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Multiple Poles at Negative Integers for $\int_A f^\lambda \square$ in the Case of an Almost Isolated Singularity

By

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Résumé

Nous donnons une condition nécessaire et suffisante topologique sur $A \in H^0(\{f \neq 0\}, \mathbb{C})$, pour un germe analytique réel $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$, dont la complexifiée présente une singularité isolée relativement à la valeur propre 1 de la monodromie, pour que le prolongement analytique de $\int_A f^\lambda \square$ présente un pôle multiple aux entiers négatifs assez "grands". On montre en particulier que si un tel pôle multiple existe, il apparaît déjà pour $\lambda = -(n+1)$ avec l'ordre maximal que nous calculons topologiquement.

Summary

We give a necessary and sufficient topological condition on $A \in H^0(\{f \neq 0\}, \mathbb{C})$, for a real analytic germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$, whose complexification has an isolated singularity relatively to the eigenvalue 1 of the monodromy, in order that the meromorphic continuation of $\int_A f^\lambda \square$ has a multiple pole at sufficiently "large" negative integers. We show that if such a multiple pole exists, it occurs already at $\lambda = -(n+1)$ with its maximal order which is computed topologically.

Content

Introduction

The aim of the present Note is to generalize the result of [5] and its converse proved in [6] to the case of the eigenvalue 1. So we shall give a

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necessary and sufficient topological condition in order that the meromorphic extension of the holomorphic current.

$$\lambda \to \int_A f^\lambda \Box$$

defined in a neighbourhood of the origin in $\mathbb{R}^{n+1}$ has a pole of order a least 2 at $\lambda = -(n+1)$, in the case of a real analytic germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ satisfying the following condition: we assume that the complexification $f_{\mathbb{C}}$ of f admits an isolated singularity at 0 for the eigenvalue 1 of the monodromy. This notion, introduced in [2], means that for any $x \neq 0$ in $f_{\mathbb{C}}^{-1}(0)$ near 0, the monodromy of $f_{\mathbb{C}}$ acting on the reduced cohomology of the Milnor fiber of $f_{\mathbb{C}}$ at x has no non zero invariant vector.

Of course this hypothesis is satisfied when $f_{\mathbb{C}}$ has an isolated singularity at 0, but it allows also much more complicated situations.

In our result the interplay between connected components of the semi-analytic set $\{f \neq 0\}$ is essential: we denote by $A = \sum_{\alpha=1}^{a} c_\alpha A_\alpha$ an element in $H^0(\{f \neq 0\}, \mathbb{C})$ so A_α are connected components of $\{f \neq 0\}$ and c_α are complex numbers (we shall precise below the meaning of $\int_{A_\alpha} f^\lambda \Box$ when $A_\alpha \subset \{f < 0\}$). Our topological necessary and sufficient condition is given on A.

The main new point here, compare to [5] and [6] is the use of [3] which explains how to compute the variation map in this context of isolated singularity for the eigenvalue 1, in term of differential forms.

I want to thank Prof. Guzein–Zade who point out to me that the orientations are not enough precise in [5]; so I shall try to take them carefully in account here. The reader will see that it is not so easy. I want also to thank Prof. B. Malgrange who suggests several improvements to the first draw of this article.

§ 1. Mellin Transform on $\mathbb{R}^*$

Let $\varphi \in C^\infty(\mathbb{R}^*)$ such that

$$\begin{cases} \text{(i)} & \operatorname{supp} \varphi \subset [-A, A] \\ \text{(ii)} & \varphi \text{ is bounded} \end{cases}.$$

We define for $\operatorname{Re} \lambda > 0$

$$M\varphi(\lambda) := \frac{1}{i\pi}\left[\int_0^{+\infty} x^\lambda \varphi(x) \frac{dx}{x} - e^{-i\pi\lambda}. \int_0^{+\infty} x^\lambda \varphi(-x) \frac{dx}{x}\right].$$

Examples. Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \geq 0$ and let $\varphi_0(x) = |x|^\alpha$ near 0 and $\varphi_1(x) = |x|^\alpha \operatorname{sgn}(x)$ near 0. Then we have

$$M\varphi_0(\lambda) = \frac{1}{i\pi}\frac{1 - e^{-i\pi\lambda}}{\lambda + \alpha} + \text{entire function of } \lambda$$

and

$$M\varphi_1(\lambda) = \frac{1}{i\pi}\frac{1 + e^{-i\pi\lambda}}{\lambda + \alpha} + \text{entire function of } \lambda.$$

So for $\alpha \notin \mathbb{N}$ we have a simple pole at $\lambda = -\alpha$. For $\alpha = 2k$ with $k \in \mathbb{N}$, $M\varphi_0$ has no pole but $M\varphi_1$ has one at $\lambda = -2k$.

