Publ. RIMS, Kyoto Univ.
35 (1999), 249-261

Canonical Isomorphism of Two Lie Algebras Arising in CR-geometry

By

Vladimir V. EZHOV* and Alexander V. ISAEV**

Abstract

We show that the maximal prolongation of a certain algebra associated with a non-degenerate Hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ with values in $\mathbb{R}^k$ is canonically isomorphic to the Lie algebra of infinitesimal holomorphic automorphisms of the corresponding quadric in $\mathbb{C}^{n+k}$. This fact creates a link between different approaches to the equivalence problem for Levi-nondegenerate strongly uniform CR-manifolds.

§0. Introduction and Formulation of Result

A *CR-structure* on a smooth real manifold M of dimension m is a smooth distribution of subspaces in the tangent spaces $T_p^c(M) \subset T_p(M)$, $p \in M$, with operators of complex structure J_p: $T_p^c(M) \to T_p^c(M)$, $J_p^2 \equiv -\text{id}$, that depend smoothly on p. A manifold M equipped with a CR-structure is called a *CR-manifold*. It follows that the number $CR\dim M := \dim_{\mathbb{C}} T_p^c(M)$ does not depend on p; it is called the *CR-dimension* of M. The number $CR\text{codim}M := m - 2CR\dim M$ is called the *CR-codimension* of M. CR-structures naturally arise on real submanifolds in complex manifolds. Indeed, if, for example, M is a real submanifold of $\mathbb{C}^k$, then one can define the distribution $T_p^c(M)$ as follows:

$$T_p^c(M) := T_p(M) \cap iT_p(M).$$

Communicated by M. Kashiwara, March 30, 1998.
1991 Mathematics Subject Classifications: 32C16, 32F40, 17B66
Keywords and Phrases: CR-manifolds, equivalence problem, isomorphism of Lie algebras
*Department of Pure Mathematics, The University of Adelaide, Box 498, G. P. O. Adelaide, South Australia 5001, Australia
e-mail: vezhov@spam.maths.adelaide.edu.au
**Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, Australia
e-mail: Alexander.Isaev@anu.edu.au

On each $T_p^c(M)$ the operator J_p is then defined as the operator of multiplication by i. Then $\{T_p^c(M), J_p\}_{p\in M}$ form a CR-structure on M, if $\dim_{\mathbb{C}} T_p^c(M)$ is constant. This is always the case, for example, if M is a real hypersurface in $\mathbb{C}^k$ (in which case $CR\mathrm{codim} M=1$). We say that such a CR–structure is *induced by* $\mathbb{C}^k$.

A mapping between two CR-manifolds $f\colon M_1 \to M_2$ is called a *CR-mapping*, if for every $p \in M_1$: (i) $df(p)$ maps $T_p^c(M_1)$ to $T_{f(p)}^c(M_2)$, and (ii) $df(p)$ is complex linear on $T_p^c(M_1)$. Two CR-manifolds M_1, M_2 are called *CR-equivalent*, if there is a CR-diffeomorphism from M_1 onto M_2. Such a CR-diffeomorphism f is called a *CR-isomorphism*.

Let M be a CR-manifold. For every $p \in M$ consider the complexification $T_p^c(M) \otimes_{\mathbb{R}} \mathbb{C}$. Clearly, this complexification can be represented as the direct sum

$$T_p^c(M) \otimes_{\mathbb{R}} \mathbb{C} = T_p^{(1,0)}(M) \oplus T_p^{(0,1)}(M),$$

where

$$T_p^{(1,0)}(M) := \{X - iJ_pX\colon X \in T_p^c(M)\},$$
$$T_p^{(0,1)}(M) := \{X + iJ_pX\colon X \in T_p^c(M)\}.$$

The CR-structure on M is called *integrable* if for any local sections Z, Z' of the bundle $T^{(1,0)}(M)$, the vector field $[Z, Z']$ is also a section of $T^{(1,0)}(M)$. It is not difficult to see that if $M \subset \mathbb{C}^K$ and the CR-structure on M is induced by $\mathbb{C}^K$, then it is integrable.

An important characteristic of a CR-structure called the *Levi form* comes from taking commutators of local sections of $T^{(1,0)}(M)$ and $T^{(0,1)}(M)$. Let $p \in M$, $z, z' \in T_p^{(1,0)}(M)$, and Z, Z' be local sections of $T^{(1,0)}(M)$ near p such that $Z(p) = z$, $Z'(p) = z'$. The Levi form of M at p is the Hermitian form on $T_p^{(1,0)}(M) \times T_p^{(1,0)}$ with values in $(T_p(M)/T_p^c(M)) \otimes_{\mathbb{R}} \mathbb{C}$ given by

$$\mathcal{L}_M(p)(z, z') := i[Z, \bar{Z}'](p) \pmod{T_p^c(M) \otimes_{\mathbb{R}} \mathbb{C}}.$$

The Levi form is defined uniquely up to the choice of coordinates in $(T_p(M)/T_p^c(M)) \otimes_{\mathbb{R}} \mathbb{C}$, and, for fixed z and z', its value does not depend on the choice of Z and Z'.

Let $H = (H^1, \cdots, H^k)$ be a Hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ with values in $\mathbb{R}^k$. For any such H there is a corresponding standard CR-manifold $Q_H \subset \mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k defined as follows:

$$Q_H := \{(z, w)\colon \operatorname{Im} w = H(z, z)\},$$

where $z := (z_1, \cdots, z_n)$, $w := (w_1, \cdots, w_k)$ are coordinates in $\mathbb{C}^{n+k}$. The manifold Q_H is often called the *quadric associated with the form H*. The Levi form of Q_H at any point is given by H.

A Hermitian form H is called *non-degenerate* if:

(i) The scalar Hermitian forms $H^1, \cdots, H^k$ are linearly independent over $\mathbb{R}$;
(ii) $H(z, z') = 0$ for all $z' \in \mathbb{C}^n$ implies $z = 0$.

