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Energy Decay for a Degenerate Hyperbolic Equation with a Dissipative Term

By

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§1. Introduction

We shall investigate the energy decay of the solutions to the following Cauchy problem for a degenerate hyperbolic equation :

$$(1)\qquad u_{tt}-a(t)\Delta u+2b(t)u_t+m^2u=0,\ t\geq 0,\ x\in\mathbb{R}^n,$$

$$(2)\qquad u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\ x\in\mathbb{R}^n,$$

where $u(t, x)$ is real valued, $a(t)\in C^1([0, \infty))$, $a(t)\geq 0$, $b(t)\in C^0([0, \infty))$, $\inf_t\{b(t)\}\equiv b_0>0$, $\sup_t\{b(t)\}\equiv b_1<\infty$, m is a positive constant and Δ is the Laplace operator in $\mathbb{R}^n$.

The purpose of this paper is to seek sufficient conditions on a, b, m and the initial data (u_0, u_1) which guarantee the exponential order decay of the total energy

$$(3)\qquad E(u(t))=\frac{1}{2}\{\|u_t(t)\|^2+a(t)\|\nabla u(t)\|^2+m^2\|u(t)\|^2\}$$

to the solution of (1)–(2) as $t\to\infty$, where $\|\cdot\|$ denotes the usual $L^2(\mathbb{R}^n)$ norm.

Energy decay problem in the whole space for the wave equation with a dissipative term has been considered by many authors (Matsumura [2], Mochizuki [3], Mochizuki-Nakazawa [4], Rauch-Taylor [5], etc.). But it seems to be very few results for the case of general hyperbolic equations as (1).

If $a(t)\equiv 1$ and $b(t)\equiv 0$, the total energy $E(u(t))$ is conserved, that is, $E(u(t))=E(u(0))$ for any t, and it is possible that the energy decays when $b\geq 0$. Indeed, it is well-known that the energy decays in exponential order if $\inf_t\{b(t)\}>0$ and $m>0$ (see [5]), and decays like $O(t^{-1})$ as $t\to\infty$ if there are positive constants b_0 and b_1 such that $b_0(1+t)^{-1}\leq b(t)\leq b_1$, $b'(t)\leq 0$ and $m=0$ (see [3]; [2] and [4] have

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gone into more particulars).

However, if $a(t)\not\equiv const.$, the energy does not always decay in spite of the existence of the dissipation, because $a(t)$ may play a part in the growth of the energy when $a'(t)>0$. Indeed, according to Reissig-Yagdjian [6], there exists $a(t) \in C^1([0, \infty))$ such that the total energy cannot be bounded for any function of $O(e^{\sigma t})$ with $\sigma<1$. Thus, it seems that the behavior of the energy for the general hyperbolic equation is more complicate than that for the wave equation. Furthermore, we shall consider the case that $a(t)$ has zero points. When $a(t)$ vanishes at a point, in general, (1) is not H^∞ well-posed in any neighborhood of the vanishing point. However, if the initial data is sufficiently regular, the Cauchy problem (1)–(2) is well-posed in the Gevrey class of order 3/2 since $a(t)\in C^1$ (see [1]). Hence we can consider a classical solution of (1)–(2) and its energy for the initial data in the Gevrey class.

§2. Preliminaries and Results

In this section we mention our main theorems.

In the first theorem is treated the case that (1) is strictly hyperbolic, that is, $a(t)$ is strictly positive. In the second one, we consider the case that (1) is weakly hyperbolic, that is, $a(t)$ has a zero point.

We define the positive constants $\alpha_0=\alpha_0(b_0, b_1, m)$ and $\beta_0=\beta_0(b_0, b_1, m)$ by

$$\alpha_0(b_0, b_1, m)=\begin{cases} b_0 & (b_0 b_1<m^2), \\ \dfrac{m^2}{b_1} & (b_0 b_1\geq m^2), \end{cases}$$

and

$$\beta_0(b_0, b_1, m)=\begin{cases} b_1-\dfrac{m(b_1-b_0)}{\sqrt{m^2-b_0^2}} & (b_0 b_1<m^2), \\ b_1-\sqrt{b_1^2-m^2} & (b_0 b_1\geq m^2). \end{cases}$$

Theorem 2.1. *Assume that* $\inf_t\{a(t)\}>0$, $\sup_t\{\frac{a'(t)}{2a(t)}\}<\infty$, *and* $(u_0, u_1)\in H^1\times L^2$. *If the following condition*

$$\sup_t\left\{\frac{a'(t)}{2a(t)}\right\}<\alpha_0(b_0, b_1, m) \tag{4}$$

holds, then there exists a positive constant C_ρ such that the following decay estimate to the solution of the Cauchy problem (1)–(2)

$$E(u(t))\leq C_{\rho}E(u(0))e^{-2\rho t}$$

holds for any $t\geq 0$ *and* $\rho<\rho_0$, *where*

$$\rho_0(I)=\beta_0-\Big[\sup_{t\in I}\Big\{\frac{a'(t)}{2a(t)}\Big\}-\alpha_0+\beta_0\Big]_+ \tag{5}$$

and $\rho_0=\rho_0([0,\infty))$.

Now, we shall introduce a class of functions, which is called the Gevrey class, to state our second theorem. Let $s>0$ and $\mu>0$. We define the Gevrey class G^s_μ and G^s by

$$G^s_\mu=\Big\{f(x)\in H^\infty;\ \sup_j\Big\{\frac{\mu^j}{j!^s}\|\nabla^j f\|\Big\}<\infty\Big\}\text{ and }G^s=\bigcup_{\mu>0}G^s_\mu,$$

where

$$\|\nabla^j f\|=\begin{cases}\|\Delta^{\frac{j}{2}}f\| & (j:\text{even}),\\ \|\nabla\Delta^{\frac{j-1}{2}}f\| & (j:\text{odd})\end{cases}$$

and $\Delta^k=(\partial^2_{x_1}+\cdots+\partial^2_{x_n})^k$. Then we have the following theorem :

Theorem 2.2. *Define the set* $\mathfrak{J}\subset[0,\infty)$ *by*

$$\mathfrak{J}=\Big\{t\in[0,\infty);\ \frac{a'(t)}{2a(t)}\geq\alpha_0\text{ or }a(t)=0\Big\} \tag{6}$$

