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$\mathscr{D}$-Modules Associated to the Group of Similitudes

By

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Abstract

We classify regular holonomic $\mathscr{D}$-modules whose characteristic variety is the union of the conormal bundles of the orbits of the group of similitudes of a non degenerate quadratic form.

Contents

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Introduction of Publication

Let V be an n-dimensional complex vector space and q a nondegenerate quadratic form on V. As usual $\mathscr{D}_V$ will refer to the sheaf of analytic differential operators on V. We shall denote by Q the quadratic cone of equation $q=0$, and by G the group of similitudes of q. Note that G has three orbits in V, $\{0\}$, $Q\backslash\{0\}$, $V\backslash Q$.

We shall denote by $\Lambda\subset T^*V$ the union of the conormal bundles of these orbits.

Our aim is to classify the regular holonomic $\mathscr{D}$-modules whose characteristic variety is contained in Λ. We give two examples of such modules:

Examples 0.1.

1) The $\mathscr{D}_V$-module which describes the equations satisfied by a homogeneous $\mathrm{SO}(q)$-invariant section with a generator u and relations $(x\cdot\partial_x-\lambda)u=0\,(\lambda\in\mathbb{C})$, $\left(\frac{\partial q}{\partial x_i}\partial_j-\frac{\partial q}{\partial x_j}\partial_i\right)u=0$, for $i,j=1,\cdots,n$.

2) The $\mathscr{D}_V$-module which describes the equations of the elementary solution of the Laplace operator Δ with generator u and relation $x_i\Delta u=0$, $(x\cdot\partial_x+n-2)u=0$, $\left(\frac{\partial q}{\partial x_j}\partial_j-\frac{\partial q}{\partial x_i}\partial_i\right)u=0$ for $i,j=1,\cdots,n$.

The classification of such modules is well known in dimension 1 and 2. In dimension 1, L. Boutet de Monvel (see. [BM]) has given a description of holonomic $\mathscr{D}_{\mathbb{C}}$-modules which are regular at the origin.

In dimension 2, Galligo, Granger and Maisonobe (see. [G-G-M]) have described regular holonomic $\mathscr{D}$-modules whose characteristic variety is contained in $\Lambda:=\{x_1=x_2=0\}\cup\{\xi_1=\xi_2=0\}\cup\{x_1=\xi_2=0\}\{x_2=\xi_1=0\}$.

The aim of this paper is the study of cases $n\geq 3$.

Let $\mathscr{G}$ be the Lie algebra of infinitesimal generators of G. As q is a nondegenerate quadratic form then $\mathscr{G}$ is generated by the Euler vector field $\theta=\sum_{i=1}^{n}x_i\partial_i$ and the vectors field $\left(\frac{\partial q}{\partial x_i}\partial_j-\frac{\partial q}{\partial x_j}\partial_i\right)_{i,j=1,\cdots,n}$.

In Paragraph 1, we show that any coherent $\mathscr{D}_V$-module $\mathscr{M}$ which has a good filtration stable under the action of the Euler vector field θ on V, is generated over $\mathscr{D}_V$ by a finite number of global sections $u_1,\cdots,u_p\in\Gamma(V,\mathscr{M})$ such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]u_i<\infty$, $i=1,\cdots,p$. We also show that if $\mathscr{M}$ is a regular holomomic $\mathscr{D}_V$-module such that char $\mathscr{M}\subset\Lambda$, the infinitesimal action of G lifts to an action of the universal covering $\tilde{G}=\mathrm{Spin}(q)\times\mathbb{C}$ (the group of spinors) of G on $\mathscr{M}$. The group of spinors has a central element denoted ϵ such that $\epsilon^2=1$; this acts trivially on V and defines an automorphism denoted ϵ_M such that $\epsilon_{\mathscr{M}}^2=1_{\mathscr{M}}$. Then $\mathscr{M}$ is decomposed into $\mathscr{M}=\mathscr{M}_+\oplus\mathscr{M}_-$ where $\mathscr{M}_+=\ker(1-\epsilon_{\mathscr{M}})$ (resp. $\mathscr{M}_-=\ker(1+\epsilon_{\mathscr{M}})$)

is the fixed points set of ϵ (resp. $-\epsilon$).

In Paragraph 2, we describe models which will be useful for the classification of regular holonomic $\mathscr{D}_V$-modules such that $\epsilon_{\mathscr{M}}=1_{\mathscr{M}}$.

In Paragraph 3, we show that if $\mathscr{M}$ is a regular holonomic $\mathscr{D}_V$-module such that $\epsilon_{\mathscr{M}}=1_{\mathscr{M}}$, then $\mathscr{M}$ is generated by a finite number of G-invariant global sections $u_1,\cdots,u_p\in\Gamma(V,\mathscr{M})$ such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]u_i<\infty$ for $i=1,\cdots,p$. The study of such modules ends by the following main result.

Let $\mathrm{Mod}_{\Lambda}^{\mathrm{rh}}(\mathscr{D}_V)$ denote the category of regular holonomic $\mathscr{D}_V$-modules whose characteristic variety is contained in Λ. The invariant modules under the action of ϵ form a full subcategory of the category $\mathrm{Mod}_{\Lambda}^{\mathrm{rh}}(\mathscr{D}_V)$; denote it by $\mathrm{Mod}_{\Lambda,+}^{\mathrm{rh}}(\mathscr{D}_V)$.

We denote by $\mathscr{W}_n$ the Weyl algebra on V.

Let $\bar{\mathscr{A}}\subset\mathscr{W}_n$ be the subalgebra of $\mathrm{SO}(q)$-invariant differential operators. Then $\bar{\mathscr{A}}$ is the algebra generated over $\mathbb{C}$ by q, θ, Δ where Δ is the Laplacian associated to q (see. Proposition 2.1). Let J be the two sided ideal of $\bar{\mathscr{A}}$ generated by the operator $q\Delta-\theta(\theta+n-2)$. One sets $\mathscr{A}:=\bar{\mathscr{A}}/J$, $\mathscr{A}$ is graded by the action of homotheties.

We shall denote by $\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$ the category of homogeneous graded $\mathscr{A}$-modules T of finite type such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]u<\infty$, for $u\in T$; in other words $T=\bigoplus_{\lambda\in\mathbb{C}}T_\lambda$ is a $\mathbb{C}$-vector space ($T_\lambda:=\bigcup_{p\in\mathbb{N}}\ker(\theta-\lambda)^p$ is of finite dimension) equipped with endomorphisms q, Δ, θ of degree 2, -2, 0 such that $[\theta,q]=2q$, $[\theta,\Delta]=-2\Delta$, $[\Delta,q]=4\theta+2n$, with $\theta-\lambda$ a nilpotent operator on T_λ, and T is generated by a finite number of T_λ.

Let $I\subset\mathscr{W}_n$ be the left ideal generated by the infinitesimal generators of $\mathrm{SO}(q)$ and sets $\mathscr{M}_0:=\mathscr{W}_n/I$, $\mathscr{M}_0$ is a $(\mathscr{W}_n,\mathscr{A})$-bimodule.

If $\mathscr{M}$ is an object of the category $\mathrm{Mod}_{\Lambda,+}^{\mathrm{rh}}(\mathscr{D}_V)$, denote by $\Psi(\mathscr{M})$ the $\mathscr{D}_V$-module of global sections u of $\mathscr{M}$ which are G-invariant and such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]u<\infty$. Then $\Psi(\mathscr{M})$ is an object of $\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$.

Conversely, if T is an object of $\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$, set $\Phi(T)=\mathscr{M}_0\otimes_{\mathscr{A}}T$, an object of $\mathrm{Mod}_{\Lambda,+}^{\mathrm{rh}}(\mathscr{D}_V)$.

Theorem 0.2. *The two functors Φ and Ψ are equivalence of categories* $\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})\simeq\mathrm{Mod}_{\Lambda,+}^{\mathrm{rh}}(\mathscr{D}_V)$ *inverse to each other.*

After this, we give a classification of such objects in terms of finite diagrams of linear applications.

Finally, the case $\epsilon_{\mathscr{M}}=-1_{\mathscr{M}}$ (corresponding to the non-invariant modules), only exists in dimension 3. We show that such a module is a direct sum of a finite number of a distinguished module described in Paragraph 4.

Note: A similar class of $\mathscr{D}$-modules, with a different point of view, has been announced, by S. I. Gelfand and S. M. Khoroshkin in the announcement [G-K].

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§1. Homogeneous Modules

We refer the reader to [K1] for the theory of analytic $\mathcal{D}$-modules.

Definition 1.1. Let $\mathcal{M}$ be a $\mathcal{D}_V$-module. We will say that $\mathcal{M}$ is homogeneous if there is a good filtration stable under the action of the Euler vector field θ on V. We will say that a section s of $\mathcal{M}$ is homogeneous if $\dim_{\mathbb{C}}\mathbb{C}[\theta]s<\infty$. The section s is said to be homogeneous of degree $\lambda\in\mathbb{C}$, if there exists an integer $j\in\mathbb{N}$ such that $(\theta-\lambda)^j s=0$.

In this Paragraph we show that any coherent homogeneous $\mathcal{D}_V$-module is generated over $\mathcal{D}_V$ by a finite number of homogeneous global sections.

Remark 1.2.

i) If $\mathcal{M}$ is a coherent $\mathcal{D}_V$-module such that char $\mathcal{M}\subset$ char(θ), in particular if $\mathcal{M}$ is homogeneous, then it is stable under the action of the group generated by the hamiltonian H_θ: $(x,\xi)\mapsto\left(\lambda x,\frac{\xi}{\lambda}\right)$, $\lambda\in\mathbb{C}^*$, $(x,\xi)\in T^*V$. So the support of $\mathcal{M}$, supp $(\mathcal{M})$, is stable by homotheties. Therefore if $\mathcal{M}$ vanishes in a neighborhood of 0, $\mathcal{M}$ vanishes everywhere. Thus if homogeneous global sections $(s_i)_{i=1,\cdots,p}$ generate $\mathcal{M}$ in a neighborhood of 0, they generate $\mathcal{M}$ everywhere.

ii) In the same way if $\mathcal{N}$ is a coherent $\mathcal{O}$-module (on $V\simeq\mathbb{C}^n$ or on a ball of V centered in 0), $\mathcal{N}$ is called homogeneous if it is equipped with a lifting $\tilde{\theta}$ of θ such that $\tilde{\theta}(f\cdot s)=(\theta f)\cdot s+f\cdot\tilde{\theta}s$ $(*)$, where $s\in\mathcal{N}$, $f\in\mathcal{O}$. The support of such a module is stable under the action of θ (i.e. supp$(\mathcal{N})$ is conic). In particular $\mathcal{N}$ vanishes if it vanishes in the neighborhood of 0.

iii) It arises from the previous remark i) that if s is an homogeneous section of a coherent module on a ball of V centered in 0, s vanishes if it vanishes in the neighborhood of 0.

One has the following Theorem.

Theorem 1.3. *Let $\mathcal{M}$ be a coherent homogeneous $\mathcal{D}_V$-module with a good filtration $(F_p\mathcal{M})_{p\in\mathbb{Z}}$ stable by the Euler vector field θ. Then*

i) *$\mathcal{M}$ is generated over $\mathcal{D}_V$ by a finite number of homogeneous global sections,*

ii) *for any $p\in\mathbb{N}$, $\lambda\in\mathbb{C}$, the $\mathbb{C}$-vector space, $\Gamma(V,F_p\mathcal{M})\cap[\cup_{k\in\mathbb{N}}\ker(\theta-\lambda)^k]$, of homogeneous global sections of $F_p\mathcal{M}$ of degree λ is of finite dimension.*

This result was proved in the case of regular holonomic $\mathcal{D}$-module by B.

Malgrange (see. [B-M-V]).

Proof. One may assume that the first term of the good filtration of $\mathcal{M}$, $F_0\mathcal{M}$, generates $\mathcal{M}$ in the neighborhood of 0.
We are going to show that $F_0\mathcal{M}$ is generated by a finite number of homogeneous global sections and the required result will be deduced. The proof is decomposed in three steps.

Denote by $\mathfrak{m}$ the defining ideal of the origin and by $B_r=\{x\in\mathbb{C}^n, |x|\leq r\}$ the ball of radius $r>0$.