A CR-structure on M is called *Levi non-degenerate*, if its Levi form at any $p \in M$ is non-degenerate. An important tool in the geometry of Levi non-degenerate integrable CR-manifolds is the *automorphism group of* Q_H. Let $\mathrm{Aut}(Q_H)$ denote the collection of all local CR-isomorphisms of Q_H to itself that we call *local CR-automorphisms*. It turns out that, if H is non-degenerate, then any local CR-automorphism extends to a rational (more precisely, a matrix fractional quadratic) map of $\mathbb{C}^{n+k}$ [8], [7], [12], [5]. Thus, for a non-degenerate H, $\mathrm{Aut}(Q_H)$ is a finite-dimensional Lie group. Let $\mathfrak{g}_H$ denote the Lie algebra of $\mathrm{Aut}(Q_H)$. As shown in [1], [10], (see also [6] for a simple proof), the algebra $\mathfrak{g}_H$ consists of polynomial vector fields on $\mathbb{C}^{n+k}$ of the form

$$\mathfrak{g}_H = \Big\{ (p + Cz + aw + A(z,z) + B(z,w)) \frac{\partial}{\partial z} + (q + 2iH(z, p) + sw + 2iH(z, a\bar{w}) + r(w, w)) \frac{\partial}{\partial w} \Big\}, \tag{0.1}$$

where $p \in \mathbb{C}^n$, $q \in \mathbb{R}^k$, C is an $n \times n$-matrix, s is a $k \times k$-matrix, $A(z, z)$ is a quadratic form on $\mathbb{C}^n \times \mathbb{C}^n$ with values in $\mathbb{C}^n$, a is an $n \times k$-matrix, $B(z, w)$ is a bilinear form on $\mathbb{C}^n \times \mathbb{C}^k$ with values in $\mathbb{C}^n$, $r(w_1, w_2)$ is a symmetric bilinear form on $\mathbb{C}^k \times \mathbb{C}^k$ with values in $\mathbb{C}^k$, and the following holds

$$2\mathrm{Re}\, H(Cz, z) = sH(z, z), \tag{0.2.a}$$
$$H(A(z, z), z) = 2iH(z, aH(z, z)), \tag{0.2.b}$$
$$\mathrm{Re}\, H(B(z, u), z) = r(H(z, z), u), \tag{0.2.c}$$
$$\mathrm{Im}\, H(B(z, H(z, z)), z) = 0, \tag{0.2.d}$$

for all $z \in \mathbb{C}^n$, $u \in \mathbb{R}^k$.

We can now make $\mathfrak{g}_H$ into a graded Lie algebra by introducing weights as follows: z has weight 1, w has weight 2, $\frac{\partial}{\partial z}$ has weight -1, $\frac{\partial}{\partial w}$ has weight -2. Then we get $\mathfrak{g}_H = \bigoplus_{l=-2}^{2} \mathfrak{g}_H^l$, where

$$\mathfrak{g}_H^{-2} := \Big\{ q \frac{\partial}{\partial w} \Big\}, \tag{0.3.a}$$
$$\mathfrak{g}_H^{-1} := \Big\{ p \frac{\partial}{\partial z} + 2iH(z, p) \frac{\partial}{\partial w} \Big\}, \tag{0.3.b}$$
$$\mathfrak{g}_H^{0} := \Big\{ Cz \frac{\partial}{\partial z} + sw \frac{\partial}{\partial w} \Big\}, \tag{0.3.c}$$
$$\mathfrak{g}_H^{1} := \Big\{ (aw + A(z, z)) \frac{\partial}{\partial z} + 2iH(z, a\bar{w}) \frac{\partial}{\partial w} \Big\}, \tag{0.3.d}$$
$$\mathfrak{g}_H^{2} := \Big\{ B(z, w) \frac{\partial}{\partial z} + r(w, w) \frac{\partial}{\partial w} \Big\}. \tag{0.3.e}$$

Note that Q_H is a homogeneous manifold since the global CR-automorphisms

$$z \mapsto z+z^0, \quad w \mapsto w+w^0+2iH(z, z^0), \tag{0.4}$$

for $(z^0, w^0) \in Q_H$, act transitively on Q_H. The subalgebra $\mathfrak{g}_H^{-1} \oplus \mathfrak{g}_H^{-2}$ is the Lie algebra of the subgroup of $\mathrm{Aut}(Q_H)$ consisting of automorphisms of the form (0.4). The subalgebra $\mathfrak{g}_H^0$ is the Lie algebra of the subgroup of $\mathrm{Aut}(Q_H)$ consisting of linear automorphisms, i.e. automorphisms of the form

$$z \mapsto Pz, \qquad w \mapsto Rw,$$

where P is a complex $n \times n$-matrix, R is a real $k \times k$-matrix such that

$$R^{-1}H(Pz, Pz)=H(z, z).$$

The components $\mathfrak{g}_H^1$, $\mathfrak{g}_H^2$ are responsible for the existence of nonlinear automorphisms of Q_H that preserve the origin.

An example of how the algebra $\mathfrak{g}_H$ is used in CR-geometry is the equivalence problem for *strongly uniform* CR-manifolds. Let H_1, H_2 be two $\mathbb{R}^k$-valued Hermitian forms on $\mathbb{C}^n \times \mathbb{C}^n$. We say that H_1 and H_2 are *equivalent*, if there exist linear transformations A of $\mathbb{C}^n$ and B of $\mathbb{R}^k$ such that

$$H_2(z, z)=BH_1(Az, Az).$$

We call a CR-manifold M strongly uniform, if the forms $\mathcal{L}_M(p)$ are equivalent for all $p \in M$. If, for example, M is Levi non-degenerate and CR codim $M=1$ then M is strongly uniform. The equivalence problem for strongly uniform Levi non-degenerate integrable CR-manifolds is usually approached by constructing a CR-invariant parallelism on certain bundles over the manifolds with values in a suitable Lie algebra. In a number of cases (see [2], [3], [9], [4]) this Lie algebra was chosen to be $\mathfrak{g}_H$, where H is a Hermitian form equivalent to any $\mathcal{L}_M(p)$, $p \in M$. In the general approach of Tanaka [11], however, a seemingly different algebra was used: Tanaka considered a certain maximal prolongation $\tilde{\mathfrak{g}}_H$ of $\mathfrak{g}_H^{-2} \oplus \mathfrak{g}_H^{-1} \oplus \mathfrak{g}_H^0$. It is therefore a reasonable question whether the algebras $\mathfrak{g}_H$ and $\tilde{\mathfrak{g}}_H$ are isomorphic. In this paper we give a positive answer to this question in the main theorem below (see [11] [9], [4] for partial results).