For $\alpha = 2k+1$ with $k \in \mathbb{N}$ $M\varphi_1$ has no pole but $M\varphi_0$ has one at $\lambda = -2k-1$. This is reasonnable because $|x|^{2k}$ is C^∞ at 0 and $|x|^{2k+1}\operatorname{sgn}(x)$ is also C^∞ at 0 for $k \in \mathbb{N}$. So poles of $M\varphi$ measure the singularity of φ at 0, as usual.

Without the condition ii) the situation is slightly more complicated: we shall use the following elementary lemma.

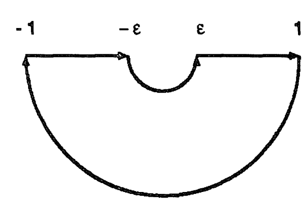
Lemma 1. *Let P and Q in $\mathbb{C}[x]$ of degrees at most $k-1$ and let*

$$\varphi(x) = \begin{cases} P(\log x) & \text{for } x > 0 \\ Q(\log|x| - i\pi) & \text{for } x < 0 \end{cases}$$

near 0, *and assume φ satisfies condition* i) *and $\varphi \in C^\infty(\mathbb{R}^*)$.*

Then $M\varphi$ has no pole at $\lambda = 0$ iff $P = Q$. Morever if $P = Q$ $M\varphi$ is entire.

Proof. For $P = Q$ we have $M\varphi(\lambda) = \frac{1}{i\pi}\int_{-1}^{+1} P(\log z) z^\lambda \frac{dz}{z}$ modulo an entire function of λ, where $\log z = \log|z| + i\operatorname{Arg} z$ with $-\pi < \operatorname{Arg} z < \pi$. From Cauchy formula on the path



this give $-\int_{-\pi}^{0} P(i\theta) e^{i\lambda\theta} d\theta$ which is an entire function of λ.

If $P \neq Q$, as we have already seen that $\int_{-1}^{+1} Q(\log z) z^\lambda \frac{dz}{z}$ is entire in λ, it is enough to show that $\int_0^1 (Q - P)(\log x) x^\lambda \frac{dx}{x}$ has a pole of order ≥ 1 at $\lambda = 0$. But we have

$$\int_0^1 (\log x)^l x^\lambda \frac{dx}{x} = \frac{d^l}{d\lambda^l}\left(\int_0^1 x^\lambda \frac{dx}{x}\right)$$
$$= (-1)^{l-1}\frac{(l-1)!}{\lambda^{l+1}}$$

which gives the conclusion.

§ 2. Statement of the Result

Let $f : X_{\mathbb{R}} \to]-\delta,\delta[$ a Milnor representative of the non zero real analytic germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$. This is, by definition, the restriction to $\mathbb{R}^{n+1}$ of a Milnor representative of the complexification $f_{\mathbb{C}} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of f.

Let $(A_\alpha)_{\alpha\in[1,a]}$ be the connected components of the relatively compact semi-analytic open set $\{f \neq 0\} \cap X_{\mathbb{R}}$ and denote by $A = \sum_{\alpha=1}^{a} c_\alpha A_\alpha$, where the c_α are complex numbers, a fixed element in $H^0(\{f \neq 0\} \cap X_{\mathbb{R}}, \mathbb{C})$.

Definition. For a compactly supported C^∞ n-form φ on $X_{\mathbb{R}}$, and for $-\delta < s < \delta$, set

$$I_\alpha(s) = \int_{(f=s)\cap A_\alpha} \varphi$$

where the orientation of $\{f = s\} \cap A_\alpha$ is choosen in such a way that we have

$$\left.\begin{aligned} i\pi M_{I_\alpha}(\lambda) &= \int_{A\alpha} f^\lambda \varphi \wedge \frac{df}{f} && \text{if } A_\alpha \subset \{f > 0\} \\ i\pi M_{I_\alpha}(\lambda) &= -e^{-i\pi\lambda}\int_{A\alpha} (-f)^\lambda \varphi \wedge \frac{df}{f} && \text{if } A_\alpha \subset \{f < 0\} \end{aligned}\right\} \tag{1}$$

where the open set A_α is oriented by the canonical orientation of $\mathbb{R}^{n+1}$ (assumed to be fixed in the sequel).

For $A = \sum_1^a c_\alpha A_\alpha$ we define

$$I_A(s) := \sum_1^a c_\alpha \int_{(f=s)\cap A_\alpha} \varphi$$

with the previous conventions. So we shall get, by definition,

$$i\pi M_{I_A}(\lambda) = \int_A f^\lambda \varphi \wedge \frac{df}{f} \qquad \text{where}$$

$$\int_A f^\lambda \varphi \wedge \frac{df}{f} := \sum_{A_\alpha \subset \{f>0\}} c_\alpha \int_{A\alpha} f^\lambda \varphi \wedge \frac{df}{f} - e^{-i\pi\lambda} \sum_{A_\alpha \subset \{f<0\}} c_\alpha \int_{A\alpha} (-f)^\lambda \varphi \wedge \frac{df}{f}$$

with the natural orientations of the open sets A_α.

