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Large Time Behavior of Solutions for Derivative Cubic Nonlinear Schrödinger Equations

By

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Abstract

We study the asymptotic behavior in time and scattering problem for the solutions to the Cauchy problem for the derivative cubic nonlinear Schrödinger equations of the following form

(A) $$iu_t+u_{xx}=\mathcal{N}(u,\bar{u},u_x,\bar{u}_x),\quad t\in\mathbf{R},x\in\mathbf{R};\quad u(0,x)=u_0(x),\quad x\in\mathbf{R},$$

where

$$\mathcal{N}(u,\bar{u},u_x,\bar{u}_x)=\mathcal{K}_1|u|^2u+i\mathcal{K}_2|u|^2u_x+i\mathcal{K}_3u^2\bar{u}_x+\mathcal{K}_4|u_x|^2u+\mathcal{K}_5\bar{u}u_x^2+i\mathcal{K}_6|u_x|^2u_x,$$

$\mathcal{K}_j=\mathcal{K}_j(|u|^2)$, $\mathcal{K}_j(z)\in C^3(\mathbf{R}^+)$; $\mathcal{K}_j(z)=\lambda_j+O(z)$, as $z\to+0$, $\mathcal{K}_1$, $\mathcal{K}_6$ are real valued functions. Here the parameters λ_1, $\lambda_6\in\mathbf{R}$, and λ_2, λ_3, λ_4, $\lambda_5\in\mathbf{C}$ are such that $\lambda_2-\lambda_3\in\mathbf{R}$ and $\lambda_4-\lambda_5\in\mathbf{R}$. If $\mathcal{K}_5(z)=\frac{\lambda_5}{1+\mu z}$ and $\lambda_5=\mu=\pm1$, $\mathcal{K}_1=\mathcal{K}_2=\mathcal{K}_3=\mathcal{K}_4=\mathcal{K}_6=0$ equation (A) appears in the classical pseudospin magnet model [9]. We prove that if $u_0\in H^{3,0}\cap H^{2,1}$ and the norm $\|u_0\|_{3,0}+\|u_0\|_{2,1}=\varepsilon$ is sufficiently small, then the solution of (A) exists globally in time and satisfies the sharp time decay estimate $\|u(t)\|_{2,0,\infty}\le C\varepsilon(1+|t|)^{-1/2}$, where $\|\varphi\|_{m,s,p}=\|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\varphi\|_{L^p}$, $H_p^{m,s}=\{\varphi\in\mathcal{S}'$; $\|\varphi\|_{m,s,p}<\infty\}$. Furthermore we prove existence of modified scattering states and nonexistence of nontrivial scattering states. Our method is based on a certain gauge transformation and an appropriate phase function.

§ 1. Introduction

In this paper we study the Cauchy problem for the derivative cubic nonlinear

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Schrödinger equation of the following form

$$(1.1)\qquad \begin{cases} iu_t+u_{xx}=\mathcal{N}(u,\bar{u},u_x,\bar{u}_x), & x\in\mathbb{R},\quad t\in\mathbb{R},\\ u(0,x)=u_0, & x\in\mathbb{R},\end{cases}$$

where

$$\mathcal{N}(u,\bar{u},u_x,\bar{u}_x)=\mathscr{K}_1|u|^2u+i\mathscr{K}_2|u|^2u_x+i\mathscr{K}_3u^2\bar{u}_x + \mathscr{K}_4|u_x|^2u+\mathscr{K}_5\bar{u}u_x^2+i\mathscr{K}_6|u_x|^2u_x,$$

$\mathscr{K}_j=\mathscr{K}_j(|u|^2)$, $\mathscr{K}_j(z)\in C^3(\mathbb{R}^+)$; $\mathscr{K}_j(z)=\lambda_j+O(z)$, as $z\to+0$, $\mathscr{K}_1$, $\mathscr{K}_6$ are real valued functions. The parameters λ_1, $\lambda_6\in\mathbb{R}$, and λ_2, λ_3, λ_4, $\lambda_5\in\mathbb{C}$ are such that $\lambda_2-\lambda_3\in\mathbb{R}$ and $\lambda_4-\lambda_5\in\mathbb{R}$. The linear part of equation (1.1) consists of the linear Schrödinger operator, while the nonlinearity involves all possible combinations of derivatives of unknown functions of cubic order with a complex-conjugate structure. Such kinds of equations are of highest interest in many areas of Physics. The nonlinear term of (1.1) does not satisfy the condition that $\partial_{u_x}\mathcal{N}(u,\bar{u},u_x,\bar{u}_x)$ is pure imaginary, which is the well known sufficient condition for the local solvability of the nonlinear Schrödinger equations of the derivative type (see [8]). Here we encounter a difficulty of the derivative loss and so the standard energy methods can not be applicable directly to such an equation. To overcome this difficulty one has therefore to use either dispersive smoothing effects of the linear part of the equation, or some gauge transformation. This last method is used in the present paper. It relies on some algebraic properties of the nonlinearity (similar to that of papers [1], [6], [12]) and is subtle. Another difficulty in the study of the large time asymptotic behavior of solutions to the Cauchy problem (1.1) is that the cubic nonlinear term of (1.1) is critical for large time values, because it does not satisfy the so called null gauge condition introduced in [13]. Indeed the nonlinearity of (1.1) in general can not be written in the form $(\nu_1u+\nu_2u_x)\partial_x|u|^2$, where ν_1, $\nu_2\in\mathbb{C}$. To treat the critical cubic nonlinearity of (1.1) we use the techniques developed in previous works [4], [5], where we introduced an appropriate phase function.

Note that in the case $\mathscr{K}_1=\mathscr{K}_6=0$, $\mathscr{K}_2=\mathscr{K}_3=-i\nu_1\in\mathbb{C}$ and $\mathscr{K}_4=\mathscr{K}_5=\nu_2\in\mathbb{C}$ the nonlinear term of equation (1.1) has the form $(\nu_1u+\nu_2u_x)\partial_x|u|^2$ and therefore satisfies the null gauge condition of Tsutsumi [13]. So the global existence of small solutions and the existence of the usual scattering states were proved in [8] under the conditions that the initial data $u_0\in H^{6,0}\cap H^{1,5}$ and the norm $\|u_0\|_{5,0}+\|u_0\|_{0,5}$ is sufficiently small. Here and below we denote by $H^{m,s}=\{\varphi\in L^2:\ \|\varphi\|_{m,s}=\|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\varphi\|_{L^2}<\infty\}$ the usual weighted Sobolev space.