In the first step, taking a surjective homomorphism $\mathcal{O}^N_{|B_r}\to F_0\mathcal{M}_{|B_r}$, we show that the action θ on $F_0\mathcal{M}$ lifts to an action $\tilde{\theta}$ on $\mathcal{O}^N$ satisfying (*) (see. Remark 1.2, ii)).

In the second step, $\mathcal{O}^N(B_r)$ is equipped with the norm defined by $\|f\|=\sup_{j\geq 0}\|f_j\|$ where $f=\sum_{j\geq 0} f_j$ is the Taylor's serie of f in B_r and $\|f_j\|=\sup_{x\in B_r}\|f_j(x)\|$. We show the following results.

(a) $\tilde{\theta}$ is invertible on $\mathfrak{m}^k\mathcal{O}^N(B_r)$ if k is large enough.

Let $E_\lambda=\bigcup_{j\in\mathbb{N}}\ker(\tilde{\theta}-\lambda)^j$ be the spectral subspace of $\tilde{\theta}$ of $\mathcal{O}^N(B_r)$ and let $E_\lambda^{(k)}$ be the spectral subspace of $\tilde{\theta}$ of $\mathcal{O}^N(B_r)/\mathfrak{m}^k\mathcal{O}^N(B_r)$. From (a) we deduced the following result:

(b) The map $E_\lambda\to E_\lambda^{(k)}$ is onto, and one to one for large enough k.

They imply similar results for the coherent $\mathcal{O}_V$-module $F_0\mathcal{M}$.

In the third step, we complete the proof.

Step 1. The Cartan's Theorem A (see. [C]) mentions that $F_0\mathcal{M}$ is generated above B_r by a finite number of global sections. This means that there exists a morphism of sheaves of $\mathcal{O}_{|B_r}$-modules: $\mathcal{O}^N_{|B_r}\to F_0\mathcal{M}_{|B_r}$, $N\in\mathbb{N}$, above B_r, which is surjective. If $e_1,\cdots,e_N$ generate $F_0\mathcal{M}$ on B_r and θ. $F_0\mathcal{M}\subset F_0\mathcal{M}$, there exists holomorphic functions on B_r, $a_{ij}\in\mathcal{O}(B_r)$, $i,j=1,\cdots,N$ such that

$$\theta e_j=\sum_{i=1}^{N}a_{ij}e_i. \tag{1}$$

Denote by $A:=(a_{ij})$ the matrix with coefficients in $\mathcal{O}(B_r)$. Let u: $\mathcal{O}^N(B_r)\to F_0\mathcal{M}(B_r)$ be the morphism such that $u(\tilde{e}_j)=e_j$, where $\tilde{e}_j$, $j=1,\cdots,N$, is the canonical basis of $\mathcal{O}^N(B_r)$. One has

$$\theta\Big(\sum_{i=1}^{N}\lambda_i e_i\Big)=u\,[(\theta+A(x))\Big(\sum_{i=1}^{N}\lambda_i\tilde{e}_i\Big)],\ \lambda_i\in\mathcal{O}(B_r). \tag{2}$$

Let

(3) $$A(x)=\sum_{j\geq 0}A_j(x)$$

be the decomposition of $A(x)$ into homogeneous components on B_r (Taylor's serie of $A(x)$): this serie converges in the neighborhood of B_r.
Set $\tilde{\theta}:=x\circ\partial_x+A(x)$ and denote by $\tilde{\theta}_0=x\cdot\partial_x+A_0$ the component of degree 0 of $\tilde{\theta}$, where $A_0=(b_{ij})$, $b_{ij}\in\mathbb{C}$.
The operators $\tilde{\theta}=x\cdot\partial_x+A(x)$ and $\tilde{\theta}_0$ act continuously on $\mathcal{O}^N(B_r)$ and preserve the filtration $(\mathfrak{m}^k\mathcal{O}^N(B_r))_{k\in\mathbb{Z}}$. Let $f\in\mathfrak{m}^k\mathcal{O}^N(B_r)$, $f=\sum_{j\geq k}f_j$ its decomposition in homogeneous components (Taylor's serie of f) in the neighborhood of B_r. One has

$$\tilde{\theta}f=\sum_{j\geq k}\Big[(j+A_0)f_j+\sum_{p\geq 1,p+q=j}A_pf_q\Big]\in\mathfrak{m}^k\mathcal{O}^N(B_r).$$

Step 2. Denote by $\mathcal{P}_j$, $j\in\mathbb{N}$, the space of homogeneous polynomials, on B_r, of degree j equipped with the following norm: $\|f_j\|=\sup_{x\in B_r}\|f_j(x)\|$, $f_j\in\mathcal{P}_j$. In fact, on each space $\mathcal{P}_j^N=\mathcal{P}_j\otimes\mathbb{C}^N$, the operator $\tilde{\theta}_0:=(x\cdot\partial_x+A_0)$ induces the operator $j+A_0$ so that $\tilde{\theta}_0$ is invertible on $\mathcal{P}_j^N$ if $j+A_0$ is invertible (i.e. if $j>\|A_0\|=\sup_{i,j}|b_{ij}|$). Moreover, one has for large enough j,

$$\|\tilde{\theta}_0^{-1}{}_{|\mathcal{P}_j^N}\|\leq\frac{1}{j-\|A_0\|}.$$

Let $\mathcal{O}^N(B_r)$ equipped with the norm defined by $\|f\|=\sup_{j\geq 0}\|f_j\|$ ($f_j\in\mathcal{P}_j$ with $\|f_j\|=\sup_{x\in B_r}\|f_j(x)\|$) for all $f\in\mathcal{O}^N(B_r)$. Then $\tilde{\theta}$ is invertible on the space $\mathfrak{m}^k\mathcal{O}^N(B_r)$ when $\tilde{\theta}_0$ is.
Indeed $\tilde{\theta}=\tilde{\theta}_0+R$ where $R=\sum_{q\geq 1}A_q$ is bounded i.e. $\|R\|\leq C$, $C=$constant. One has $\tilde{\theta}=\tilde{\theta}_0+R=\tilde{\theta}_0[1+\tilde{\theta}_0^{-1}R]$ and as on $\mathcal{P}_j$ one has

$$\|\tilde{\theta}_0^{-1}{}_{|\mathcal{P}_j^N}\|\leq\frac{1}{j-\|A_0\|}\quad\text{if}\quad j>\|A_0\|,$$

on $\mathfrak{m}^k\mathcal{O}^N(B_r)$ one has

$$\|\tilde{\theta}_0^{-1}{}_{|\mathfrak{m}^k\mathcal{O}^N(B_r)}\|\leq\sup_{j\geq k}\|\tilde{\theta}_0^{-1}{}_{|\mathcal{P}_j^N}\|\leq\sup_{j\geq k}\Big(\frac{1}{j-\|A_0\|}\Big),$$
$$\|\tilde{\theta}_0^{-1}{}_{|\mathfrak{m}^k\mathcal{O}^N(B_r)}\|\leq\frac{1}{k-\|A_0\|}\text{ if }k>\|A_0\|$$

and

$$\|\tilde{\theta}_0^{-1}R_{|\mathfrak{m}^k\mathcal{O}^N(B_r)}\|\leq\|\tilde{\theta}_0^{-1}{}_{|\mathfrak{m}^k\mathcal{O}^N(B_r)}\|\cdot\|R\|\leq\frac{C}{k-\|A_0\|},$$

C=constant.

As R: $\mathfrak{m}^k\mathcal{O}^N(B_r)\to\mathfrak{m}^{(k+1)}\mathcal{O}^N(B_r)$, by iteration one has

$$\|(\tilde{\theta}_0^{-1}R)^n\|\leq C^n\frac{1}{(k-\|A_0\|)}\cdot\frac{1}{(k+1-\|A_0\|)\cdots(k+n-\|A_0\|)}.$$

Therefore the serie $\sum_{n\geq 1}(\tilde{\theta}_0^{-1}R)^n$ converges on $\mathfrak{m}^k\mathcal{O}^N(B_r)$ (as soon as $\tilde{\theta}_0^{-1}$ is defined on) for large enough k. Finally $\tilde{\theta}$ is invertible on the space $\mathfrak{m}^k\mathcal{O}^N(B_r)$ (of inverse $\tilde{\theta}^{-1}=(1+\tilde{\theta}_0^{-1}R)^{-1}\tilde{\theta}_0^{-1}=\sum_{n\geq 1}(\tilde{\theta}_0^{-1}R)^n\tilde{\theta}_0^{-1}$) for $k>>0$ (as soon as $\tilde{\theta}_0$ is).

In the sequel, one denotes by $E_\lambda=E_\lambda(B_r)=\bigcup_{j\in\mathbb{N}}\ker(\tilde{\theta}-\lambda)^j$, $\lambda\in\mathbb{C}$, the spectral subspace of $\tilde{\theta}$ of $\mathcal{O}^N(B_r)$, that is the subspace of homogeneous sections on B_r of degree λ. Let $E_\lambda^{(k)}$ be the spectral subspace of $\tilde{\theta}$ of $\mathcal{O}^N(B_r)/\mathfrak{m}^k\mathcal{O}^N(B_r)$.

For a large enough k, $(\tilde{\theta}-\lambda)$ is invertible on $\mathfrak{m}^k\mathcal{O}^N(B_r)$. Therefore the map from E_λ to the quotient $E_\lambda^{(k)}$ module $\mathfrak{m}^k$: $E_\lambda\to E_\lambda^{(k)}$ is one to one for $k>>0$, and E_λ is a finite dimensional $\mathbb{C}$-vector space. For any $k\geq 1$, the map $E_\lambda\to E_\lambda^{(k)}$ is surjective (because $\mathcal{O}^N(B_r)/\mathfrak{m}^k\mathcal{O}^N(B_r)$ is of finite dimension and the map $E_\lambda^{(k)}\to E_\lambda^{(k')}$ is surjective $k'\leq k$).

In the same way, let $\mathcal{M}_\lambda$ be the special subspace of θ of $F_0\mathcal{M}$ and let $\mathcal{M}_\lambda^{(k)}$ be the spectral subspace of θ of $F_0\mathcal{M}(B_r)/\mathfrak{m}^kF_0\mathcal{M}(B_r))$, then $\theta-\lambda$ is invertible on $\mathfrak{m}^kF_0\mathcal{M}(B_r)$ if k is largde enough. Therefore the map $\mathcal{M}_\lambda\to\mathcal{M}_\lambda^{(k)}$ is one to one if $k>>0$ and $\mathcal{M}_\lambda$ is a finite dimensional $\mathbb{C}$-vector space. This map is surjective in all cases.

It remains to check that the spectral subspaces $E_\lambda^{(k)}$ and $\mathcal{M}_\lambda^{(k)}$ do not depend on the choice of the ball B_r. It comes that for all $r\geq r'$, the restriction map from B_r to $B_{r'}$ induced an isomorphism, denoted by Res, from $E_\lambda(B_r)$ to $E_\lambda(B_{r'})$. Indeed Res is surjective: $F_0\mathcal{M}_{|B_r}$ is generated by a finite family of homogeneous sections $(s_j)_{j=1,\cdots,p}$. Let s be a homogeneous section on $B_{r'}$, then s is a linear combination of sections $(s_j)_{j=1,\cdots,p}$, which are homogeneous on B_r i.e. $s=\sum_{j=1}^p f_js_j$, $f_j\in\mathcal{O}(B_{r'})$. As s is homogeneous, one may choose f_j a homogeneous polynomial in $\mathcal{P}_j$ (one replaces f_j by the term of degree (degree of s − degree of s_j) in its Taylor's serie), therefore s extends on B_r.

Res is injective because any section which vanishes in the neighborhood of 0 vanishes everywhere (see. Remark 1.2 ii)), in particular on B_r. Then a section on the small ball $B_{r'}$ is the restriction of an unique section on the large ball B_r for all $r\leq\infty$.

Step 3. One has the surjective map u: $\mathfrak{m}^k\mathcal{O}^N(B_r)\to\mathfrak{m}^kF_0\mathcal{M}(B_r)$. One sees that the spectral subspace $\mathcal{M}_\lambda^{(k)}$ of θ is the image by u of the spectral subspace

$E_\lambda^{(k)}$ of $\widetilde{\theta}$ i.e. $u(E_\lambda^{(k)})=\mathcal{M}_\lambda^{(k)}$; indeed

-The image of $E_\lambda^{(k)}$ by u is contained in $\mathcal{M}_\lambda^{(k)}$.