Define now, for any $\alpha \in [1, a]$

$$F_{A_\alpha} := f^{-1}(s_0) \cap A_\alpha \qquad \text{if } A_\alpha \subset \{f > 0\}$$

and

$$F_{A_\alpha} := f^{-1}(-s_0) \cap A_\alpha \qquad \text{if } A_\alpha \subset \{f < 0\}$$

where s_0 is a base point in D_δ^* choosen in $D_\delta^* \cap \mathbb{R}^{+*}$. Here we assume that we have a Milnor fibration for $f_{\mathbb{C}}$:

$$f_{\mathbb{C}} : X_{\mathbb{C}} - f_{\mathbb{C}}^{-1}(0) \to D_\delta^*$$

and we shall denote by $F_{\mathbb{C}}$ the complex Milnor fiber (that is to say $F_{\mathbb{C}} := f_{\mathbb{C}}^{-1}(s_0)$). We define then $F_A := \sum_{\alpha=1}^{a} c_\alpha F_{A_\alpha}$ as a closed oriented n-cycle of $X_{\mathbb{R}}$, the orientation of the F_{A_α} being given by (1).

We define $\theta_\alpha : F_{A_\alpha} \to F_{\mathbb{C}}$ as the obvious inclusion if $A_\alpha \subset \{f > 0\}$; and for $A_\alpha \subset \{f < 0\}$ θ_α is given by the closed embedding of $F_{A_\alpha} = f^{-1}(-s_0) \cap A_\alpha$ in $f_{\mathbb{C}}^{-1}(s_0) = F_{\mathbb{C}}$ given by a C^∞ trivialisation of $F_{\mathbb{C}}$ along the half-circle $|s| = s_0$ and $\mathrm{Arg}(s) \in [-\pi, 0]$.

For $A = \sum_{\alpha=1}^{a} c_\alpha A_\alpha$ define the closed oriented n-cycle G_A of $F_{\mathbb{C}}$

$$G_A = G_{A^+} - G_{A^-} = \sum_{A_\alpha \subset \{f>0\}} (\theta_\alpha)_* F_{A_\alpha} - \sum_{A_\alpha \subset \{f<0\}} (\theta_\alpha)_* F_{A_\alpha}.$$

The minus sign in this definition comes from the following facts:

In our definition of Mellin transform, $\mathbb{R}^*$ is oriented by the natural orientation coming from $\mathbb{R}$. Using the monodromy brings the orientation of $\mathbb{R}^{*-}$ we have chosen to the opposite orientation of $\mathbb{R}^{*+}$. If we want to keep the global orientation of $\mathbb{R}^{n+1}$ in this transfert (we push the Milnor fiber $F_{\mathbb{R}} := f^{-1}(-s_0) \coprod f^{-1}(s_0)$ in $F_{\mathbb{C}}$) we have to change the orientation in $f^{-1}(-s_0)$. This explains our definition of the cycle G_A in $F_{\mathbb{C}}$.

When $\varphi \in C_c^\infty(F_{\mathbb{C}})$ is a n-form, we have

$$\int_{G_A} \varphi := \sum_{A_\alpha \subset \{f>0\}} c_\alpha \int_{F_{A_\alpha}} \theta_\alpha^*(\varphi) - \sum_{A_\alpha \subset \{f<0\}} c_\alpha \int_{F_{A_\alpha}} \theta_\alpha^*(\varphi)$$

where F_{A_α} is oriented as before.

This gives a linear form on $H_c^n(F_{\mathbb{C}}, \mathbb{C})$ associated to the oriented n-cycle G_A in $F_{\mathbb{C}}$:

$$\varphi \to \int_{G_A} \varphi$$

where $\varphi \in C_c^\infty(F_{\mathbb{C}})$ is a d-closed n form.

We shall denote by $\delta(A)$ the cohomology class in $H^n(F_{\mathbb{C}}, \mathbb{C})$ which gives this linear form on $H_c^n(F_{\mathbb{C}}, \mathbb{C})$ via the Poincare duality: $H^n(F_{\mathbb{C}}, \mathbb{C}) \times H_c^n(F_{\mathbb{C}}, \mathbb{C}) \to \mathbb{C}$. Our result is the following analogue of [5] and its converse [6].

Theorem. *Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ a non zero real analytic germ. Assume that $0 \in \mathbb{C}^{n+1}$ is an isolated singularity relative to the eigenvalue 1 of the monodromy of $f_{\mathbb{C}}$ the complexification of f.*

Let $A = \sum_{\alpha=1}^{a} c_\alpha A_\alpha$ an element in $H^0(\{f \neq 0\}, \mathbb{C})$ and $\delta(A)$ the corresponding class in $H^n(F_{\mathbb{C}}, \mathbb{C})$ (see the definition above). Then we have an equivalence between:

(i) *$\delta(A)$ has a non zero component on $H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$ in the spectral decomposition of the monodromy acting on $H^n(F_{\mathbb{C}}, \mathbb{C})$*