We will now give the precise definition of the algebra $\tilde{\mathfrak{g}}_H$ from [11]. It is defined as an a priori infinite-dimensional graded Lie algebra

$$\tilde{\mathfrak{g}}_H=\mathfrak{g}_H^{-2} \oplus \mathfrak{g}_H^{-1} \oplus \mathfrak{g}_H^0 \oplus (\oplus_{l=1}^{\infty} \tilde{\mathfrak{g}}_H^l)$$

which is maximal among all Lie algebras of the above form that satisfy the conditioons:

(i) For $l \geq 0$ and $X \in \tilde{\mathfrak{g}}_H^l$, $[X, \mathfrak{g}_H^{-1}]=0$ implies $X=0$;

(ii) $\mathfrak{g}_H^{-2}\oplus\mathfrak{g}_H^{-1}\oplus\mathfrak{g}_H^0$ is a subalgebra of $\tilde{\mathfrak{g}}_H$.

It is shown in [11] that $\tilde{\mathfrak{g}}_H$ is unique and can be constructed by the following inductive procedure. First we define vector spaces $\tilde{\mathfrak{g}}_H^l$ and brackets $[X_l, X_{-1}]\in\tilde{\mathfrak{g}}_H^{l-1}$, $[X_l, X_{-2}]\in\tilde{\mathfrak{g}}_H^{l-2}$, where $X_p\in\tilde{\mathfrak{g}}_H^p$ (we set $\tilde{\mathfrak{g}}_H^l:=\mathfrak{g}_H^l$ for $l=-2, -1, 0$). Suppose that these spaces and brackets have been defined for $0\leq l\leq L-1$ in such a way that the following holds

$$[[X_l, X_{-1}], Y_{-1}]-[[X_l, Y_{-1}], X_{-1}]=[X_l, [X_{-1}, Y_{-1}]], \tag{0.5.a}$$
$$[[X_l, X_{-2}], X_{-1}]=[[X_l, X_{-1}], X_{-2}], \tag{0.5.b}$$

for all $X_l\in\tilde{\mathfrak{g}}_H^l$, X_{-1}, $Y_{-1}\in\tilde{\mathfrak{g}}_H^{-1}$. Then we define $\tilde{\mathfrak{g}}_H^L$ to be the vector space of all linear mappings X_L: $\tilde{\mathfrak{g}}_H^{-1}\rightarrow\tilde{\mathfrak{g}}_H^{L-1}$ for which there exist linear mappings X_L': $\tilde{\mathfrak{g}}_H^{-2}\rightarrow\tilde{\mathfrak{g}}_H^{L-2}$ such that

$$[X_L(X_{-1}), Y_{-1}]-[X_L(Y_{-1}), X_{-1}]=X_L'([X_{-1}, Y_{-1}]), \tag{0.6.a}$$
$$[X_L'(X_{-2}), X_{-1}]=[X_L(X_{-1}), X_{-2}], \tag{0.6.b}$$

for all X_{-1}, $Y_{-1}\in\tilde{\mathfrak{g}}_H^{-1}$, $X_{-2}\in\tilde{\mathfrak{g}}_H^{-2}$. We set $[X_L, X_{-1}]:=X_L(X_{-1})$ for all $X_{-1}\in\tilde{\mathfrak{g}}_H^{-1}$. Since H is non-degenerate, we have $\tilde{\mathfrak{g}}_H^{-2}=[\tilde{\mathfrak{g}}_H^{-1}, \tilde{\mathfrak{g}}_H^{-1}]$, and therefore X_L' is uniquely determined by X_L. Then we set $[X_L, X_{-2}]:=X_L'(X_{-2})$ for all $X_{-2}\in\tilde{\mathfrak{g}}_H^{-2}$. We also set $[X_{-1}, X_l]:=-[X_l, X_{-1}]$ and $[X_{-2}, X_l]:=-[X_l, X_{-2}]$. Clearly, (0.6) then gives equations (0.5) for $l=L$.

Note that equations (0.5) imply

$$[[X_l, X_{-2}], Y_{-2}]=[[X_l, Y_{-2}], X_{-2}], \tag{0.7}$$

for all $X_l\in\tilde{\mathfrak{g}}_H^l$, $l\geq 0$, and X_{-2}, $Y_{-2}\in\tilde{\mathfrak{g}}_H^{-2}$.