-Conversely if σ is a section of $\mathcal{M}_\lambda^{(k)}$, it lifts in the neighborhood of 0. If s is the lifting $s=\sum_{\mu\in\mathbb{C},\alpha\geq 0}z^\alpha s_{\mu,\alpha}$ (where $s_{\mu,\alpha}$ is homogeneous of degree μ) one may replace it by $s_\lambda=\sum_{\mu+\alpha=\lambda}z^\alpha s_{\mu,\alpha}$ in $E_\lambda^{(k)}$. Therefore $F_0\mathcal{M}/\mathrm{m}F_0\mathcal{M}$ is a finite dimensional vector space over $\mathbb{C}$. If the $(\lambda_i)_{i=1,\cdots,n_0}$, are the eigenvalues of A_0 then $F_0\mathcal{M}/\mathrm{m}F_0\mathcal{M}=\sum_{i=1}^{n_0}\mathcal{M}_{\lambda_i}^1$. This finite sum lifts in $F_0\mathcal{M}$ in the following finite sum $\sum_{i=1}^{n_0}\mathcal{M}_{\lambda_i}$, and the Nakayama's Lemma shows that the sum $\sum_{i=1}^{n_0}\mathcal{M}_{\lambda_i}$ generates in 0 the given $\mathcal{O}_V$-coherent module $F_0\mathcal{M}$; it also generates $F_0\mathcal{M}$ everywhere because the support of $F_0\mathcal{M}$ is conic (see. Remark 1.2 ii)). Therefore the same finite family of global sections generates $\mathcal{M}$ as a $\mathcal{D}_V$-module (see. Remark 1.2 i)). □

Let $\mathcal{M}$ be a coherent homogeneous $\mathcal{D}_V$-module. We have the following Lemma:

Lemma 1.4. *Let $\mathcal{M}^{\mathrm{pol}}\subset\mathcal{M}$ be the sub $\mathcal{W}_n$-module generated by homogeneous global sections. Then one has $\mathcal{M}\simeq\mathcal{D}_V\otimes_{\mathcal{W}_n}\mathcal{M}^{\mathrm{pol}}$ and $\mathcal{M}^{\mathrm{pol}}$ is, up to isomorphism, the unique homogeneous $\mathcal{W}_n$-module which describes this property.*

Proof. It suffices to prove that the canonical morphism $\mathcal{D}_V\otimes_{\mathcal{W}_n}\mathcal{M}^{\mathrm{pol}}\xrightarrow{h}\mathcal{M}$ is an isomorphism. One has $\mathcal{M}^{\mathrm{pol}}:=\bigoplus_{\lambda\in\mathbb{C}}\mathcal{M}_\lambda$ where $\mathcal{M}_\lambda:=\Gamma(V,\mathcal{M})\cap[\bigcup_{k\in\mathbb{N}}\ker(\theta-\lambda)^k]$.

The $\mathcal{D}_V$-modules $\mathcal{M}$ and $\mathcal{D}_V\otimes_{\mathcal{W}_n}\mathcal{M}^{\mathrm{pol}}$ are homogeneous, so that ker h and coker h are homogeneous. They vanishe at 0. Indeed the morphism h is surjective (because the $\mathcal{M}_\lambda$ generate $\mathcal{M}$ see. Theorem 1.3). The morphism h is injective: if $\underline{m}=\sum_{\lambda\in\mathbb{C}}P_\lambda\underline{m}_\lambda$ $(\underline{m}_\lambda\in\mathcal{M}_\lambda,\ P_\lambda\in\mathcal{W}_n)$ is a section of ker h at 0 and if we decompose the P_λ in the Taylor's serie in the neighborhood of 0 $(P_\lambda=\sum_{r\geq 0}P_{r,\lambda})$ that is $\underline{m}=\sum_{\mu\in\mathbb{C}}(\sum_{\lambda+r=\mu}P_{r,\lambda}\otimes\underline{m}_\lambda)=\sum_{\mu\in\mathbb{C}}(\sum_{\lambda+r=\mu}1\otimes P_{r,\lambda}\underline{m}_\lambda)$, its image in $\mathcal{M}$ is $m=\sum_{\mu\in\mathbb{C}}(\sum_{\lambda+r=\mu}1\otimes P_{r,\lambda}m_\lambda)=0$ where $m_\lambda=h(\underline{m}_\lambda)$ (the $\mathcal{M}_\lambda$ are not linearly depending on so that the homogeneous components m_λ of m in $\mathcal{M}$ are exactly the images of the homogeneous components $\underline{m}_\lambda$ of $\underline{m}$ i.e. $\mathrm{m}_\lambda=h(\underline{m}_\lambda)$).

Then each homogeneous component of m vanishes i.e. $\sum_{\lambda+r=\mu}1\otimes P_{r,\lambda}m_\lambda=0$ for all μ, so that the homogeneous components of m vanishe also i.e. $\sum_{\lambda+r=\mu}1\otimes P_{r,\lambda}\underline{m}_\lambda=0$ for all μ; therefore $\underline{m}=0$. In other words ker h vanishes at 0. As ker h is homogeneous and vanishes at 0, it vanishes everywhere and h is injective (see. Remark 1.2 i)). □

Remark 1.5. The action of G (preserving the good filtration) on a $\mathcal{D}_V$-module $\mathcal{M}$ is given by an isomorphism u: $p_1^+(\mathcal{M})\xrightarrow{\sim}p_2^+(\mathcal{M})$ where p_1: $G\times V\to V$ is the projection on V, and p_2: $G\times V\to V$, $(g, x)\mapsto g\cdot x$ defined the action of G on V (satisfying the associativity conditions). In fact u is an isomorphism above the isomorphism of algebras u: $p_1^+(\mathcal{D}_V)\xrightarrow{\sim}p_2^+(\mathcal{D}_V)$.

Recall that $\tilde{G}:=\mathrm{Spin}(q)\times\mathbb{C}$ denote the universal covering of the group G. The result above leads to the following Proposition.

Proposition 1.6. *Let $\mathcal{M}$ be an object of the category* $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)$. *The infinitesimal action of G on $\mathcal{M}$ lifts to an action of $\tilde{G}$ on $\mathcal{M}$, compatible with the action of G on V and $\mathcal{D}_V$.*

Proof. One knows that $\mathcal{M}$ admits a good filtration $(\mathcal{M}_k)_{k\in\mathbb{Z}}$ stable under the action of the Lie algebra $\mathcal{A}$. In fact, one has seen that the $\mathbb{C}$-vector spaces $\mathcal{M}_{k,\lambda}:=\Gamma(V, \mathcal{M}_k)\cap[\bigcup_{p\in\mathbb{N}}\ker(\theta-\lambda)^p]$ are of finite dimension and generate $\mathcal{M}$ (see. Theorem 1.3). Each $\mathcal{M}_{k,\lambda}$ is stable under the action of the Lie algebra $\mathcal{G}$ of G, thus this action lifts to an action of the group $\tilde{G}$ on each $\mathcal{M}_{k,\lambda}$ (see. [W]). According to the previous Lemma there is an unique homogeneous $\mathcal{W}_n$-module denoted by $\mathcal{M}^{\mathrm{pol}}$ such that $\mathcal{M}\simeq\mathcal{D}_V\otimes_{\mathcal{W}_n}\mathcal{M}^{\mathrm{pol}}$. The module $\mathcal{M}^{\mathrm{pol}}=\bigoplus_{k\in\mathbb{Z},\lambda\in\mathbb{C}}\mathcal{M}_{k,\lambda}$ is stable under the action of $\mathcal{G}$ (because the $\mathcal{M}_{k,\lambda}$ are). This action lifts to an action of the group $\tilde{G}$ on $\mathcal{M}^{\mathrm{pol}}$ so that $\tilde{G}$ acts also on $\mathcal{M}$. □

Remark 1.7. Let $\mathcal{M}$ be an object of $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)$, $U\subset V$ an open subset, $g\in\tilde{G}$, $P\in\mathcal{D}_V$. If $s\in\Gamma(U, \mathcal{M})$ then $gs\in\Gamma(gU, \mathcal{M})$ and $g\cdot(Ps)=\bar{g}P.\,gs$ where $\bar{g}$ is the image of g in G. In particular ϵ acts trivially on V thus $\epsilon(Ps)=P.\epsilon s$ that is the action of ϵ on $\Gamma(U, \mathcal{M})$ (resp. $\mathcal{M}$) defines an automorphism of $\Gamma(U, \mathcal{D})$ (resp. $\mathcal{D}$)-modules denoted $\epsilon_{\mathcal{M}}$ such that $\epsilon_{\mathcal{M}}^2=1_{\mathcal{M}}$. Therefore $\mathcal{M}$ is decomposed into $\mathcal{M}=\mathcal{M}_+\oplus\mathcal{M}_-$ with $\mathcal{M}_\pm=\ker(1_{\mathcal{M}}\pm\epsilon_{\mathcal{M}})$.

§2. Description of Models

Recall that $\mathcal{G}$ denotes the Lie algebra of infinitesimal generators of G; denote by $\mathcal{U}$ its envelopping algebra. If E is a finite dimensional representation of G, we may associate to it a $\mathcal{D}_V$-module $\mathcal{D}_E:=\mathcal{D}_V\otimes_{\mathcal{U}}E$. The $\mathcal{D}_V$-module $\mathcal{D}_E$ is equipped with a natural filtration, quotient of the canonical filtration of $\mathcal{D}_V\otimes_{\mathbb{C}}E$, which is stable under the action of infinitesimal generators. Then the module $\mathcal{D}_E$ is an object of the category $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)$ (see [B-A]). In particular we denote by $\mathcal{D}_{W,\lambda,N}$ the module corresponding $W\otimes_{\mathbb{C}}E_{\lambda,N}$ where W is a simple representation of $\mathrm{SO}(q)$, $E_{\lambda,N}$ is the $\mathbb{C}$-module generated by one generator e subjected to the relation $(\theta-\lambda)^Ne=0$.

The result of Paragraph 1 (see. Theorem 1.3) shows that if $\mathcal{M}$ is an object of the category $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)$ there is a finite dimensional representation of G, E, as above, such that the morphism $\mathcal{H}om_{\mathcal{D}_V}(\mathcal{D}_E, \mathcal{M})\otimes\mathcal{D}_E\to\mathcal{M}$ is surjective and E is a finite sum of modules such as $W\otimes_{\mathbb{C}}E_{\lambda,N}$ described above.

2.1. Invariant operators and invariant sections

In this section we determine the algebra of $\mathrm{SO}(q)$-invariant differential operators. Let (φ_{ij}) be the inverse matrix of $\left(\frac{\partial^2 q}{\partial x_i \partial x_j}\right)$. Denote by $\Delta := \frac{1}{2}\sum_{i,j=1}^n \varphi_{ij}\partial_i\partial_j$ the Laplacian associated to the nondegenerate quadratic form q.

Let $\tilde{\mathcal{A}}\subset\mathcal{W}_n$ be the subalgebra of $\mathrm{SO}(q)$-invariant differential operators. We have the following Proposition:

Proposition 2.1. *The subalgebra $\tilde{\mathcal{A}}$ is generated over $\mathbb{C}$ by the operators q, Δ, θ satisfying the following relations $[\theta, q]=2q$, $[\theta, \Delta]=-2\Delta$, $[q, \Delta]=-(4\theta+2n)$.*

Proof. Let $P := \sum_{|\alpha|\leq m} a_\alpha\partial^\alpha$ $(a_\alpha\in\mathbb{C}[x])$ be an $\mathrm{SO}(q)$-invariant differential operator with polynomial coefficients of degree m. Its principal symbol $\sigma_m(P)$ is also invariant. Therefore $\sigma_m(P)$ is a polynomial of $(q(x), \frac{1}{2}\sum_{i,j=1}^n\varphi_{ij}\xi_i\xi_j, x\circ\xi)$, $(x, \xi)\in T^*V$, according to Herman Weyl (see. [W]). Since we may take averages on the real group $\mathrm{SO}(n, \mathbb{R})$ (maximal compact subgroup of $\mathrm{SO}(q)$), denote by $\tilde{P}=\int_{\mathrm{SO}(n,\mathbb{R})} gPdg$ the average of P on $\mathrm{SO}(n, \mathbb{R})$, and by $\sigma_m(\tilde{P})$ its principal symbol. Then one has $\sigma_m(\tilde{P})=\sigma_m(P)$ because $\sigma_m(P)$ is $\mathrm{SO}(n, \mathbb{R})$-invariant. Thus there exist a polynomial $Q := Q(q, \Delta, \theta)$, invariant by $\mathrm{SO}(q)$, such that $P_m - Q$ (where $P_m=\sum_{|\alpha|=m}a_\alpha\partial^\alpha$ is the principal part of P) is of degree $m-1$. By induction on the degree of P this shows the result. □

One sets $J := \tilde{\mathcal{A}}\cap I$ where I is the left ideal of infinitesimal generators of G. Put $\mathcal{A}=\tilde{\mathcal{A}}/J$.

