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On Totally Characteristic Type Non-linear Partial Differential Equations in the Complex Domain

By

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Abstract

The paper deals with a singular non-linear partial differential equation $t\partial u/\partial t = F(t,x,u,\partial u/\partial x)$ with two independent variables $(t,x) \in \boldsymbol{C}^2$ under the assumption that $F(t,x,u,v)$ is holomorphic and $F(0,x,0,0) \equiv 0$. Set $\gamma(x) = (\partial F/\partial v)(0,x,0,0)$. In case $\gamma(x) \equiv 0$ the equation was investigated quite well by Gérard–Tahara [3]. In case $\gamma(0) = 0$ and $\operatorname{Re}\gamma'(0) < 0$ the existence of holomorphic solution was proved in Chen–Tahara [2] under a non-resonance condition. The present paper proves the existence of holomorphic solution under the same non-resonance condition but using the following weaker condition: $\gamma(0) = 0$ and $\gamma'(0) \in \boldsymbol{C}\backslash[0,\infty)$. The result is extended to higher order equations.

§ 1. Introduction and Main Result

Let $(t,x) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$, and let us consider the following non-linear singular partial differential equation:

$$(\mathrm{E}_1) \qquad t\frac{\partial u}{\partial t} = F\left(t,x,u,\frac{\partial u}{\partial x}\right)$$

with $u = u(t,x)$ as an unknown function, where $F(t,x,u,v)$ is a function with respect to the variables (t,x,u,v) defined in an open polydisk Δ centered at the origin of $\boldsymbol{C}_t \times \boldsymbol{C}_x \times \boldsymbol{C}_u \times \boldsymbol{C}_v$. Denote: $\Delta_0 = \Delta \cap \{t=0,u=0,v=0\}$. We assume the following conditions:

(H-1) $F(t,x,u,v)$ is holomorphic on Δ;
(H-2) $F(0,x,0,0) \equiv 0$ on Δ_0.

Thus the function $F(t,x,u,v)$ may be expressed in the form

$$(1.1) \qquad F(t,x,u,v) = \alpha(x)t + \beta(x)u + \gamma(x)v + \sum_{p+q+\alpha\geq 2} a_{p,q,\alpha}(x)t^p u^q v^\alpha,$$

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and the coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $a_{p,q,\alpha}(x)$ are all holomorphic functions on Δ_0.

In general, in case $\gamma(x) \equiv 0$ the equation (E_1) is called *non-linear Fuchsian type* (or *Briot-Bouquet type*); and in case $\gamma(x) \not\equiv 0$ the equation (E_1) is called *non-linear totally characteristic type*. These names come from the following facts: if $\gamma(x) \equiv 0$ the linearized equation of (E_1) is linear Fuchsian type (in Baouendi-Goulaouic [1], Tahara [8]); on the other hand, if $\gamma(x) \not\equiv 0$ the linearized equation of (E_1) is not linear Fuchsian type but is linear totally characteristic type (in the sense of Hörmander [7, section 18.3]).

Note that in case $\gamma(x) \not\equiv 0$ we have $\gamma(x) = x^p c(x)$ with $c(0) \neq 0$ for some $p \in \boldsymbol{Z}_+$. Denote: $\boldsymbol{Z}_+ = \{0, 1, 2, \dots\}$ and $\boldsymbol{N} = \{1, 2, \dots\}$.

The main theme is:

Problem. Under (H-1) and (H-2), find a holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ satisfying $u(0,x) \equiv 0$ near $x = 0$.

We already know the following results.

(1) (Gérard–Tahara [3]). When $\gamma(x) \equiv 0$, if $\beta(0) \notin \boldsymbol{N}$, the equation (E_1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ satisfying $u(0,x) \equiv 0$.

(2) (by Cauchy–Kowalewski theorem). When $\gamma(0) \neq 0$, for any holomorphic function $\phi(t)$ with $\phi(0) = 0$ the equation (E_1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ satisfying $u(0,x) \equiv 0$ and $u(t,0) \equiv \phi(t)$.

(3) (Chen–Tahara [2]). When $\gamma(x) = xc(x)$ with $c(0) \neq 0$, if

i) $i - \beta(0) - jc(0) \neq 0$ for any $(i,j) \in \boldsymbol{N} \times \boldsymbol{Z}_+$,

ii) $\mathrm{Re}\, c(0) < 0$

hold, the equation (E_1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ satisfying $u(0,x) \equiv 0$.

Remark 1. Yamane [9] has also discussed some problem concerning holomorphic solutions of (E_1) under the condition: $\gamma(x) \equiv 0$ and $\beta(0) \in \boldsymbol{N}$.

In this paper, by using an argument quite different from that in [2] we shall improve the above result (3) into the following form.

Theorem 1. *Assume* (H-1), (H-2) *and that* $\gamma(x) = xc(x)$ *with* $c(0) \neq 0$. *Then, if*

$$|i - \beta(0) - jc(0)| \geq \sigma(j+1) \qquad \textit{for any } (i,j) \in \boldsymbol{N} \times \boldsymbol{Z}_+ \tag{1.2}$$

holds for some $\sigma > 0$, *the equation* (E_1) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ *satisfying* $u(0,x) \equiv 0$ *near* $x = 0$.