Assume that there exist monotonically increasing sequences of non-negative real numbers $\{t_j\}$ *and* $\{t'_j\}$ *satisfying*

$$\mathfrak{J}\subset\begin{Bmatrix}\bigcup_{j\geq 1}(t_j,t'_j) & (t_1\neq 0)\\ [0,t_1)\cup\bigcup_{j\geq 2}(t_j,t'_j) & (t_1=0),\end{Bmatrix}\equiv\mathfrak{J}_0, \tag{7}$$

$$\lim_{t\to\infty}\Big\{\frac{\sum_{\{k;\,t_k\leq t\}}a(t_k)^{-1}+\sum_{\{k;\,t'_k\leq t\}}a(t'_k)^{-1}}{t}\Big\}=0 \tag{8}$$

and

$$\sum_j(t'_j-t_j)\exp\Big(M\sum_{l=1}^{j}\int_{t_l}^{t'_l}[a'(s)]_+ds\Big)<\infty \tag{9}$$

for any $M>0$. *If* $u_0,u_1\in G^s$ *for* $0<s<3/2$, *then there exists a positive constant* C_{ρ,u_0,u_1} *such that the following decay estimate to the solution of* (1)–(2)

$$E(u(t)) \leq C_{\rho, u_0, u_1} e^{-2\rho t}$$

holds for any $\rho < \rho_0([0, \infty) \backslash \mathfrak{J}_0)$. *Moreover, if*

$$\sum_{l \geq 1} \int_{t_l}^{t'_l} [a'(s)]_+ ds < \infty, \tag{10}$$

then there exist positive constants μ *and* C_ρ *such that the following decay estimate of the infinity order energy holds*

$$\sum_{j=0}^{\infty} \frac{\mu^j}{j!^{\frac{3}{2}}} E(\nabla^j u(t)) \leq C_{\rho, u_0, u_1} e^{-2\rho t}$$

for any $\rho < \rho_0([0, \infty) \backslash \mathfrak{J}_0)$.

Furthermore, we can prove the following theorem by a little modification of the proof of Theorem 2.2.

Theorem 2.3. *Under the same assumptions as in Theorem* 2.2, *there exists a positive constant* C_{l, ρ, u_0, u_1} *such that the following decay estimate of the higher order energy*

$$E(\nabla^l u(t)) \leq C_{l, \rho, u_0, u_1} e^{-2\rho t}$$

holds for any $\rho < \rho_0([0, \infty) \backslash \mathfrak{J}_0)$ *and for any positive integer* l.

Remark 2.1. If the assumption (9) holds, then the measure of $\mathfrak{J}_0$ must be finite.

The proofs of our theorems are based on a well-known method to obtain the exponential order energy decay to the dissipative wave equation of Klein–Gordon type, which is introduced in Zuazua [7], for instance.

Let us consider the case that $a(t) \equiv 1$ and that $u(t, x)$ is real valued. Let α be a positive constant satisfying $m > \alpha$ and define $E_{(\alpha)}(u(t))$ by

$$\begin{aligned} E_{(\alpha)}(u(t)) &= E(u(t)) + \alpha(u(t), u_t(t)) \\ &= \frac{1}{2} \{\|u_t(t) + \alpha u(t)\|^2 + \|\nabla u(t)\|^2 + (m^2 - \alpha^2)\|u(t)\|^2\}, \end{aligned} \tag{11}$$

where $(\bullet, \bullet)$ denotes the usual $L^2(\mathbb{R}^n)$ inner product.

Differentiating $E_{(\alpha)}(u(t))$ with respect to t, using the equation (1) and applying Schwarz' inequality (1), we have

$$\frac{d}{dt} E_{(\alpha)}(u(t)) = -(b(t) - \alpha)\|u_t(t)\|^2 - \alpha\|\nabla u(t)\|^2 - \alpha m^2\|u(t)\|^2$$

$$-\alpha b(t)(u(t),\ u_t(t))$$
$$\leq -\left\{b(t)-\alpha\left(1+\frac{b(t)}{2\alpha}\right)\right\}\|u_t(t)\|^2$$
$$-\alpha\|\nabla u(t)\|^2-\alpha\left(m^2-\frac{\alpha b(t)}{2}\right)\|u(t)\|^2,$$

where α is a positive constant satisfying $\frac{2m^2}{b(t)}>\alpha$. Now, we choose α satisfying

$$\inf_t\left\{\frac{b(t)}{1+\frac{b(t)}{2\delta}}\right\}>\alpha>0.$$

Then, there exists a positive constant ρ such that

$$\frac{d}{dt}E_{(\alpha)}(u(t))\leq -\rho E_{(\alpha)}(u(t)),$$

hence we obtain

$$E_{(\alpha)}(u(t))\leq E_{(\alpha)}(u(0))e^{-\rho t}.$$

Recalling the definition of $E_{(\alpha)}(u(t))$ and applying Schwarz' inequality, there exists a positive constant C_α such that we have from the above inequality

$$E(u(t))\leq CE(u(0))e^{-\rho t}$$

for any $t\in[0,\ \infty)$.

Thus, choosing $\alpha>0$ suitably, we obtain the energy decay of exponential order to the Cauchy problem (1)–(2) with $a(t)\equiv 1$.

Here we note the method above cannot be applied in case that $a(t)$ is not constant, but considering the choice of constant α, it is possible to prove the exponential order decay to the solution of a general hyperbolic equation like (1).

§3. Proof of Theorem 2.1

Let α be a positive constant satisfying $m>\alpha$ and define $E_{(\alpha)}(u(t))$ by

$$(12)\quad E_{(\alpha)}(u(t))=E(u(t))+\alpha(u(t),\ u_t(t))$$
$$=\frac{1}{2}\{\|u_t(t)+\alpha u(t)\|^2+a(t)\|\nabla u(t)\|^2+(m^2-\alpha^2)\|u(t)\|^2\},$$

where $(\cdot,\ \cdot)$ denotes the usual $L^2(\mathbb{R}^n)$ inner product. Here we remark that the energy norm $E_{(\alpha)}(u(t))$ is equivalent to the usual energy norm $E(u(t))$ since $\alpha<m$, that is, there exists a positive constant C_α independent of j such that the

following inequality

$$C_\alpha^{-1}E_{(\alpha)}(\nabla^j u(t)) \le E(\nabla^j u(t)) \le C_\alpha E_{(\alpha)}(\nabla^j u(t)) \tag{13}$$

holds for any $j \ge 0$.

