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The Furuta Inequality and an Operator Equation for Linear Operators

By

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Abstract

We show that a special form of the Furuta inequality is equivalent to an operator equation $H^{\frac{p-2rn}{2(n+1)}} T (H^{\frac{p+2r}{n+1}} T)^n H^{\frac{p-2rn}{2(n+1)}} = K^p$. This result also generalize Lemma 1 in [3] which is about the operator equation $T(H^{1/n}T)^n = K$. A new characterization of the Löwner-Heinz formula and some applications are given.

§1. Notation and Introduction

Throughout this note the capital letters mean bounded linear operators on a Hilbert space H. T is positive (written $T \geq O$) in case $(Tx, x) \geq 0$ for all $x \in$ H. If S and T are Hermitian, we write $T \geq S$ in case $T - S \geq O$. I will denote the identity operator. Pedersen and Takesaki [6] proved that if $H, K \geq O$ and H is nonsingular, then $(H^{1/2}KH^{1/2})^{1/2} \leq aH$ holds for some $a > 0$, if and only if there exists a unique $T \geq O$ such that $THT = K$. Nakamoto [5] showed the necessary condition by using Douglas's majorization theorem [1], and it turned out to be a very simple proof. Furuta [3] extended and characterized the operator equation to the equation $T(H^{1/n}T)^n = K$ for any natural number n. In this paper we shall use the remarkable Furuta inequality [2] to give a further generalization (as the equation in abstract), which is also a new characterization of a special form of the Furuta inequality. Consequently, a new characterization of the Löwner-Heinz formula and some applications are given.

We recall the following two celebrated results. Firstly, the Douglas theorem [1], i.e., the inequality $AA^* \leq \lambda^2 BB^*$ holds for some $\lambda \geq 0$, if and only if there exists a C so that $A = BC$. Moreover, if these statements are valid, then there exists a unique C so that $\|C\| \leq \lambda$. Secondly, the Furuta inequality [2], i.e., if

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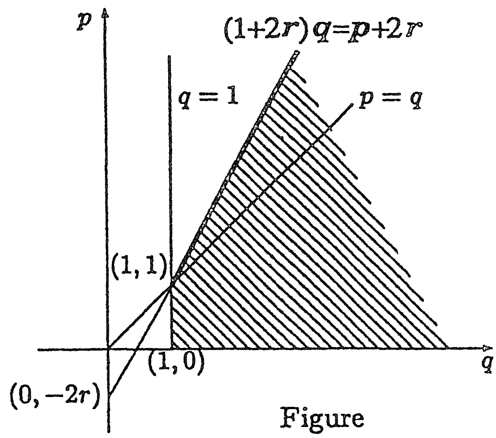
$A \geq B \geq O$, then both inequalities $A^{\frac{p+2r}{q}} \geq (A^rB^pA^r)^{1/q}$ and $(B^rA^pB^r)^{1/q} \geq B^{\frac{p+2r}{q}}$ hold for $p, r \geq 0$, and $q \geq 1$ such that $(1+2r)q \geq p+2r$.

Remark that the conditions on p, r, q, and the expression $(1+2r)q \geq p+2r$ are the best possible with respect to the Furuta inequality [7] (See Figure). More precisely, for p, $r \geq 0$, if $q \in (0,1)$ or $(1+2r)q < p+2r$, then there are operators A, B: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $A \geq B \geq O$, but $A^{\frac{p+2r}{q}} \not\geq (A^rB^pA^r)^{1/q}$. The above two results have a beautiful relationship as we will see in the proof of Theorem below.



Figure

§2. Main Result

We shall make frequent use of the Löwner-Heinz formula throughout the paper, viz. $A^\alpha \geq B^\alpha$ if $A \geq B \geq O$ for $\alpha \in [0,1]$. If q in the Furuta inequality is a natural number instead, then the inequality may be characterized in terms of an operator equation. We now proceed to derive the main result.

Theorem 1. Let $H \geq K \geq O$, and assume that H is nonsigular. Then the following are equivalent for p, $r \geq 0$, and an integer $n \geq 0$ with $(1+2r)(n+1) \geq p+2r$.