Theorem 1 has the following obvious corollary:

Corollary 1. *Assume* (H-1), (H-2) *and that* $\gamma(x)=xc(x)$ *with* $c(0)\neq 0$. *Then, if*

i) $i-\beta(0)-jc(0)\neq 0$ *for any* $(i,j)\in \boldsymbol{N}\times \boldsymbol{Z}_+$,

ii) $c(0)\in \boldsymbol{C}\backslash[0,\infty)$

hold, the equation (E_1) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0)\in \boldsymbol{C}_t\times \boldsymbol{C}_x$ *satisfying* $u(0,x)\equiv 0$ *near* $x=0$.

Note that i) is a kind of non-resonance condition and ii) is the Poincaré condition on the vector field $t\partial/\partial t-c(0)x\partial/\partial x$.

We shall prove Theorem 1 in the next section, and in sections 3 and 4 we shall extend Theorem 1 to higher order totally characteristic type non-linear partial differential equations.

In sections 2 and 3 we shall use the following notations. We denote by $\boldsymbol{C}[[t,x]]$ (resp. $\boldsymbol{C}[[x]]$) the ring of formal power series in the variables (t,x) (resp. in the variable x). For formal power series

$$f(t,x)=\sum_{i,j\geq 0} f_{i,j}t^i x^j,\qquad g(t,x)=\sum_{i,j\geq 0} g_{i,j}t^i x^j$$

in $\boldsymbol{C}[[t,x]]$, we write $f(t,x)\ll g(t,x)$ if $|f_{i,j}|\leq g_{i,j}$ holds for all $(i,j)\in \boldsymbol{Z}_+\times \boldsymbol{Z}_+$, and we say that $g(t,x)$ is a majorant series of $f(t,x)$. Also we write

$$|f|(t,x)=\sum_{i,j\geq 0}|f_{i,j}|t^i x^j,$$

$$S(f)(t,x)=\sum_{i,j\geq 0} f_{i,j+1}t^i x^j.$$

Clearly we have $f(t,x)\ll |f|(t,x)$,

$$S(f)(t,x)=\frac{f(t,x)-f(t,0)}{x} \tag{1.3}$$

and the following: if $f(t,x)$ is convergent then $|f|(t,x)$ and $S(f)(t,x)$ are also convergent.

§2. Proof of Main Result

Under the condition $\gamma(x)=xc(x)$ with $c(0)\neq 0$ the equation (E_1) is written as

$$t\frac{\partial u}{\partial t}=\alpha(x)t+\beta(x)u+c(x)\left(x\frac{\partial u}{\partial x}\right)+H_2\left(t,x,u,\frac{\partial u}{\partial x}\right)+R_3\left(t,x,u,\frac{\partial u}{\partial x}\right), \tag{2.1}$$

where

$$H_2(t,x,u,v) = \sum_{p+q+\alpha=2} a_{p,q,\alpha}(x)t^p u^q v^\alpha,$$

$$R_3(t,x,u,v) = \sum_{p+q+\alpha\geq 3} a_{p,q,\alpha}(x)t^p u^q v^\alpha.$$

First, let us find a formal power series solution of the form

$$u(t,x) = \sum_{i\geq 1} u_i(x)t^i, \qquad u_i(x) \in \mathbf{C}[[x]] \quad (\text{for } i \geq 1). \tag{2.2}$$

Substituting this into (2.1) and comparing the coefficients of t^i (for $i \geq 1$) in the both sides of the equation, we have the following recursive formula:

$$\left(i - \beta(x) - c(x)x\frac{\partial}{\partial x}\right)u_i = f_{i-1}\left(x, u_1, \ldots, u_{i-1}, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_{i-1}}{\partial x}\right) \tag{2.3$_i$}$$

for $i = 1, 2, \ldots,$

where $f_0(x) = \alpha(x)$ and f_{i-1} (for $i \geq 2$) is a polynomial of $u_1, \ldots, u_{i-1}$ and $\partial u_1/\partial x, \ldots, \partial u_{i-1}/\partial x$. In particular, $f_1(x, u_1, \partial u_1/\partial x)$ is given by

$$f_1\left(x, u_1, \frac{\partial u_1}{\partial x}\right) = H_2\left(1, x, u_1, \frac{\partial u_1}{\partial x}\right). \tag{2.4}$$

Note that $(2.3)_i$ is expressed in the form

$$\left(i - \beta(0) - c(0)x\frac{\partial}{\partial x}\right)u_i = xS(\beta)(x)u_i + xS(c)(x)\left(x\frac{\partial u_i}{\partial x}\right) + f_{i-1}. \tag{2.5$_i$}$$

If $f_{i-1} \in \mathbf{C}[[x]]$ is known and if $i - \beta(0) - c(0)j \neq 0$ for all $j \in \mathbf{Z}_+$, by a simple calculation we see that $(2.5)_i$ has a unique formal solution $u_i(x) \in \mathbf{C}[[x]]$.

Thus, under the condition (1.2) we can solve $(2.3)_i$ inductively on i and obtain a unique formal solution $u(t,x)$ of the form (2.2).