Let β be a positive constant to be chosen later. Differentiating $E_{(\alpha)}(u(t))$ with respect to t, we have

$$\begin{aligned}\frac{d}{dt}E_{(\alpha)}(u(t)) &= \frac{1}{2}(a_t(t)-2\alpha a(t))\|\nabla u(t)\|^2\\ &\quad -(2b(t)-\alpha)\|u_t(t)+\alpha u(t)\|^2-\alpha(m^2-\alpha^2)\|u(t)\|^2\\ &\quad +2\alpha(b(t)-\alpha)(u(t),\, u_t(t)+\alpha u(t))\\ &= -2\beta E_{(\alpha)}(u(t))+\frac{1}{2}\{a_t(t)-2(\alpha-\beta)a(t)\}\|\nabla u(t)\|^2\\ &\quad -(2b(t)-\alpha-\beta)\left\|u_t(t)+\alpha u(t)+\frac{\alpha(\alpha-b(t))}{2b(t)-\alpha-\beta}u(t)\right\|^2\\ &\quad -\frac{\phi(b(t),\alpha,\beta)}{2b(t)-\alpha-\beta}\|u(t)\|^2,\end{aligned}$$

where

$$\phi(b(t),\alpha,\beta) = -\alpha^2 b(t)^2+2\{m^2(\alpha-\beta)+\alpha^2\beta\}b(t)-m^2(\alpha^2-\beta^2)-\alpha^2\beta^2.$$

If $2b(t)-\alpha-\beta>0$ and $\phi(b(t),\alpha,\beta)\ge 0$ then we have

$$\begin{aligned}\frac{d}{dt}E_{(\alpha)} &\le -2\beta E_{(\alpha)}(u(t))+\frac{1}{2}\{a_t(t)-2(\alpha-\beta)a(t)\}\|\nabla u(t)\|^2\\ &\le -2\rho(\alpha,\beta;I)E_{(\alpha)}(u(t)),\end{aligned}$$

where

$$\rho(\alpha,\beta;I)=\beta-\left[\sup_{t\in I}\left\{\frac{a'(t)}{2a(t)}\right\}-\alpha+\beta\right]_+$$

Now, we shall show that there exist $\alpha<\alpha_0$ and $\beta<\beta_0$ such that $\phi(b(t),\alpha,\beta)\ge 0$ and $\rho_0(I)-\rho(\alpha,\beta;I)<\varepsilon$ for any $\varepsilon>0$.

We can easily see the following inequalities $2b(t)-\alpha-\beta>0$, $\phi(b_0,\alpha_0,\beta_0)>0$, $\phi(b_1,\alpha_0,\beta_0)=0$ and $\partial_\beta\phi(b_1,\alpha,\beta)=-2(b_1-\beta)(m^2-\alpha^2)<0$ for any $\alpha<\alpha_0$ and $\beta<\beta_0$. Here, noting that $\phi(b,\alpha,\beta)$ is quadratic and convex with respect to b, hence we have $\phi(b(t),\alpha,\beta)\ge\min\{\phi(b_0,\alpha,\beta),\,\phi(b_1,\alpha,\beta)\}$. Therefore, by the continuity of $\phi(b,\alpha,\beta)$ with respect to α and β, we can take $\alpha<b$ and $\beta<b$ such that

$$\frac{d}{dt}E_{(\alpha)} \leq -2\rho(\alpha, \beta; I)E_{(\alpha)}(u(t))$$

and

$$\rho_0(I) - \rho(\alpha, \beta; I) < \varepsilon$$

for any given positive constant ε. Thus by Gronwall's inequality we have

$$E_{(\alpha)}(u(t)) \leq E_{(\alpha)}(u(t_0))\exp(-2(\rho_0(I) - \varepsilon)(t'_0 - t))$$

for any $t \in [t_0, t'_0] \equiv I$ and $\varepsilon > 0$. Recalling (13), we obtain

$$E(u(t)) \leq C_\varepsilon E(u(t_0))\exp(-2(\rho_0(I) - \varepsilon)(t'_0 - t))$$

and

$$\partial_\beta \phi(b_1, \alpha, \beta) = -2(b_1 - \beta)(m^2 - \alpha^2) < 0$$

for any $\alpha < \alpha_0$ and $\beta < \beta_0$. Here, noting that $\phi(b, \alpha, \beta)$ is quadratic and convex with respect to b, that is, $\phi(b(t), \alpha, \beta) \geq \min\{\phi(b_0, \alpha, \beta), \phi(b_1, \alpha, \beta)\}$. Thus, on the analogy of CASE 1, we see that there exist $\alpha < \alpha_0$ and $\beta < \beta_0$ such that $\frac{d}{dt}E_{(\alpha)} \leq -2\rho(\alpha, \beta; I)E_{(\alpha)}(u(t))$ and $\rho_0(I) - \rho(\alpha, \beta; I) < \varepsilon$ for any $\varepsilon > 0$.

CASE 3. When $b_0 b_1 \geq m^2$ and $b_0 < b_1$, repeating a similar proof as in CASE 2, we can also take $\alpha < \alpha_0$ and $\beta < \beta_0$ such that $\frac{d}{dt}E_{(\alpha)} \leq -2\rho(\alpha, \beta; I)E_{(\alpha)}(u(t))$ and $\rho_0(I) - \rho(\alpha, \beta; I) < \varepsilon$ for any $\varepsilon > 0$.

Therefore, by Gronwall's lemma, we obtain

$$E_{(\alpha)}(u(t)) \leq E_{(\alpha)}(u(t_0))\exp(-2(\rho_0(I) - \varepsilon)(t'_0 - t))$$

for any $t \in [t_0, t'_0] \equiv I$ and $\varepsilon > 0$. Recalling (13), we obtain

$$E(u(t)) \leq C_\varepsilon E(u(t_0))\exp(-2(\rho_0(I) - \varepsilon)(t'_0 - t)$$

for any positive constant C_ε. Thus, by putting $I = [0, \infty)$, the proof of Theorem 2.1 is concluded.

Remark 3.1. On the analogy of the proof above, we have also the decay estimate of higher order energy :

$$E(\nabla^j(u(t)) \leq C_\varepsilon E(\nabla^j u(t_0))\exp(-2(\rho_0(I) - \varepsilon)(t'_0 - t)) \tag{14}$$

for any $t\in I$ since $I\subset[0, \infty)\backslash\mathfrak{J}$.