Lemma 2.2. *The left ideal J is a two sided ideal generated by the central element $P_0 := q\Delta-\theta(\theta+n-2)$.*

Proof. One has $P_0=\frac{1}{2}\sum_{i,j=1}^n(x_i\partial_j - x_j\partial_i)^2$ (if $q(x)=\sum_{j=1}^n x_j^2$) thus $P_0\in J$. One should remark that the operator P_0 is homogeneous with respect to homotheties and belongs to the center of $\tilde{\mathcal{A}}$. Conversely if $P\in\tilde{\mathcal{A}}$, one decomposes it into homogeneous components with respect to homotheties $P=\sum_{m\in\mathbb{Z}}H_{2m}$; dividing each homogeneous components H_{2m} by P_0 we obtain

$$H_{2m}=\begin{cases} q^m Q \bmod P_0 & \text{if } m\geq 0 \\ \Delta^m Q \bmod P_0 & \text{if } m\leq 0. \end{cases}$$

where $Q=Q(q\Delta, \theta)$ is an operator of degree 0.

Then if $P\in J$, P annihilates q^k for all $k\in\mathbb{N}$, its homogeneous components that is q^mQ if $m\geq 0$ (resp. Δ^mQ if $m\leq 0$) annihilate also q^k. Thus implies that the polynomial in k, $Q(2k(2k-n+2), 2k)=0$ for $k>m$; we deduce that the polynomial in $\lambda\in\mathbb{C}$, $Q(2\lambda(2\lambda-n+2), \lambda)=0$, therefore Q is a multiple of P_0. $\square$

Remark 2.3. The Lie algebra $\mathrm{sl}_2(\mathbb{C})$ is generated by e, f, h sastisfying the relations $[h, e]=2e$, $[h, f]=-2f$, $[e, f]=h$. The map which associates e, f, h to $q/2$, $-\Delta/2$, $\theta+n/2$ identifies the algebra $\mathscr{A}$ with the quotient of the envelopping algebra of $\mathrm{sl}_2(\mathbb{C})$ by the left ideal generated by $C-\frac{n}{2}\left(\frac{n}{2}-2\right)$ where C is the Casimir operator.

In the sequel we will use the following remark:

Remark 2.4. (averages over $\mathscr{D}$-modules). Let $\mathscr{M}$ be a coherent sheaf of $\mathscr{D}$-modules equipped with a good fitration $(\mathscr{M}_k)_{k\in\mathbb{Z}}$. Each sheaf $\mathscr{M}_k$ is, as every coherent sheaf over $\mathscr{O}$, a sheaf of Frechet spaces. If G is a compact group acting continuously on V and $\mathscr{D}_V$, and if this action lifts to $\mathscr{M}$ (preserving the good filtration $(\mathscr{M}_k)_{k\in\mathbb{Z}}$) there is a notion of average on $\Gamma(U, \mathscr{M})$ for any invariant open set $U\subset V$: for any section $s\in\Gamma(U, \mathscr{M})$, one denote by $\tilde{s}=\int_G g\cdot s\,dg$ its average over G where dg is the Haar measure of mass 1. Here, for the modules we are interested in, one knows from Theorem 1.3 that for each $\lambda\in\mathbb{C}$, $k\in\mathbb{Z}$, the complex vector space $\mathscr{M}_{k,\lambda}(U)=\Gamma(U, \mathscr{M}_k)\cap[\cup_{j\in\mathbb{N}}\ker(\theta-\lambda)^j]$ is of finite dimension; it is stable under the continuous action of $\widetilde{G_{\mathbb{R}}}=\mathrm{Spin}(n, \mathbb{R})$ (or $\mathrm{SO}(n, \mathbb{R})$). Therefore we are just taken averages in a finite dimensional $\mathbb{C}$-vector space on which G acts.

§3. Invariant Modules

3.1. Inverse images

The quadratic form q defines a map $V\to\mathbb{C}$, which we still denote by q, which is submersive outside of 0 since q is nondegenerate. In this section, we are interested in the inverse image, $q^+\mathscr{N}$, of a $\mathscr{D}_{\mathbb{C}}$-module $\mathscr{N}$.

3.1.1. Flatness of the transfer module

Lemma 3.1. *The transfer module* $\mathscr{D}_{V\to\mathbb{C}}$ *is flat over* $q^{-1}(\mathscr{D}_V)$.

Proof. The transfer module $\mathscr{D}_{V\xrightarrow{q}\mathbb{C}}$ is generated over $\mathscr{D}_V\otimes\mathscr{D}_{\mathbb{C}}$ by a generator e and relations

$$\begin{cases} qe=et \\ \partial_{x_j}e=\dfrac{\partial q}{\partial x_j}e\partial_t \quad \text{for } j=1,\cdots,n \end{cases}$$

where t is the coordinate of $\mathbb{C}$ and $x=(x_1,\cdots,x_n)$ is a local coordinate system of V. Thus it is free over $\mathscr{D}(V,\partial_t)=\mathscr{O}_V\otimes\mathbb{C}[\partial_t]$. Since $\mathscr{O}_t$ is of dimension 1 this means that $\mathscr{O}_V$ is torsion free over $\mathscr{O}_{\mathbb{C}}$; this is obvious because $q(x)=t$. That is to say that $\mathscr{O}_V$ is flat over $\mathscr{O}_{\mathbb{C}}$. Then the module $\mathscr{D}_{V\xrightarrow{q}\mathbb{C}}$ is flat over $q^{-1}(\mathscr{D}_{\mathbb{C}})$. □

Therefore the inverse image functor $\mathscr{N}\rightarrow q^+\mathscr{N}$ is reduced to its first term that is the module $\mathscr{D}_{V\xrightarrow{q}\mathbb{C}}\otimes_{q^{-1}(\mathscr{D}_V)}q^{-1}\mathscr{N}$, and it is an exact functor.

3.1.2. Characterisation of $q^+\mathscr{N}$

Let $\mathscr{N}$ be a $\mathscr{D}_{\mathbb{C}}$-module. We have the following Proposition:

Proposition 3.2. *Let $U\subset\mathbb{C}$ be an open subset. The G-invariant sections of $q^+\mathscr{N}$ over $q^{-1}U$ are exactly the inverse image of sections of $\mathscr{N}$.*

Proof. The sections of $q^+\mathscr{N}$ are of the form $s=\sum_{j\in I}f_j(x)q^{-1}(\eta_j)$ where the θ_j are the sections of $\mathscr{N}$ over U. If s is a germ at 0 of a G-invariant section, we may replace each $f_j(x)$ by its average (over $SO(n,\mathbb{R})$ the compact maximal subgroup of $SO(q)$ denoted by $\tilde{f}_j(q(x))$, so that $s=\sum_{j\in I}\tilde{f}_j(q(x))q^{-1}(\eta_j)=q^{-1}(\sum_{j\in I}\tilde{f}_j(t)\eta_j)$ with $t=q(x)$. Thus we can see that the invariant sections of $q^+\mathscr{N}$ are exactly inverse image of sections of $\mathscr{N}$. □

In particular we have the following Corollary:

Corollary 3.3. *The module $q^+\mathscr{N}$ has no section supported by* 0.

Proof. Otherwise it should contain a section δ (the dirac mass δ is invariant by rotation and if q^+n $(n\in N)$ vanishes outside of 0, it vanishes everywhere). □

If $\mathscr{M}$ is a $\mathscr{D}_V$-module, we set $\overline{\mathscr{M}}=j_*j^*(\mathscr{M})$ [1] where j is the embedding $V\setminus\{0\}\rightarrow V$. We have a canonical homorphism $\mathscr{M}\rightarrow\overline{\mathscr{M}}$; the functor $\mathscr{M}\mapsto\overline{\mathscr{M}}$ is left exact (as j_*).

3.2. Isomorphism between $q^+\mathscr{N}$ and $\overline{q^+\mathscr{N}}$

In this section we intend to show that the inverse image by the map $q\colon X\rightarrow\mathbb{C}$, of a regular holonomic $\mathscr{D}_{\mathbb{C}}$-module $\mathscr{N}$ is isomorphic to $\overline{q^+\mathscr{N}}=j_*j^*(q^+\mathscr{N})$ that is the canonical homomorphism $q^+\mathscr{N}\rightarrow\overline{q^+\mathscr{N}}$ is an isomorphism. It is the subject of the following Theorem:

[1] $j_*j^*(\mathscr{M})$ is the module of holomorphic sections outside of $\{0\}$ of $\mathscr{M}$.

Theorem 3.4. *If $\mathcal{N}$ is a holonomic $\mathcal{D}_{\mathbb{C}}$-module regular singular at* 0, *the canonical homomorphism $q^+\mathcal{N}\to\underline{q^+}\mathcal{N}$ is an isomorphism.*

Proof. We are going to show it at first when $\mathcal{N}$ is simple, that is of the form $\mathcal{O}_{\mathbb{C}}$ or $\delta=\mathcal{O}\left(\frac{1}{t}\right)/\mathcal{O}$ or $\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(t\partial_t-\lambda)$ with $\lambda\notin\mathbb{Z}$, $t\in\mathbb{C}$. The module $q^+\mathcal{N}$ is generated by the sections $q^+\partial_t^j\eta_\lambda$, $j\in\mathbb{N}$, where the η_λ are generators of $\mathcal{N}$ such that the degree of each η_λ differs form $-\frac{n-2}{2}$ (because of the relation $\Delta q^{-1}\eta_\lambda=2q^{-1}\partial_t(n-2+2t\partial_t)\eta_\lambda$).

- If $\mathcal{N}=\mathcal{O}_{\mathbb{C}}$, we have $q^+\mathcal{N}\simeq\mathcal{O}_V$.
- If $\mathcal{N}=\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(t\partial_t-\lambda)$ with $\lambda\notin\mathbb{Z}$, as an $\mathcal{O}_{\mathbb{C}}$-module it is isomorphic to $\mathcal{O}\left(\frac{1}{t}\right)\cdot e$ where e is a generator such that $t\partial_t e=\lambda e$, $\lambda\notin\mathbb{Z}$. We see that $q^+\mathcal{N}=\mathcal{D}_V/\mathcal{D}_V(x\cdot\partial_x-2\lambda)\simeq\mathcal{O}\left(\frac{1}{q}\right)\cdot f$ (as an $\mathcal{O}$-module), with $f=q^{-1}(e)$ a SO(q)-invariant generator such that $x.\partial_x f=2\lambda f$. Indeed the module $\mathcal{N}=\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(t\partial_t-\lambda)$ is generated by the η_μ ($\mu=\lambda$ mod $\mathbb{Z}$) and the relations $t\eta_\mu=\eta_{\mu+1}$, $\partial_t\eta_\mu=\eta_{\mu-1}$ (for λ integer, this is true only for $\lambda<0$).

As an $\mathcal{O}_{\mathbb{C}}$-module $\mathcal{N}\simeq\mathcal{O}\left(\frac{1}{t}\right)\cdot\eta_{\lambda_0}$ with $\lambda_0=\lambda$ mod $\mathbb{Z}$ (and $\lambda_0<<0$ if it is integer).