(1) $H^{\frac{p+2r}{n+1}} \geq (H^rK^pH^r)^{\frac{1}{n+1}}$ (Furuta inequality);

(2) There exists a unique operator $T \geq O$ with $\|T\| \leq 1$ such that

$$K^p = H^{\frac{p-2rn}{2(n+1)}} T (H^{\frac{p+2r}{n+1}} T)^n H^{\frac{p-2rn}{2(n+1)}}.$$

Proof. (1) implies (2). As both sides of (1) are positive, and also by Douglas's theorem the inequality (1) implies that there exists a unique S with $\|S\| \leq 1$ such that

$$(H^rK^pH^r)^{\frac{1}{2(n+1)}} = H^{\frac{p+2r}{2(n+1)}} S = S^* H^{\frac{p+2r}{2(n+1)}}.$$

If we put $T = SS^*$, then

$$(H^rK^pH^r)^{\frac{1}{n+1}} = H^{\frac{p+2r}{2(n+1)}} T H^{\frac{p+2r}{2(n+1)}}.$$

It follows that

$$H^rK^pH^r = (H^{\frac{p+2r}{2(n+1)}}TH^{\frac{p+2r}{2(n+1)}})^{n+1} = H^{\frac{p+2r}{2(n+1)}}T(H^{\frac{p+2r}{n+1}}T)^nH^{\frac{p+2r}{2(n+1)}}.$$

As H is nonsingular we obtain the required equality in (2).

To show the uniqueness of T, for some Z assume $H^{\frac{p-2rn}{2(n+1)}}T(H^{\frac{p+2r}{n+1}}T)^nH^{\frac{p-2rn}{2(n+1)}}$ $=H^{\frac{p-2rn}{2(n+1)}}Z(H^{\frac{p+2r}{n+1}}Z)^nH^{\frac{p-2rn}{2(n+1)}}$, then $T(H^{\frac{p+2r}{n+1}}T)^n = Z(H^{\frac{p+2r}{n+1}}Z)^n$, and

$$\begin{aligned}(H^{\frac{p+2rn}{2(n+1)}}TH^{\frac{p+2r}{2(n+1)}})^{n+1} &= H^{\frac{p+2r}{2(n+1)}}T(H^{\frac{p+2r}{n+1}}T)^nH^{\frac{p+2r}{2(n+1)}}\\ &= H^{\frac{p+2r}{2(n+1)}}Z(H^{\frac{p+2r}{n+1}}Z)^nH^{\frac{p+2r}{2(n+1)}} = (H^{\frac{p+2r}{2(n+1)}}ZH^{\frac{p+2r}{2(n+1)}})^{n+1},\end{aligned}$$

and the nonsingularity of H yields $Z=T$. And $\|T\| = \|SS^*\| = \|S\|^2 \le 1$.

(2) implies (1).
$$\begin{aligned}(H^rK^pH^r)^{\frac{1}{n+1}} &= [H^rH^{\frac{p-2rn}{2(n+1)}}T(H^{\frac{p+2r}{n+1}}T)^nH^{\frac{p-2rn}{2(n+1)}}H^r]^{\frac{1}{n+1}}\\ &= H^{\frac{p+2r}{2(n+1)}}TH^{\frac{p+2r}{2(n+1)}}\\ &\le H^{\frac{p+2r}{n+1}},\end{aligned}$$

since $T \le \|T\|I \le I$, and H is nonsingular. Q.E.D.

It was proved in [3, Lemma 1] that $aH^{1/n} \ge (H^{1/2n}KH^{1/2n})^{\frac{1}{n+1}}$ holds for some $a \ge 0$, if and only if there exists a unique $T \ge O$ such that $T(H^{1/n}T)^n = K$. This is indeed a special case of our Theorem 1 if $a=1$, in which $p=1$, $r=\frac{1}{2n}$, and a natural number $n \ge 1$. Notice that in the proof of Theorem 1 the hypothesis that $H \ge K$ was not used, but it is made only to ensure the validity of the inequality (1) under imposed conditions on p, r, and n. In fact, all we need is the condition that $H, K \ge O$.

§3. Applications

The next result is a new characterization of the Löwner-Heinz formula, and the proof is trivial; let $n=r=0$ in Theorem 1.

Corollary 1. Let $H \ge K \ge O$, $p \in [0,1]$, and assume that H is nonsingular. Then the following are equivalent.

(1) $H^p \ge K^p$ (Löwner-Heinz formula);

(2) There exists a unique operator $T \ge O$ with $\|T\| \le 1$ such that $K^p = H^{p/2}TH^{p/2}$.

Recall that T is a p-hyponormal operator for $0 < p \le 1$ if $(T^*T)^p \ge (TT^*)^p$, and it is hyponormal when $p=1$. It is easily seen that T is p-hyponormal, if and only if $|T^*|^{2p} \leq |T|^{2p}$. We write $T=U|T|$ the polar decomposition of T with U the partial isometry, and $|T|$ the positive square root of the positive operator T^*T. The next result shows some properties of such operator.