If we choose $\mathscr{K}_5(z)=\frac{\lambda_5}{1+\mu z}$ and $\lambda_5=\mu=\pm1$, and the rest functions $\mathscr{K}_1=\mathscr{K}_2=\mathscr{K}_3=\mathscr{K}_4=\mathscr{K}_6=0$, then equation (1.1) appears in the classical pseudospin magnet model [9]. For this case the almost global existence of solutions to (1.1) was obtained in [8]. More precisely, the existence time T was shown in [8] to be greater than $\exp((\|u_0\|_{4,0}+\|u_0\|_{0,4})^{-2})$, if the initial data $u_0\in H^{5,0}\cap H^{1,4}$ have sufficiently small norm $\|u_0\|_{4,0}+\|u_0\|_{0,4}$.

When $\mathscr{K}_2=2$, $\mathscr{K}_3=1$, and the rest functions $\mathscr{K}_1=\mathscr{K}_4=\mathscr{K}_5=\mathscr{K}_6=0$, then equation (1.1) reduces to the derivative nonlinear Schrödinger equation

$$iu_t+u_{xx}=i(|u|^2u)_x, \tag{1.2}$$

which was studied in [5], [11]. Note that equation (1.2) also does not satisfy the null gauge condition of [13]. However, by the gauge transformation technique (see [3]) equation (1.2) can be translated into a system of nonlinear Schrödinger equations which do not involve derivatives of unknown functions in the nonlinear terms. So one can treat equation (1.2) similar to the cubic nonlinear Schrödinger equation $iu_t+u_{xx}=|u|^2u$, namely, we can apply the method of papers [2], [4], [10] to equation (1.2). Thus in the case of equation (1.2), the modified scattering states were constructed in [5] and the existence of the modified wave operators was proved in paper [7]. However for the case of equation (1.1) we do not know the existence of a gauge transformation translating it into a system of nonlinear Schrödinger equations without derivatives of unknown function in the nonlinear term and as far as we know the existence of modified wave operators and modified scattering states for equation (1.1) are still open problems.

In this paper we prove the global existence of solutions to the Cauchy problem (1.1) in the weighted Sobolev spaces for small initial data as well as the existence of the modified scattering states (see Theorem 1.1 below). Our result is sharp because in Theorem 1.2 below we provide a non-existence result of ordinary scattering state. Furthermore we obtain the large time asymptotics of solutions (involving the sharp L^∞ time decay estimates). We now introduce

Notation and function spaces. Let $\mathscr{F}\varphi$ or $\hat{\varphi}$ denote the Fourier transform of φ defined by $\hat{\varphi}(\xi)=\frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}\varphi(x)dx$. The inverse Fourier transform $\mathscr{F}^{-1}\varphi$ or $\check{\varphi}$ is given by $\check{\varphi}(x)=\frac{1}{\sqrt{2\pi}}\int e^{ix\xi}\varphi(\xi)d\xi$. The free Schrödinger evolution group $\mathscr{U}(t)$ is written as $\mathscr{U}(t)\varphi=\mathscr{F}^{-1}e^{-it\xi^2}\hat{\varphi}$ and also can be represented in the following form $\mathscr{U}(t)=M(t)\mathscr{D}(t)\mathscr{F}M(t)$, where $M=M(t)=\exp(ix^2/4t)$, the dilation operator is $(\mathscr{D}(t)\varphi)(x)=\frac{1}{\sqrt{2it}}\varphi(\frac{x}{2t})$. The inverse free Schrödinger evolution group is $\mathscr{U}(-t)=-M(-t)i\mathscr{F}^{-1}\mathscr{D}(\frac{1}{2t})M(-t)$, where $\mathscr{D}^{-1}(t)=-\frac{i}{\sqrt{2}}D(\frac{1}{2t})$ is the inverse dilation operator. Using the above identities we easily see that $\mathscr{T}(t)=x+2it\partial_x=\mathscr{U}(t)x\mathscr{U}(-t)=M(t)(2it\partial_x)M(-t)$. We also widely use the following identities $[\mathscr{T},\partial_x]=-1$, $[\mathscr{L},\mathscr{T}]=0$, where $\mathscr{L}=i\partial_t+\partial_x^2$, $\partial_t=\frac{\partial}{\partial t}$ and $\partial_x=\frac{\partial}{\partial x}$.

We introduce some function spaces. The Lebesgue space is $L^p=\{\varphi\in\mathscr{S}' : \|\varphi\|_p<\infty\}$, where $\|\varphi\|_p=(\int|\varphi(x)|^pdx)^{1/p}$ if $1\le p<\infty$ and $\|\varphi\|_\infty=\text{ess. sup}\{|\varphi(x)|\,;x\in$

$\mathbf{R}\}$ if $p=\infty$. For simplicity we let $\|\varphi\|=\|\varphi\|_2$. Weighted Sobolev space is $H_p^{m,s}=\{\varphi\in\mathscr{S}' : \|\varphi\|_{m,s,p}=\|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\varphi\|_p<\infty\}$, $m, s\in\mathbf{R}$, $1\le p\le\infty$; we denote also $H^{m,s}=H_2^{m,s}$ and $\|\varphi\|_{m,s}=\|\varphi\|_{m,s,2}$. Let $C(I\,;\,B)$ be the space of continuous functions from a time interval I to a Banach space B. Different positive constants might be denoted by the same letter C.