Denote by $\mathcal{M}$ the $\mathcal{D}_V$-module generated by the homogeneous SO(q)-invariant section f_μ (with $\mu=\lambda$ mod $2\mathbb{Z}$) and the relations $qf_\mu=f_{\mu+2}$, $x\cdot\partial_x f_\mu=\mu f_\mu$, $\Delta f_\mu=\mu(\mu+n-2)f_{\mu-2}$. As an $\mathcal{O}_V$-module $\mathcal{M}\simeq\mathcal{O}\left(\frac{1}{q}\right).f_\mu$ if $\mu=\lambda$ mod2 $\mathbb{Z}$ and $\mu<-n+2$. Then $\mathcal{M}$ is a free $\mathcal{O}\left(\frac{1}{q}\right)$-module of rank 1.

We know that $\mathcal{N}$ is generated by the η_μ ($\mu=\lambda$ mod $\mathbb{Z}$) this implies that $q^+\mathcal{N}$ is generated by the $q^+(\eta_\mu)$ and the $q^+(\partial_t^j\eta_\mu)=\mu(\mu-1)\cdots(\mu-j+1)$ $q^+(\eta_{j-\mu})$, $J\geq 1$. The morphism h: $q^+\mathcal{N}\to\mathcal{M}$ defined by $h(q^+(\eta_\lambda))=f_\lambda$ (it satisfies the good relations) is surjective since $\mathcal{M}$ is a finite type module (it is generated by the f_μ if Re $\mu\leq -n$).

Then h is one to one outside of 0 thus ker h and coker h are supported by $\{0\}$. But $q^+\mathcal{N}$ has no submodules supported by $\{0\}$ (see. Corollary 3.3) therefore h is one to one. The Theorem is true in this case because meromorphic functions with poles in the quadric cone $\{q=0\}$ extend at the origin if $n\geq 2$.

If $\mathcal{N}=\delta=\mathcal{O}\left(\frac{1}{t}\right)/\mathcal{O}$, $q^+\mathcal{N}$ is isomorphic to $\mathcal{O}\left(\frac{1}{q}\right)/\mathcal{O}$ (polar parts). The Theorem is true because the cone Q: $q=0$ is normal and the origin $\{0\}$ is of codimension great than 3 in the cone Q that is any meromorphic section with poles in the cone extends at 0.

We prove the general case by induction on the length of $\mathcal{N}$: if $\mathcal{N}$ is not simple there exists an exact sequence $0\to\mathcal{N}'\to\mathcal{N}\to\mathcal{N}''\to 0$ where $\mathcal{N}'$, $\mathcal{N}''$ are of lengh less that the lengh of $\mathcal{N}$. Hence one has the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}' & \xrightarrow{a} & \mathcal{N} & \xrightarrow{b} & \mathcal{N}'' & \longrightarrow & 0 \\ & & u\downarrow & & v\downarrow & & w\downarrow & & . \\ 0 & \longrightarrow & \bar{\mathcal{N}}' & \xrightarrow{\bar{a}} & \bar{\mathcal{N}} & \xrightarrow{\bar{b}} & \bar{\mathcal{N}}'' & \longrightarrow & 0. \end{array}$$

In this diagram the line (a, b) is exact because q^+ is an exact functor (see 3.1.1). Moreover u and w are isomorphisms (by induction hypothesis). The line $(\bar{a}, \bar{b})$ is left exact ($\mathcal{M}\to\bar{\mathcal{M}}$ is left exact) and $\bar{b}$ is surjective because $\bar{b}v=wb$ is. Therefore it is exact. Finally, by the "five Lemma", v is an isomorphism.

3.2.1. Comparison of $\mathcal{M}$ and $q^+i^+(\mathcal{M})$

Let e_1, e_2 be two isotropic vectors (i.e. $q(e_1)=q(e_2)=0$) in V such that $2q(e_1, e_2)=1$ where q is, again, the associated bilinear form (for example $e_2=\frac{1}{\sqrt{2}}(1, i, 0, \cdots,0)$, $e_2=\frac{1}{\sqrt{2}}(1, -i, 0,\cdots,0)$ if $q(x)=\sum_{j=1}^{n}x^2{}_j)$. Denote by $D=\mathbb{C}e_1+e_2$ the isotropic affine line parametrized by $i(t)=te_1+e_2$ where t is a coordinate of $\mathbb{C}$, one has $qi(t)=t$ i.e. the map i is a section of q. One has the following Lemma:

Lemma 3.5. *The line D is non characteristic for $\mathcal{M}$ i.e. $T_D^*V\cap \mathrm{char}\mathcal{M}\subset T_V^*V$.*

Proof. On the line D one has $q=t$, this implies that D is transversal to the fibers of q, especially to the singular fiber Q: $q=0$. □

Let $\mathcal{M}$ be an object of $\mathrm{Mod}_\Lambda^{\mathrm{rh}}(\mathcal{D}_V)$. As D is non characteristic for $\mathcal{M}$, $\mathcal{M}$ is canonically isomorphic to $q^+i^+\mathcal{M}$ on a neighborhood of the line D (see. [BM-M], §3). One knows, according to Kashiwara (see. [K2]) that the sheaf $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, q^+i^+\mathcal{M})$ is constructible. This implies that the homomorphism of sheaves of $\mathcal{D}_V$-module are all locally constant sheaves in the fibers Q_c: $q=c$, $c\in\mathbb{C}^*$, in particular in $Q-\{0\}$. As the group $\tilde{G}$ acts on $\mathcal{M}$ and $q^+\mathcal{N}$ it acts also on $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, q^+i^+\mathcal{M})$ and because of the action of the group G the stratas are the orbits of G that is to say $\{0\}$, $Q\backslash\{0\}$, $V\backslash Q$ (see. [K-K]). The sheaf $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, q^+i^+\mathcal{M})$ has a canonical section u defined in the neighborhood of the line D (corresponding with the isomorphism $\mathcal{M}\tilde{\to}q^+i^+\mathcal{M}$ which induces the identity on D).

Let us recall that ϵ_M is the action of the central spinor on the $\mathcal{D}_V$-module $\mathcal{M}$. One has the following Proposition.

Proposition 3.6. *If $\epsilon_M = 1_M$, the canonical isomorphism $\mathcal{M} \tilde{\rightarrow} q^+ i^+ \mathcal{M}$ defined in the neighborhood of D such that $i^+ . u = Id$, extends to $V \backslash \{0\}$.*

Proof. One sets $\mathcal{N} := i^+ \mathcal{M}$. One has in any case $\epsilon_{q^+\mathcal{N}} = 1_{q^+\mathcal{N}}$ because $q^+\mathcal{N}$ is generated by invariant sections.

Let γ be a path of the group Spin (q) (or a lifting of a path of SO(q)) joining identity 1 to ϵ i.e. $\gamma(0) = 1$, $\gamma(1) = \epsilon$. Then for all x in the non singular quadric (resp. the singular quadric) $Q_c := \{x \in V \backslash \{0\}, q(x) = c\} c \in \mathbb{C}$ $c \neq 0$ (resp. $Q_0 = Q \backslash \{0\}$ if $q(x) = 0$) the closed path $\gamma \cdot x$: $t \mapsto \gamma(t) x$, $t \in [0, 1]$ generates the homotopy group $\Pi_1(Q_c, x)$. Let us recall that this group is trivial except for $n = 3$, $c = 0$ (see. [Spa]): one has

a) $\Pi_1(Q \backslash \{0\}, x_0) = \begin{cases} \{1\} & \text{if } n \geq 4 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 3 \end{cases}$ and the generator is the image of the closed path $t \mapsto \gamma(t) x_0$.

b) $\Pi_1(Q_c, x)_{c \neq 0} = \{1\}$ if $n \neq 2$

c) $\Pi_1(V \backslash Q) = \mathbb{Z}$ if $n \neq 2$.

This means that the quadric Q_c is simply connected if $n \geq 4$, or if $n = 3$ and $c \neq 0$. The homotopy group $\Pi_1(Q_c, x)$ acts on the constructible sheaves we are interested in. The sheaf $\mathrm{Hom}_{\mathscr{D}_V}(\mathcal{M}, q^+ i^+ \mathcal{M})$ is locally constant over the stratas $Q - \{0\}$, $V - Q$ and it has a section u. The path γ defines on $\mathrm{Hom}_{\mathscr{D}_V}(\mathcal{M}, q^+ i^+ \mathcal{M})$ a path γ^h: $t \mapsto u_t = \gamma_t u \gamma_t^{-1}$ which lifts the path $t \mapsto \gamma(t) . x_0$ of $Q - \{0\}$. Therefore the action of the generator of the homotopy group $\Pi_1(Q_0, x_0)$ on $\mathrm{Hom}_{\mathscr{D}_V}(\mathcal{M}, q^+ i^+ \mathcal{M})$ is the action of the central spinor ϵ that is $\gamma^h(1)$. The action of ϵ on $\mathrm{Hom}_{\mathscr{D}_V}(\mathcal{M}, q^+ i^+ \mathcal{M})$ is trivial. Indeed, as $\epsilon_{q^+\mathcal{N}} = 1_{q^+\mathcal{N}}$, one has $\gamma^h(1) . u = \epsilon_{\mathcal{M}} u \epsilon_{q^+\mathcal{N}}^{-1} = u$.

The sheaf $\mathrm{Hom}_{\mathscr{D}_V}(\mathcal{M}, q^+ i^+ \mathcal{M})$ is trivial outside of the cone Q, and trivial over the singular quadric $Q - \{0\}$. Consequently the section u extends, on a unique way to the union of quadrics $\cup_{c \in \mathbb{C}} Q_c = V - \{0\}$. □

3.3. General case

Let $\mathcal{M}$ be an objecct of $\mathrm{Mod}_\Lambda^{rh}(\mathscr{D}_V)$. One knows that (see. Remark 1.7) ϵ defines an automorphism of $\mathscr{D}_V$-modules of $\mathcal{M}$, thus $\mathcal{M}$ is decomposed into $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$ with $\epsilon_{\mathcal{M}} = \pm 1_{\mathcal{M}}$.

Let $i(t) = te_1 + e_2$ be the parametrisation of an affine line D as in n°3.2.1. One has seen that the module $\mathcal{M}$ is not characteristic for i (see. Lemma 3.5) and one sets $\mathcal{N} = i^+ \mathcal{M}$; the $\mathscr{D}_V$-module $q^+\mathcal{N}$ is isomorphic to $\mathcal{M}$ in the neighborhood of D: denote by u the canonical isomorphism (such that $i^+ u = Id_{|D}$). If $\epsilon_{\mathcal{M}} = 1_{\mathcal{M}}$, it arises from Proposition 3.6 that u extends along $V - \{0\}$, hence there exists a morphism $\bar{u}$: $\mathcal{M} \rightarrow q^+\mathcal{N}$ $(= \overline{q^+\mathcal{N}})$ (see. Theorem 3.4) which is an isomorphism outside of 0.

We are going to see that any regular holonomic $\mathscr{D}_V$-module invariant under the action of ϵ, $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathscr{D}_V)$, is generated by a finite family of invariant global sections $(u_i)_{i=1,\cdots,p}$ such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]s_i<\infty$.

3.3.1. Invariant sections

Theorem 3.7. *If $\epsilon_{\mathscr{M}}=1_{\mathscr{M}}$, $\mathscr{M}$ is generated by G-invariant global sections.*

Proof. Let us recall that $\mathscr{A}\subset\mathscr{W}_n$ is the algebra of differential operators with polynomial coefficients which are $\mathrm{SO}(q)$-invariant.

Let $\mathscr{M}^G$ (resp. $(q^+\mathscr{N})^G$) be the module over $\mathscr{A}$ of global sections of $\mathscr{M}$ (resp. $q^+\mathscr{N}$) which are G-invariant. Recall that $\mathscr{M}_\lambda:=\Gamma(V,\mathscr{M})\cap[\cup_{p\in\mathbb{N}}\ker(\theta-\lambda)^p]$ is the complex vector space of homogeneous global sections of $\mathscr{M}$ of degree $\lambda\in\mathbb{C}$. One knows that there is a morphism $\mathscr{M}\xrightarrow{\bar{u}}q^+\mathscr{N}$ (see. 3.3). This morphism induces a natural morphism $\mathscr{M}^G\to(q^+\mathscr{N})^G\simeq q^{-1}\mathscr{N}$ (inverse image of global sections of $\mathscr{N}$ see. Proposition 3.2) which sends bijectively $\mathscr{M}^G_{2\lambda}$ on $q^{-1}\mathscr{N}$ if $\lambda\notin-n-\mathbb{N}$ (see. Corollary 3.3).