Let us now define brackets $[X_p, X_q]\in\tilde{\mathfrak{g}}_H^{p+q}$, $X_p\in\tilde{\mathfrak{g}}_H^p$, $X_q\in\tilde{\mathfrak{g}}_H^p$, p, $q\geq 0$, inductively as follows. Suppose that these brackets have been defined for p, $q\geq 0$, $p+q\leq L-1$, in such a way that for any $X_p\in\tilde{\mathfrak{g}}_H^p$, $X_q\in\tilde{\mathfrak{g}}_H^q$ the following holds

$$[[X_p, X_q], X_{-1}]=[[X_p, X_{-1}], X_q]+[X_p, [X_q, X_{-1}]], \tag{0.8.a}$$
$$[[X_p, X_q], X_{-2}]=[[X_p, X_{-2}], X_q]+[X_p, [X_q, X_{-2}]], \tag{0.8.b}$$

for all $X_{-1}\in\tilde{\mathfrak{g}}_H^{-1}$, $X_{-2}\in\tilde{\mathfrak{g}}_H^{-2}$. We take any $X_p\in\tilde{\mathfrak{g}}_H^p$, $X_q\in\tilde{\mathfrak{g}}_H^q$ with p, $q\geq 0$ and $p+q=L$ and define linear mappings X_L and X_L' from $\tilde{\mathfrak{g}}_H^{-1}$ and $\tilde{\mathfrak{g}}_H^{-2}$ to $\tilde{\mathfrak{g}}_H^{L-1}$ and $\tilde{\mathfrak{g}}_H^{L-2}$ respectively by

$$X_L(X_{-1}):=[[X_p, X_{-1}], X_q]+[X_p, [X_q, X_{-1}]],$$
$$X_L'(X_{-2}):=[[X_p, X_{-2}], X_q]+[X_p, [X_q, X_{-2}]].$$

Then we see that X_L, X_L' so defined satisfy (0.6) and therefore $X_L\in\tilde{\mathfrak{g}}_H^L$. We then define $[X_p, X_q]:=X_L$. Clearly, this definition gives identities (0.8) for all p, $q\geq 0$, $p+q=L$. Thus $[X_p, X_q]$ have been defined for all p, $q\geq 0$. Note that $[X_p, X_q]=$

$-[X_q, X_p]$ for all $p, q \geq 0$. By induction, we can also prove

$$[[X_p, X_q], X_r] + [[X_q, X_r], X_p] + [[X_r, X_p], X_q] = 0, \qquad (0.9)$$

for all $X_p \in \tilde{\mathfrak{g}}_H^p$, $X_q \in \tilde{\mathfrak{g}}_H^q$, $X_r \in \tilde{\mathfrak{g}}_H^r$, $p, q, r \geq 0$. By (0.5), (0.7), (0.8), (0.9) the brackets defined above give a Lie algebra structure on $\tilde{\mathfrak{g}}_H$. This completes the construction of $\tilde{\mathfrak{g}}_H$ in [11].

We now define a mapping Φ: $\mathfrak{g}_H \to \tilde{\mathfrak{g}}_H$ as follows:

Φ is identical on $\mathfrak{g}_H^{-2} \oplus \mathfrak{g}_H^{-1} \oplus \mathfrak{g}_H^0$,
$[\Phi(X)](X_{-1}) := [X, X_{-1}]$ for $X \in \mathfrak{g}_H^1$.
$[[\Phi(X)](X_{-1})](Y_{-1}) := [[X, X_{-1}], Y_{-1}]$ for $X \in \mathfrak{g}_H^2$.

It follows that Φ is a Lie algebra homomorphism and $\ker \Phi = \{0\}$. Moreover, $\Phi(\mathfrak{g}_H^p) \subset \tilde{\mathfrak{g}}_H^p$ for $p=1, 2$.

We are now ready to formulate the main result of the paper.

Theorem 0.1. *The mapping Φ is an isomorphism.*

We will prove the theorem in the next section. Before proceeding, we would like to thank G. Schmalz for useful discussions.

§1. Proof of Theorem

It is clear from the preceding discussion that to prove the theorem it is sufficient to show that $\dim \mathfrak{g}_H^p = \dim \tilde{\mathfrak{g}}_H^p$ for $p=1, 2$, and $\tilde{\mathfrak{g}}_H^3 = \{0\}$.

Lemma 1.1. $\dim \mathfrak{g}_H^1 = \dim \tilde{\mathfrak{g}}_H^1$.

Proof. Let $X_1 \in \tilde{\mathfrak{g}}_H^1$. Since $\mathfrak{g}_H^{-1}, \mathfrak{g}_H^0$ are given in the form as in (0.3.b), (0.3.c), X_1 can be written as

$$X_1\left(p\frac{\partial}{\partial z} + 2iH(z, p)\frac{\partial}{\partial w}\right) = \phi(p) z\frac{\partial}{\partial z} + \psi(p) w\frac{\partial}{\partial w},$$

$p \in \mathbb{C}^n$, where ϕ, ψ are real-linear mapppings from $\mathbb{C}^n$ to the spaces $M(n, \mathbb{C})$ of complex $n \times n$- and $M(k, \mathbb{R})$ of real $k \times k$-matrices respectively such that, for any $p, z \in \mathbb{C}^n$,

$$\psi(p) H(z, z) = 2\mathrm{Re}\, H(\phi(p) z, z) \qquad (1.1)$$

(see (0.2.a)). Let X_1' be the linear mapping from $\mathfrak{g}_H^{-2}$ to $\mathfrak{g}_H^{-1}$ corresponding to X_1 as in the definition of $\tilde{\mathfrak{g}}_H^{-1}$. It then follows from (0.3.a), (0.3.b) that X_1' can be written in the form

$$X_1'\left(q\frac{\partial}{\partial w}\right) = \mu(q)\frac{\partial}{\partial z} + 2iH(z, \mu(q))\frac{\partial}{\partial w},$$

$q\in\mathbb{R}^k$, where μ is a linear mapping from $\mathbb{R}^k$ to $\mathbb{C}^n$. Next, conditions (0.6) for $L=1$ are equivalent to

$$4\mu(\operatorname{Im} H(p_1,p_2))=\phi(p_1)p_2-\phi(p_2)p_1, \tag{1.2.a}$$
$$4\operatorname{Im} H(\mu(q),p)=\psi(p)q, \tag{1.2.b}$$

for all $p, p_1, p_2\in\mathbb{C}^n$, $q\in\mathbb{R}^k$.