Next, let us prove the convergence of this formal solution.

Consider the following equation with respect to $Y = Y(t,x)$:

$$\sigma Y = A(x)t + xB(x)Y + |H_2|(t,x,Y,S(Y)) + |R_3|(t,x,Y,S(Y)) \tag{2.6}$$

where $\sigma > 0$ is the constant in (1.2) and

$$A(x) = |\alpha|(x), \qquad B(x) = |S(\beta)|(x) + |S(c)|(x).$$

It is easy to see that (2.6) has a unique formal power series solution

$$Y(t,x) = \sum_{i\geq 1} Y_i(x)t^i, \qquad Y_i(x) \in \mathbf{C}[[x]] \quad (\text{for } i \geq 1) \tag{2.7}$$

and the coefficients $Y_i(x)$ $(i \geq 1)$ are determined by the following recursive formula:

(2.8) $\sigma Y_i = xB(x)Y_i + |f_{i-1}|(x, Y_1, \dots, Y_{i-1}, S(Y_1), \dots, S(Y_{i-1}))$ for $i = 1, 2, \dots$

which yields

$$(2.9) \qquad Y_i = \frac{|f_{i-1}|(x, Y_1, \dots, Y_{i-1}, S(Y_1), \dots, S(Y_{i-1}))}{\sigma - xB(x)} \qquad \text{for } i = 1, 2, \dots .$$

Moreover we have:

Lemma 1. *Under the condition* (1.2) *we have for all* $i = 1, 2, \dots$

$$(2.10)_i \qquad u_i(x) \ll Y_i(x),$$

$$(2.11)_i \qquad \frac{\partial u_i}{\partial x}(x) \ll S(Y_i)(x).$$

Proof. Let us prove this by induction on i. Put

$$u_i(x) = \sum_{j \geq 0} u_{i,j} x^j.$$

First let us check the case $i = 1$. By (1.2) and $(2.5)_1$ we have

$$\begin{aligned}
&\sigma \sum_{j \geq 0} (j+1)|u_{1,j}|x^j \\
&\quad \ll \sum_{j \geq 0} |(1 - \beta(0) - c(0)j)u_{1,j}|x^j = \left| \left(1 - \beta(0) - c(0)x\frac{\partial}{\partial x} \right) u_1 \right| (x) \\
&\quad \ll x|S(\beta)|(x)|u_1| + x|S(c)|(x)\left| x\frac{\partial u_1}{\partial x} \right| + |f_0|(x) \\
&\quad \ll xB(x) \sum_{j \geq 0} (j+1)|u_{1,j}|x^j + |f_0|(x)
\end{aligned}$$

and therefore by (2.9) we have

$$\sum_{j \geq 0} (j+1)|u_{1,j}|x^j \ll \frac{|f_0|(x)}{\sigma - xB(x)} = Y_1(x).$$

This easily leads us to $(2.10)_1$ and $(2.11)_1$.

Next, let $i \geq 2$ and suppose that $(2.10)_p$ and $(2.11)_p$ are already proved for all $p < i$. Then, by (1.2), $(2.5)_i$ and the induction hypothesis we have

$$\sigma\sum_{j\geq 0}(j+1)|u_{i,j}|x^j$$

$$\ll \sum_{j\geq 0}|(i-\beta(0)-c(0)j)u_{i,j}|x^j = \left|\left(i-\beta(0)-c(0)x\frac{\partial}{\partial x}\right)u_i\right|(x)$$

$$\ll x|S(\beta)|(x)|u_i| + x|S(c)|(x)\left|x\frac{\partial u_i}{\partial x}\right|$$

$$+ |f_{i-1}|\left(x, |u_1|, \dots, |u_{i-1}|, \left|\frac{\partial u_1}{\partial x}\right|, \dots, \left|\frac{\partial u_{i-1}}{\partial x}\right|\right)$$

$$\ll xB(x)\sum_{j\geq 0}(j+1)|u_{i,j}|x^j + |f_{i-1}|(x, Y_1, \dots, Y_{i-1}, S(Y_1), \dots, S(Y_{i-1})).$$

Combining this with (2.9) we obtain

$$\sum_{j\geq 0}(j+1)|u_{i,j}|x^j \ll \frac{|f_{i-1}|(x, Y_1, \dots, Y_{i-1}, S(Y_1), \dots, S(Y_{i-1}))}{\sigma - xB(x)} = Y_i(x).$$

This leads us to $(2.10)_i$ and $(2.11)_i$.

Lemma 1 implies that $Y(t,x)$ is a majorant series of the formal solution $u(t,x)$. Therefore, to complete the proof of Theorem 1 it is sufficient to prove that $Y(t,x)$ is convergent near $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$.

To do so, we write $Y(t,x)$ in the form

$$Y(t,x) = tY_1(x) + t^2Y_2(x) + t^2W(t,x),$$

where

$$W(t,x) = \sum_{i\geq 1} Y_{i+2}(x)t^i.$$

By the condition $|f_0|(x) = |\alpha|(x) = A(x)$, (2.4) and (2.9) we have

$$Y_1(x) = \frac{A(x)}{\sigma - xB(x)} \qquad (\gg 0),$$

$$Y_2(x) = \frac{|H_2|(1, x, Y_1, S(Y_1))}{\sigma - xB(x)} \qquad (\gg 0).$$

This implies that $Y_1(x)$ and $Y_2(x)$ are holomorphic functions in a neighborhood of $x = 0$ and therefore $S(Y_1)(x)$ and $S(Y_2)(x)$ are also holomorphic near $x = 0$.