Our main results are

Theorem 1.1. *We assume that the initial data* $u_0\in H^{3,0}\cap H^{2,1}$ *and the norm* $\varepsilon=\|u_0\|_{3,0}+\|u_0\|_{2,1}$ *is sufficiently small. Then there exists a unique global solution of the Cauchy problem* (1.1) *such that* $u\in C(\mathbf{R}\,;\,H^{2,0})\cap L^\infty_{loc}(\mathbf{R}\,;\,H^{3,0})$, $\mathscr{T}u\in C(\mathbf{R}\,;\,H^{1,0})\cap L^\infty_{loc}(\mathbf{R}\,;\,H^{2,0})$, *and the following time decay estimate*

$$\|u(t)\|_{2,0,\infty}\le C\varepsilon(1+|t|)^{-1/2} \tag{1.3}$$

is valid for all t. Moreover there exist unique functions $u_j^\pm\in L^2\cap L^\infty$ *and real valued functions* $\mathscr{G}^\pm\in L^\infty$, *such that*

$$\left\|\mathscr{U}(-t)\partial_x^j u(t)-\mathscr{F}^{-1}(u_j^\pm\exp(\pm i\mathscr{G}^\pm\log|t|))\right\|\le C|t|^{-\alpha}, \tag{1.4}$$

as $t\to\pm\infty$, *where* $j=0, 1, 2,\ 0<\alpha<\frac{1}{4}$. *And the following asymptotic formula*

$$\partial_x^j u(t,x)=\frac{1}{\sqrt{t}}u_j^\pm\Big(\frac{x}{2t}\Big)\exp\Big(\frac{ix^2}{4t}\pm i\mathscr{G}^\pm\Big(\frac{x}{2t}\Big)\log|t|\Big)+O(|t|^{-1/2-\alpha}), \tag{1.5}$$

is valid as $t\to\pm\infty$ *uniformly in* $x\in\mathbf{R}$, *where* $0<\alpha<\frac{1}{4}$.

Next we present the nonexistence of the usual scattering states.

Theorem 1.2. *In addition to the conditions of Theorem* 1.1 *we assume that* $|\lambda_1|+|\lambda_2-\lambda_3|+|\lambda_4-\lambda_5|+|\lambda_6|\ne0$ *and there exists a final state* $u^+\in H^{1,\delta}$, $\delta>\frac{1}{2}$ *such that* $\|u(t)-\mathscr{U}(t)u^+\|\to0$ *as* $t\to\infty$. *Then the final state* u^+ *is identically zero. Furthermore if the solution satisfies* L^2 *conservation law, then the solution u is also identically zero.*

We organize our paper as follows. Section 2 is devoted to some preliminary estimates of solutions to the Cauchy problem (1.1). The local existence of solutions for the Cauchy problem (1.1) in a space $\mathbf{X}^{m,s}=\{\varphi\in C([0,T]\,;H^{m-1,0})\cap L^\infty(0,T\,;H^{m,0})\,;\,\mathscr{T}^s\varphi\in C([0,T]\,;H^{m-s-1,0})\cap L^\infty(0,T\,;H^{m-s,0})\}$, where $m\ge3$, $0\le s\le m$, is stated in Theorem 2.1. Then in our key Lemma 2.1 we prove the optimal time decay estimate of global solutions to the Cauchy problem (1.1) in the uniform norm $\sup_{t\ge0}\sqrt{1+t}\,\|u(t)\|_{2,0,\infty}\le C$, while the norm of solutions $\|u(t)\|_{\mathbf{Y}}=\|u(t)\|_{3,0}+\|\mathscr{T}u(t)\|_{2,0}$ can slightly grow with respect to time : $\sup_{t\ge0}(1+t)^{-\gamma}\|u(t)\|_{\mathbf{Y}}\le C$, where $\gamma \in (0, \frac{1}{12})$. Section 3 is devoted to the proof of Theorems 1.1–1.2. In what follows we consider the case of the positive time only since the negative time can be treated in the same manner.

§ 2. Preliminary A Priori Estimates

By virtue of the method of papers [1], [6] and [12] (see also the proof of a-priory estimates in the norm $\mathbb{Y}$ below in Lemma 2.1) we easily obtain the existence of local solutions in the functional spaces $\mathbb{X}^{m,s}$, with any integers $m \geq 3$, and $0 \leq s \leq m$. Below we will use this result taking $m=3$ and $s=1$.

Theorem 2.1. *Let the initial data $u_0 \in H^{m,0} \cap H^{m-s,s}$ with some $m \geq 3$, $0 \leq s \leq m$, $m, s \in \mathbf{N}$. Then for some time $T>0$ there exists a unique solution $u \in \mathbb{X}^{m,s}$ of the Cauchy problem (1.1). If we assume in addition that the norm of the initial data $\|u_0\|_{m,0} + \|u_0\|_{m-s,s} = \varepsilon$ is sufficiently small, then there exists a unique solution $u \in \mathbb{X}^{m,s}$ of (1.1) on a finite time interval $[0, T]$ with $T > 1/\varepsilon$, such that the following estimate $\sup_{t \in [0,T]} (\|u\|_{m,0} + \|\mathscr{T}^s u\|_{m-s,0}) < 2\varepsilon$ is valid.*

We now prove in the next lemma the optimal time decay estimate $\|u(t)\|_{2,0,\infty} \leq C(1+t)^{-1/2}$ of global solutions to the Cauchy problem (1.1) along with a-priori estimates of solutions in the norm $\mathbb{Y}$.

Lemma 2.1. *Let the initial data $u_0 \in H^{3,0} \cap H^{2,1}$ and the norm $\|u_0\|_{3,0} + \|u_0\|_{2,1} = \varepsilon$ be sufficiently small. Then there exists a unique global solution of the Cauchy problem (1.1) such that $u \in C(\mathbf{R}; H^{2,0}) \cap L^\infty(\mathbf{R}; H^{3,0})$, and $\mathscr{T} u \in C(\mathbf{R}; H^{1,0}) \cap L^\infty(\mathbf{R}; H^{2,0})$. Moreover the following estimate*

$$(1+t)^{-\gamma} \|u(t)\|_{\mathbb{Y}} + \sqrt{1+t}\, \|u(t)\|_{2,0,\infty} < 15\varepsilon \tag{2.1}$$

is valid for all $t \geq 0$, where $\gamma \in (0, \frac{1}{12})$.