§4. Proof of Theorem 2.2

When $a(t)$ degenerates at $t=t_0$, the usual hyperbolic energy $E(u(t))$ in general cannot be bounded by $E(u(0))$ in any neighborhood of $t=t_0$, so we shall introduce the infinity order energy.

Let $\{\tau_k\}$ and $\{\tau'_k\}$ be sequences of non-negative real numbers satisfying $\tau'_{k-1}<\tau_k<t_k<t'_k<\tau'_k$ for any k (if $t_1=0$, we put $\tau_1=t_1$).

Let δ be a positive real number and $\chi_\delta(t)\in C^1([0, \infty))$ be a non-negative function such that supp $\chi_\delta(t)\subset\cup_{k\geq 1}[\tau_k, \tau'_k]$, $\chi_\delta(t)=1$ on $\mathfrak{J}_0$ and that $\chi_\delta(t)$ is monotonically increasing on $[\tau_k, t_k]$ and monotonically decreasing on $[t'_k, \tau'_k]$ for each k. We define $e_j(u(t))=e_j(u(t);\alpha, \delta)$ $(j=0, 1, \cdots)$ by

$$e_j(u(t);\alpha,\delta)\begin{cases}=E_{(\alpha)}(u(t)) & (j=0),\\ =E_{(\alpha)}(\nabla^j u(t))+j^{-1}\chi_\delta(t)\|\nabla^{j+1}u(t)\|^2 & \\ \quad +j^3(E_{(\alpha)}(u(t))+j^{-1}\chi_\delta(t)\|\nabla^j u(t)\|^2) & (j\geq 1).\end{cases}$$

Let $\mu(t)\in C^1([0, \infty))$ be positive function to be chosen later. We define the infinity order energy $\mathscr{E}(u(t))=\mathscr{E}(u(t);\mu(t))=\mathscr{E}(u(t);\mu(t), \alpha, \delta)$ by

$$\mathscr{E}(u(t);\mu(t),\alpha,\delta)=\sum_{j=0}^{\infty}\frac{\mu(t)^j}{j!^3}e_j(u(t);\alpha,\delta). \tag{15}$$

Differentiating $\mathscr{E}(u(t))$ with respect to t, we have

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(u(t))&=\sum_{j=0}^{\infty}\frac{\mu(t)^j}{j!^3}\frac{d}{dt}e_j(u(t))+\sum_{j=0}^{\infty}j\frac{\mu'(t)}{\mu(t)}\frac{\mu(t)^j}{j!^3}e_j(u(t))\\ &=\frac{d}{dt}e_0(u(t))+\sum_{j=1}^{\infty}\frac{\mu(t)^j}{j!^3}\frac{d}{dt}e_j(u(t))+\sum_{j=1}^{\infty}j\frac{\mu'(t)}{\mu(t)}\frac{\mu(t)^j}{j!^3}e_j(u(t)).\end{aligned}$$

Now we shall estimate the infinity order $\mathscr{E}(u(t))$.
Applying Schwarz' inequality, we have

$$\begin{aligned}\frac{d}{dt}e_j(u(t))=&-2\rho E_{(\alpha)}(\nabla^j u(t))-2j^3\rho E_{(\alpha)}(\nabla^{j-1}u(t))\\ &-2\alpha j^{-1}\chi_\delta(t)\|\nabla^{j+1}u(t)\|^2-2\alpha j^2\chi_\delta(t)\|\nabla^j u(t)\|^2\\ &+j^{-1}\chi'_\delta(t)\|\nabla^{j+1}u(t)\|^2+j^2\chi'_\delta(t)\|\nabla^j u(t)\|^2\\ &+2j^{-1}\chi_\delta(t)(\nabla^{j+1}u(t), \nabla^{j+1}(u_t(t)+\alpha u(t)))\\ &+2j^2\chi_\delta(t)(\nabla^j u(t), \nabla^j(u_t(t)+\alpha u(t)))\\ \leq&-2\rho e_j(u(t))+j^{-1}[\chi'_\delta(t)]_+(\|\nabla^{j+1}u(t)\|^2+j^3\|\nabla^j u(t)\|^2)\end{aligned}$$

$$
\begin{aligned}
&+j^{-1}\chi_\delta(t)\{(j+1)\|\nabla^{j+1}u(t)\|^2+(j+1)^{-1}\|\nabla^{j+1}(u_t(t)+\alpha u(t))\|^2\} \\
&+j^2\chi_\delta(t)\{j\|\nabla^j u(t)\|^2+j^{-1}\|\nabla^j(u_t(t)+\alpha u(t))\|^2\} \\
\le& -2\Big(\rho-\frac{[\chi'_\delta(t)]_+}{ja(t)}\Big)e_j(u(t)) \\
&+\frac{\chi_\delta(t)}{j(j+1)}\{\|\nabla^{j+1}(u_t(t)+\alpha u(t))\|^2+j^2(j+1)\|\nabla^j(u_t(t)+\alpha u(t))\|^2\} \\
&+\chi_\delta(t)\Big(\frac{j+1}{j}\|\nabla^{j+1}u(t)\|^2+j^3\|\nabla^j u(t)\|^2\Big) \\
\le& -2\Big(\rho-\frac{[\chi'_\delta(t)]_++2\chi_\delta(t)}{a(t)}\Big)e_j(u(t))+\frac{4\chi_\delta(t)}{j(j+1)}e_{j+1}(u(t)),
\end{aligned}
$$

where we have assumed that $\alpha\le\beta$. Therefore, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathscr{E}(u(t);\mu(t))\le& -2(\rho-\eta_\delta(t))\mathscr{E}(u(t);\mu(t)) \\
&+\sum_{j=1}^{\infty}\frac{\mu(t)^j}{j!^3}\frac{j}{\mu(t)}(\mu'(t)+8\chi_\delta(t))e_j(u(t)),
\end{aligned}
$$

where

$$\eta_\delta(t)=\frac{[\chi'_\delta(t)]_++2\chi_\delta(t)}{a(t)},$$

and note the inequality

$$\sum_{j=1}^{\infty}\frac{\mu(t)^j}{j!^3}\frac{1}{j(j+1)}e_{j+1}(u(t))\le 2\sum_{j=2}^{\infty}\frac{\mu(t)^j}{j!^3}\frac{j}{\mu(t)}e_j(u(t)).$$