Let $\mathscr{M}'\subset\mathscr{M}$ be the submodule generated, over $\mathscr{D}_V$, by $\mathscr{M}^G$. Then $\mathscr{M}'$ contains the module $\mathscr{M}^G_{2\lambda}\simeq q^{-1}\mathscr{N}_\lambda$ if $\lambda\notin-n-\mathbb{N}$. Seeing that $q^+\mathscr{N}$ is generated by the $q^{-1}\mathscr{N}_{\frac{\lambda}{2}}$ with $\mathrm{Re}\left(\frac{\lambda}{2}\right)\geq-1$, the restriction of $\mathscr{M}'$ on the line D, $i^+(\mathscr{M}')$ is isomorphic to $\mathscr{N}$ and one has $\mathscr{M}'\simeq q^+\mathscr{N}\simeq\mathscr{M}$ outside of the origin.

Let K be the cokernel of the embedding $\mathscr{M}'\subset\mathscr{M}$, one has the exact sequence $0\to\mathscr{M}'\to\mathscr{M}\to K\to0$. The quotient K is coherent and supported by the origin so that it is generated by invariant homogeneous global sections $(s_i)_{i=1,\cdots,n_1}$. Let σ_i be a lifting of s_i. Denote by $\tilde{\sigma}_i=\int_{\mathrm{SO}(n,\mathbb{R})}g\cdot\sigma_i dg$ the average of σ_i over the compact maximal subgroup $\mathrm{SO}(n,\mathbb{R})$ of $\mathrm{SO}(q)$ (see. Paragrah 2, Remark 2.3 for the calculus of averages in $\mathscr{D}$-modules). Then $\tilde{\sigma}_i$ is also an invariant lifting of s_i. Therefore $\tilde{\sigma}_i\in\Gamma(V,\mathscr{M}')$ and $s_i=0$ for $i=1,\cdots,n_1$. Consequently $K=0$ and $\mathscr{M}'\simeq\mathscr{M}$. □

Since ϵ acts trivially on all submodules and quotients of $\mathscr{M}$, we have the following Corollary:

Corollary 3.8. *If $\epsilon_{\mathscr{M}}=1_{\mathscr{M}}$, any subquotient of $\mathscr{M}$ is generated by G-invariant global sections.*

3.3.2. Diagrams associated to the $\mathscr{D}$-module $\mathscr{M}$

Let us reall that $\mathscr{W}_n$ indicates the Weyl algebra on V and that $\bar{\mathscr{A}}\subset\mathscr{W}_n$ indicates the algebra of differential operators with polynomial coefficients which are invariant by rotation (see. Paragraph 2). Recall also that the $\mathbb{C}$-algebra $\bar{\mathscr{A}}$ is generated by the operators q, Δ, θ satisfying the following relations $[\theta, q]=2q$, $[\theta, \Delta]=-2\Delta$, $[\Delta, q]=4\theta+2n$ (see. Proposition 2.1). One has set $\mathscr{A}:=\bar{\mathscr{A}}/\bar{\mathscr{A}}\cap I$ and that $\bar{\mathscr{A}}/\bar{\mathscr{A}}\cap I=\bar{\mathscr{A}}/J$ where I (resp. J) is the left ideal sheaf of $\mathscr{D}_V$ generated by infinitesimal generators of $\mathrm{SO}(q)$ (resp. $q\Delta-\theta(\theta+n-2)$) (see. Lemma 2.2). The algebra $\mathscr{A}$ is graded by the action of homotheties and it acts naturally on sections which are invariant by rotation.

Denote by $\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$ the category of finite type graded $\mathscr{A}$-module T such that $\dim_{\mathbb{C}}\mathbb{C}[\theta]u<\infty$, for $u\in T$. It amounts to the same thing to give a graded vector space $T=\bigoplus_{\lambda\in\mathbb{C}}T_\lambda (T_\lambda:=\mathrm{T}\cap[\cup_{k\in\mathbb{N}}\ker(\theta-\lambda)^k])$ equipped with three endomorphisms θ, q, Δ of degree 0, $+2$, -2 respectively such that $[\theta, q]=2q\,[\theta, \Delta]=-2\Delta$, $[\Delta, q]=4\theta+2n$. Each T_λ is a finite dimensional $\mathbb{C}$-vector space such that $\theta-\lambda$ is a nilpotent operator over T_λ and T is generated by a finite number of T_λ.

Let us recall that $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathscr{D}_V)$ indicates the category of regular holonomic $\mathscr{D}_V$-modules $\mathscr{M}$ such that char $\mathscr{M}\subset\Lambda$ and $\epsilon_{\mathscr{M}}=1_{\mathscr{M}}$.
If $\mathscr{M}$ is an object of $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathscr{D}_V)$, one denotes by $\Psi(\mathscr{M})$ the $\mathscr{A}$-module formed by homogeneous global sections of $\mathscr{M}$ which are invariant by rotation. Recall that $\Psi_\lambda(\mathscr{M}):=[\Psi(\mathscr{M})]\cap[\cup_{k\in\mathbb{N}}\ker(\theta-\lambda)^k]$ is the $\mathbb{C}$-vector space of the homogeneous global sections of $\Psi(\mathscr{M})$ of degree λ and that $\Psi(\mathscr{M})=\bigoplus_{\lambda\in\mathbb{C}}\Psi_\lambda(\mathscr{M})$ (see. Theorem 1.3). It is easy to see that $\Psi(\mathscr{M})$ is a finite type graded $\mathscr{A}$-module i.e. $\Psi(\mathscr{M})\in\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$. Indeed, if the $\mathscr{D}_V$-module $\mathscr{M}$ is generated by a finite family of sections $s_1, \cdots, s_N$ which are homogeneous and invariant by rotation then $s_1,\cdots,s_N$ generate $\Psi(\mathscr{M})$ as a $\mathscr{A}$-module: Let $s=\sum p_i(x, \partial_x)s_i$ be an invariant section of $\mathscr{M}$ ($p_i\in\Gamma(V, \mathscr{D}_V)$) and let $\tilde{p}_i$ be the average of p_i over $\mathrm{SO}(n, \mathbb{R})$, maximal compact subgroup of $\mathrm{SO}(q)$, then $\tilde{p}_i\in\tilde{\mathscr{A}}$. Denoting by g_i the class of $\tilde{p}_i$ modulo J (i.e. $g_i\in\mathscr{A}$), one also has $s=\sum\tilde{p}_is_i=\sum g_is_i$.
Such a module $\Psi(\mathscr{M})$ is characterized by a diagram of $\mathbb{C}$-vector spaces

$$\cdots\rightleftarrows\Psi_\lambda(M)\underset{\Delta}{\overset{q}{\rightleftarrows}}\Psi_{\lambda+2}(M)\rightleftarrows\cdots$$

equipped with operators q (of degree 2), Δ (of degree -2), $\theta=x\cdot\partial_x$ (of degree 0) satisfying the relations $[\theta, q]=2q, [\theta, \Delta]=-2\Delta, [\Delta, q]=4\theta+2n$. Moreover one has $q\Delta=\theta(\theta+n-2)$ over invariant sections, and the operator $(\theta-\lambda)$ is nilpotent over each $\Psi_\lambda(\mathscr{M})$.

Conversely, if T is an object of $\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$, one associates it the module

$$\Phi(T)=\mathscr{M}_0\otimes_{\mathscr{A}}T$$

where $\mathscr{M}_0=\mathscr{W}_n/\sum_{i,j}\mathscr{W}_n\left(\frac{\partial q}{\partial x_i}\partial_j-\frac{\partial q}{\partial x_j}\partial_i\right)=\mathscr{W}_n/I$. It is a left $\mathscr{W}_n$-module and a right $\mathscr{A}$-module.

Therefore we have defined two functors $\mathcal{M}\to\Psi(\mathcal{M})$ and $T\to\Phi(T)$. We will prove the following Theorem:

Theorem 3.9. *The functors* $\mathcal{M}\to\Psi(\mathcal{M})$ (resp. $T\to\Phi(T)$) *are equivalence of categories of the category* $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathcal{D}_V)$ *of regular holonomic* $\mathcal{D}_V$*-modules* $\mathcal{M}$ *such that* $\epsilon_{\mathcal{M}}=1_{\mathcal{M}}$, char $\mathcal{M}\subset\Lambda$ *over the category* $\mathrm{Mod}^{\mathrm{h}}(\mathcal{A})$ *of homogeneous graded* $\mathcal{A}$*-modules of finite type such that* $T=\bigoplus_{\lambda\in\mathbb{C}}T_\lambda$.

Remark 3.10. Let us recall that the algebra $\mathcal{A}$ becomes identified with $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))/\left(C-\frac{n}{2}\left(\frac{n}{2}-2\right)\right)$ where C denote the Casimir operator $\Big($i.e the quotient of the envelopping algebra of $\mathrm{sl}_2(\mathbb{C})$ by the ideal generated by $C-\frac{n}{2}\left(\frac{n}{2}-2\right)\Big)$ (see. Remark 2.3). In other words the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathcal{D}_V)$ is equivalent to the category of finite type modules over $\mathcal{U}(\mathrm{sl}_2(\mathbb{C}))/\left(C-\frac{n}{2}\left(\frac{n}{2}-2\right)\right)$.

The previous Theorem follows immediatly from the two Lemmas:

Lemma 3.11. *The canonical morphism* $T\to\Psi(\Phi(T))$ $(t\mapsto 1\otimes t)$ *is an isomorphism, and defines an isomorphism of functors* $\mathrm{Id}_{\mathrm{Mod}^{\mathrm{h}}(\mathcal{A})}\to\Psi\circ\Phi$.

Proof. Let us recall that $\mathcal{M}_0=\mathcal{W}_n/I$. Denote by ϵ_0 (the class of $1_{\mathcal{W}_n}$ modulo I) the canonical generator of $\mathcal{M}_0$. Let $f\in\mathcal{W}_n$, denote by $\tilde{f}\in\bar{\mathcal{A}}$ its average on $\mathrm{SO}(n,\mathbb{R})$ and by φ the class of $\tilde{f}$ in $\bar{\mathcal{A}}$ modulo $\bar{\mathcal{A}}\cap I$. As ϵ_0 is invariant by rotation one has $\widetilde{f\epsilon_0}=\tilde{f}\epsilon_0=\epsilon_0\varphi$. Moreover $\tilde{f}\varphi=0$ if and only if $\tilde{f}\in I$, in other words $\varphi=0$. Therefore the average operator (over $\mathrm{SO}(n,\mathbb{R})$) $f\mapsto\tilde{f}$, $\mathcal{W}_n\to\bar{A}$ induces a surjective homomorphism of $\mathcal{A}$-modules $v_0:\mathcal{M}_0\to\mathcal{A}$. More generally for all $\mathcal{A}$-module T in $\mathrm{Mod}^{\mathrm{h}}(\mathcal{A})$ the morphism $v_0\otimes 1_T$ is a surjection $v_T:\mathcal{M}_0\otimes_{\mathcal{A}}T\to\mathcal{A}\otimes_{\mathcal{A}}T=T$ which is left inverse of the morphism $u_T: T\to\mathcal{M}_0\otimes_{\mathcal{A}}T$, $t\mapsto\epsilon_0\otimes t$ i.e. $(v_0\otimes 1_T)\circ\epsilon_0\otimes 1_T=v_0\epsilon_0=1_T$; that is u_T is injective. The image of u_T is exactly the set of invariant sections of $\mathcal{M}_0\otimes_{\mathcal{A}}T$ that is to say $\Psi(\Phi(T))$; indeed if $s=\sum_{i=1}^{N_0}f_i\otimes t_i$ is an invariant section of $\mathcal{M}_0\otimes_{\mathcal{A}}T$, we can replace each f_i by their average $\tilde{f}_i\in\mathcal{A}$, then $s=\sum_{i=1}^{N_0}\tilde{f}_i\otimes t_i=\epsilon_0\otimes\sum_{i=1}^{N_0}\tilde{f}_it_i\in\epsilon_0\otimes T$ i.e. $\sum_{i=1}^{N_0}\tilde{f}_it_i\in T$. Thus the morphism u_T is an isomorphism from T to $\Psi(\Phi(T))$ and defines a functorial isomorphism. □

Lemma 3.12. *The canonical morphism* $w:\Phi(\Psi(\mathcal{M}))\to\mathcal{M}$ *is an isomorphism and defines an isomorphism of functors* $\Phi\circ\Psi\to\mathrm{Id}_{\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}}$.