Proof. Applying the result of Theorem 2.1 and using a standard continuation argument we can find a maximal time $T>0$ such that the nonstrict inequality

$$(1+t)^{-\gamma} \|u(t)\|_{\mathbb{Y}} + \sqrt{1+t}\, \|u(t)\|_{2,0,\infty} \leq 15\varepsilon \tag{2.2}$$

is true for all $t \in [0, T]$. If we prove the strict estimate (2.1) on the whole time interval $[0, T]$, then by contradiction arguments we obtain the desired result of the lemma. By the usual energy method (i.e. multiplying (1.1) by $\bar{u}$, integrating over $\mathbf{R}$ and taking the imaginary part of the result) we get

$$\frac{d}{dt}\|u(t)\|^2 \leq C\|u\|_{2,0,\infty}^2\|u\|^2,$$

whence in view of the estimate (2.2) by the Gronwall inequality we obtain $\|u(t)\| \leq 2\varepsilon(1+t)^{\gamma}$. Applying the operator $\mathscr{T}=x+2it\partial_x=M(t)(2it\partial_x)M(-t)$ to both sides of equation (1.1) and using the structure of the nonlinearity $\mathscr{N}(u, \bar{u}, u_x, \bar{u}_x)$ and the identity $[\mathscr{L}, \mathscr{T}]=0$, we find

$$\begin{aligned} \mathscr{L}\mathscr{T}u=\mathscr{T}\mathscr{N}(u, \bar{u}, u_x, \bar{u}_x)&=M2it\partial_x\mathscr{N}(Mu, \overline{M\bar{u}}, Mu_x, \overline{M\bar{u}_x}) \\ &=\mathscr{N}_u\mathscr{T}u+\mathscr{N}_{\bar{u}}\overline{\mathscr{T}u}+\mathscr{N}_{u_x}\mathscr{T}u_x+\mathscr{N}_{\bar{u}_x}\overline{\mathscr{T}u_x}, \end{aligned} \tag{2.3}$$

whence the energy method with estimate (2.2) yield

$$\frac{d}{dt}\|\mathscr{T}u(t)\|^2 \leq C\|u\|_{1,0,\infty}^2(\|\mathscr{T}u\|^2+\|\mathscr{T}u_x\|^2) \leq C\varepsilon^4(1+t)^{2\gamma-1}. \tag{2.4}$$

Integrating (2.4) with respect to time we get $\|\mathscr{T}u\| \leq 2\varepsilon(1+t)^{\gamma}$. Differentiating equation (1.1) three times with respect to x we get

$$\mathscr{L}\partial_x^3 u=\mathscr{N}_{u_x}\partial_x^4 u+\mathscr{N}_{\bar{u}_x}\partial_x^4\bar{u}+\mathscr{R}_1, \tag{2.5}$$

where the remainder term $\mathscr{R}_1$ has the estimate $\|\mathscr{R}_1\| \leq C\|u\|_{2,0,\infty}^2\|u\|_{3,0} \leq C\varepsilon^3(1+t)^{\gamma-1}$ in view of (2.2). Analogously differentiating equation (2.3) two times with respect to x we obtain

$$\mathscr{L}(\mathscr{T}u)_{xx}=\mathscr{N}_{u_x}(\mathscr{T}u)_{xxx}+\mathscr{N}_{\bar{u}_x}(\overline{\mathscr{T}u})_{xxx}+\mathscr{R}_2. \tag{2.6}$$

It is easy to see that the remainder term $\mathscr{R}_2$ in view of (2.2) has the following estimate $\|\mathscr{R}_2\| \leq C\|u\|_{2,0,\infty}^2(\|\mathscr{T}u\|_{2,0}+\|u\|_{2,0}) \leq C\varepsilon^3(1+t)^{\gamma-1}$. We can not apply the usual energy method to (2.5) and (2.6) because of the lost of derivatives, so we use a gauge transformation similar to that applied in papers [1], [6], [12]. We compute the multiplication factor

$$\mathscr{A}=\exp\left(-\frac{1}{2}\int_{-\infty}^{x}\mathrm{Re}\mathscr{N}_{u_x}dx\right),$$

thus the function $\mathscr{A}$ satisfies the condition $\partial_x\mathscr{A}=-\frac{1}{2}\mathscr{A}\mathrm{Re}\mathscr{N}_{u_x}$ with the commutator relation $[\mathscr{L}, \mathscr{A}]=i\mathscr{A}_t+\mathscr{A}_{xx}+2\mathscr{A}_x\partial_x=-\mathscr{A}\mathrm{Re}\mathscr{N}_{u_x}\partial_x+\mathscr{A}(\frac{i}{2}\int_{-\infty}^{x}\mathrm{Re}(\mathscr{N}_{u_x})_t dx+\frac{1}{4}(\mathrm{Re}\mathscr{N}_{u_x})^2-\frac{1}{2}(\mathrm{Re}\mathscr{N}_{u_x})_x)$. Multiplying both sides of (2.5) and (2.6) by $\mathscr{A}$, we get

$$\mathscr{L}\mathscr{A}\partial_x^3 u=[\mathscr{L}, \mathscr{A}]\partial_x^3 u+\mathscr{A}\mathscr{L}\partial_x^3 u=i\,\mathrm{Im}\,\mathscr{N}_{u_x}\mathscr{A}\partial_x^4 u+\mathscr{N}_{\bar{u}_x}\mathscr{A}\partial_x^4\bar{u}+\mathscr{R}_3 \tag{2.7}$$

and

$$\mathscr{L}\mathscr{A}(\mathscr{T}u)_{xx}=i\,\mathrm{Im}\,\mathscr{N}_{u_x}\mathscr{A}(\mathscr{T}u)_{xxx}+\mathscr{N}_{\bar{u}_x}\mathscr{A}(\overline{\mathscr{T}u})_{xxx}+\mathscr{R}_4, \tag{2.8}$$

where the remainder terms $\mathscr{R}_3$ and $\mathscr{R}_4$ have the estimate $\|\mathscr{R}_3\|+\|\mathscr{R}_4\|\le C\|u\|^2_{2,0,\infty}(\|u\|_{3,0}+\|\mathscr{T}u\|_{2,0})\le C\varepsilon^3(1+t)^{\gamma-1}$. Now we can apply the usual energy method to (2.7) and (2.8) to get