Let us show that $W(t,x)$ is convergent near $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$. By (2.4), (2.6) and (2.8) we see that $W(t,x)$ is the unique formal solution with $W(0,x) \equiv 0$ of the equation

$$\sigma W = xB(x)W + G(t, x, tW, tS(W)), \tag{2.12}$$

where

(2.13) $G(t, x, X_1, X_2)$

$$= (|H_2|(1, x, Y_1 + tY_2 + X_1, S(Y_1) + tS(Y_2) + X_2) - |H_2|(1, x, Y_1, S(Y_1)))$$
$$+ \frac{1}{t^2}|R_3|(t, x, tY_1 + t^2Y_2 + tX_1, tS(Y_1) + t^2S(Y_2) + tX_2).$$

In (2.13) we may regard $Y_1(x)$, $Y_2(x)$, $S(Y_1)(x)$, and $S(Y_2)(x)$ as known holomorphic functions. Since $W(t,x) \gg 0$, to have the convergence of $W(t,x)$ it is sufficient to prove

Lemma 2. *If $\varepsilon > 0$ is sufficiently small, the power series $W(\varepsilon\rho, \rho)$ is convergent in a neighborhood of $\rho = 0 \in \boldsymbol{C}$.*

Proof. Set

$$W_\varepsilon(\rho) = W(\varepsilon\rho, \rho).$$

Then, by substituting $t = \varepsilon\rho$, $x = \rho$ into (2.12) and by using the relation

$$\rho S(W)(\varepsilon\rho, \rho) = W(\varepsilon\rho, \rho) - W(\varepsilon\rho, 0) \ll W(\varepsilon\rho, \rho) = W_\varepsilon(\rho),$$

we have

$$\sigma W_\varepsilon = \rho B(\rho) W_\varepsilon + G(\varepsilon\rho, \rho, \varepsilon\rho W_\varepsilon, \varepsilon\rho S(W)(\varepsilon\rho, \rho)) \tag{2.14}$$
$$\ll \rho B(\rho) W_\varepsilon + G(\varepsilon\rho, \rho, \varepsilon\rho W_\varepsilon, \varepsilon W_\varepsilon).$$

Hence, instead of considering (2.14) we shall consider the following analytic equation with respect to $Z(\rho)$:

$$\sigma Z = \rho B(\rho) Z + G(\varepsilon\rho, \rho, \varepsilon\rho Z, \varepsilon Z). \tag{2.15}$$

Then, the proof of Lemma 2 can be reduced to the following lemma.

Lemma 3. *If $\varepsilon > 0$ is sufficiently small, the equation* (2.15) *has a unique formal power series solution of the form*

$$Z(\rho) = \sum_{i \geq 1} Z_i \rho^i \in \boldsymbol{C}[[\rho]] \tag{2.16}$$

and it is convergent in a neighborhood of $\rho = 0 \in \boldsymbol{C}$. Moreover we have $W_\varepsilon(\rho) \ll Z(\rho)$.

Proof. By (2.13) we see that $G(t, x, X_1, X_2)$ is an analytic function with respect to (t, x, X_1, X_2) in a neighborhood of the origin of $\boldsymbol{C}^4$ and that $G(0,0,0,0) = 0$ holds. Put

$$K = \frac{\partial G}{\partial X_2}(0,0,0,0).$$

Then the equation (2.15) is written in the form

$$(2.17)\qquad (\sigma-\varepsilon K)Z=R(\rho,Z),$$

where $R(\rho,Z)$ is a holomorphic function with respect to (ρ,Z) in a neighborhood of $(0,0)\in \boldsymbol{C}_\rho\times\boldsymbol{C}_Z$ which depends on ε and satisfies the following: $R(\rho,Z)\gg 0$, $R(0,0)=0$ and $(\partial R/\partial Z)(0,0)=0$.

Hence, if we choose $\varepsilon>0$ so that

$$\sigma-\varepsilon K>0,$$

we can easily see that (2.15) has a unique formal solution $Z(\rho)$ of the form (2.16). The convergence of $Z(\rho)$ is obtained by the implicit function theorem.

Moreover, by (2.14) we have

$$(2.18)\qquad (\sigma-\varepsilon K)W_\varepsilon\ll R(\rho,W_\varepsilon),$$

and therefore by comparing (2.17) and (2.18) we obtain the result: $W_\varepsilon(\rho)\ll Z(\rho)$.

Thus, the proof of Lemma 3 is completed.

The proof of Theorem 1 is also completed at last.

§3. Case of Higher Order Totally Characteristic PDE

In this section, we shall extend the result of Theorem 1 to the case of higher order totally characteristic partial differential equations.

