in O(W)$ is a complex structure. Finally, if dim $W^{\perp} > 1$ then we may enlarge F by adding to (W, L) a complex plane, so dim $W^{\perp} \leq 1$.

In summary, maximally isotropic subspaces of $V_{\mathbb{C}}$ come in two varieties: those of the (Lagrangian) form $F_K = \{v + iKv : v \in V\}$ for $K \in O(V)$ a complex structure; those of the (non-Lagrangian) form $F_{W,L} = \{w + iLw : w \in W\}$ for $W \subset V$ a closed real hyperplane and $L \in O(W)$ a complex structure.

Pure spinors for the Lagrangian $F_K \subset V_{\mathbb{C}}$ are readily determined. In fact, we may choose $g \in O(V)$ so that $K = gJg^{-1}$ and then $F_K = \{gv + igJv : v \in V\}$ whence (5.8), (5.9) and (5.14) together with the formula

$$v \in V \Rightarrow \frac{1}{2}\pi(gv + igJv) = a_g(v)$$

establish that $(\wedge V')^F$ is nonzero if and only if Ker C_g is finite-dimensional if and only if Ker$(J + K)$ is finite-dimensional. Moreover, if (say) $J + K$ is an isomorphism then the discussion after (5.12) shows that $(\wedge V')^F$ is a complex line.

Pure spinors for the non-Lagrangian maximal isotropic subspace $F_{W,L} \subset V_{\mathbb{C}}$ are somewhat mysterious. Here we offer only a couple of remarks. On the one hand, $(\wedge V')^F$ is invariant under both the parity operator $\Gamma: \wedge V' \to \wedge V'$ and the operator $\pi(u)$ when $u \in W^{\perp}$ is a unit vector; from $\{\Gamma, \pi(u)\} = 0$, $\Gamma^2 = I$ and $\pi(u)^2 = I$ it follows in particular that $(\wedge V')^F$ cannot be odd-dimensional. On the other hand, as W has real codimension one so the complex structure $L \in O(W)$ can have little to do with the original complex structure J on V. Further investigation of these pure spinors is postponed to a future publication.

Spin$_{\mathbb{C}}$ transformations

For some purposes, considering only (real) orthogonal transformations of V

itself is too restrictive: (complex) orthogonal transformations of $V_{\mathbb{C}}$ should also be considered. [4] addresses the implementation in fermionic Fock space $\wedge[V]$ of those transformations that lie in a certain subgroup of $O(V_{\mathbb{C}})$. Here we offer some remarks along similar lines for more general transformations in the generalized sense.

It is convenient to express a complex orthogonal transformation $G \in O(V_{\mathbb{C}})$ relative to the decomposition $V_{\mathbb{C}} = V^+ \oplus V^-$ in block form

$$G = \begin{bmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{bmatrix}.$$

Here, if $v \in V$ then

$$\begin{aligned} G^{++}(\gamma^+ v) &= \gamma^+(C_G^{++} v) \\ G^{+-}(\gamma^- v) &= \gamma^+(A_G^{+-} v) \\ G^{-+}(\gamma^+ v) &= \gamma^-(A_G^{-+} v) \\ G^{--}(\gamma^- v) &= \gamma^-(G_G^{--} v) \end{aligned}$$

where C_G^{++}, C_G^{--} are complex-linear and A_G^{+-}, A_G^{-+} are antilinear. For example, if $G = g_{\mathbb{C}}$ for $g \in O(V)$ then $C_G^{++} = C_G^{--} = C_g$ and $A_G^{+-} = A_G^{-+} = A_g$.

Now, when $G \in O(V_{\mathbb{C}})$ we ask for a linear operator $U: \wedge V \to \wedge V'$ such that

$$v \in V_{\mathbb{C}} \Rightarrow U\pi(v) = \pi(Gv)U$$

or equivalently

$$v \in V \Rightarrow \begin{cases} Uc(v) = c_G(v)U \\ Ua(v) = a_G(v)U \end{cases}$$

where

$$v \in V \Rightarrow \begin{cases} c_G(v) = c(C_G^{++} v) + a(A_G^{-+} v) \\ a_G(v) = a(C_G^{--} v) + c(A_G^{+-} v). \end{cases}$$

Again, such (nonzero) generalized Fock implementers U correspond to (nonzero) displaced Fock vacua $\Phi \in \wedge V'$ satisfying

$$v \in V \Rightarrow a_G(v)\Phi = 0$$

according to $U1 = \Phi$ and the rule that if $v_1, \cdots, v_n \in V$ then

$$U(v_1 \cdots v_n) = c_G(v_1) \cdots c_G(v_n)\Phi.$$

In the one direction, suppose that $G \in O(V_{\mathbb{C}})$ has a generalized Fock implementer $U: \wedge V \to \wedge V'$ and hence a displaced Fock vacuum $\Phi = U1 \in \wedge V'$. If $v \in \operatorname{Ker} C_G^{--}$ then the displaced vacuum condition reads $c(A_G^{+-} v)\Phi = 0$ whence the remark prior to (5.9) implies that $A_G^{+-}(\text{Ker } C_G^{--})$ is finite-dimensional; orthogonality of G on $V_{\mathbb{C}}$ entails (among other things) that

$$(C_G^{++})^* C_G^{--} + (A_G^{-+})^* A_G^{+-} = I$$

whence A_G^{+-} is injective on Ker C_G^{--}. Thus: if $G \in O(V_{\mathbb{C}})$ admits a generalized Fock implementer then Ker C_G^{--} is finite-dimensional.

In the opposite direction, we are content to state here a simple case: each $G \in O(V_{\mathbb{C}})$ for which Ker C_G^{--} is even-dimensional and Ran C_G^{--} is closed admits a generalized Fock implementer; a proof of this proceeds by first reducing to the case in which C_G^{--} is invertible and then arguing along lines similar to those for (5.12). A fuller account of these and related matters will be presented in due course.

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