(ii) *the meromorphic extension to the complex plane of the distribution $\lambda \to \int_A f^\lambda \square$ (holomorphic in λ for $\operatorname{Re} \lambda > 0$), admits a pole of order ≥ 2 at $\lambda = -(n+1)$. Moreover, the order of the pole $-v$ for $v \in \mathbb{N}$ and $v \geq n+1$ of the meromorphic continuation of $\frac{1}{\Gamma(\lambda)} \int_A f^\lambda \square$ is exactly the nilpotency order of $T - 1$ acting on $\delta(A)_1$, the component of $\delta(A)$ on $H^n(F, \mathbb{C})_{\lambda=1}$.* ∎

Remarks. 1) The notion of an isolated singularity relative to the eigenvalue 1 of the monodromy has been introduced in [2]. It means that for any $x \neq 0$ near 0 in $\mathbb{C}^{n+1}$ such that $f_{\mathbb{C}}(x) = 0$, the monodromy acting on the **reduced** cohomology of the Milnor fiber of $f_{\mathbb{C}}$ at x has no non zero invariant vector.

2) In the case where A is a connected component of the open set $\{f \neq 0\}$, (ii) is equivalent, in term of asymptotic expansion of integrals $s \to \int_{A \cap f^{-1}(s)} \varphi$ when $s \to 0$, with $\varphi \in C_c^\infty(x)$ is a n-form, to the non vanishing of the coefficient of $s^p (\log|s|)^j$ for some $p \in \mathbb{N}$ and some $j \in \mathbb{N}^*$ (for some choice of φ).

3) The precise order of poles at large negative integers is describe in a purely topological way.

§3. The Proof

We shall use here the notations of [3]. For A given, let e be the component of $\delta(A)$ on $H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$, the spectral subspace of $H^n(F_{\mathbb{C}}, \mathbb{C})$ associated to eigenvalue 1 of the monodromy.

Assume $e \neq 0$ and let us prove that (i) $\Rightarrow$ (ii). As the canonical hermitian form h is non degenerated on $H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$ (see [2]) there exists $e' \in H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$ such that $h(e', e) \neq 0$.

From [3] we know that h is topological and can be computed by the following formula:

$$h(e', e) = I(\widetilde{\operatorname{var}}(e'), e)$$

where I is the (hermitian) intersection form on $F_{\mathbb{C}}$ which gives the Poincare duality

$$I : H_c^n(F_{\mathbb{C}}, \mathbb{C}) \times H^n(F_{\mathbb{C}}, \mathbb{C}) \to \mathbb{C}$$

which is invariant by the monodromy and where

$$\widetilde{\text{var}} : H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1} \to H_c^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$$

is the composition of the "ordinary variation map" (built in this context in [3]) and of the automorphism

$$\Theta(x) := \frac{1}{x}\log(1+x) \qquad \text{with}$$

$$1 + x := T|_{H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}}, \qquad \text{here } T \text{ is the monodromy.}$$

So, if $e'' := \Theta(e')$, we have

$$I(\text{var}(e''), \delta(A)) \neq 0,$$

using the fact that I is monodromy invariant, which implies that the spectral decomposition of $H^n(F_{\mathbb{C}}, \mathbb{C})$ is I-orthogonal.

If now $\varphi \in C_c^\infty(F_{\mathbb{C}})$ if a closed n-form representing $\text{var}(e'') = \widetilde{\text{var}}(e')$ in $H_c^n(F_{\mathbb{C}}, \mathbb{C})$ we shall have

$$\int_{G_A} \varphi \neq 0 \tag{2}$$

But in [3] it is explained how to represent $\widetilde{\text{var}}(e') = \text{var}(e'')$ by a de Rham representative (that is to say how to build such a φ) for a given class $e' \in H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$. Let us give a direct construction of the variation map in this context (as suggested by B. Malgrange) following [9].

Let Ψ_1 and Φ_1 the spectral parts for eigenvalue 1 of the monodromy of respectively nearby and vanishing-cycle sheaves of f. The assumption says that Φ_1 is concentrated at 0 and so we have an isomorphism

$$R\Gamma_{\{0\}}\Phi_1 \xrightarrow{\sim} \Phi_1.$$

Now the variation map $\text{var} : \Phi_1 \to \Psi_1$ gives a map $R\Gamma_{\{0\}}\Phi_1 \to R\Gamma_{\{0\}}\Psi_1$. The composition

$$\Psi_1 \xrightarrow{\text{can}} \Phi_1 \xrightarrow{\sim} R\Gamma_{\{0\}}\Phi_1 \longrightarrow R\Gamma_{\{0\}}\Psi_1 \longrightarrow R\Gamma_c\Psi_1$$

induces our variation map (see [3])

$$H^n(F, \mathbb{C})_{\lambda=1} \to H_c^n(F, \mathbb{C})_{\lambda=1}.$$

Let $\mathscr{E}$ the complex of semi-meromorphic forms with poles in $f = 0$ and

$\mathscr{E}^{\cdot}[\log f]$ the complex given by polynomial in $\log f$ with coefficients in $\mathscr{E}^{\cdot}$ and the differential

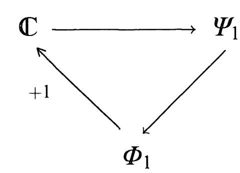
$$D(u.(\log f)^{(j)}) = du.(\log f^{(j)}) + \frac{df}{f} \wedge u.(\log f)^{(j-1)})$$

where $(\log f)^{(j)} := \frac{1}{j!}(\log f)^{j}$.