$$\frac{d}{dt}(\|\mathscr{A}\partial_x^3 u\|^2+\|\mathscr{A}(\mathscr{T}u)_{xx}\|^2)$$
$$\le C\|u\|^2_{2,0,\infty}(\|\mathscr{A}\partial_x^3 u\|^2+\|\mathscr{A}(\mathscr{T}u)_{xx}\|^2)+C\varepsilon^4(1+t)^{\gamma-1},$$

whence the Gronwall inequality yields $(\|\mathscr{A}\partial_x^3 u\|^2+\|\mathscr{A}(\mathscr{T}u)_{xx}\|^2)<2\varepsilon(1+t)^{\gamma}$. Thus we have the estimate

$$\|u(t)\|_{\mathrm{Y}}<3\varepsilon(1+t)^{\gamma} \tag{2.9}$$

for all $t\in[0,T]$. By the Sobolev inequality $\|u\|_\infty\le\sqrt{2}\|u\|_{1,0}$ and estimate (2.9) we obtain

$$\sup_{0\le t\le 1}\sqrt{1+t}\,\|u(t)\|_{2,0,\infty}\le 2\|u(t)\|_{\mathrm{Y}}<12\varepsilon. \tag{2.10}$$

So now let us consider the case $t\ge 1$. As in paper [4] we change the dependent variable $u(t,x)=\frac{1}{\sqrt{t}}e^{it\chi^2}v(t,\chi)$, where $\chi=\frac{x}{2t}$. Then since $u_x(t,x)=\frac{1}{\sqrt{t}}e^{it\chi^2}\mathscr{I}v(t,\chi)$ with $\mathscr{I}=i\chi+\frac{1}{2t}\partial_\chi$, taking into account the complex-conjugate structure of the nonlinearity in equation (1.1) we obtain

$$\begin{cases} iv_t+\dfrac{1}{4t^2}v_{\chi\chi}+\sqrt{t}\,\mathscr{N}\Big(\dfrac{v}{\sqrt{t}},\dfrac{\bar{v}}{\sqrt{t}},\dfrac{\mathscr{I}v}{\sqrt{t}},\dfrac{\overline{\mathscr{I}v}}{\sqrt{t}}\Big)=0, & \chi\in\mathbf{R},\quad t\ge 1,\\ v(1,\chi)=u(1,\chi)e^{-i\chi^2}, & \chi\in\mathbf{R}. \end{cases} \tag{2.11}$$

Note also that $\mathscr{T}u(t,x)=\frac{i}{\sqrt{t}}e^{it\chi^2}v_\chi(t,\chi)$, therefore we have the following relations $\sqrt{t}\,\|u\|_{2,0,\infty}=\|(1-\mathscr{I}^2)v\|_\infty$, $\|\mathscr{T}u\|_{2,0}=\|(1-\mathscr{I}^2)v_\chi\|$ and $\|u\|+\|u_{xxx}\|=\|v\|+\|\mathscr{I}^3v\|$. So we need to prove the estimate $\|(1-\mathscr{I}^2)v\|_\infty<12\varepsilon$. In order to get it we multiply equation (2.11) by operators $\mathscr{I}$ and $\mathscr{I}^2$. Then we need to extract from the nonlinearity the main term which diverges for large values of time. Using the identity

$$\bar{\varphi}\mathscr{I}\phi=\frac{1}{2}(\bar{\varphi}\mathscr{I}\phi-\phi\overline{\mathscr{I}\varphi})+\frac{1}{4t}(\bar{\varphi}\phi)_\chi, \tag{2.12}$$

where we take $\varphi=\phi=v$, we write the following representations

$$i\lambda_2|v|^2\mathscr{I}v=i\lambda_2 v\Big(\frac{1}{2}(\bar{v}\mathscr{I}v-v\overline{\mathscr{I}v})+\frac{1}{4t}(|v|^2)_x\Big)=\lambda_2 v\mathrm{Im}(v\overline{\mathscr{I}v})+\frac{i\lambda_2}{4t}v(|v|^2)_x$$

and

$$i\lambda_3 v^2\overline{\mathscr{I}v}=i\lambda_3 v(\overline{\bar{v}\mathscr{I}v})=-\lambda_3 v\mathrm{Im}(v\overline{\mathscr{I}v})+\frac{i\lambda_3}{4t}v(|v|^2)_x.$$

Applying the identity

$$\bar{\varphi}\mathscr{I}\phi=-\phi\overline{\mathscr{I}\varphi}+\frac{1}{2t}(\bar{\varphi}\phi)_x \tag{2.13}$$

with $\varphi=\phi=v$, we get $\lambda_5(\bar{v}\mathscr{I}v)\mathscr{I}v=-\lambda_5|\mathscr{I}v|^2v+\frac{\lambda_5}{2t}(|v|^2)_x\mathscr{I}v$ and in the same manner taking $\varphi=\mathscr{I}v$, $\phi=v$ in the formula (2.13) and after that choosing $\varphi=\phi=\mathscr{I}v$ in the identity (2.12), we obtain

$$\begin{aligned}i\lambda_6|\mathscr{I}v|^2\mathscr{I}v&=i\lambda_6\Big(-v\overline{\mathscr{I}^2v}+\frac{1}{2t}(v\overline{\mathscr{I}v})_x\Big)\mathscr{I}v\\&=-i\lambda_6\Big(\frac{1}{2}(\overline{\mathscr{I}^2v}\mathscr{I}v-\overline{\mathscr{I}v}\mathscr{I}^2v)+\frac{1}{4t}(|\mathscr{I}v|^2)_x\Big)v+\frac{i\lambda_6}{2t}(v\overline{\mathscr{I}v})_x\mathscr{I}v\\&=-\lambda_6 v\mathrm{Im}(\overline{\mathscr{I}v}\mathscr{I}^2v)-\frac{i\lambda_6}{4t}(|\mathscr{I}v|^2)_xv+\frac{i\lambda_6}{2t}(v\overline{\mathscr{I}v})_x\mathscr{I}v.\end{aligned}$$