We set

$$A(p,p):=\frac{1}{2}\phi(p)p-i\mu(H(p,p)),\quad a:=\mu,$$

$p\in\mathbb{C}^n$. We will show that the following holds (cf. (0.2.b))

$$H(A(p,p),p)=2iH(p,aH(p,p)), \tag{1.3}$$

for all $p\in\mathbb{C}^n$. We write ϕ in the most general form

$$\phi(p)=Rp+Q\bar{p},$$

where R, Q are constant vectors of length n with entries from $M(n,\mathbb{C})$. Formulas (1.1), (1.2) then give

$$Rp_1p_2=Rp_2p_1 \tag{1.4.a}$$
$$Q\bar{p}_1p_2=2i\mu(H(p_2,p_1)), \tag{1.4.b}$$
$$H(Rp_1p_2,p_3)=2iH(p_1,\mu(H(p_2,p_3)))-H(p_2,Q\bar{p}_1p_3), \tag{1.4.c}$$

for all $p_1, p_2, p_3\in\mathbb{C}^n$, where μ is complex-linearly extended from $\mathbb{R}^k$ to $\mathbb{C}^k$. Identities (1.3) easily follow from (1.4.b), (1.4.c). Identity (1.4.b) in addition gives

$$A(p,p)=\frac{1}{2}Rpp,$$

thus showing that A is a quadratic form on $\mathbb{C}^n\times\mathbb{C}^n$.

It is clear from (1.1), (1.4.b), (1.4.c) that a uniquely determines X_1 (also note that (1.3) implies that A and a uniquely determine each other), and the lemma is proved. □

Lemma 1.2. $\dim\mathfrak{g}_H^2=\dim\tilde{\mathfrak{g}}_H^2$.

Proof. Let $X_2\in\tilde{\mathfrak{g}}_H^2$. It follows from (0.3.a) - (0.3.c) that there exist real-bilinear mappings $\phi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ from $\mathbb{C}^n\times\mathbb{C}^n$ to the spaces $M(n,\mathbb{C})$ and $M(k,\mathbb{R})$ respectively, and a real-bilinear mapping $\mu(\cdot,\cdot)$ from $\mathbb{C}^n\times\mathbb{R}^k$ to $\mathbb{C}^n$ such thaat

$$X_2\left(p_1\frac{\partial}{\partial z}+2iH(z,p_1)\frac{\partial}{\partial w}\right)\left(p_2\frac{\partial}{\partial z}+2iH(z,p_2)\frac{\partial}{\partial w}\right)=\phi(p_1,p_2)z\frac{\partial}{\partial z}+\psi(p_1,p_2)w\frac{\partial}{\partial w},$$
$$\left[X_2\left(p\frac{\partial}{\partial z}+2iH(z,p)\frac{\partial}{\partial w}\right)\right]'\left(q\frac{\partial}{\partial w}\right)=\mu(p,q)\frac{\partial}{\partial z}+2iH(z,\mu(p,q))\frac{\partial}{\partial w},$$

$p, p_1, p_2 \in \mathbb{C}^n$, $q \in \mathbb{R}^k$, where $\left[X_2\left(p\frac{\partial}{\partial z}+2iH(z,p)\frac{\partial}{\partial w}\right)\right]'$ corresponds to $X_2\left(p\frac{\partial}{\partial z}+\right.$ $\left.2iH(z,p)\frac{\partial}{\partial w}\right)$ as an element of $\tilde{\mathfrak{g}}_H^1$. Let X_2' be the corresponding linear mapping from $\mathfrak{g}_H^{-2}$ to $\mathfrak{g}_H^0$. It follows from (0.3.a), (0.3.c) that it can be written in the form

$$X_2'\left(q\frac{\partial}{\partial w}\right)=\eta(q)z\frac{\partial}{\partial z}+\nu(q)w\frac{\partial}{\partial w},$$

$q \in \mathbb{R}^k$, where η and ν are linear mappings from $\mathbb{R}^k$ to the spaces $M(n, \mathbb{C})$ and $M(k, \mathbb{R})$ respectively. Equation (0.2.a) gives that the following conditions are satisfied

$$\phi(p_1, p_2)H(z,z)=2\mathrm{Re}\, H(\phi(p_1,p_2)z,z), \tag{1.5.a}$$
$$\nu(q)H(z,z)=2\mathrm{Re}\, H(\eta(q)z,z), \tag{1.5.b}$$

for all $p_1, p_2, z \in \mathbb{C}^n$, $q \in \mathbb{R}^k$. Next, analogously to (1.2), the following holds

$$4\mu(p_1, \mathrm{Im}\, H(p_2,p_3))=\phi(p_1,p_2)p_3-\phi(p_1,p_3)p_2, \tag{1.6.a}$$
$$4\mathrm{Im}\, H(\mu(p_1,q),p_2)=\phi(p_1,p_2)q, \tag{1.6.b}$$

for all $p, p_1, p_2 \in \mathbb{C}^n$, $q \in \mathbb{R}^k$. Further, conditions (0.6) for $L=2$ are equivalent to

$$4\eta(\mathrm{Im}\, H(p_1,p_2))=\phi(p_2,p_1)-\phi(p_1,p_2), \tag{1.7.a}$$
$$\mu(p,q)=-\eta(q)p, \tag{1.7.b}$$

for all $p, p_1, p_2 \in \mathbb{C}^n$, $q \in \mathbb{R}^k$.

