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Coincidence Points for Perturbations of Linear Fredholm Maps of Index Zero

By

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Abstract

Coincidence points for single and set valued maps are discussed in this paper. We show if F is essential and $F \cong G$ then G has a coincidence point.

§1. Introduction

The notion of an essential map was introduced by Granas in [4]. He showed in [4] that if F is essential and $F \cong G$ then G is essential. Since the property of being essential is quite general Granas was only able to show this homotopy property for particular classes of maps. However from an application point of view he was asking too much. What one needs usually in applications is the following question to be answered: if F is essential and $F \cong G$, does G have a fixed (or more generally a coincidence) point? Recall two maps $F: X \to 2^Y$ and $G: X \to 2^Y$ have a coincidence if $F(x_0) \cap G(x_0) \neq \emptyset$ for some $x_0 \in X$; the point x_0 is called a coincidence point. In this paper we discuss this question in detail. In Section 2 we discuss single valued maps which satisfy the Mönch-Precup condition and in Section 3 multivalued k-set contractive maps. Our results extend those in Precup [9] and Volkmann [11].

For the remainder of this section we present some concepts which will be needed in Section 2 and in Section 3. Let (Z, d) be a metric space and let Ω_Z be the bounded subsets of Z. The Kuratowskii measure of noncompactness is the map $\alpha: \Omega_Z \to [0, \infty]$ defined by (here $B \in \Omega_Z$),

$$\alpha(B) = \inf\{r > 0 : B \subseteq \bigcup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq r\}.$$

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Let S be a nonempty subset of Z and suppose $G:S\to 2^Z$ (here 2^Z denotes the family of nonempty subsets of Z). Then (i). $G:S\to 2^Z$ is k-set contractive (here $k\geq 0$) if $\alpha(G(A))\leq k\alpha(A)$ for all nonempty, bounded sets A of S (here $G(A)=\bigcup_{x\in A}G(x)$), and (ii). $G:S\to 2^Z$ is condensing if G is 1-set contractive and $\alpha(G(A))<\alpha(A)$ for all bounded sets A of S with $\alpha(A)\neq 0$.

Let X and E be Fréchet spaces and $L:dom\,L\subseteq X\to E$ ($dom\,L$ is a vector subspace of X) is a linear Fredholm map of index zero i.e. L is a linear (not necessarily continuous) single valued map with $Im\,L$ closed and $dim(ker\,L)=codim(Im\,L)<\infty$. Let $X=X_1\oplus X_2$ and $E=E_1\oplus E_2$ (topological direct sums) where $X_1=ker\,L$ and $E_2=Im\,L$. Let $P:X\to X_1$, $Q:E\to E_1$ be continuous linear projections and $J:X_1\to E_1$ a linear isomorphism (i.e. a linear homeomorphism). Finally $\Phi:X\to E_1$ will be a linear, continuous single valued map with $L+\Phi$: $dom\,L\to E$ an isomorphism; for convenience we say $\Phi\in H_L(X,E_1)$.

§2. Single Valued Maps

Let X and E be Fréchet spaces, U an open subset of X, $0\in U$ and $\Phi\in H_L(X,E_1)$ is fixed (here L and E_1 are as described in Section 1).

Definition 2.1. We let $M_{\partial U}(\bar{U},E;L,\Phi)$ denote the set of all continuous maps $F:\bar{U}\to E$ which satisfy the Mönch-Precup condition (i.e. if $C\subseteq\bar{U}$ is countable, $W\subseteq ker\,L$ is compact and $C\subseteq\overline{co}(\{0\}\cup(L+\Phi)^{-1}(F+\Phi)(C))+W$ then $\bar{C}$ is compact) and with $(L-F)(x)\neq 0$ for $x\in\partial U\cap dom\,L$; here ∂U is the boundary of U in X, $\bar{U}$ the closure of U in X and $\overline{co}(A)$ denotes the closed convex hull of A.