Therefore we can represent the nonlinearity in the form $\sqrt{t}\mathscr{N}\big(\frac{v}{\sqrt{t}},\frac{\bar{v}}{\sqrt{t}},\frac{\mathscr{I}v}{\sqrt{t}},\frac{\overline{\mathscr{I}v}}{\sqrt{t}}\big)=\frac{1}{t}\mathscr{G}v+\mathscr{P}_0$, where the coefficient at the main term is

$$\mathscr{G}=\lambda_1|v|^2+(\lambda_2-\lambda_3)\mathrm{Im}(v\overline{\mathscr{I}v})+(\lambda_4-\lambda_5)|\mathscr{I}v|^2-\lambda_6\mathrm{Im}(\overline{\mathscr{I}v}\mathscr{I}^2v)$$

and the remainder term is

$$\mathscr{P}_0=\mathscr{L}-\frac{i\lambda_6}{4t^2}(|\mathscr{I}v|^2)_xv+\frac{i\lambda_6}{2t^2}(v\overline{\mathscr{I}v})_x\mathscr{I}v,$$

where

$$\begin{aligned}\mathscr{L}=\frac{1}{t}\Big(&(\mathscr{K}_1-\lambda_1)|v|^2v+i(\mathscr{K}_2-\lambda_2)|v|^2\mathscr{I}v+i(\mathscr{K}_3-\lambda_3)v^2\overline{\mathscr{I}v}\\&+(\mathscr{K}_4-\lambda_4)|\mathscr{I}v|^2v+(\mathscr{K}_5-\lambda_5)\bar{v}(\mathscr{I}v)^2+i(\mathscr{K}_6-\lambda_6)|\mathscr{I}v|^2\mathscr{I}v\Big)\\&+\frac{i(\lambda_2+\lambda_3)}{4t^2}(|v|^2)_xv+\frac{\lambda_5}{2t^2}(|v|^2)_x\mathscr{I}v.\end{aligned}$$

Here the functions $\mathscr{K}_j=\mathscr{K}_j\big(\frac{|v|^2}{t}\big)$, $j=1,\dots,6$. Now we apply the following formula $\mathscr{I}(\varphi\phi)=\phi\mathscr{I}\varphi+\frac{1}{2t}\varphi\phi_x$ with $\varphi=\mathscr{I}v$, $\phi=|\mathscr{I}v|^2$ and then (2.12) with $\varphi=\phi=\mathscr{I}v$ to get

$$\mathscr{I}(i\lambda_6|\mathscr{I}v|^2\mathscr{I}v)=i\lambda_6|\mathscr{I}v|^2\mathscr{I}^2v+\frac{i\lambda_6}{2t}(|\mathscr{I}v|^2)_\chi\mathscr{I}v$$
$$=-\lambda_6\operatorname{Im}(\overline{\mathscr{I}v}\mathscr{I}^2v)\mathscr{I}v+\frac{3i\lambda_6}{4t}(|\mathscr{I}v|^2)_\chi\mathscr{I}v.$$

Finally using the identity $\mathscr{I}(\varphi\phi\bar{\phi})=\phi\bar{\phi}\mathscr{I}\varphi+\varphi\bar{\phi}\mathscr{I}\phi+\varphi\phi\overline{\mathscr{I}\phi}$ we represent the nonlinearity in the form $\sqrt{t}\mathscr{I}^j\mathscr{N}(\frac{v}{\sqrt{t}},\frac{\bar{v}}{\sqrt{t}},\frac{\mathscr{I}v}{\sqrt{t}},\frac{\overline{\mathscr{I}v}}{\sqrt{t}})=\frac{1}{t}\mathscr{G}\mathscr{I}^jv+\mathscr{P}_j, j=1, 2$, where the remainder terms are

$$\mathscr{P}_1=\frac{1}{2t^2}v(\lambda_1|v|^2+(\lambda_2-\lambda_3)\operatorname{Im}(\overline{v\mathscr{I}v})$$
$$+(\lambda_4-\lambda_5)|\mathscr{I}v|^2)_\chi+\mathscr{I}\mathscr{Z}+\frac{3i\lambda_6}{4t^2}(|\mathscr{I}v|^2)_\chi\mathscr{I}v$$

and $\mathscr{P}_2=\frac{1}{2t^2}\mathscr{G}_\chi\mathscr{I}v+\mathscr{I}\mathscr{P}_1$. Via inequality (2.2) we have $\|\mathscr{G}\|_\infty\le C\varepsilon^2$. The difference $\mathscr{K}_j-\lambda_j$ has an additional time decay $\|\mathscr{K}_j-\lambda_j\|_\infty\le\frac{C\varepsilon^2}{t}, j=1, \ldots, 6$, therefore in view of (2.2) we obtain the following estimate $\|\mathscr{P}_j\|_{L^1(\mathbf{R}_\chi)}+\|\mathscr{P}_j\|_{L^2(\mathbf{R}_\chi)}\le Ct^{-2}\|u\|_Y^3 \le C\varepsilon^3t^{3\gamma-2}, j=0, 1, 2$ for the remainder terms. Taking into account the commutator relation $[\tilde{\mathscr{L}}, \mathscr{I}]=0$, where $\tilde{\mathscr{L}}=i\partial_t+\frac{1}{4t^2}\partial_\chi^2$, we get from equation (2.11)

$$\tilde{\mathscr{L}}\mathscr{I}^jv+\frac{1}{t}\mathscr{G}\mathscr{I}^jv+\mathscr{P}_j=0,\quad j=0, 1, 2. \tag{2.14}$$