Let $(t,x)\in\boldsymbol{C}_t\times\boldsymbol{C}_x$, $m\in\boldsymbol{N}$, put $N=\#\{(j,\alpha)\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+;\, j+\alpha\le m,\, j<m\}$, and denote

$$Z=\{Z_{j,\alpha}\}_{\substack{j+\alpha\le m\\ j<m}}\in\boldsymbol{C}^N.$$

Let us consider

$$(\mathrm{E}_m)\qquad \left(t\frac{\partial}{\partial t}\right)^m u=F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+\alpha\le m\\ j<m}}\right)$$

with $u=u(t,x)$ as an unknown function, where $F(t,x,Z)$ is a function with respect to the variables (t,x,Z) defined in an open polydisk Δ centered at the origin of $\boldsymbol{C}_t\times\boldsymbol{C}_x\times\boldsymbol{C}_Z^N$. Denote: $\Delta_0=\Delta\cap\{t=0,\,Z=0\}$. We assume the following conditions:

(H_m-1) $F(t,x,Z)$ is holomorphic on Δ;
(H_m-2) $F(0,x,0)\equiv 0$ on Δ_0.

Thus we rewrite $F(t,x,Z)$ near the origin as

$$F(t,x,Z)=a(x)t+\sum_{\substack{j+\alpha\le m\\ j<m}}b_{j,\alpha}(x)Z_{j,\alpha}+\sum_{p+|\nu|\ge 2}g_{p,\nu}(x)t^pZ^\nu,$$

where

$$\nu = \{\nu_{j,\alpha}\}_{\substack{j+\alpha\le m\\ j<m}} \in \mathbf{Z}_+^N, \qquad |\nu| = \sum_{\substack{j+\alpha\le m\\ j<m}} \nu_{j,\alpha}, \qquad Z^\nu = \prod_{\substack{j+\alpha\le m\\ j<m}} (Z_{j,\alpha})^{\nu_{j,\alpha}}$$

and the coefficients $a(x)$, $b_{j,\alpha}(x)$, $g_{p,\nu}(x)$ are all holomorphic functions on Δ_0.

For simplicity, we write

$$C\left(x, t\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \left(t\frac{\partial}{\partial t}\right)^m - \sum_{\substack{j+\alpha\le m\\ j<m}} b_{j,\alpha}(x)\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha,$$

$$Du = \{D_{j,\alpha}u\}_{\substack{j+\alpha\le m\\ j<m}}, \qquad \text{and} \qquad D_{j,\alpha}u = \left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u.$$

Then the equation (E_m) is expressed in the form

$$C\left(x, t\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u = a(x)t + \sum_{p+|\nu|\ge 2} g_{p,\nu}(x)t^p(Du)^\nu. \tag{3.1}$$

If $b_{j,\alpha}(x) \equiv 0$ for all (j,α) with $\alpha > 0$, $C(x, t\partial/\partial t, \partial/\partial x)$ is nothing but an ordinary differential operator in t with a parameter x. In this case, the equation (E_m) was studied quite well in Gérard–Tahara [4]. This is the higher order version of *non-linear Fuchsian type* partial differential equations.

If $b_{j,\alpha}(x) \not\equiv 0$ for some (j,α) with $\alpha > 0$, the equation (E_m) is called *non-linear totally characteristic type* partial differential equations. This case is divided into the following two cases:

Case (I) $b_{j,\alpha}(0) \neq 0$ for some $\alpha > 0$.
Case (II) $b_{j,\alpha}(0) = 0$ for all (j,α) with $\alpha > 0$, but $b_{j,\alpha}(x) \not\equiv 0$ for some $\alpha > 0$.

Gérard–Tahara [6] discussed the case (I) and proved the existence of holomorphic solutions and also singular solutions of (E_m).

Here, we shall consider a particular class of the case (II) under the following assumption:

(H_m-3) $b_{j,\alpha}(x) = O(x^\alpha)$ (as $x \to 0$) for all (j,α).

Then, $b_{j,\alpha}(x)$ is expressed in the form $b_{j,\alpha}(x) = x^\alpha c_{j,\alpha}(x)$ for some holomorphic function $c_{j,\alpha}(x)$, and $C(x, t\partial/\partial t, \partial/\partial x)$ is written as

$$C\left(x, t\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$$
$$= \left(t\frac{\partial}{\partial t}\right)^m - \sum_{\substack{j+\alpha\le m\\ j<m}} c_{j,\alpha}(x)\left(t\frac{\partial}{\partial t}\right)^j\left(x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x}-1\right)\cdots\left(x\frac{\partial}{\partial x}-\alpha+1\right).$$

We write

$$L(\lambda,\rho)=\lambda^m-\sum_{\substack{j+\alpha\le m\\ j<m}} c_{j,\alpha}(0)\lambda^j\rho(\rho-1)\cdots(\rho-\alpha+1).$$

Theorem 2. *Assume* (H_m-1), (H_m-2) *and* (H_m-3). *Then, if*

$$|L(k,l)|\ge\sigma(k+l+1)^{m-1}(l+1)\qquad \text{for any } (k,l)\in \boldsymbol{N}\times\boldsymbol{Z}_+ \tag{3.2}$$

holds for some $\sigma>0$, *the equation* (E_m) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0)\in \boldsymbol{C}_t\times\boldsymbol{C}_x$ *satisfying* $u(0,x)\equiv 0$ *near* $x=0$.