Remark 2.1. If $(L+\Phi)^{-1}\Phi(\bar{U})$ is a bounded set in $ker\,L$ then it is well known [9] (note $dim(ker\,L)<\infty$) that $(L+\Phi)^{-1}\Phi(\bar{U})$ is relatively compact, so as a result in this case we could define the Mönch-Precup condition in Definition 2.1 as: if $C\subseteq\bar{U}$ is countable, $W\subseteq ker\,L$ is compact and $C\subseteq\overline{co}(\{0\}\cup(L+\Phi)^{-1}F(C))+W$ then $\bar{C}$ is compact. To see why we need only note that $(L+\Phi)^{-1}\Phi(C)\subseteq ker\,L$ is relatively compact since $(L+\Phi)^{-1}\Phi(\bar{U})$ is relatively compact.

Definition 2.2. A map $F\in M_{\partial U}(\bar{U},E;L,\Phi)$ is essential if for every $G\in M_{\partial U}(\bar{U},E;L,\Phi)$ with $G|_{\partial U}=F|_{\partial U}$ we have that there exists $x\in\bar{U}\cap dom\,L$ with $Lx=G(x)$.

Theorem 2.1. *Let X and E be Fréchet spaces, U an open subset of X, $0\in U$*

and $\Phi \in H_L(X, E_1)$. *Suppose* $F \in M_{\partial U}(\bar{U}, E; L, \Phi)$ *is an essential map and* $H: \bar{U} \times [0,1] \to E$ *a continuous map with the following properties*:

$$H(x,0) = F(x) \text{ for } x \in \bar{U} \tag{2.1}$$

$$Lx \neq H_t(x) \text{ for any } x \in \partial U \cap \operatorname{dom} L \text{ and } t \in (0,1] \ (\text{here } H_t(x) = H(x,t)) \tag{2.2}$$

and

$$\begin{cases} \text{for any continuous } \mu: \bar{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \text{ the map} \\ R_\mu: \bar{U} \to E \text{ defined by } R_\mu(x) = H(x, \mu(x)) \text{ satisfies the Mönch-Precup} \\ \text{condition (i.e. if } C \subseteq \bar{U} \text{ is countable, } W \subseteq \ker L \text{ is compact and} \\ C \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1}(R_\mu + \Phi)(C)) + W \text{ then } \bar{C} \text{ is compact).} \end{cases} \tag{2.3}$$

Then there exists $x \in U \cap \operatorname{dom} L$ *with* $Lx = H_1(x)$.

Remark 2.2. If $(L+\Phi)^{-1}\Phi(\bar{U})$ is a bounded set in X we could define the Mönch-Precup condition in (2.3) as: if $C \subseteq \bar{U}$ is countable, $W \subseteq \ker L$ is compact and $C \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1} R_\mu(C)) + W$ then $\bar{C}$ is compact.

Proof. Let

$$B = \{x \in \bar{U} \cap \operatorname{dom} L : Lx = H_t(x) \text{ for some } t \in [0,1]\}.$$

It is immediate that

$$B = \{x \in \bar{U} : x = (L+\Phi)^{-1}(H_t + \Phi)\ (x) \text{ for some } t \in [0,1]\}.$$

When $t=0$, $H_0 = F$ and since $F \in M_{\partial U}(\bar{U}, E; L, \Phi)$ is essential there exists $x \in \bar{U} \cap \operatorname{dom} L$ with $Lx = F(x)$. Thus $B \neq \emptyset$. The continuity of H, Φ and $(L+\Phi)^{-1}$ guarantees that B is closed. In addition (2.2) (together with $F \in M_{\partial U}(\bar{U}, E; L, \Phi)$) implies $B \cap \partial U = \emptyset$. Thus there exists a continuous $\mu: \bar{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R: \bar{U} \to E$ by

$$R(x) = H(x, \mu(x)).$$

Now R is continuous and satisfies the Mönch-Precup condition (see (2.3)). Moreover for $x \in \partial U \cap \operatorname{dom} L$,

$$(L-R)(x) = (L-H_0)(x) = (L-F)(x) \neq 0$$

so $R \in M_{\partial U}(\bar{U}, E; L, \Phi)$. Also notice $R|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ and since $F \in M_{\partial U}(\bar{U}, E; L, \Phi)$ is essential there exists $x \in \bar{U} \cap \operatorname{dom} L$ with $(L-R)(x) = 0$ (i.e. $(L - H_{\mu(x)}(x) = 0)$. Thus $x \in B$ and so $\mu(x) = 1$. Consequently $(L-H_1)(x) = 0$ and we are finished (since (2.2) implies $x \in U \cap \operatorname{dom} L$). □

We now use Theorem 2.1 to obtain a nonlinear alternative of Leray-Schauder type for Mönch-Precup maps. To prove our result we need the following well known result from the literature [2]. For the remainder of this section X and E will be Banach spaces.