Proof. As $\epsilon_{\mathcal{M}}=1_{\mathcal{M}}$, $\mathcal{M}$ is generated by a finite family of invariant sections $(s_i)\in\Psi(\mathcal{M})$ (see. Theorem 3.7) so that w is surjective.

Anyway w is injective. Indeed if K is the kernel of the morphism $w: \Phi(\Psi(\mathscr{M}))\to\mathscr{M}$, one has $\epsilon_K = 1_K$ because $\epsilon_{\Phi(\Psi(\mathscr{M}))} = 1_{\mathscr{M}}$; in other words the $\mathscr{D}_V$-module K is also generated by its invariant sections i.e. $\Psi(K)$ (see. Corollary 3.8). Thus one has $\Psi(K)\subset\Psi[\Phi(\Psi(\mathscr{M}))] = \Psi(\mathscr{M})$ (see. the previous Lemma) and as $\Psi(\mathscr{M})\to\mathscr{M}$ is injective ($\Psi(\mathscr{M})\subset\Gamma(V, \mathscr{M})$) one has $\Psi(K)=0$. Therefore $K=0$ (because $\Psi(K)$ generates K). □

Remark 3.13. From Lemma 1.4, one deduces that the results above are also true for analytics $\mathscr{D}_V$-modules.

3.4. Classification of homogeneous graded $\mathscr{A}$-modules

An $\mathscr{A}$-module $T\in\mathrm{Mod}^{\mathrm{h}}(\mathscr{A})$ defines an infinite diagram whose objects are finite dimensional vector spaces T_λ and arrows are linear maps deduced from q, Δ, θ:

$$\cdots T_\lambda \underset{\Delta}{\overset{q}{\rightleftarrows}} T_{\lambda+2} \underset{\Delta}{\overset{q}{\rightleftarrows}} \cdots$$

We shall remark that the operator θ over T_λ is completly determined by q and Δ because of the relation $[\Delta, q] = 4\left(\theta+\frac{n}{2}\right)$. We should forget it in the diagram, adding appropriate conditions over $q\Delta$ $\left(\text{i.e. } \frac{1}{4}[\Delta, q]-\frac{n}{2}-\lambda \text{ is nilpotent over } T_\lambda \text{ and } q\Delta=\left(\frac{1}{4}[\Delta, q]+\frac{n}{2}\right)\left(\frac{1}{4}[\Delta, q]+\frac{n}{2}-2\right)\right)$. This diagram is completly determined by a finite subset of objects and arrows: indeed we see at once that

(i) for $\sigma\in\mathbb{C}/2\mathbb{Z}$, if we denote $T^\sigma\subset T$ the submodule $T^\sigma=\bigoplus_{\lambda=\sigma \bmod 2\mathbb{Z}}T_\lambda$ then T is generated by the direct sum of T^σ.

$$T=\bigoplus_{\sigma\in\mathbb{C}/2\mathbb{Z}}T^\sigma=\bigoplus_{\sigma\in\mathbb{C}/2\mathbb{Z}}\left(\bigoplus_{\lambda=\sigma \bmod 2\mathbb{Z}}T_\lambda\right).$$

(ii) if $\sigma\not\equiv 0$, n, mod $2\mathbb{Z}$ ($\sigma\equiv\lambda$ mod $2\mathbb{Z}$), the maps Δ, q are bijectives. Then T^σ is completly determined up isomorphism by one of the T_λ and the action of $\theta =\left(\frac{1}{4}[\Delta, q]-\frac{n}{2}\right)$. In other words the functor

$$T^\sigma\mapsto\left(T_\lambda, \left(\frac{1}{4}[\Delta, q]-\frac{n}{2}-\lambda\right)\right)$$

is an equivalence of categories between the category of the T^σ's and the category of $\mathbb{C}$-vector spaces T_λ equipped with a nilpotent endomorphism $\left(\frac{1}{4}[\Delta, q]-\frac{n}{2}-\lambda\right)$.

(iii) if n is odd, $\sigma\equiv 0$ (resp. n) mod $2\mathbb{Z}$, the functor

$$T^\sigma\mapsto\left(T_{-2}\underset{\Delta}{\overset{q}{\rightleftarrows}}T_0\right) \text{ (resp. } T_{-n}\rightleftarrows T_{-n+2})$$

is an equivalence of categories between the category of the T^{σ}'s such that $T^{\sigma}=\bigoplus_{\lambda\in 2\mathbb{Z}}T_{\lambda}$ (resp. $\bigoplus_{\lambda=n \bmod 2\mathbb{Z}}T_{\lambda}$) and the category of diagrams of the form above. The operators $q\Delta$ (resp. Δq) on T_{λ} is with one only eigenvalue $\lambda(\lambda+n-2)$ (resp. $(\lambda+2)(\lambda+n)$) in such a way that the equation $q\Delta=\theta(\theta+n-2)$ (resp. $\Delta q=(\theta+2)(\theta+n)$) admits one unique solution θ of eigenvalue λ if $\lambda\neq\frac{-n+2}{2}\left(\text{resp. } \frac{-n-2}{2}\right)$ critical value; here $\lambda=0,\ -2,\ -n,\ -n+2$ and $\frac{-n+2}{2}$ is $\frac{1}{2}$ integer, thus it is always the case. In the others degrees q or Δ is bijective and determines the remaining by induction.

(iv) if n is even, $\sigma\equiv 0 \bmod 2\mathbb{Z}$: We may consider either all the diagram

$$T_{-n}\rightleftarrows T_{-n+2}\rightleftarrows\cdots\rightleftarrows T_0$$

with the operator θ (which we cannot reconstitute from q, Δ on $T_{\frac{-n-2}{2}}$ if $\frac{-n-2}{2}$ is even $(n\equiv 2 \bmod 4)$)
or only one diagram with three elements

$$T_{-n}\underset{b}{\overset{a}{\rightleftarrows}}T_{-2}\underset{\Delta}{\overset{q}{\rightleftarrows}}T_0$$

with $a=q^{(n-2)/2}$, $b=\Delta^{(n-2)/2}$ and the relations θ, $q\Delta$, Δq, etc$\cdots$); and there is an equivalence of categories between the category of the $T^{\sigma}=\bigoplus_{\lambda\in 2\mathbb{Z}}T_{\lambda}$ and the category of finite diagrams with $\frac{n}{2}$ (or 3) vertices $T_{-n}\underset{b}{\overset{a}{\rightleftarrows}}T_{-2}\underset{\Delta}{\overset{q}{\rightleftarrows}}T_0$.

In any case, except the last one, the number of isomorphism class of diagrams, for a given dimension and fixed λ, is finite. In the last case, there is continuous family of non isomorphic diagrams (see the example below).

3.4.1. A continuous family of non isomorphic $\mathcal{D}_V$-modules with "fixed monodromy"

Example 3.14. For $n=4$ and for $\lambda\in\mathbb{C}$, $T(\lambda)$: $T_{-4}\underset{\Delta}{\overset{q}{\rightleftarrows}}T_{-2}\underset{\Delta}{\overset{q}{\rightleftarrows}}T_0$ be the diagram constructed as follows:

1) T_j is of dimension 2 and has a basis consisting of e_j, f_j $(j=0,\ -2,\ -4)$.

2) On T_0 $\quad \theta(f_0)=\frac{1}{2}e_0,\quad \theta(e_0)=0.$

On T_{-2} $\quad (\theta+2)=0.$

On T_{-4} $\quad (\theta+4)f_{-4}=-\frac{1}{2}e_{-4}\quad (\theta+4)e_{-4}=0.$

3) $qe_{-2}=0\quad \Delta e_0=0.$

$qf_{-2}=e_0\quad \Delta f_0=e_{-2}+f_{-2}$

(one has $\Delta q=0$ over T_{-2}, $q\Delta=2\theta=\theta(\theta+4-2)$ over T_0).

$\Delta e_{-2}=e_{-4} \quad qe_{-4}=0$
$\Delta f_{-2}=0 \quad qf_{-4}=e_{-2}+\lambda f_{-2}$
(one has again $q\Delta=0$ on T_{-2}, $\Delta q=-2(\theta+4)(\theta+2)$).

The diagramm $T(\lambda)$ as constructed above corresponds to a module $\mathcal{M}(\lambda)\in \mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathscr{D}_V)$. The four lines $\mathbb{C}e_{-2}=T_{-2}\cap \ker q$, $\mathbb{C}f_{-1}=T_{-2}\cap \ker \Delta$, $\mathbb{C}(e_{-2}+f_{-2})=T_{-2}\cap \mathrm{Im}\Delta=\Delta(T_0)$, $\mathbb{C}(e_{-2}+\lambda f_{-2})=T_{-2}\cap \mathrm{Im}\, q$ are obviously the invariant of $\mathcal{M}(\lambda)$. Then the module $\mathcal{M}(\lambda)$ form a 1-parameter (algebraic) family of pairwise of non isomorphic modules.

§4. Odd Modules ($n=3$, $\epsilon_{\mathcal{M}}=-1_{\mathcal{M}}$)

In this Paragraph, we study the $\mathscr{D}$-modules $\mathcal{M}$ such that $\epsilon_{\mathcal{M}}=-1_{\mathcal{M}}$. That case only exists in dimension 3.

4.1. Description of the model

Let $(x, y, z)\in\mathbb{C}^3$ be a system of coordinates in which the cone Q is defined by the equation $xy=z^2$. The map i: $(u,v)\mapsto(u^2, v^2, uv)$ is a proper morphism of degree 2, surjective from $\mathbb{C}^2$ to Q and the restriction to $\mathbb{C}^2-\{0\}$ is the universal covering of $Q-\{0\}$. This last induces the covering $\mathrm{SL}(2,\mathbb{C})\simeq\mathrm{Spin}(q)\to\mathrm{SO}(q)$. Denote by $E:=i_+(\mathcal{O}_{\mathbb{C}^2-\{0\}})$ the direct image by i of the sheaf of holomorphic sections outside of $\{0\}$. The $\mathscr{D}_V$-module E is decomposed into $E=E_+\oplus E_-$ as above under the action of ϵ. The even part E_+ belongs to category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda,+}(\mathscr{D}_V)$ of modules studied in Paragraph 3, and one checks easily that it is isomorphic to the module of meromorphic sections with poles in the cone $\delta(q):=\mathcal{O}\left(\frac{1}{q}\right)/\mathcal{O}$. Let us study the $\mathscr{D}_V$-module E_-; it is supported by the cone Q and it contains the two sections $f:=i_+(u)$ and $g:=i_+(v)$, which satisfy the following relations

$(r_1)\quad (x\partial_x+y\partial_y+z\partial_z)f=-\frac{3}{2}f,\ (x\partial_x+y\partial_y+z\partial_z)g=-\frac{3}{2}g$

$(r_2)\quad (x\partial_x-y\partial_y)f=\frac{1}{2}f,\quad (x\partial_x-y\partial_y)g=-\frac{1}{2}g$

$(r_3)\quad (2z\partial_y-x\partial_x)f=0,\quad (2z\partial_y-x\partial_x)g=f$

$(r_4)\quad (2z\partial_y-x\partial_x)f=g,\quad (2z\partial_y-x\partial_x)g=0$

$(r_5)\quad yf=zg,\ zf=xg.$

One has the following Theorem.