Denote by $c_1,\dots,c_m$ the roots of the equation in X:

$$X^m-\sum_{\substack{j+\alpha=m\\ j<m}} c_{j,\alpha}(0)X^j=0.$$

Then, if we factorize $L(\lambda,\rho)$ into the form

$$L(\lambda,\rho)=(\lambda-\xi_1(\rho))\cdots(\lambda-\xi_m(\rho)),$$

we see that

$$\lim_{\rho\to\infty}\frac{\xi_i(\rho)}{\rho}=c_i\qquad \text{for } i=1,\dots,m.$$

Thus, as a corollary to Theorem 2 we obtain

Corollary 2. *Assume* (H_m-1), (H_m-2) *and* (H_m-3). *Then, if*

i) $L(k,l)\neq 0$ *for any* $(k,l)\in\boldsymbol{N}\times\boldsymbol{Z}_+$,

ii) $c_i\in\boldsymbol{C}\setminus[0,\infty)$ *for* $i=1,\dots,m$

hold, the equation (E_m) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0)\in\boldsymbol{C}_t\times\boldsymbol{C}_x$ *satisfying* $u(0,x)\equiv 0$ *near* $x=0$.

Remark 2. The above corollary 2 is an improvement of Theorem 2 in Chen–Tahara [2]. Recall that in [2] we have assumed the conditions (H_m-1), (H_m-2), (H_m-3), i) and

ii)′ $\mathrm{Re}\, c_i<0$ for $i=1,\dots,m$.

Proof of Theorem 2. Denote

$$I=\{(j,\alpha)\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+;\, j+\alpha\le m,\, j<m\},$$

$$H_2(t,x,Z)=\sum_{p+|\nu|=2} g_{p,\nu}(x)t^pZ^\nu,$$

$$R_3(t,x,Z)=\sum_{p+|\nu|\ge 3} g_{p,\nu}(x)t^pZ^\nu.$$

Assume the conditions (H_m-3) and that $b_{j,\alpha}(x) = x^\alpha c_{j,\alpha}(x)$ for $(j,\alpha) \in I$. Then, the equation (E_m) is written in the form

$$(3.3)\quad L\left(t\frac{\partial}{\partial t}, x\frac{\partial}{\partial x}\right)u$$
$$= x\sum_{(j,\alpha)\in I} S(c_{j,\alpha})(x)\left(t\frac{\partial}{\partial t}\right)^j\left(x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x}-1\right)\cdots\left(x\frac{\partial}{\partial x}-\alpha+1\right)u$$
$$+ a(x)t + H_2(t,x,Du) + R_3(t,x,Du).$$

First, let us find a formal solution of the form

$$(3.4)\qquad u(t,x) = \sum_{k\geq 1} u_k(x)t^k, \qquad u_k(x) \in \boldsymbol{C}[[x]] \quad (\text{for } k \geq 1).$$

Substituting this into (3.3) and comparing the coefficients of t^k (for $k \geq 1$) we have the following recursive formula:

$$(3.5)\qquad L\left(k, x\frac{\partial}{\partial x}\right)u_k$$
$$= x\sum_{(j,\alpha)\in I} S(c_{j,\alpha})(x)k^j\left(x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x}-1\right)\cdots\left(x\frac{\partial}{\partial x}-\alpha+1\right)u_k$$
$$+ f_{k-1}\left(x, \left\{p^j\left(\frac{\partial}{\partial x}\right)^\alpha u_p;\ 1\leq p\leq k-1, (j,\alpha)\in I\right\}\right),$$
$$\text{for } k = 1,2,\ldots,$$

where $f_0(x) = a(x)$ and f_{k-1} (for $k \geq 2$) is a polynomial of $\{p^j(\partial/\partial x)^\alpha u_p;$ $1 \leq p \leq k-1,\ (j,\alpha) \in I\}$.

Since $L(k,l) \neq 0$ for any $k \in \boldsymbol{N}$ and any $l \in \boldsymbol{Z}_+$, we can solve (3.5) uniquely and formally in $\boldsymbol{C}[[x]]$ by induction on k. Thus, we have obtained a unique formal solution $u(t,x)$ of the form (3.4). It remains to prove the convergence of this formal solution.

Next, we consider the following equation with respect to $Y = Y(t,x)$:

$$(3.6)\qquad \sigma Y = xC(x)Y + A(x)t$$
$$+ |H_2|(t,x,\{S^\alpha(Y)\}_{(j,\alpha)\in I}) + |R_3|(t,x,\{S^\alpha(Y)\}_{(j,\alpha)\in I}),$$

where $\sigma > 0$ is the constant in (3.2),

$$A(x) = |a|(x), \qquad C(x) = \sum_{(j,\alpha)\in I} |S(c_{j,\alpha})|(x)$$

and $S^\alpha(Y)$ is defined by $S^2(Y) = S(S(Y)), \ldots, S^\alpha(Y) = S(S^{\alpha-1}(Y))$. It is easy to see that (3.6) has a unique formal solution $Y(t,x)$ of the form

$$Y(t,x)=\sum_{k\geq 1} Y_k(x)t^k, \qquad Y_k(x)\in \boldsymbol{C}[[x]] \quad (\text{for } k\geq 1),$$

and the coefficients $Y_k(x)$ $(k\geq 1)$ are determined by the following recursive formula:

$$\text{(3.7)} \qquad \sigma Y_k = xC(x)Y_k + |f_{k-1}|(x,\{S^\alpha(Y_p);\ 1\leq p\leq k-1, (j,\alpha)\in I\}),$$
$$\text{for } k=1,2,\dots.$$