Then the exact sequence of complexes

$$0 \to C^{\infty} \to \mathscr{E}^{\cdot}[\log f] \to \mathscr{E}^{\cdot}[\log f]/C^{\infty} \to 0$$

corresponds to the distinguished triangle



and $\mathscr{E}^{\cdot}[\log f]$ is a complex of fine sheaves representing Ψ_1.

Let consider now a n-cycle x in $\mathscr{E}^{n}[\log f]$

$$x = x_k + x_{k-1}.(\log f) + \cdots + x_1.(\log f)^{(k-1)}$$

(this strange way of indexation will be compatible with notations in [3]!).

Then $Dx = 0$ gives $dx_k + \frac{df}{f} \wedge x_{k-1} = 0, \ldots, dx_2 + \frac{df}{f} \wedge x_1 = 0$ and $dx_1 = 0$. To compute var on $[x]$ we have to write

$$x = y + z + Dt.$$

Where $t \in \mathscr{E}^{n-1}[\log f], z$ is C^{∞} of degree n and y is in $\mathscr{E}^{n}[\log f]$ with compact support. So that $D(y+z) = 0$ and $\widetilde{\mathrm{var}}[x]$ (see above $\widetilde{\mathrm{var}} = \Theta \circ \mathrm{var}$) is given by

$$N(y+z) = N(y) = y_{l-1} + y_{l-2}.\log f + \cdots + y_1.(\log f)^{(l-2)}$$

if

$$y = y_l + y_{l-1}.\log f + \cdots + y_1.(\log f)^{(l-1)}.$$

Now in [3] this is performed in an "explicit" way for a given w $(=x)$

$$w = w_k + w_{k-1}.\log f + \cdots + w_1.(\log f)^{(k-1)}$$

such that $w_k|_{F_c} = e'$ in $H^n(F, \mathbb{C})_{\lambda=1}$.

In a first step w is replaced by a cocycle $\hat{w}$ in $\mathscr{E}^{n}[\log f]$ with degree k in $\log f$ such that $N\hat{w}$ has compact support in the Milnor ball X and such that

$\hat{w}_k = w_k + \frac{df}{f} \wedge \xi_k$ still induces e' in $H^n(F, \mathbb{C})_{\lambda=1}$. For a C^∞ function σ on X which is equal to 1 on a large enough compact set, we will have

$$D(\sigma\hat{w}) = W = d\sigma \wedge \left(w_k + \frac{df}{f} \wedge \xi_k\right)$$

which is in $\mathscr{E}^{n+1}$ and has compact support near ∂X and is d-closed. Using a Leray residu on $\{f = 0\}$ near ∂X (where 1 is not an eigenvalue of the monodromy of f in positive degrees) we write

$$\begin{aligned} W &= \omega + D(\alpha + \eta . \log f) \\ &= \omega + \frac{df}{f} \wedge \eta + d\alpha \end{aligned} \tag{3}$$

where η is C^∞ d-closed of degree n with compact support near $\partial X, \omega$ is C^∞ of degree $n+1$ with compact support near ∂X and also d-closed, and where $\alpha \in \mathscr{E}^n$ has compact support near ∂X.

Then $\widetilde{\mathrm{var}}(e')$ is given by $\tilde{w}_k$ where

$$\tilde{w} = \sigma\hat{w} - \alpha + \eta . \log f$$

induces a n-cocycle with compact support in $\mathscr{E}^\cdot[\log f]/C^\infty$; so that $\tilde{w} \in \mathscr{E}^n[\log f]$ has degree $k \geq 1$ in $\log f$ and coincide with $\hat{w}$ on a large compact set ($N\tilde{w} = N\hat{w}$ and $\tilde{w}_k = \sigma\hat{w}_k + \eta = v_k + \eta$ with the notation in [3] p. 20).