Now, taking $\mu(t)$ as

$$
\mu(t)=\begin{cases}\mu(t'_k)-8\int_{t'_k}^{t}\chi_\delta(s)ds & (t\in[t'_k,\tau'_k]),\\ \mu(\tau_k)-8\int_{\tau_k}^{t}\chi_\delta(s)ds & (t\in[\tau_k,t_k]),\end{cases}
$$

we have

$$
\mathscr{E}(u(t);\mu(t))\le\begin{cases}\mathscr{E}(u(t'_k);\mu(t'_k))\exp(-2\{\rho(t-t'_k)-\int_{t'_k}^{t}\eta_\delta(s)ds\})\,(t\in[t'_k,\tau'_k]),\\ \mathscr{E}(u(\tau_k);\mu(\tau_k))\exp(-2\{\rho(t-\tau_k)-\int_{\tau_k}^{t}\eta_\delta(s)ds\})\;\;(t\in[\tau_k,t_k]).\end{cases}
$$

<u>CASE 3 ($t\in[t'_k,t_k]$).</u> Repeating a similar estimate as in CASE 2, we have

$$
\begin{aligned}
\frac{d}{dt}e_0(u(t))\le& -2\beta e_0(u(t))+\Big[\sup_{t\notin\mathfrak{J}_0}\Big\{\frac{a'(s)}{2a(s)}\Big\}-\alpha+\beta\Big]a(t)\|\nabla u(t)\|^2 \\
&+\frac{1}{2}a'(t)\|\nabla u(t)\|^2-\sup_{t\notin\mathfrak{J}_0}\Big\{\frac{a'(s)}{2a(s)}\Big\}a(t)\|\nabla u(t)\|^2
\end{aligned}
$$

$$\leq -2\rho e_0(u(t)) + \frac{[a'(t)]_+}{2\delta} e_1(u(t))$$

and

$$\frac{d}{dt} e_j(u(t)) \leq -2\rho e_j(u(t)) + \frac{j[a'(t)]_+}{2\delta} e_j(u(t))$$
$$+\frac{\chi_\delta(t)}{j(j+1)} \{\|\nabla^{j+1}(u_t(t)+\alpha u(t))\|^2 + j^2(j+1)\|\nabla^j(u_t(t)+\alpha u(t))\|^2\}$$
$$+\chi_\delta(t)\Big(\frac{j+1}{j}\|\nabla^{j+1}u(t)\|^2 + j^3\|\nabla^j u(t)\|^2\Big)$$
$$\leq -2\rho e_j(u(t)) + \frac{j[a'(t)]_+}{2\delta} e_j(u(t))$$
$$+\frac{4\chi_\delta(t)}{j(j+1)} e_{j+1}(u(t)) + 2(j+1)e_j(u(t))$$

Therefore, we obtain

$$\frac{d}{dt}\mathscr{E}(u(t);\mu(t)) \leq -2\rho\mathscr{E}(u(t);\mu(t))$$
$$+\sum_{j=1}^{\infty}\frac{\mu(t)^j}{j!^3}\frac{j}{\mu(t)}\Big(\mu'(t)+\Big(\frac{[a'(t)]_+}{2\delta}+4\Big)\mu(t)+8\chi_\delta(t)\Big\}e_j(u(t))$$

for any $t \in [t_k, t'_k]$. Here, taking $\mu(t)$ as

$$\mu(t) = \Big(u(t_k) - 8\int_{t_k}^{t}\chi_\delta(s)\exp\Big(\int_{t_k}^{s}\frac{[a'(\tau)]_+}{2\delta}+4dr\Big)ds\Big)\exp\Big(-\int_{t_k}^{t}\frac{[a'(s)]_+}{2\delta}+4ds\Big),$$

we obtain

$$\mathscr{E}(u(t);\mu(t) \leq \mathscr{E}(u(t_k);\mu(t_k))e^{-2\rho(t-t_k)}$$

for any $t \in [t_k, t'_k]$.

Thus, we have the following decay estimate for the infinity order energy on $t \in [\tau'_k, \tau_{k+1}]$.

$$\mathscr{E}(u(t);\mu(t)) \leq \mathscr{E}(u(\tau'_k);\mu(\tau'_k))e^{-2\rho(t-\tau'_k)}$$
$$\leq \mathscr{E}(u(t'_k);\mu(t'_k))\exp\Big(-2\rho(t-t'_k)+\int_{t'_k}^{\tau'_k}\eta_\delta(s)ds\Big)$$
$$\leq \mathscr{E}(u(t_k);\mu(t_k))\exp\Big(-2\rho(t-t_k)+\int_{t'_k}^{\tau'_k}\eta_\delta(s)ds\Big)$$
$$\leq \mathscr{E}(u(\tau_k);\mu(\tau_k))\exp\Big(-2\rho(t-\tau_k)+\int_{[\tau_k,t_k]\cup[t'_k,\tau'_k]}\eta_\delta(s)ds\Big)$$

$$\vdots$$

$$\leq \mathscr{E}(u(0);\mu(0))\exp\Big(-2\rho t+\sum_{l=1}^{k}\int_{[\tau_l,t_l]\cup[t'_l,\tau'_l]}\eta_\delta(s)ds\Big)$$

$$\leq \mathscr{E}(u(0);\mu(0))\exp\Big(-2\rho t+\sum_{l=1}^{k}a_l^{-1}\delta\{1+2(t_l-\tau_l+\tau'_l-t'_l)\}\Big),$$

where

$$a_l^{-1}=\frac{1}{\inf_{s\in[\tau_l,t_l]}\{a(s)\}}+\frac{1}{\inf_{s\in[t'_l,\tau'_l]}\{a(s)\}}$$

and we note that

$$\int_{[\tau_k,t_k]\cup[t'_k,\tau'_k]}[\chi'_\delta(s)]_+ +2\chi_\delta(s)ds\leq\chi_\delta(t_l)+2\delta\int_{[\tau_k,t_k]\cup[t'_k,\tau'_k]}ds$$

$$\leq\delta\{1+2(t_l-\tau_l+\tau'_l-t'_l)\}.$$

Now, taking $\{\tau_l\}$ and $\{\tau'_l\}$ satisfying that $(\tau'_l-t'_l)+(t_l-\tau_l)\leq 1/2$ for any l, we have

$$\mathscr{E}(u(t);\mu(t))\leq\mathscr{E}(u(0);\mu(0))\Big(-2\rho t+2\delta\sum_{l=1}^{k}a_l^{-1}\Big). \tag{16}$$

Here we remark that the decay estimate (16) holds for any $t\in[\tau_k,\tau_{k+1}]$.