Moreover we have:

Lemma 4. *Under the condition* (3.2) *we have for any* $k=1,2,\dots$

$$\text{(3.8)}_k \qquad k^j\left(\frac{\partial}{\partial x}\right)^\alpha u_k(x) \ll S^\alpha(Y_k)(x) \qquad \textit{for any } (j,\alpha)\in I.$$

Proof. We will prove this by induction on k. It is easy to prove that $(3.8)_k$ holds for $k=1$. Let $k\geq 2$ and suppose that $(3.8)_p$ is already proved for all $p<k$. Denote:

$$u_k(x)=\sum_{l\geq 0} u_{k,l}\,x^l, \qquad Y_k(x)=\sum_{l\geq 0} Y_{k,l}\,x^l.$$

Also denote:

$$U_k(x)=\sum_{l\geq 0}(k+l+1)^{m-1}(l+1)|u_{k,l}|x^l.$$

Then, by (3.2), (3.5) and the induction hypothesis we have

$$\begin{aligned}\sigma U_k(x) &\ll \left|L\left(k,x\frac{\partial}{\partial x}\right)u_k\right|(x)\\ &\ll x\sum_{(j,\alpha)\in I}|S(c_{j,\alpha})|(x)k^j\sum_{l\geq 0} l(l-1)\cdots(l-\alpha+1)|u_{k,l}|(x)x^l\\ &\quad + |f_{k-1}|\left(x,\left\{p^j\left|\left(\frac{\partial}{\partial x}\right)^\alpha u_p\right|(x);\ 1\leq p\leq k-1,\ (j,\alpha)\in I\right\}\right)\\ &\ll xC(x)U_k(x)+|f_{k-1}|(x,\{S^\alpha(Y_p);\ 1\leq p\leq k-1,\ (j,\alpha)\in I\}).\end{aligned}$$

Hence, combining this with (3.7) we can obtain

$$U_k(x)\ll Y_k(x),$$

which immediately leads us to $(3.8)_k$.

By Lemma 4 we see that $Y(t,x)$ is a majorant series of the formal solution $u(t,x)$ of (3.4). Thus, to complete the proof of Theorem 2 it is sufficient to prove the convergence of $Y(t,x)$ in a neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$.

Now, we divide $Y(t,x)$ into

$$Y(t,x) = tY_1(x) + t^2 Y_2(x) + t^2 W(t,x),$$

where

$$W(t,x) = \sum_{k \geq 1} Y_{k+2}(x) t^k.$$

Then we have

$$Y_1(x) = \frac{A(x)}{\sigma - xC(x)},$$

$$Y_2(x) = \frac{|H_2|(1,x,\{S^\alpha(Y_1)\}_{(J,\alpha)\in I})}{\sigma - xC(x)},$$

and $W(t,x)$ is the unique formal power series solution with $W(0,x) \equiv 0$ of the equation

$$\sigma W = xC(x)W + G(t,x,\{tS^\alpha(W)\}_{(J,\alpha)\in I}), \tag{3.9}$$

where

$$\begin{aligned} G(t, x, &\{X_{J,\alpha}\}_{(J,\alpha)\in I}) \\ &= \{|H_2|(1,x,\{S^\alpha(Y_1) + tS^\alpha(Y_2) + X_{J,\alpha}\}_{(J,\alpha)\in I}) \\ &\qquad\qquad - |H_2|(1,x,\{S^\alpha(Y_1)\}_{(J,\alpha)\in I})\} \\ &\quad + \frac{1}{t^2}|R_3|(t,x,\{tS^\alpha(Y_1) + t^2 S^\alpha(Y_2) + tX_{J,\alpha}\}_{(J,\alpha)\in I}). \end{aligned}$$

Since $Y_1(x)$ and $Y_2(x)$ are holomorphic in a neighborhood of $x = 0$, to prove the convergence of $Y(t,x)$ we only need to prove

Lemma 5. *If $\varepsilon > 0$ is sufficiently small, the power series $W(\varepsilon\rho^m, \rho)$ is convergent in a neighborhood of $\rho = 0 \in \boldsymbol{C}$.*