Theorem 2.2. Let *X be a Banach space, D a closed, convex set of X with $0\in D$. Suppose $J_0:D\to D$ is a continuous map which satisfies Mönch's condition (i.e. if $C\subseteq \bar{U}$ is countable and $C\subseteq \overline{co}(\{0\}\cup J_0(C))$ then $\bar{C}$ is compact). Then J_0 has a fixed point in D.*

Theorem 2.3. *Let X and E be Banach spaces, U an open subset of X, $0\in U\cap dom\, L$ and $\Phi\in H_L(X,E_1)$ is such that $(L+\Phi)^{-1}\Phi(\bar{U})$ is a bounded set in $ker\, L$. Suppose $G:\bar{U}\to E$ is a continuous map which satisfies the Mönch-Precup condition (i.e. if $C\subseteq \bar{U}$ is countable, $W\subseteq ker\, L$ is compact and $C\subseteq \overline{co}(\{0\}\cup(L+\Phi)^{-1}G(C))+W$ then $\bar{C}$ is compact) and assume*

$$(2.4)\qquad Lx\neq tG(x)+(1-t)(-\Phi(x)) \quad \text{for } x\in\partial U\cap dom\, L \text{ and } t\in(0,1)$$

is satisfied. Then there exists $x\in\bar{U}\cap dom\, L$ with $Lx=G(x)$.

Proof. We assume $Lx\neq G(x)$ for $x\in\partial U\cap dom\, L$ (otherwise we are finished). Then

$$(2.5)\qquad Lx\neq tG(x)+(1-t)(-\Phi(x)) \text{ for } x\in\partial U\cap dom\, L \text{ and } t\in[0,1].$$

(Note if $t=0$ and if $Lx_0=-\Phi(x_0)$ for $x_0\in\partial U\cap dom\, L$, then $(L+\Phi)(x_0)=0$ so $x_0=0$, which is a contradiction since $0\in U\cap dom\, L$). Let $H(x,t)=tG(x)+(1-t)(-\Phi(x))$ for $(x,t)\in\bar{U}\times[0,1]$ and $F(x)=-\Phi(x)$ for $x\in\bar{U}$. Notice (2.1) and (2.2) hold. To see (2.3) let $C\subseteq\bar{U}$ be countable and $W\subseteq ker\, L$ compact with

$$(2.6)\qquad C\subseteq\overline{co}(\{0\}\cup(L+\Phi)^{-1}(R_\mu+\Phi)(C))+W.$$

Notice for $x\in C$, $(R_\mu+\Phi)(x)=\mu(x)[G(x)+\Phi(x)]$ and as a result

$$(L+\Phi)^{-1}(R_\mu+\Phi)(C)\subseteq co((L+\Phi)^{-1}(G+\Phi)(C)\cup\{0\}).$$

In addition since $\{0\}\cup co((L+\Phi)^{-1}(G+\Phi)(C)\cup\{0\})=co((L+\Phi)^{-1}(G+\Phi)(C)\cup\{0\})$ and $co((L+\Phi)^{-1}(G+\Phi)(C)\cup\{0\})$ is convex we have

$$C\subseteq\overline{co}(\{0\}\cup(L+\Phi)^{-1}(R_\mu+\Phi)(C))+W\subseteq\overline{co}\big(co((L+\Phi)^{-1}(G+\Phi)(C)\cup\{0\})\big)+W$$
$$=\overline{co}\big((L+\Phi)^{-1}(G+\Phi)(C)\cup\{0\}\big)+W.$$