Now, we shall estimate $\sum_{l=1}^{k}a_l^{-1}$ by (8).

If $a'(t_l)<0$, we can take τ_l such that $a'(t)<0$ for any $t\in[\tau_l,t_l]$, then we have $\inf_{s\in[\tau_l,t_l]}a(s)=a(t_l)$. When $a'(t'_l)>0$, we can also take τ'_l such that $\inf_{s\in[t'_l,\tau'_l]}a(s)=a(t'_l)$.

If $a'(t_l)>0$, we can take τ_l such that $a'(t)>0$ for any $t\in[\tau_l,t_l]$. Then, noting that $\inf_{t\in[t_l,\tau_l]}\{a(t)\}=a(\tau_l)$ and $a'(t)\leq 2\alpha_0 a(t)$ for any $t\in[\tau_l,t_l]$ by (7), applying the mean value theorem, there exixts $s_l\in[\tau_l,t_l]$ such that

$$\frac{1}{\inf_{t\in[\tau_l,t_l]}\{a(t)\}}=\frac{1}{a(t_l)}+\frac{a(t_l)-a(\tau_l)}{a(t_l)a(\tau_l)}$$

$$\leq\frac{1}{a(t_l)}+\frac{a'(s_l)(t_l-\tau_l)}{a(t_l)a(\tau_l)}$$

$$\leq\frac{1}{a(t_l)}+\frac{2\alpha_0(t_l-\tau_l)}{a(t_l)}.$$

By taking τ_l like $2\alpha_0(t_l-\tau_l)\leq\frac{a(\tau_l)}{a(t_l)}$, we have

$$\frac{1}{\inf_{t\in[\tau_l,t_l]}\{a(t)\}}\leq\frac{2}{a(t_l)}. \tag{17}$$

When $a'(t'_l)<0$, we can also take τ'_l satisfying

$$\frac{1}{\inf_{t\in[t'_l,\,\tau'_l]}\{a(t)\}} \leq \frac{2}{a(t_l)}. \tag{18}$$

If $a'(t_l)=0$ and $a'(t)\neq 0$ near $t<t_l$, or $a'(t'_l)=0$ and $a'(t)\neq 0$ near $t>t'_l$, we can also take τ_l and τ'_l satisfying (17) and (18) respectively.

Therefore, there exist sequences $\{\tau_l\}$ and $\{\tau'_l\}$ of non-negative numbers such that

$$\sum_{l=1}^{k} a_l^{-1} \leq \sum_{l=1}^{k} 2a(t_l)^{-1} + \sum_{l=1}^{k} 2a(t_l)^{-1},$$

and $\lim_{t\to\infty}\delta(\sum_{t_k\leq t} a(t_k)^{-1} + \sum_{t'_k\leq t} a(t'_k)^{-1})/t = 0$ for any $\delta>0$ by (8).

Thus, there exists a positive constant C_δ independent of t, we have the following decay estimate of $\mathscr{E}(u(t);\mu(0))$:

$$\mathscr{E}(u(t);\mu(t)) \leq C_\delta \mathscr{E}(u(0);\mu(0))e^{-2\rho t} \tag{19}$$

for any $t\geq 0$ and $\rho<\rho_0$.

Next, we shall consider the positivity of $\mu(t)$. Let $t\in[\tau'_k, \tau_{k+1}]$ and put $\phi_l(t) = 8\int_{\tau_l}^{t}\chi_\delta(s)e^{\varphi_l(s)}ds$, $\phi'_l(t) = 8\int_{t'_l}^{t}\chi_\delta(s)e^{\varphi_l(s)}ds$, $\varphi_l(t) = \int_{t_l}^{t}\frac{[a'(s)]_+ + 8\delta}{2\delta}ds$ and $\Psi_l(t) = 8\int_{t_l}^{t}\chi_\delta(s)e^{\varphi_l(s)}ds$. Recalling the definition of $\mu(t)$;

$$\mu(t) = \begin{cases} \mu(\tau'_1) & (t\in[\tau'_1, \tau_{l+1}]), \\ \mu(t'_l) - \phi'_l(t) & (t\in[t'_l, \tau'_l]), \\ e^{-\varphi_l(t)}(\mu(t_l) - \Psi_l(t)) & (t\in[t_l, t'_l]), \\ \mu(\tau_l) - \phi_l(t) & (t\in[\tau_l, t_l]), \end{cases}$$

we have

$$\begin{aligned} \mu(\tau_{l+1}) &= \mu(\tau'_l) \\ &= \mu(t'_l) - \phi'_l \\ &= e^{-\varphi_l}\{\mu(t_l) - (\Psi_l + \phi'_l e^{\varphi_l})\} \\ &= e^{-\varphi_l}\{\mu(\tau_l) - (\Psi_l + \phi'_l e^{\varphi_l} + \phi_l)\} \end{aligned}$$

where $\phi_l=\phi_l(t_l)$, $\phi'_l=\phi'_l(\tau'_l)$, $\varphi_l=\varphi_l(t'_l)$ and $\Psi_l=\Psi(t'_l)$. Therefore, we have

$$\begin{aligned} \mu(t) &= \mu(\tau'_k) \\ &= e^{-\varphi_k}\{\mu(\tau_k) - (\Psi_k + \phi'_k e^{\varphi_k} + \phi_k)\} \\ &= e^{-\varphi_k}[e^{-\varphi_{k-1}}\{\mu(\tau_{k-1}) - (\Psi_{k-1} + \phi'_{k-1}e^{\varphi_{k-1}} + \phi_{k-1})\} - (\Psi_k + \phi'_k e^{\varphi_k} + \phi_k)] \end{aligned}$$