We set

$$B(p,s):=\eta(s)p, \quad r(s_1,s_2):=\frac{1}{2}\nu(s_2)s_1,$$

$p \in \mathbb{C}^n$, $s, s_1, s_2 \in \mathbb{C}^k$, where η, ν are complex-linearly extended to $\mathbb{C}^k$. Then (1.5.b) implies

$$\mathrm{Re}\, H(B(p,q),p)=r(H(p,p),q), \tag{1.8}$$

for all $p \in \mathbb{C}^n$, $q \in \mathbb{R}^k$, which is analogous to (0.2.c). It is follows from (1.6) that ϕ is uniquely determined by μ (as in (1.4) above). Therefore, by (1.5.a) and (1.7.b), X_2 is uniquely determined by B (note that B also uniquely determines r by (1.8)). Thus, it is clear from (0.2.d) that to prove the lemma, we need to show that

$$\mathrm{Im}\, H(B(p,H(p,p)),p)=0, \tag{1.9}$$

for all $p \in \mathbb{C}^n$ and that $r(s_1, s_2)$ is symmetric. We write ϕ in the most general form

$$\phi(p_1, p_2) = Mp_1p_2 + N\overline{p_1}p_2 + Tp_1\overline{p_2} + S\overline{p_1p_2}.$$

Then (1.5.a), (1.6), (1.7) give

$$M=0, \tag{1.10.a}$$
$$S=0, \tag{1.10.b}$$
$$N\overline{p_1}p_2p_3 = N\overline{p_1}p_3p_2, \tag{1.10.c}$$
$$N\overline{p_1}p_2p_3 - Tp_2\overline{p_1}p_3 = 2i\mu(p_3, H(p_2, p_1)), \tag{1.10.d}$$
$$Tp_1\overline{p_2}p_3 = 2i\mu(p_1, H(p_3, p_2)), \tag{1.10.e}$$
$$H(N\overline{p_1}p_2p_3, p_4) = 2iH(p_2, \mu(p_1, H(p_3, p_4))) - H(p_3, Tp_1\overline{p_2}p_4), \tag{1.10.f}$$

for all $p_1, p_2, p_3, p_4 \in \mathbb{C}^n$, where μ is extended in the last argument to a complex-linear mapping on $\mathbb{C}^k$. Calculating Im $H(\eta(H(p, p))p, p)$ from (1.7.a) we get

$$\text{Im } H(\eta(H(p, p))p, p) = \frac{1}{2}\text{Re } H(N\overline{p}pp - Tp\overline{p}p, p).$$

On the other hand, (1.5.a), (1.6.b), (1.7.b) give

$$\text{Im } H(\eta(H(p, p))p, p) = -\frac{1}{2}\text{Re } H(N\overline{p}pp + Tp\overline{p}p + Mppp + S\overline{pp}p, p).$$

Comparing the last two expressions and using (1.10.a), (1.10.b), (1.10.d), (1.10.e) yields (1.9).

To show that $r(s_1, s_2)$ is symmetric, by (1.5.b), we need to prove that

$$\text{Re } H(\eta(H(p_1, p_1))p_2, p_2) = \text{Re } H(\eta(H(p_2, p_2))p_1, p_1), \tag{1.11}$$

for all $p_1, p_2 \in \mathbb{C}^n$. It follows from (1.7a) that

$$\text{Re } H(\eta(H(p_1, p_1))p_2, p_2) = -\frac{1}{2}\text{Im } H(N\overline{p_1}p_1p_2 - Tp_1\overline{p_1}p_2, p_2), \tag{1.12}$$

for all $p_1, p_2 \in \mathbb{C}^n$. On the other hand, (1.5.a), (1.6.b), (1.7.b), (1.10.a), (1.10.b) give

$$\text{Re } H(\eta(H(p_1, p_1))p_2, p_2) = -\frac{1}{2}\text{Im } H(N\overline{p_2}p_2p_1 - Tp_2\overline{p_2}p_1, p_1),$$

for all $p_1, p_2 \in \mathbb{C}^n$, which together with (1.12) implies (1.11).

The lemma is proved. □

Lemma 1.3. $\tilde{\mathfrak{g}}^3_H = \{0\}$.

Proof. Let $X_3 \in \tilde{\mathfrak{g}}^3_H$. Then there exist real-trilinear mappings $\phi(\cdot,\cdot,\cdot)$, $\psi(\cdot,\cdot,\cdot)$ from $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$ to the spaces $M(n,\mathbb{C})$ and $M(k,\mathbb{R})$ respectively, real-bilinear mappings $\eta(\cdot,\cdot)$, $\nu(\cdot,\cdot)$ from $\mathbb{C}^n \times \mathbb{R}^k$ to the above spaces of matrices respectively, and a real-trilinear mapping $\mu(\cdot,\cdot,\cdot)$ from $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}^k$

to $\mathbb{C}^n$ such that

$$X_3\Big(p_1\frac{\partial}{\partial z}+2iH(z,p_1)\frac{\partial}{\partial w}\Big)\Big(p_2\frac{\partial}{\partial z}+2iH(z,p_2)\frac{\partial}{\partial w}\Big)\Big(p_3\frac{\partial}{\partial z}+2iH(z,p_3)\frac{\partial}{\partial w}\Big)=$$
$$\phi(p_1,p_2,p_3)z\frac{\partial}{\partial z}+\psi(p_1,p_2,p_3)w\frac{\partial}{\partial w},$$
$$\Big[X_3\Big(p_1\frac{\partial}{\partial z}+2iH(z,p_1)\frac{\partial}{\partial w}\Big)\Big(p_2\frac{\partial}{\partial z}+2iH(z,p_2)\frac{\partial}{\partial w}\Big)\Big]'\Big(q\frac{\partial}{\partial w}\Big)=$$
$$\mu(p_1,p_2,q)\frac{\partial}{\partial z}+2iH(z,\mu(p_1,p_2,q))\frac{\partial}{\partial w},$$
$$\Big[X_3\Big(p\frac{\partial}{\partial z}+2iH(z,p)\frac{\partial}{\partial w}\Big)\Big]'\Big(q\frac{\partial}{\partial w}\Big)=$$
$$\eta(p,q)z\frac{\partial}{\partial z}+\nu(p,q)w\frac{\partial}{\partial w},$$