Now since G satisfies the Mönch-Precup condition we have that $\bar{C}$ is compact. Thus (2.3) holds. We can apply Theorem 2.1 if we show $F \in M_{\partial U}(\bar{U}, E; L, \Phi)$ is essential. First notice $F \in M_{\partial U}(\bar{U}, E; L, \Phi)$ [It is immediate that F satisfies the Mönch-Precup condition in Remark 2.1. Also note if $(L-F)(x_0)=0$ for some $x_0 \in \partial U \cap dom\, L$ then $x_0=0$, a contradiction]. To show F is essential let $\theta \in M_{\partial U}(\bar{U}, E; L, \Phi)$ with $\theta|_{\partial U} = F|_{\partial U} = -\Phi|_{\partial U}$. We must show that there exists $x \in \bar{U} \cap dom\, L$ with $Lx = \theta(x)$. Let $D = \overline{co}((L+\Phi)^{-1}(\theta+\Phi)(\bar{U}))$ and let $J_0 : D \to D$ be defined by

$$J_0(x) = \begin{cases} (L+\Phi)^{-1}(\theta+\Phi)(x), & x \in \bar{U} \\ 0, \quad x \notin \bar{U}. \end{cases}$$

Note $0 \in D$ and $J_0 : D \to D$ is continuous. We now show J_0 satisfies Mönch's condition. To see this let $C \subseteq D$ be countable with $C \subseteq \overline{co}(\{0\} \cup J_0(C))$. Then

(2.7) $$C \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1}(\theta+\Phi)(\bar{U} \cap C)).$$

Note as well that $(L+\Phi)^{-1}\Phi(\bar{U} \cap C) \subseteq ker\, L$ is relatively compact and this together with (2.7) gives

$$C \cap \bar{U} (\subseteq C) \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1}\theta(\bar{U} \cap C)) + W$$

where $W \subseteq ker\, L$ is a compact set. Since θ satisfies the Mönch-Precup condition we have $\overline{C \cap \bar{U}}$ compact. Thus since $(L+\Phi)^{-1}\theta$ is continuous, $(L+\Phi)^{-1}\theta(\overline{C \cap \bar{U}})$ is compact and Mazur's Theorem implies $\overline{co}(\{0\} \cup (L+\Phi)^{-1}\theta(\overline{C \cap \bar{U}})) + W$ is compact. Now since $C \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1}\theta(\bar{U} \cap C)) + W$ we have that $\bar{C}$ is compact. Consequently $J_0 : D \to D$ is continuous and satisfies Mönch's condition. Theorem 2.2 implies that there exists $x \in D$ with $J_0(x) = x$. Now if $x \notin U$, we have $0 = J_0(x) = x$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x = J_0(x) = (L+\Phi)^{-1}(\theta+\Phi)(x)$ i.e. $x \in U \cap dom\, L$ and $Lx = \theta(x)$. Hence F is essential and we may apply Theorem 2.1 to deduce the result. □

Theorem 2.3 gives us a nice criteria for recognizing essential maps (see Remark 2.4.). Our next result is particularly useful in applications.

Theorem 2.4. *Let X and E be Banach spaces. U an open subset of X and $0 \in U \cap dom\, L$. Let P, Q, J be as in Section* 1 *with $\Phi = JP$ and assume $(L+JP)^{-1}JP(\bar{U})$ is a bounded set in ker L. Suppose $G : \bar{U} \to E$ is a continuous map with $(L+JP)^{-1}G : \bar{U} \to X$ k-set contractive (here $0 \le k < 1$) and $(L+JP)^{-1}G(\bar{U})$ a bounded set in X. Also assume*

(2.8) $$QG \in M_{\partial U}(\bar{U}, E; L, JP) \text{ is an essential map}$$

(2.9) $$QG(x) \neq 0 \text{ for all } x \in \partial U \cap X_1$$

and

(2.10) $$Lx \neq tG(x) \text{ for } x \in \partial U \cap (dom\, L \backslash X_1) \text{ and } t \in (0,1)$$

are satisfied. Then there exists $x \in \bar{U} \cap dom\, L$ with $Lx = G(x)$.