$$=e^{-(\phi_k+\phi_{k-1})}\{\mu(\tau_{k-1})-(\Psi_k e^{\phi_{k-1}}+\Psi_{k-1}+\phi'_k e^{\phi_k+\phi_{k-1}}+\phi'_{k-1}e^{\phi_{k-1}}$$
$$+\phi_k e^{\phi_{k-1}}+\phi_{k-1})\}$$
$$=e^{-(\phi_k+\phi_{k-1}+\phi_{k-2})}\{\mu(\tau_{k-2})-(\Psi_k e^{\phi_{k-1}+\phi_{k-2}}+\Psi_{k-1}e^{\phi_{k-2}}+\Psi_{k-2}+\phi'_k e^{\phi_k+\phi_{k-1}+\phi_{k-2}}$$
$$+\phi'_{k-1}e^{\phi_{k-1}+\phi'_{k-2}}+\phi'_{k-2}e^{\phi_{k-2}}+\phi_k e^{\phi_{k-1}+\phi_{k-2}}+\phi_{k-1}e^{\phi_{k-1}}+\phi_{k-2})\}$$
$$=e^{-(\phi_k+\cdots+\phi_1)}\{\mu(0)-(\Psi_k e^{\phi_{k-1}+\cdots+\phi_1}+\cdots+\Psi_2 e^{\phi_1}+\Psi_1+\phi'_k e^{\phi_k+\cdots+\phi_1}$$
$$+\cdots+\phi'_2 e^{\phi_2+\phi_1}+\phi'_1 e^{\phi_1}+\phi_k e^{\phi_{k-1}+\cdots+\phi_1}+\cdots+\phi_2 e^{\phi_1}+\phi_1)\}.$$

From the representation of $\mu(t)$, we can prove that $\mu(t)>0$ for any k and t if $\mu(0)=\mu_0$ is sufficiently large. Here we remark that $\mu(t)$ is monotonically increasing, hence $\mu(\tau_{k+1})\leq\mu(t)$ for any $t\in[\tau_k,\tau_{k+1}]$.

Now, we choose $\{\tau_l\}$ and $\{\tau'_l\}$ satisfying $t_l-\tau_l=\tau'_l-t'_l\equiv\nu_l$ and $\nu_l\leq t'_l-t_l$. Noting that $\phi_l\leq 8\delta\nu_l$, $\phi'_l\leq 8\delta\nu_l$ and

$$\sum_{h=1}^{l}\phi_h\leq\frac{1}{2\delta}\sum_{h=1}^{l}\int_{t_h}^{t'_h}[a'(s)]_+ds+4\sum_{h=1}^{l}(t'_h-t_h),$$

we have

$$\sum_{l=1}^{k}\phi'_l e^{\phi_l+\cdots+\phi_1}+\sum_{l=1}^{k}\phi_l e^{\phi_{l-1}+\cdots+\phi_1}$$
$$\leq\sum_{l=1}^{k}(\phi'_l+\phi_l)\exp\Big(\frac{1}{2\delta}\sum_{h=1}^{l}\int_{t_h}^{t'_h}[a'(s)]_+ds+4\sum_{h=1}^{l}(t'_h-t_h)\Big)$$
$$\leq 16\delta e^{4\,\mathrm{meas}\{\mathfrak{J}_0\}}\sum_{l=1}^{k}\nu_l\exp\Big(\frac{1}{2\delta}\sum_{h=1}^{l}\int_{t_h}^{t'_h}[a'(s)]_+ds\Big)$$
$$\leq 16\delta e^{4\,\mathrm{meas}\{\mathfrak{J}_0\}}\sum_{l=1}^{\infty}\nu_l\exp\Big(\frac{1}{2\delta}\sum_{h=1}^{l}\int_{t_h}^{t'_h}[a'(s)]_+ds\Big)$$
$$\equiv S_\delta<\infty$$

and

$$\sum_{l=1}^{k}\Psi_l e^{\phi_{l-1}+\cdots+\phi_1}=8\sum_{l=1}^{k}\int_{t_h}^{t'}\chi_\delta(s)e^{\phi_l(s)}ds\,e^{\phi_{l-1}+\cdots+\phi_1}$$
$$\leq 8\sum_{l=1}^{k}(t'_l-t_l)e^{\phi_l+\cdots+\phi_1}$$
$$\leq 8e^{4\,\mathrm{meas}\{\mathfrak{J}_0\}}\sum_{l=1}^{k}(t'_l-t_l)\exp\Big(\frac{1}{2\delta}\sum_{h=1}^{l}\int_{t_h}^{t'_h}[a'(s)]_+ds\Big)$$
$$\equiv S'_\delta<\infty$$

by (9), where meas$\{\mathfrak{J}_0\}$ is the measure of $\mathfrak{J}_0$. Therefore, we obtain

$$\mu(t) \geq e^{-\Sigma_k \phi_k}\{\mu_0 - (S_\delta + S'_\delta)\} > 0$$

for any $t \in [\tau_k, \tau_{k+1}]$ $(k=1, 2, \cdots)$ since $\mu_0 > S_\delta + S'_\delta$.

Here we remark that $\mu(t)$ is positive for any $t \geq 0$, but $\mu(t)$ is not uniformly positive, because $\sum_{l=1}^{k} \phi_l \to \infty$ as $k \to \infty$ in general. On the other hand, if (10) is satisfied, then $\sum_{l=1}^{\infty} \phi_l < \infty$, hence, $\mu(t)$ is uniformly positive, that is, there exists a positive constant μ_∞ such that $\inf_t \mu(t) = \lim_{t\to\infty} \mu(t) = \mu_\infty$.

Now, we shall conclude the proof of Theorem 2.2 from the decay estimate (19).

We introduce the following lemma.

Lemma 4.1. *Let* $0 < s \leq 3/2$ *and* $\mu > 0$.

(i) *If* $u_0, u_1 \in G^s$, *then the infinity order energy* $\mathscr{E}(u(0); \mu_0; \alpha, \delta)$ *is finite for any* $\mu_0 > 0$, $\alpha < m$ *and* $\delta > 0$.

(ii) *There exists a positive constant* $C_{\alpha,\delta}$ *such that* $\sum_{j=0}^{\infty} \frac{\mu^j}{j!^3} E(\nabla^j u(t)) = \mathscr{E}_0(u(t); \mu, 0, 0) \leq C_{\alpha,\delta} \mathscr{E}(u(t); \mu(t), \alpha, \delta)$.