Now (2) gives

$$\int_{G_A} v_k + \eta \neq 0 \tag{4}$$

This will show that the meromorphic extension of $\int_A f^\lambda (v_k + \eta) \frac{df}{f}$ will have at $\lambda = 0$ a pole of order ≥ 1 (see lemma 2 below). Consider now the meromorphic function

$$\int_A f^\lambda \tilde{w}_{k+1} \wedge \frac{df}{f} = \int_A f^\lambda \sigma w_k \wedge \frac{df}{f}$$

as

$$\frac{1}{\lambda} d(f^\lambda \tilde{w}_{k+1}) = f^\lambda \frac{df}{f} \wedge \tilde{w}_{k+1} + \frac{1}{\lambda} f^\lambda d\tilde{w}_{k+1} \qquad \text{for } \mathrm{Re}(\lambda) \gg 0,$$

Stokes formula and the analytic continuation give

$$\int_A f^\lambda \sigma w_k \wedge \frac{df}{f} = \frac{(-1)^{n-1}}{\lambda} \int_A f^\lambda (v_k + \eta) \wedge \frac{df}{f} - \frac{1}{\lambda} \int_A f^\lambda \omega \tag{5}$$

using $d\tilde{w}_{k+1} = d\sigma \wedge \left(w_k + \frac{df}{f} \wedge \xi_k\right) + \sigma \frac{df}{f} \wedge v_k$, (3) and the fact that $\sigma v_k = v_k$ ($\sigma \equiv 1$ on the support of v_k). As ω is C^∞ the meromorphic function $\int_A f^\lambda \omega$ has no pole at $\lambda = 0$, and so $\frac{1}{\lambda}\int f^\lambda \omega$ has at most a simple pole at 0. We conclude from (3) and (4) that $\int_A f^\lambda \sigma w_k \wedge \frac{df}{f}$ has at least a pole of order 2 at $\lambda = 0$ from the following lemma:

Lemma 2. *Let $\tilde{v} \in H^0_{c/f}(X_{\mathbb{C}}, \mathscr{E}^n(k))$ such that $\delta\tilde{v} = 0$ and $\int_{G_A} \tilde{v}_k \neq 0$. Then the meromorphic extension of $\int_A f^\lambda \tilde{v}_k \wedge \frac{df}{f}$ has a pole of order ≥ 1 at $\lambda = 0$.*

Proof. For $x \in \mathbb{R}$ near 0 define $\varphi(x) = \int_{(f=x)\cap A} \tilde{v}_k$. Then we shall have $\left(x\frac{d}{dx}\right)^k \varphi \equiv 0$ on $\mathbb{R}^*$ near 0 because of the assumption $\delta\tilde{v} = 0$. So we can apply lemma 1 to φ. The main point is now to show that if $P, Q \in \mathbb{C}[x]$ of degre $\leq k-1$ are such that

$$\varphi(x) = P(\log|x|) \qquad \text{for } 0 < x \ll 1$$
$$\varphi(x) = Q(\log|x|) - i\pi) \qquad \text{for } -1 \ll x < 0$$

we have $P \neq Q$!

The hypothesis $\int_{G_A} \tilde{v}_k \neq 0$ can be written $\int_{G_{A^+}} \tilde{v}_k - \int_{G_{A^-}} \tilde{v}_k \neq 0$ if $A = A^+ + A^-$ with $A^+ = \sum_{A_\alpha \subset \{f>0\}} c_\alpha A_\alpha$ and $A^- = \sum_{A_\alpha \subset \{f<0\}} c_\alpha A_\alpha$. We have $\int_{G_{A^+}} \tilde{v}_k = \varphi(s_0)$ by definition. To compute $\int_{G_{A^-}} \tilde{v}_k$ we have to follow, along the half-circle $s_0 e^{i\theta}, \theta \in [-\pi, 0]$, the holomorphic multivalued function given by $\int_{(f=s)\cap A^-} \tilde{v}_k$ where $(f = s) \cap A^-$ is a notation for the horizontal family of oriented, closed n-cycles in the fibers of $f_{\mathbb{C}}$ with value

$$(f = -s_0) \cap A^- \qquad \text{at } s = -s_0 = s_0 e^{-i\pi}.$$

From the fact that $\varphi(x) = Q(\log|x| - i\pi)$ for $-1 \ll x < 0$, this multivalued function is $Q(\log s)$ for the choice $-\pi \leq \operatorname{Arg} s \leq 0$. So we get $\int_{G_A^-} \tilde{v}_k = Q(\log s_0)$ and then $\int_{G_A} \tilde{v}_k = (P - Q)(\log s_0) \neq 0$.

So we have $P \neq Q$ and by lemma 1 we get the desired pole of order ≥ 1 at $\lambda = 0$. ∎

So (i) ⇒ (ii) is proved if we can choose $\tilde{v}$ in order to have

$$f^{n+1}\tilde{v}_k \wedge \frac{df}{f} \in C^\infty(X_{\mathbb{C}}).$$

In fact $\tilde{v}_k = v_k + \eta$ where η is C^∞ so we only need to satisfy $f^{n+1} v_k \wedge \frac{df}{f} \in$

$C^\infty(X_{\mathbb{C}})$. But from [3] p. 20 we have

$$v_k = w_{k-1} - d\xi_k + \frac{df}{f} \wedge \xi_{k-1}$$

and so $\frac{df}{f} \wedge v_k = \frac{df}{f} \wedge w_{k-1} - \frac{df}{f} \wedge d\xi_k$. Now

$$\int_A f^\lambda \frac{df}{f} \wedge d\xi_k = \int_A d\left(\frac{f^{\lambda+1}}{\lambda+1} d\xi_k\right) \equiv 0$$

by Stokes formula (for $\operatorname{Re}\lambda \gg 0$ so everywhere) and it is enough to choose w such that $f^n w$ is holomorphic.