$p, p_1, p_2, p_3\in\mathbb{C}^n$, $q\in\mathbb{R}^k$, where $\Big[X_3\Big(p_1\frac{\partial}{\partial z}+2iH(z,p_1)\frac{\partial}{\partial w}\Big)\Big(p_2\frac{\partial}{\partial z}+2iH(z,p_2)\frac{\partial}{\partial w}\Big)\Big]'$ corresponds to $X_3\Big(p_1\frac{\partial}{\partial z}+2iH(z,p_1)\frac{\partial}{\partial w}\Big)\Big(p_2\frac{\partial}{\partial z}+2iH(z,p_2)\frac{\partial}{\partial w}\Big)$ as an element of $\tilde{\mathfrak{g}}^1_H$ and $\Big[X_3\Big(p\frac{\partial}{\partial z}+2iH(z,p)\frac{\partial}{\partial w}\Big)\Big]'$ corresponds to $X_3\Big(p\frac{\partial}{\partial z}+2iH(z,p)\frac{\partial}{\partial w}\Big)$ as an element of $\tilde{\mathfrak{g}}^2_H$. Let X_3' be the corresponding linear mapping from $\mathfrak{g}_H^{-2}$ to $\tilde{\mathfrak{g}}^1_H$. Then there exist real-bilinear mappings $\lambda(\cdot,\cdot)$ and $\rho(\cdot,\cdot)$ from $\mathbb{C}^n\times\mathbb{R}^k$ to the spaces $M(n,\mathbb{C})$ and $M(k,\mathbb{R})$ respectively such that

$$X_3'\Big(q\frac{\partial}{\partial w}\Big)\Big(p\frac{\partial}{\partial z}+2iH(z,p)\frac{\partial}{\partial w}\Big)=\lambda(p,q)z\frac{\partial}{\partial z}+\rho(p,q)w\frac{\partial}{\partial w},$$

$p\in\mathbb{C}^n$, $q\in\mathbb{R}^k$. Equation (0.2.a) gives

$$\psi(p_1,p_2,p_3)H(z,z)=2\mathrm{Re}\,H(\phi(p_1,p_2,p_3)z,z), \qquad (1.13.a)$$
$$\nu(p,q)H(z,z)=2\mathrm{Re}\,H(\eta(p,q)z,z), \qquad (1.13.b)$$
$$\rho(p,q)H(z,z)=2\mathrm{Re}\,H(\lambda(p,q)z,z), \qquad (1.13.c)$$

for all $p, p_1, p_2, p_3, z\in\mathbb{C}^n$, $q\in\mathbb{R}^k$. Next, analogously to (1.6), we have

$$4\mu(p_1,p_2,\mathrm{Im}\,H(p_3,p_4))=\phi(p_1,p_2,p_3)p_4-\phi(p_1,p_2,p_4)p_3, \qquad (1.14.a)$$
$$4\mathrm{Im}\,H(\mu(p_1,p_2,q),p_3)=\psi(p_1,p_2,p_3)q, \qquad (1.14.b)$$

for all $p_1,p_2,p_3\in\mathbb{C}^n$, $q\in\mathbb{R}^k$. Further, there are the following analogues of identities (1.7)

$$4\eta(p_1,\mathrm{Im}\,H(p_2,p_3))=\phi(p_1,p_3,p_2)-\phi(p_1,p_2,p_3), \qquad (1.15.a)$$
$$\mu(p_1,p_2,q)=-\eta(p_1,q)p_2, \qquad (1.15.b)$$

for all $p_1, p_2\in\mathbb{C}^n$, $q\in\mathbb{R}^k$. Finally, conditions (0.6) for $L=3$ are equivalent to

$$4\lambda(p_1,\mathrm{Im}\,H(p_2,p_3))=\phi(p_3,p_2,p_1)-\phi(p_2,p_3,p_1), \qquad (1.16.a)$$
$$\lambda(p,q)=\eta(p,q), \qquad (1.16.b)$$

for all $p, p_1, p_2, p_3 \in \mathbb{C}^n$, $q \in \mathbb{R}^k$.

We will now show that identities (1.13) - (1.16) imply that $X_3 = 0$. It follows by the argument in Lemma 1.2 from (1.13.a) and identities (1.14), (1.15) that ϕ has the form (see (1.10.a), (1.10.b))

$$\phi(p_1, p_2, p_3) = Gp_1\overline{p_2}p_3 + K\overline{p_1}\overline{p_2}p_3 + Fp_1p_2\overline{p_3} + L\overline{p_1}p_2\overline{p_3},$$

and (see (1.10.c) - (1.10.f))

$$Gp_1p_2p_3p_4 = Gp_1p_2p_4p_3, \tag{1.17.a}$$
$$(Gp_1 + K\overline{p_1})\overline{p_2}p_3p_4 - (Fp_1 + L\overline{p_1})p_3\overline{p_2}p_4 = 2i\mu(p_1, p_4, H(p_3, p_2)), \tag{1.17.b}$$
$$(Fp_1 + L\overline{p_1})p_2\overline{p_3}p_4 = 2i\mu(p_1, p_2, H(p_4, p_3)), \tag{1.17.c}$$
$$H((Gp_1 + K\overline{p_1})\overline{p_2}p_3p_4, p_5) = 2iH(p_3, \mu(p_1, p_2, H(p_5, p_4))) - H(p_4, (Fp_1 + L\overline{p_1})p_2\overline{p_3}p_5), \tag{1.17.d}$$
$$Kp_1p_2p_3p_4 = Kp_1p_2p_4p_3, \tag{1.17.e}$$

for all $p_1, p_2, p_3, p_4 \in \mathbb{C}^n$, where μ is extended complex-linearly to $\mathbb{C}^k$ in the last argument. Further, (1.15.b), (1.16) imply

$$L\overline{p_1}p_2\overline{p_3}p_4 - Gp_2\overline{p_1}p_3p_4 = 2i\mu(p_3, p_4, H(p_2, p_1)), \tag{1.18}$$

for all $p_1, p_2, p_3, p_4, p_5 \in \mathbb{C}^n$. From (1.17.b) - (1.17.d), (1.18) we obtain