Proof. (i) Noting the inequality (12), we have

$$\begin{aligned}\mathscr{E}(u(0); \mu_0, \alpha, \delta) &\leq C_\alpha \sum_{j=0}^{\infty} \frac{\mu_0^j}{j!^3} E(\nabla^j u(0)) \\ &\quad + \frac{1}{2} \delta \sum_{j=1}^{\infty} \frac{\mu_0^j}{j!^3} (j^{-1} \|\nabla^j u(0)\|^2 + j^2 \|\nabla^{j-1} u(0)\|^2) \\ &\leq C_1 C_\alpha \max\left\{1, m^2 - \alpha^2, a(0), \frac{\delta}{2}\right\} \sum_{j=0}^{\infty} \left(\frac{\mu_0}{\mu^2}\right)^j j!^{-3+2s} \\ &< \infty,\end{aligned}$$

where μ is a positive constant and we used the inequality $\frac{\kappa^j}{j!} \leq e^\kappa$ for any positive integer j and any positive real number κ.

(ii) is trivial from the inequality (12). □

Therefore, we obtain

$$E(u(t)) \leq \sum_{j=0}^{\infty} \frac{\mu(t)^j}{j!^3} E(\nabla^j u(t)) \leq C_{\rho,\delta} \mathscr{E}(u(0); \mu(0), \alpha) e^{-2\rho t}. \tag{20}$$

In particular, if $\mu(t)$ is uniformly positive, then we have the following decay estimate for the infinity order energy :

$$\sum_{j=0}^{\infty} \frac{\mu_\infty^j}{j!^3} E(\nabla^j u(t)) \leq C_{\rho,\delta} \mathscr{E}(u(0); \mu(0), \alpha) e^{-2\rho t} \tag{21}$$

for any $\rho < \rho_0$. Thus, the proof of Theorem 2.2 is concluded. □

Theorem 2.3 does not follow from the decay estimate (20) without the assumption (10). However, by considering the infinity order energy $\mathscr{E}(\nabla^k u(t); \mu(t))$, we have a decay estimate of the higher order energy $E(\nabla^k u(t))$ by analogy with the proof of Theorem 2.2.

§5. Examples

Finally, we shall introduce some examples which can be applied our theorems.

Example 1. Let $p>1$ and $q>0$, and define $a(t)$ by

$$a(t)=\sin(qt)+p, \quad b(t)\equiv 1 \text{ and } m=1.$$

If $\sup_t\left\{\frac{a'(t)}{2a(t)}\right\}=\sup_t\left\{\frac{q\cos(qt)}{2(\sin(qt)+p)}\right\}=\frac{q}{2(p-1)}<\beta_0=1$, then the total energy of the solution satisfies the following decay estimate :

$$E(u(t))\leq C_\rho E(u(0))e^{-2\rho t}$$

for any $t\geq 0$ and $\rho<\rho_0=1-q/2(p-1)$.

Example 2. Let $p\geq 1$ and define

$$a(t)=t^p, \quad b(t)=1 \text{ and } m=1.$$

Noting that $\sup_{t>p/2}\left\{\frac{a'(t)}{2a(t)}\right\}=\sup_{t>p/2}\left\{\frac{p}{2t}\right\}<\beta_0=1$, we see that $\mathfrak{J}=[0,p/2]$. Now we let $\mathfrak{J}_0=[0, p\tau/2]$, then $\mathfrak{J}_0$ satisfies from (7) to (10) for any $\tau>1$. If u_0, $u_1\in G^s(s<3/2)$, then there exists a positive constant C_{ρ, u_0, u_1} such that the total energy of the solution satisfies the following decay estimate :

$$E(u(t))\leq C_{\rho, u_0, u_1}E(u(0))e^{-2\rho t}$$

for any $t\geq 0$ and $\rho<\rho_0=1-\tau^{-1}$. Moreover, we can take $\tau>1$ arbitrary and the decay estimate above holds for any $\rho<1$.

Example 3. Let p and q be positive constants satisfying $a+q/p<p$. Define $a(t)\in C^1([0, \infty))$ by

$$a(t)=\begin{cases}\dfrac{\cos((2\pi e^{j}(t-j))+1}{[p]\sqrt{1+t}} & (t\in[j,j+e^{-j}]\ \ (j=1,2,\cdots)),\\ \dfrac{1}{[p]\sqrt{1+t}} & (t\notin[j,j+e^{-j}]\ \ (j=1,2,\cdots)),\end{cases}$$

$b(t)=2$ and $m=3$. Put $\{t_j\}=\{j^q\}$ and $\{t_j'\}=\{j^q+e^{-j}\}$. Then, the assumptions in Theorem 2.2 (7)–(9) are fulfilled. Indeed, $\mathfrak{J}\subset\cup(t_j,t_j')\equiv\mathfrak{J}_0$,

$$\text{meas}\{\mathfrak{J}_0\}=\sum_{j=1}^{\infty}e^{-j}=\frac{1}{e-1}<\infty,$$

$$\frac{\sum_{\{k;\,t_k\le t\}}a(t_k)^{-1}+\sum_{\{k;\,t_k'\le t\}}a(t_k')^{-1}}{t}\le\frac{2\sum_{j=1}^{k}(j^{\frac{q}{p}}+2)}{k^p}=\frac{\mathcal{O}(1+k^{\frac{q}{p}+1})}{k^p}\to 0$$

$$(k\to\infty)$$

for any $t\in[k^q,(k+1)^q)$, and

$$\sum_{j=1}^{\infty}(t_j'-t_j)\exp\left(M\sum_{l=1}^{j}\int_{t_l}^{t_l'}[a'(s)]_+ds\right)=\sum_{j=1}^{\infty}\exp\left(-j+M\sum_{l=1}^{j}\int_{t_l}^{t_l'}[a'(s)]_+ds\right)$$

$$\le\left.\begin{cases}\sum_{j=1}^{\infty}e^{-j+Mj^{1-\frac{q}{p}}} & \left(\frac{q}{p}\neq1\right)\\ \sum_{j=1}^{\infty}j^{M}e^{-j}<\infty & \left(\frac{q}{p}=1\right)\end{cases}\right\}<\infty$$

for any $M>0$. By noting that $\sup_{t\in[0,\infty)\setminus\mathfrak{J}_0}\frac{a'(t)}{2a(t)}\le0$, there exists a positive constant C_{ρ,u_0,u_1} such that the total energy of the solution to (1)–(2) has the decay estimate of exponential order $E(u(t))\le C_{\rho,u_0,u_1}e^{-2\rho t}$ for any $t\ge0$ and $\rho<2$ since $u_0,u_1\in G^s$ with $s<3/2$.

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