Proposition 4.1. *The $\mathscr{D}_V$-module E_- is generated by the generators f, g and the relations r_i, $i=1,\cdots,5$.*

Proof. Let $\mathcal{M}'$ be the $\mathscr{D}_V$-module defined by the relations $(r_i)_{i=1,\cdots,5}$. One knows that the module E_- is the odd parts of $i_+(\mathcal{O}_{\mathbb{C}^2-\{0\}})$ and it contains the sections $f=i_+(u)$, $i_+(v)$. One has an homomorphism b of $\mathscr{D}_V$-modules, b: $\mathcal{M}'\to E_-$

such that $b(h)=i_+(u)$, $b(l)=i_+(v)$ where h and l satisfy the relations $(r_i)_{i=1,\cdots,5}$. It is enough to show that b is an isomorphism.

-The morphism b is injective: indeed it is immediate that b is an isomorphism at the regular points of the cone (i.e. outside of $\{0\}$); in such a point E_- (the germ) is isomorphic to $i_+(\mathcal{O}_{\mathbb{C}^2-\{0\}})$, as $\mathcal{M}'$ which is holonomic of multiplicity 1 (so that simple) every where in $Q-\{0\}$ because the group $\widetilde{G}$ acts. Thus $\mathcal{N}:=\ker b$ is coherent and supported by $\{0\}$, that is $\epsilon_{\mathcal{N}}=1_{\mathcal{N}}$, as one also has $\epsilon_{\mathcal{N}}=\epsilon_{\mathcal{M}'}=-1$ (because $\mathcal{M}'$ and $\mathcal{N}$ are supported by the cone Q), then $\mathcal{N}$ is zero and b is injective.

-The morphism b is surjective: one sets $H=\operatorname{coker} b=E_-/\operatorname{Im} b$, it is a coherent $\mathcal{D}_V$-module supported by $\{0\}$ that implies $\epsilon_H=1_H$. But H is also supported by the cone Q (because E_- is). One deduces that $H=\{0\}$ and b is surjective.

4.2. Modules $\mathcal{M}$ such that $\epsilon_{\mathcal{M}}=-1_{\mathcal{M}}$

Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_V$-module such that $\epsilon_{\mathcal{M}}=-1_{\mathcal{M}}$; necessarily $\mathcal{M}$ is supported by the quadratic cone Q because outside of the singular quadric the homotopy group $\Pi_1(Q_c)=\{1\}$ with $Q_c=\{(x,y,z)\in\mathbb{C}^3/xy-z^2=c\}$, $c\neq 0$ and $\epsilon_{\mathcal{M}}=1_{\mathcal{M}}$. Then $\mathcal{M}$ is locally isomorphic to the direct sum of copies of $\delta(q):=\mathcal{O}\left(\frac{1}{q}\right)/\mathcal{O}$; it comes to the same thing to say that locally, outside of $\{0\}$, $\mathcal{M}\simeq\bigoplus_{i=1}^N E_i$ where $E_i\simeq E_-$. As the monodromy $\epsilon_{\mathcal{M}}$ is diagonalisable (because $\epsilon_{\mathcal{M}}^2=1_{\mathcal{M}}$) then $\mathcal{M}$ is isomorphic, globally outside of $\{0\}$, to a direct sum of a finite number of copies of E_-. Let us show that this isomorphism extends at $\{0\}$.

Theorem 4.2. *If $\epsilon_{\mathcal{M}}=-1_{\mathcal{M}}$, the module $\mathcal{M}$ is isomorphic to a direct sum of copies of E_-.*

We propose two elementary methods. In the sequel one will denote again by E the odd part E_- of $i_+(\mathcal{O}_{\mathbb{C}^2-\{0\}})$. The shortest here is to use the module of meromorphic sections with poles in $\{0\}$: if $\mathcal{M}$ is an holonomic $\mathcal{D}_V$-module, one denotes by $\widetilde{\mathcal{M}}:=\varinjlim_p \mathcal{H}om_{\mathcal{O}}(\mathfrak{m}^p,\mathcal{M})$, where $\mathfrak{m}$ is the defining ideal of the origin (it is the first term of the functor defined by ([K2]), the module of meromorphic sections outside of $\{0\}$; it is an holonomic $\mathcal{D}_V$-module (in particular coherent) if $\mathcal{M}$ is. One knows, according to Kashiwara that if $\mathcal{M}$ and $\mathcal{N}$ are regular holonomic, a morphism w: $\mathcal{M}\to\mathcal{N}$ defined outside of $\{0\}$ extends in $\widetilde{w}$: $\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ ($\widetilde{w}$ is an isomorphism if w is). If $n=3$, $\epsilon_{\mathcal{M}}=-1_{\mathcal{M}}$, one has outside of $\{0\}$ an isomorphism w: $W\otimes_{\mathbb{C}}E\to\mathcal{M}$ where W is the complex vector space of sections of $\mathcal{H}om_{\mathcal{D}_V}(E,\mathcal{M})$ (constant bundle on the cone $Q-\{0\}$), thus an isomorphism w: $W\otimes_{\mathbb{C}}E\to\mathcal{M}$. Indeed the kernels and the cokernels of the canonical arrows $E \to \tilde{E}$ and $\mathcal{M} \to \tilde{\mathcal{M}}$ are coherents and supported by the origin, thus on these $\epsilon=1$. But one has also $\epsilon=-1$ since they are subquotients of E or $\mathcal{M}$ so that they are zero and w is an isomorphism $w: W \otimes_{\mathbb{C}} E \xrightarrow{\sim} \mathcal{M}$.

Remark 4.3. Things are intuitively clear if we translate in terms of perverse sheaves: the perverse sheaf associated to $\mathcal{M}$ is $R\mathcal{H}om_{\mathcal{D}_V}(\mathcal{M}, \mathcal{O})$. It is zero outside of the cone Q, (pur in degree $+1$) locally constant with $\epsilon_{R\mathcal{H}om_{\mathcal{D}_V}(\mathcal{M},\mathcal{O})} = \epsilon_{\mathcal{O}} \otimes \epsilon_{\mathcal{M}}^{-1} = -1$, and there is nothing at the origin which is a fix point of the group $\tilde{G}$ (the action of $\epsilon = -1$ comes also from the trivial path). On $Q - \{0\}$ there is an unique locally constant bundle of rank 1 with $\epsilon = -1$ and any locally constant bundle of rank 1 is a multiple of this one; and (because $\{0\}$ is of codimension ≥ 2 in Q) there is an unique perverse sheaf, except for -1-shift, which extends it.

This is an other method more algebraic.

-Preliminaries calculus: The Lie algebra $\mathrm{sl}_2(\mathbb{C})$ is generated by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. It acts on the vector space $\mathbb{C}[u, v]$ generated by u, v and on the space W_k $(k \in \mathbb{N})$ of homogeneous polynomial of degree k by $h = u\partial_u - v\partial_v$, $e = u\partial_v$, $f = v\partial_u$ (one then obtains all the irreducible representations of finite dimension).

If x, y, z are three variables, the eigenvalues of $zh + ye - xf$ acting on W_1 are $\pm\lambda$ where $\lambda = (xy - z^2)^{1/2}$. On W_k these are the numbers $p\lambda - (k-p)\lambda = (2p-k)\lambda$, $0 \leq p \leq k$. The determinant (product of eigenvalues) is zero if k is even, and if k is odd it is equal to $(1.3 \cdots k)^2 (xy - z^2)^{(k+1)/2}$.

Let $\mathcal{M}$ be a $\mathcal{D}_V$-module, W a subspace of finite dimension of $\Gamma(U, \mathcal{M})$ ($U \subset V$ an open connected subset). One supposes that W is stable under the action of the Lie algebra $\mathcal{G} = \mathrm{so}(q)$, simple of type k with k odd.

Recall that if $q = xy - z^2$, $\mathcal{G}$ is generated by $\mathcal{H} := 2x\partial_x - 2y\partial_y$, $\mathcal{E} := x\partial_z + 2z\partial_x$, $\mathcal{F} := y\partial_z + 2z\partial_x$ and one has $z\mathcal{H} + y\mathcal{E} - x\mathcal{F} = 0$. From the last calculus it results that $q^{(k+1)/2}$ annihilates W. The situation is completly symetric in x and d and one sees as well, from the relation $\partial_z\mathcal{H} - 2\partial_x\mathcal{E} + 2\partial_y\mathcal{F}$, that $\Delta^{(k+1)/2}$ annihilates V $(\Delta = 4\partial_x\partial_y - \partial_z^2)$.

The operators $\mathcal{H}_1 = \theta + 3/2$ $(\theta = x\partial_x + y\partial_y + \partial_z)$, $\mathcal{E}_1 = q/2$ $(q = xy - z^2)$, $\mathcal{F}_1 = -\Delta/2$ $(\Delta = 4\partial_x\partial_y - \partial_z^2)$ generate the Lie algebra $\mathcal{G}_1 \sim \mathrm{sl}_2(\mathbb{C})$ (the algebra $\mathcal{U}_1$ that they generate over $\mathcal{D}_V$ is a quotient of the envelopping algebra of $\mathrm{sl}_2(\mathbb{C})$ see. Remark 2.3). The Casimir operator of the algebra $\mathcal{G}$ is

$$C = \mathcal{H}^2 + 2(\mathcal{E}\mathcal{F} + \mathcal{F}\mathcal{E}) = 4\theta(\theta + 1 - 4q\Delta)$$

and the Casimir operator of $\mathcal{G}_1$ is

$$C_1 = (\theta + 3/2)^2 - 1/2(q\Delta + \Delta q) = C - 3/4$$

If W is as above it is annihilated by Δ^N and q^N for N large enough, thus the $\mathscr{G}_1$-module that it generates is also of finite dimension, necessarily of type $\gg m$ with $k=1+2m$ (so that one has $C_1=m(m+2)=(k(k+1)-3)/4$). Therefore $\mathscr{H}_1$ is diagonalisable on W_1 and the eigenvalues are the integers j of the same parity with m, such that $-m\leq j\leq m$ in other words the elements $s\in W_1$ are sum of homogeneous components $s_i\in W_1$, of degree $-3/2+m-2i$, $0\leq i\leq m$.

Let now $\mathscr{M}$ be a regular holonomic $\mathscr{D}_V$-module, with char $\mathscr{M}\subset\Lambda$ such that $\epsilon_{\mathscr{M}}=-1_{\mathscr{M}}$, and let s be a section of $\mathscr{M}$ outside of $\{0\}$. Anyway s is a sum of series of homogeneous sections (Fourier series):

$$s=\sum_j s_j \text{ with } s_j=\left(\frac{1}{2i\pi}\right)\int s(\lambda^2 x)\lambda^{-2j}d\lambda/\lambda \quad (j\in\mathbb{Z}/2)$$

(s_j is homogeneous of degree j, zero if j is an integer since $s(e^{2i\pi})=\epsilon s(x)=-s(x)$). As in the general case one may decomposed s_j into components $s_{j,k}$ of finite type k under the action of so(q). Since $\epsilon=-1$, k is necessary odd. That preceeds shows that $s_{j,k}$ is the image of a global section (or on U) of E by a morphism $E\to\bar{\mathscr{M}}$ where one has denoted $\bar{\mathscr{M}}:=j_*j^*\mathscr{M}$ (j: $V-\{0\}\to V$). If one applies this to $\mathscr{M}=E$ one obtains $E=\bar{E}$; in general one has an isomorphism $\bar{\mathscr{M}}\to W\otimes_{\mathbb{C}}\bar{E}=W\otimes_{\mathbb{C}}E$, this proves that $\bar{\mathscr{M}}$ is coherent, since it is isomorphic to $\mathscr{M}$.

(In abstract: if $\epsilon=-1$, any homogeneous section of E is annihilated by a large enough power of q or Δ. A homogeneous section of $\bar{E}$ which is not in E cannot be annihilated by a power of Δ, therefore one has $\bar{E}=E$.

One has an isomorphism $w_1:\bar{\mathscr{M}}\to W\otimes\bar{E}=W\otimes E$; thus $\bar{\mathscr{M}}$ is coherent. If K and K' are the kernels and the cokernels of the canonical homomorphism w_2: $\mathscr{M}\to\bar{\mathscr{M}}$, one has $\epsilon_K=\epsilon_{K'}=-1$ because K and K' are subquotients of $\mathscr{M}$ and $\bar{\mathscr{M}}$, and one has also $\epsilon_K=\epsilon_{K'}=1$ because K and K' are coherents, supported by $\{0\}$, therefore $K=K'=0$, w_2 is an isomorphism as $w_1\circ w_2$.)

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