Corollary 2. Let $T=U|T|$ be p-hyponormal for $0<p\leq 1$. Then, for $q,r\geq 0$, and a natural number n with $(1+2r)(n+1)\geq q+2r$, we have

(1) $|T|^{\frac{2p(q+2r)}{n+1}} \geq (|T|^{2pr}|T^*|^{2pq}|T|^{2pr})^{\frac{1}{n+1}}$;

(2) There exists a unique operator $S\geq O$ with $\|S\|\leq 1$ such that

$$|T^*|^{2pq}=|T|^{\frac{2q(q-2rn)}{2(n+1)}}S(|T|^{\frac{2p(q+2r)}{n+1}}S)^n|T|^{\frac{2p(q-2rn)}{2(n+1)}}.$$

Moreover, the above two statements are equivalent.

Proof. Since T is p-hyponormal let $H=|T|^{2p}$ and $K=|T^*|^{2p}$ in Theorem 1 so that $H \geq K \geq O$. We may assume without loss of generality that $|T|$ is nonsingular. Q.E.D.

Corollary 3. Let H, $K\geq O$, H be nonsingular, and p, $r\geq 0$, and let n be a natural number with $(1+2r)(n+1)\geq p+2r$. Then,

(1) if there exists a $T\geq O$ such that $K^p=H^{\frac{p-2rn}{2(n+1)}}T(H^{\frac{p+2r}{n+1}}T)^nH^{\frac{p-2rn}{2(n+1)}}$, then, for any natural number $m\geq n$, there exists a unique $T'\geq O$ such that $K^p = H^{\frac{p-2rm}{2(m+1)}}T'(H^{\frac{p+2r}{m+1}}T')^mH^{\frac{p-2rm}{2(m+1)}}$;

(2) in the statement (1) if $n>m$ instead, then in general there does not exist a $T'\geq O$ such that $K^p=H^{\frac{p-2rm}{2(m+1)}}T'(H^{\frac{p+2r}{m+1}}T')^mH^{\frac{p-2rm}{2(m+1)}}$.

Proof. (1) The given equality implies the relation $H^{\frac{p+2r}{n+1}}\geq (H^rK^pH^r)^{\frac{1}{n+1}}$ by Theorem 1. Since $m\geq n$, the inequality

$$H^{\frac{p+2r}{m+1}}\geq (H^rK^pH^r)^{\frac{1}{m+1}}$$

holds by the Löwner-Heinz formula, and the conclusion is due to Theorem 1, again.

(2) Since $H^{\frac{p+2r}{n+1}}\geq (H^rK^pH^r)^{\frac{1}{n+1}}\geq O$, in view of the Furuta inequality the relation

$$H^{\frac{p+2r}{n+1}\cdot\frac{a+2c}{b}}\geq [H^{\frac{(p+2r)c}{n+1}}(H^rK^pH^r)^{\frac{a}{n+1}}H^{\frac{(p+2r)c}{n+1}}]^{\frac{1}{b}}$$

holds for a, $c\geq 0$, $b\geq 1$ with $(1+2c)b\geq a+2c$. Put $a=n+1$, $b=m+1$, and $c=0$. Then

$$H^{\frac{p+2r}{m+1}}\geq (H^rK^pH^r)^{\frac{1}{m+1}},$$

but in this case $(1+2c)b<a+2c$ as $n>m$. By the best possibility argument mentioned before the above inequality does not exists in general. Consequently, by Theorem 1 there does not exist a $T'\geq O$ in general such that $K^p=H^{\frac{p-2rm}{2(m+1)}}T'(H^{\frac{p+2r}{m+1}}T')^m H^{\frac{p-2rm}{2(m+1)}}$. Q.E.D.

Remark. It should be pointed out at this stage that if there exists a $T\geq O$ such that $T(H^{1/n}T)^n=K$ for some natural number n, then for any natural number $m\leq n$, there exists a unique T' such that $T'(H^{1/m}T')^m=K$ [3]. However, if $m>n$ instead, then in general there does not exists such T' satisfying the equation [4]. It may be of interest to compare opposite properties in the above statement and Corollary 3.

Finally, we may use the second inequality of Furuta to produce a result which is similar to Theorem 1. Notice that all conditions are exactly the same as in Theorem 1, except assuming nonsingularity of both H and K. We shall omit the proof since it may be carried out as in the case of Theorem 1.

Theorem 2. Let $H\geq K\geq O$, and both H and K be nonsingular. Then the following are equivalent for p, $r\geq 0$, and a natural number n with $(1+2r)(n+1)\geq p+2r$.

(1) $(K^rH^pK^r)^{\frac{1}{n+1}}\geq K^{\frac{p+2r}{n+1}}$ (Furuta inequality);
(2) There exists a unique operator $T\geq O$ with $\|T\|\leq 1$ such that

$$K^{p+2r}=(K^rH^pK^r)^{\frac{1}{2(n+1)}}T\left[(K^rH^pK^r)^{\frac{1}{(n+1)}}T\right]^n(K^rH^pK^r)^{\frac{1}{2(n+1)}}.$$

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