This is possible from [3] (see the begining of the proof of theorem 2) and this complete the proof of (i) $\Rightarrow$ (ii).

We shall prove now that $\delta(A)_1 = 0$ implies in fact that

$$\frac{1}{\Gamma(\lambda)} \int_A f^\lambda \square$$

has no pole at negative integers.

Proposition. *Let* $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ *a non zero real analytic germ such that* 1 *is not an eigenvalue of the monodromy of* $f_{\mathbb{C}}$ *acting on the reduced cohomology of the Milnor fiber of* $f_{\mathbb{C}}$ *at any* $x \in f_{\mathbb{C}}^{-1}(0)$ *close enough to the origine.*

Let A_0 *be a connected component of the open set* $\{f \neq 0\}$ *in* $X_{\mathbb{R}}$.

Then, the meromorphic extension of $\frac{1}{\Gamma(\lambda)} \int_{A_0} |f|^\lambda \square$ *has no pole at a negative integers.*

Proof. The point is to explain that the Bernstein–Sato polynomial b of $f_{\mathbb{C}}$ at 0 has only one simple root in $\mathbb{Z}$ which is -1. For that propose, remark that our hypothesis implies that the vanishing cycles sheaf Φ of f satisfies $\Phi_1 = 0$, and so Ψ, the nearby-cycles sheaf of f satisfies $\Psi_1 \xrightarrow{\sim} (\mathbb{C}, T = 1)$.

From [8] or [7] we conclude that all integral roots of b are simple (using that 0 is a simple root of b' and the final inequalities of [8]). If b has two different integral roots, then using the De Rham functor, we obtain a non trivial decomposition of $(\mathbb{C}, T = 1) \simeq \Psi_1$. Of course this allows us to conclude that b has exactly one integral (simple) root. But of course -1 is a root of b. So we obtain that $b(s) = (s+1)b_1(s)$ where b_1 has no integral root. Using now a Bernstein identity to perform the analytic continuation of $\int_{A_0} |f|^\lambda \square$ leads to, at most, simple poles at negative integers (because $b(\lambda) \dots b(\lambda + k)$ has, at most, a simple root at $-\delta$ for $\delta \in \mathbb{N}^*$). ■

Corollary. *If* 0 *is an isolated singularity for the eigenvalue* 1 *of* $f_{\mathbb{C}}$, *for any* $A \in H^0(\{f \neq 0\}, \mathbb{C})$ *the Laurent coefficients of the poles of* $\frac{1}{\Gamma(\lambda)}\int_A f^\lambda \Box$ *at negative integers have there supports in* $\{0\}$.

Proof. This is an obvious consequence of the proposition. ■

Assume now that we have a pole of order $j \geq 2$ at $\lambda = -k$ $(k \in \mathbb{N}^*)$ for $\int_A f^\lambda \Box$. Let $\mathfrak{T}$ be the coefficient of $\frac{1}{(\lambda+k)^j}$ in the Laurent expansion at $\lambda = -k$ of $\int_A f^\lambda \Box$. Then $\mathfrak{T} \neq 0$ by assumption.

Let $N = \text{order}(\mathfrak{T})$ (recall that $\operatorname{supp} \mathfrak{T} \subset \{0\}$ by the corollary) ant let $\varphi \in C_c^\infty(X_{\mathbb{R}})^{n+1}$ such that $\langle \mathfrak{T}, \varphi \rangle \neq 0$.

Using *a* Taylor expansion at order N at 0 for φ, we get a $\omega \in \Omega_{X_{\mathbb{C}}}^{n+1}$ such that $\langle \mathfrak{T}, \omega|_{X_{\mathbb{R}}} \rangle = \langle \mathfrak{T}, \varphi \rangle \neq 0$. Let $\rho \in C_c^\infty(X_{\mathbb{C}})$ with $\rho \equiv 1$ near 0. So the meromorphic extension of $\int_A f^\lambda \rho\omega$ has a pole of order $j \geq 2$ at $\lambda = -k$. Now using the fact that $f^l \Omega^{n+1} \subset \frac{df}{f} \wedge \Omega^n$ near 0 in $\mathbb{C}^{n+1}$ for some $l \in \mathbb{N}$, we can assume that there exist $\alpha \in \Omega^n$ such that $\int_A f^\lambda \frac{df}{f} \wedge \rho\alpha$ has a pole of order $j \geq 2$ at $\lambda = -k - l$.

Let $\omega_1 \dots \omega_\mu$ be a meromorphic Jordan basis form the Gauss-Manin system in degree n near 0 for $f_{\mathbb{C}}$. We can write

$$\alpha = \sum_{p=1}^{\mu} a_p \omega_p + df \wedge \xi + d\eta.$$

Where $a_p \in \mathbb{C}\{f\}[f^{-1}]$ and where ξ and η are meromorphic $(n-1)$-forms with poles in $\{f_{\mathbb{C}} = 0\}$.