We define the evolution operator $\mathscr{W}(t)\varphi=\mathscr{F}^{-1}e^{i\xi^2/4t}\hat{\varphi}=\frac{\sqrt{t}}{\sqrt{i\pi}}\int e^{-it(\chi-y)^2}\varphi(y)dy$. First of all we note that $\|\mathscr{W}(t)\varphi\|=\|\varphi\|$. Also it is easy to see that the estimates

$$\|\mathscr{W}(t)\varphi\|_\infty\le\sqrt{t}\|\varphi\|_1,\quad \|(\mathscr{W}(t)-1)\varphi\|_\infty\le\left\|\left(e^{i\xi^2/4t}-1\right)\hat{\varphi}(\xi)\right\|_1\le Ct^{-\alpha/2}\|\varphi\|_{1,0}$$

and analogously

$$\|(\mathscr{W}(t)-1)\varphi\|\le\left\|(e^{i\xi^2/4t}-1)\hat{\varphi}(\xi)\right\|\le Ct^{-\alpha}\|\varphi\|_{1,0}$$

are valid, where $\alpha\in[0, \frac{1}{2})$. Multiplying (2.14) by $\mathscr{W}(-t)$ we obtain

$$i(\mathscr{W}(-t)\mathscr{I}^jv)_t+\frac{1}{t}\mathscr{G}(\mathscr{W}(-t)\mathscr{I}^jv)+\mathscr{Q}_j=0,\quad j=0, 1, 2, \tag{2.15}$$

where the remainder terms $\mathscr{Q}_j=\frac{1}{t}(\mathscr{W}(-t)-1)\mathscr{G}\mathscr{I}^jv+\frac{1}{t}\mathscr{G}(1-\mathscr{W}(-t))\mathscr{I}^jv+\mathscr{W}(-t)\mathscr{P}_j$ via inequality (2.2) have the estimate $\|\mathscr{Q}_j\|_\infty+\|\mathscr{Q}_j\|\le C\varepsilon^3t^{3\gamma-5/4}$. To prove the estimate $\|\mathscr{W}(-t)\mathscr{I}^jv\|_\infty<3\varepsilon$ we change the dependent variable

$$\mathscr{W}(-t)\mathscr{I}^jv(t,\chi)=w_j(t,\chi)\exp\left(i\int_1^t\mathscr{G}(\tau,\chi)\frac{d\tau}{\tau}\right)$$

in the equation (2.15) to get

$$i\partial_t w_j + \exp\Big(-i\int_1^t \mathscr{G}(\tau, \chi)\frac{d\tau}{\tau}\Big)\mathscr{Q}_j = 0, \quad j=0, 1, 2. \tag{2.16}$$

Integration of (2.16) with respect to time t yields

$$\|w_j(t) - w_j(s)\|_p \leq C\varepsilon^3 \int_s^t \tau^{3\gamma - 5/4} d\tau \leq C\varepsilon^3 s^{-\mu}, \tag{2.17}$$

for all $1 < s < t$, where $p=2, \infty, j=0, 1, 2, \mu = \frac{1}{4} - 3\gamma > 0$. By virtue of (2.17) we obtain $\sup_{t\in[0,T]}\|w_j(t)\|_\infty < 3\varepsilon$. Therefore in view of (2.2) we find

$$\|(1-\mathscr{I}^2)v\|_\infty \leq \|(1-\mathscr{W}(-t))(1-\mathscr{I}^2)v\|_\infty + \|w_0 - w_2\|_\infty < 7\varepsilon.$$

The contradiction obtained gives us the result of the lemma.

Q. E. D.

§ 3. Proofs of Theorems 1.1–1.2

Proof of Theorem 1.1. We have the existence result and estimate (1.3) by Lemma 2.1. Via inequality (2.17) there exist unique limits $w_j^+ \in L^\infty \cap L^2$ such that $\lim_{t\to\infty} w_j(t) = w_j^+$ in $L^\infty \cap L^2$. Hence there exists a unique limit

$$\mathscr{G}^+ = \lim_{t\to\infty}\mathscr{G}(t) = \lambda|w_0^+|^2 + (\lambda_2 - \lambda_3)\mathrm{Im}(w_0^+\bar{w}_1^+) + (\lambda_4 - \lambda_5)|w_1^+|^2 - \lambda_6 \mathrm{Im}(w_2^+\bar{w}_1^+)$$

in L^∞. Thus we get

$$\begin{aligned}\partial_x^j u(t,x) &= \frac{1}{\sqrt{t}}e^{it\chi^2}\mathscr{I}^j v = \frac{1}{\sqrt{t}}e^{it\chi^2}\mathscr{W}(-t)\mathscr{I}^j v + \frac{1}{\sqrt{t}}e^{it\chi^2}(1-\mathscr{W}(-t))\mathscr{I}^j v \\ &= \frac{1}{\sqrt{t}}e^{it\chi^2}w_j\Big(t, \frac{x}{2t}\Big)\exp\Big(i\int_1^t \mathscr{G}\Big(\tau, \frac{x}{2t}\Big)\frac{d\tau}{\tau}\Big) + O(t^{-1/2-\mu}) \\ &= \frac{1}{\sqrt{t}}e^{it\chi^2}w_j^+\Big(\frac{x}{2t}\Big)\exp\Big(i\int_1^t \mathscr{G}\Big(\tau, \frac{x}{2t}\Big)\frac{d\tau}{\tau}\Big) + O(t^{-1/2-\mu}) \end{aligned} \tag{3.1}$$

uniformly with respect to $x \in \mathbf{R}$, here $\mu = \frac{1}{4} - 3\gamma > 0$. For the phase of the asymptotic representation (3.1) we write the identity

$$\int_1^t \mathscr{G}(\tau)\frac{d\tau}{\tau} = \mathscr{G}^+ \log t + \Phi(t),$$

where $\Phi(t) = (\mathscr{G}(t) - \mathscr{G}^+)\log t + \int_1^t (\mathscr{G}(\tau) - \mathscr{G}(t))\frac{d\tau}{\tau}$. We have

$$\Phi(t)-\Phi(s)=\int_s^t(\mathscr{G}(\tau)-\mathscr{G}(t))\frac{d\tau}{\tau}+(\mathscr{G}(t)-\mathscr{G}^+)\log\frac{t}{s}, \tag{3.2}$$

for all $1<s<t$. Applying estimates (2.1) of Lemma 2.1 and inequality (2.17) to (3.2) we get $\|\Phi(t)-\Phi(s)\|_\infty\leq C\varepsilon s^{-\mu}$ for $1<s<t$. This implies that there exists a unique limit $\Phi^+=\lim_{t\to\infty}\Phi(t)\in L^\infty$ such that

$$\|\Phi(t)-\Phi^+\|_\infty\leq Ct^{-\mu}. \tag{3.3}$$

By virtue of (3.3) we find

$$\left\|\int_1^t\mathscr{G}\Big(\tau,\frac{x}{2t}\Big)\frac{d\tau}{\tau}-\mathscr{G}^+\log t-\Phi^+\right\|_\infty\leq C\varepsilon t^{-\mu}. \tag{3.4}$$