Proof. Set

$$W_\varepsilon(\rho) = W(\varepsilon\rho^m, \rho).$$

Note that the definition of $S^\alpha(W)$ implies

$$S^\alpha(W)(t,x) = \frac{1}{x^\alpha}\left(W(t,x) - \sum_{j=0}^{\alpha-1} \frac{1}{j!}\left(\left(\frac{\partial}{\partial x}\right)^j W\right)(t,0)x^j\right),$$

and therefore we have

$$\begin{aligned}
&\rho^\alpha S^\alpha(W)(\varepsilon\rho^m,\rho)\\
&\quad = W(\varepsilon\rho^m,\rho)-\sum_{j=0}^{\alpha-1}\frac{1}{j!}\left(\left(\frac{\partial}{\partial x}\right)^j W\right)(\varepsilon\rho^m,0)\rho^j\\
&\quad \ll W(\varepsilon\rho^m,\rho)=W_\varepsilon(\rho).
\end{aligned}$$

Combining this with (3.9) we obtain

$$\sigma W_\varepsilon \ll \rho C(\rho)W_\varepsilon + G(\varepsilon\rho^m,\rho,\{\varepsilon\rho^{m-\alpha}W_\varepsilon\}_{(j,\alpha)\in I}).$$

Thus, if we consider the equation

$$\sigma Z = \rho C(\rho)Z + G(\varepsilon\rho^m,\rho,\{\varepsilon\rho^{m-\alpha}Z\}_{(j,\alpha)\in I}), \tag{3.10}$$

and if we choose $\varepsilon>0$ so that

$$\sigma-\varepsilon\left(\frac{\partial G}{\partial X_{0,m}}(0,0,0)\right)>0,$$

then by the implicit function theorem we can prove that (3.10) has a unique holomorphic solution $Z(\rho)$ in a neighborhood of $\rho=0\in \boldsymbol{C}$ with $Z(0)=0$; moreover we can prove that $W_\varepsilon(\rho)\ll Z(\rho)$ holds. This complete the proof of Lemma 5.

Thus, the proof of Theorem 2 is completed.

§4. A Generalization

Let us consider the following two examples:

$$\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}\right)\left(t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+1\right)u=G\left(t,x,u,t\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},t\frac{\partial^2 u}{\partial t\partial x},\frac{\partial^2 u}{\partial x^2}\right), \tag{4.1}$$

$$\left(t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+1\right)u=G\left(t,x,u,t\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},t\frac{\partial^2 u}{\partial t\partial x}\right), \tag{4.2}$$

where $G(t,x,X)$ with $X=(X_0,X_1,\dots,X_4)$ or $X=(X_0,X_1,\dots,X_3)$ is a holomorphic function defined in a neighborhood of $(t,x,X)=(0,0,0)$ satisfying $G(0,x,0)\equiv 0$ and $(\partial G/\partial X)(0,x,0)\equiv(0,\dots,0)$ near $x=0$.

We have:

(1) By Theorem 2 we see that (4.1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0)\in \boldsymbol{C}_t\times\boldsymbol{C}_x$ with $u(0,x)\equiv 0$.

(2) The equation (4.2) does not satisfy the condition (3.2) and so we cannot apply Theorem 2 to (4.2). Though, by a calculation we can see that

(4.2) has also a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ with $u(0,x) \equiv 0$.

Being motivated by this, let us give here a slight generalization of Theorem 2 so that we can apply our result to the equation (4.2).

Let $m \in \boldsymbol{N}$, let $\mathscr{M}$ be a subset of $\{(j,\alpha) \in \boldsymbol{Z}_+ \times \boldsymbol{Z}_+ ; j+\alpha \leq m, j<m\}$, and denote

$$Z = \{Z_{j,\alpha}\}_{(j,\alpha)\in \mathscr{M}}, \qquad Z_{j,\alpha} \in \boldsymbol{C}.$$

Let us consider

$$\left(t\frac{\partial}{\partial t}\right)^m u = F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha)\in\mathscr{M}}\right). \tag{4.3}$$

This equation is a particular case of equations of type (E_m) and so the assumptions $(\mathrm{H}_m\text{-}1)$, $(\mathrm{H}_m\text{-}2)$ and $(\mathrm{H}_m\text{-}3)$ make sense. We set

$$L(\lambda,\rho) = \lambda^m - \sum_{(j,\alpha)\in\mathscr{M}} c_{j,\alpha}(0)\lambda^j\rho(\rho-1)\cdots(\rho-\alpha+1),$$

$$\phi(k,l) = \max_{(j,\alpha)\in\mathscr{M}}((k+1)^j(l+1)^\alpha).$$

Then, by the same argument as in section 3 we obtain

Theorem 3. *Assume* $(\mathrm{H}_m\text{-}1)$, $(\mathrm{H}_m\text{-}2)$ *and* $(\mathrm{H}_m\text{-}3)$. *Then, if*

$$|L(k,l)| \geq \sigma\phi(k,l) \qquad \text{for any } (k,l) \in \boldsymbol{N}\times\boldsymbol{Z}_+ \tag{4.4}$$

holds for some $\sigma > 0$, *the equation* (4.3) *has a unique holomorphic solution* $u(t,x)$ *in a neighborhood of* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x$ *satisfying* $u(0,x) \equiv 0$ *near* $x = 0$.

Note that in case (4.2) we have $\mathscr{M} = \{(0,0),(1,0),(0,1),(1,1)\}$ and $L(\lambda,\rho) = \lambda(\lambda+\rho+1)$ and therefore we can apply Theorem 3 to (4.2).

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