Now

$$\int_A f^\lambda \frac{df}{f} \wedge \rho(df \wedge \xi + d\eta) = \pm \int_A f^\lambda \frac{df}{f} \wedge d\rho \wedge \eta$$

will have, at most, simple poles at negative integers because $d\rho \equiv 0$ near 0 (and the corollary). As is it enough to consider the case $a_p = f^m$ where $m \in \mathbb{Z}$ and this only shift λ by an integer, we are left only with integrals like $\int_A f^\lambda \frac{df}{f} \wedge \rho\omega$ where ω is an element of the Jordan basis (*) for the monodromy acting on $H^n(F_{\mathbb{C}}, \mathbb{C})$ where $F_{\mathbb{C}}$ is the Milnor fiber of $f_{\mathbb{C}}$ at 0. If ω belongs to an eigenvalue $\neq 1$ we can assume $\omega = w_k$ with

(*) see the computations with the sheaves $\Omega(k)$ in [1]

$$dw_k = u\frac{df}{f} \wedge w_k + \frac{df}{f} \wedge w_{k-1}$$

$$dw_{k-1} = u\frac{df}{f} \wedge w_{k-1} + \frac{df}{f} \wedge w_{k-2}, \text{ etc}\ldots,$$

$$\text{and} \qquad w_0 = 0, \qquad 0 < u < 1.$$

But

$$u\int_A f^\lambda \frac{df}{f} \wedge \rho w_1 = \int_A f^\lambda \rho dw_1 = -\lambda \int_A f^\lambda \frac{df}{f} \wedge \rho w_1 - \int_A f^\lambda d\rho \wedge w_1$$

gives

$$(\lambda + u)\int_A f^\lambda \frac{df}{f} \wedge \rho w_1 = -\int_A f^\lambda d\rho \wedge w_1$$

and $d\rho \equiv 0$ near 0 with $u \in]0,1[$ gives that $\int_A f^\lambda \frac{df}{f} \wedge \rho w_1$ has at most simple poles at negative integers $\left(\text{as } \frac{1}{\lambda + u} \text{ is holomorphic near } \mathbb{Z}\right)$. An easy induction leads to the same result for $\int_A f^\lambda \frac{df}{f} \wedge \rho w_k$.

So we are left with the eigenvalue 1 Jordan blocs, that is to say the $u = 0$ case; but then, we are back to the computation made in the direct part of the theorem. The point is now that $\int_A f^\lambda d\rho \wedge w_k$ will not have (simple) pole at $\lambda = 0$ because $\delta(A)_1 = 0$ will gives $I(\widetilde{\text{var}}(e'), \delta(A)) = 0$. So these Jordan blocks for the eigenvalue 1 does not give pole, for $\frac{1}{\Gamma(\lambda)}\int_A f^\lambda \square$ at negative integers from our assumption $\delta(A)_1 = 0$ and the equivalence of i) and ii) is proved because we have contradicted our assumption $\mathfrak{T} \neq 0$. Let us prove now the last statement of the theorem:

Let $e = \delta(A)_1$ and let $h \in \mathbb{N}^*$ be the nilpotency order of $T - 1$ acting on $\delta(A)_1$. So we have $N^{h-1}(e) \neq 0$ and $N^h(e) = 0$ $(N = T - 1)$.

Then we choose e' such that

$$h(e', N^{h-1}(e)) \neq 0$$

and so $I(\text{var}(e''), N^{h-1}(e)) \neq 0$.

Then, as var commutes with N, we have

$$I(\text{var}[N^{h-1}(e'')], \quad \delta(A)) \neq 0.$$

So we get now for $h \geq 2$

$$\int_{G_A} v_{k-h+1} \neq 0 \qquad \text{(notations as above)}$$

and then a pole of order ≥ 2 at $\lambda = 0$ for $\int_A f^\lambda \tilde{w}_{k-h+1} \wedge \frac{df}{f}$.

Now, using $\delta\tilde{w} = \begin{pmatrix} \omega \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we conclude that $\int_A f^\lambda \sigma w_k \wedge \frac{df}{f}$ has a pole of order $\geq 2 + h - 1 = h + 1$ at $\lambda = 0$. So we obtain that the order of poles of $\frac{1}{\Gamma(\lambda)} \int_A f^\lambda \square$ at (big) negative integers is at least the nilpotency order of $T - 1$ acting on $\delta(A)_1$. The fact that this happens for $v = -(n+1)$ is obtained as in the case $h = 1$. Conversly, if we have a nilpotency order equal to $h \geq 1$, arguing in the same way that in the proof of ii) $\Rightarrow$ i), we conclude that the poles of $\frac{1}{\Gamma(\lambda)} \int_A f^\lambda \square$ are of order at most h. ∎

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