We now put $u_j^+=w_j^+\exp(i\Phi^+)$. Therefore we obtain the asymptotics (1.5) for $t\to\infty$ uniformly with respect to $x\in\mathbf{R}$. Via (3.4) and (2.17) we have

$$\|\mathscr{J}^jv-u_j^+\exp(i\mathscr{G}^+\log t)\|\leq\left\|\mathscr{W}(-t)\mathscr{J}^jv-u_j^+e^{i\mathscr{G}^+\log t}\right\|+\|(1-\mathscr{W}(t))\mathscr{J}^jv\|$$

$$\leq\left\|w_j\exp\Big(i\int_1^t\mathscr{G}\frac{d\tau}{\tau}\Big)-w_j^+\exp(i\mathscr{G}^+\log t+i\Phi^+)\right\|+O(t^{-\mu})=O(t^{-\mu}),$$

whence we get

$$\left\|\mathscr{U}(-t)\partial_x^ju(t)-\mathscr{F}^{-1}(u_j^+\exp(i\mathscr{G}^+\log t))\right\|$$

$$\leq\left\|\mathscr{U}(-t)\partial_x^ju(t)-\mathscr{U}(-t)\sqrt{-2i}M\mathscr{D}\mathscr{J}^jv\right\|+\sqrt{2}\|\mathscr{U}(-t)M\mathscr{D}\mathscr{F}(M-1)\mathscr{F}^{-1}\mathscr{J}^jv\|$$

$$+\sqrt{2}\|\mathscr{F}^{-1}(\mathscr{J}^jv-u_j^+\exp(i\mathscr{G}^+\log t))\|\leq O(t^{-\mu}),$$

whence (1.4) follows. This completes the proof of Theorem 1.1.

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Proof of Theorem 1.2. We prove the theorem by contradiction, so we assume that u^+ is not identically zero. Multiplying equation (1.1) by $\mathscr{U}(-t)$ and integrating with respect to time we find

$$\mathscr{U}(-t)u(t)-\mathscr{U}(-s)u(s)=i\int_s^t\mathscr{U}(-\tau)\mathscr{N}(u,\bar{u},u_x,\bar{u}_x)d\tau. \tag{3.5}$$

We decompose $\mathscr{N}(u,\bar{u},u_x,\bar{u}_x)$ as follows

$$\mathscr{N}(u,\bar{u},u_x,\bar{u}_x)=(\mathscr{N}(u,\bar{u},u_x,\bar{u}_x)-\mathscr{N}(\mathscr{U}(t)u^+,\overline{\mathscr{U}(t)u^+},\mathscr{U}(t)u_x^+,\overline{\mathscr{U}(t)u_x^+}))$$

$$+(\mathscr{N}(\mathscr{U}(t)u^+,\overline{\mathscr{U}(t)u^+},\mathscr{U}(t)u_x^+,\overline{\mathscr{U}(t)u_x^+})$$

$$-\mathcal{N}(\mathcal{U}(t)\overline{M}u^{+}, \overline{\mathcal{U}(t)\overline{M}u^{+}}, \mathcal{U}(t)\overline{M}u_x^{+}, \overline{\mathcal{U}(t)\overline{M}u_x^{+}})) \tag{3.6}$$
$$+\mathcal{N}(M\mathcal{D}\widehat{u^{+}}, \overline{M\mathcal{D}\widehat{u^{+}}}, M\mathcal{D}\widehat{u_x^{+}}, \overline{M\mathcal{D}\widehat{u_x^{+}}}),$$

where we have used the identity $\mathcal{U}(t)\bar{M}=M\mathcal{D}\mathcal{F}$. Since

$$\left\| \mathcal{N}\left(M\mathcal{D}\widehat{u^{+}}, \overline{M\mathcal{D}\widehat{u^{+}}}, M\mathcal{D}\widehat{u_x^{+}}, \overline{M\mathcal{D}\widehat{u_x^{+}}}\right)\right\| \geq \frac{1}{t}\left\|\Lambda |\widehat{u^{+}}|^2 \widehat{u^{+}}\right\| - \frac{C}{t^2}\|u^{+}\|_{1,\delta}^5,$$

where $\Lambda(\xi)=\lambda_1-(\lambda_2-\lambda_3)\xi+(\lambda_4-\lambda_5)\xi^2-\lambda_6\xi^3$ is not equal to zero identically, we have by (3.5), (3.6) and estimates for the solution $u(t)$ provided in Theorem 1.1

$$\|\mathcal{U}(-t)u(t)-\mathcal{U}(-s)u(s)\| \geq \left\|\Lambda |\widehat{u^{+}}|^2 \widehat{u^{+}}\right\| \int_s^t \frac{d\tau}{\tau} - C\|u^{+}\|_{1,\delta}^5 \int_s^t \frac{d\tau}{\tau^2}$$
$$-C(\varepsilon^2+\|u^{+}\|_{1,\delta}^2)\int_s^t (\|u(\tau)-\mathcal{U}(\tau)u^{+}\|_{1,0}+\|(\overline{M}-1)u^{+}\|_{1,0})\frac{d\tau}{\tau}.$$

This implies that for any $\theta>0$ there exists $T(\theta)$ such that for any $t>s>T(\theta)$

$$\|\mathcal{U}(-t)u(t)-\mathcal{U}(-s)u(s)\| \geq \left(\left\|\Lambda |\widehat{u^{+}}|^2 \widehat{u^{+}}\right\| - \theta\right)\int_s^t \frac{d\tau}{\tau}$$

which means $u^{+}=0$ which is the desired contradiction. If the solution satisfies the conservation of the L^2 norm, we have $u\equiv 0$. Theorem 1.2 is proved.

Q. E. D.

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