$$H(K\overline{p_1}\overline{p_2}p_3p_4, p_5) = H(p_3, Gp_5\overline{p_4}p_1p_2) - H(p_4, Fp_1p_2\overline{p_3}p_5), \tag{1.19.a}$$
$$H(Gp_1\overline{p_2}p_3p_4, p_5) = -H(p_3, L\overline{p_4}p_5\overline{p_1}p_2) - H(p_4, L\overline{p_1}p_2\overline{p_3}p_5), \tag{1.19.b}$$
$$Fp_1p_2p_3p_4 = -Gp_4p_3p_1p_2, \tag{1.19.c}$$
$$Lp_1p_2p_3p_4 = Lp_3p_4p_1p_2, \tag{1.19.d}$$
$$Kp_1p_2p_3p_4 = Lp_1p_3p_2p_4 + Lp_1p_4p_2p_3, \tag{1.19.e}$$
$$Gp_1p_2p_3p_4 + Gp_3p_2p_1p_4 = Fp_3p_1p_2p_4, \tag{1.19.f}$$

for all $p_1, p_2, p_3, p_4, p_5 \in \mathbb{C}^n$.

We set

$$D(p_1, p_2, H(p_3, p_4)) := iGp_3\overline{p_4}p_2p_1, \tag{1.20.a}$$
$$t(H(p_1, p_2), H(p_3, p_4)) := -\frac{1}{2}L\overline{p_2}p_1\overline{p_4}p_3, \tag{1.20.b}$$

for all $p_1, p_2, p_3, p_4 \in \mathbb{C}^n$. It follows from (1.17.a), (1.18), (1.19.d) and the non-degeneracy of H that (1.20.a) defines a complex-trilinear form D on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^k$ symmetric with respect to the first two variables and (1.20.b) defines a complex-bilinear symmetric form t on $\mathbb{C}^k \times \mathbb{C}^k$, both valued in $\mathbb{C}^n$. It follows from (1.13.a), (1.19.c), (1.19.e) that X_3 is uniquely determined by t (note that it follows from (1.19.b) that D is uniquely determined by t). The forms D and t satisfy the following relations

$$H(D(p, p, q), p) = 4iH(p, t(q, H(p, p))), \tag{1.21.a}$$
$$H(D(p, p, H(p, p)), p) = 0, \tag{1.21.b}$$

for all $p \in \mathbb{C}^n$, $q \in \mathbb{R}^k$. Indeed, (1.21.a) follows from (1.19.b), (1.19.d); to prove (1.21.b) we note that it follows from (1.19.c), (1.19.f) that $Gp\bar{p}\bar{p}p \equiv 0$.

We will now show that equations (1.21) can have only zero solutions. For this we note that a polynomial vector field X on $\mathbb{C}^{n+k}$ of the form

$$X=\Big(D'(z, z, w)+t'(w, w)\Big)\frac{\partial}{\partial z}+2iH\Big(z, t'(\bar{w}, \bar{w})\Big)\frac{\partial}{\partial w}$$

defines an infinitesimal holomorphic automorphism of Q_H (i.e. $X \in \mathfrak{g}_H$) if and only if the following conditions are satisfied

$$H(D'(z, z, u), z)=4iH(z, t'(u, H(z, z))), \qquad (1.22.a)$$

$$H(D'(z, z, H(z, z)), z)=0, \qquad (1.22.b)$$

for all $z \in \mathbb{C}^n$, $u \in \mathbb{R}^k$, where D' is a complex-trilinear form on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^k$ symmetric with respect to the first two variables, and t' is a complex-bilinear symmetric form on $\mathbb{C}^k \times \mathbb{C}^k$, both valued in $\mathbb{C}^n$. Since the vector field X has weight 3, it must be zero by (0.1). This means that equations (1.22) can have only zero solutions and therefore equations (1.21) have only zero solutions also.

Thus, $X_3=0$, and the lemma is proved. □

References

[1] Beloshapka, V., A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space (translated from Russian), *Math. Notes*, **47** (1990), 239-242.

[2] Cartan, É., Sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes: I, *Ann. Math. Pura Appl.* **11** (1932), 17-90; II, *Ann. Scuola Norm. Sup. Pisa*, **1** (1932), 333-354.

[3] Chern, S. S. and Moser, J. K., Real hypersurfaces in complex manifolds, *Acta Math*, **133** (1974), 219-271.

[4] Ezhov, V., Isaev, A. and Schmalz, G., Invariants of elliptic and hyperbolic *CR*-structures of codimension 2, *Internat. J. Math.*, to appear.

[5] Ezhov, V. and Schmalz, G., Holomorphic automorphisms of nondegenerate *CR*-quadrics: explicit description, *J. Geom. Analysis*, to appear.

[6] ______, A simple proof of Beloshapka's theorem on parametrisation of the automorphism groups of *CR*-manifolds (Russian), *Mat. Zametki*, to appear.

[7] Forstnerič, F., Mappings of quadric Cauchy-Riemann manifolds, *Math. Ann.*, **292** (1992), 163-180.

[8] Khenkin, G., Tumanov, A., Local characterization of holomorphic automorphisms of Siegel domains (translated from Russian), *Funct. Anal. Appl.*, **17** (1983), 285-294.

[9] Lai, H.-F., Real submanifolds of codimension two in complex manifolds, *Trans. Amer. Math. Soc.*, **264** (1981), 331-352.

[10] Satake, I., *Algebraic Structures of Symmetric Domains*, Kanô Memorial Lectures 4, Iwanami Shoten, Tokyo; Princeton University Press, 1980.

[11] Tanaka, N., On generalized graded Lie algebras and geometric structures I, *J. Math. Soc.*

Japan, **19** (1967), 215-254.

[12] Tumanov, A., Finite-dimensionality of the group of *CR* automorphisms of a standard *CR* manifold, and proper holomorphic mappings of Siegel domains (translated from Russian), *Math. USSR. Izv.*, **32** (1989), 655-662.