Proof. Assume $Lx \neq G(x)$ for $x \in \partial U \cap dom\, L$. Let $H(x,t) = tG(x) + (1-t)QG(x)$. To see (2.2) notice if $Lx = H_t(x)$ for some $x \in \partial U \cap dom\, L$ and $t \in (0,1]$ then

(2.11) $$Lx = tG(x) + (1-t)QG(x).$$

It is easy to see that (2.11) is equivalent to

(2.12) $$Lx = t(I-Q)G(x) \text{ and } QG(x) = 0.$$

This together with (2.9) gives $Lx = tG(x)$ for $x \in \partial U \cap (dom\, L \backslash X_1)$, a contradiction. As a result (2.2) holds. To see (2.3) let $C \subseteq \bar{U}$ be countable and $W \subseteq ker\, L$ compact with

(2.13) $$C \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1}(R_\mu + \Phi)(C)) + W;$$

here $\Phi = JP$. Now since

$$\begin{aligned}(R_\mu + \Phi)(x) &= \mu(x)G(x) + (1-\mu(x))QG(x)\Phi(x)\\ &= \mu(x)[G+\Phi](x) + (1-\mu(x))[QG+\Phi](x)\end{aligned}$$

we have

(2.14) $$(L+\Phi)^{-1}(R_\mu+\Phi)(C) \subseteq co\big((L+\Phi)^{-1}(G+\Phi)(C) \cup (L+\Phi)^{-1}(QG+\Phi)(C)\big).$$

Now (2.13), (2.14), $(L+\Phi)^{-1}G: \bar{U} \to X$ k-set contractive, $(L+JP)^{-1}QG = P(L+JP)^{-1}G$ with P having finite dimensional range (so P is completely continuous), immediately guarantees that $\bar{C}$ is compact. Thus (2.3) holds so we may apply Theorem 2.1 to deduce the result. □

Remark 2.3. It is also easy to establish, under extra assumptions, the analogue of Theorem 2.4 with general Φ and $(L+\Phi)^{-1}G$ being k-set contractive replaced by the more general assumption that G satisfies the Mönch-Precup condition. We leave the details to the reader.

Remark 2.4. It is reasonably easy to put conditions on G in Theorem 2.4 to guarantee that (2.8) is satisfied. For example if

(2.15) $$\langle QG(x),\ J(x)\rangle \le 0 \text{ for } x\in\partial U\cap X_1$$

then (2.8) is satisfied; here $\langle .,. \rangle$ denotes the euclidean inner product on E_1 (note $dim\, E_1<\infty$). To see this let $\theta\in M_{\partial U}(\bar{U},E;L,JP)$ with $\theta|_{\partial U}=QG|_{\partial U}$. We will now use Theorem 2.3 to show that there exists $x\in\bar{U}\cap dom\, L$ with $Lx=\theta(x)$. We need only check (2.4). Suppose there exists $x\in\partial U\cap dom\, L$ and $t\in(0,1)$ with

$$Lx=t\theta(x)+(1-t)(-JP(x))=tQG(x)+(1-t)(-JP(x)).$$

Then

$$(L+JP)(x)=t(QG+JP)(x).$$

It is easy to see that

$$Lx=0 \text{ (i.e. } x\in X_1 \text{ so } P(x)=x) \text{ and } tQG(x)=(1-t)J(x).$$

Thus $x\in\partial U\cap X_1$ and

$$t\langle QG(x),\ J(x)\rangle=(1-t)\ |J(x)|^2>0.$$

This contradicts (2.15), so (2.4) holds. Theorem 2.3 guarantees that there exists $x\in\bar{U}\cap dom\, L$ with $Lx=\theta(x)$, so (2.8) holds.

As an application of the results above consider the system of n first order differential equations

(2.16) $$\begin{cases} y'=f(t,y) & \text{for} \quad t\in[0,1] \\ y(0)=y(1). \end{cases}$$

Let $C_P[0,1]=\{u\in C[0,1]:u(0)=u(1)\}$, $C_0=\{u\in C[0,1]:u(0)=0\}$, and let $L:C_P\to C_0$ be given by

$$Lu(t)=u(t)-u(0).$$

For each $a\in\mathbf{R}^n$ let $\bar{a}$ denote the constant function in C_P with value a. It is immediate that

$$ker\, L=\{\bar{a}:a\in\mathbf{R}^n\} \text{ and } Im\, L=\{v\in C_0:v(1)=0\}.$$

Since each $v\in C_0$ can be expressed as $v(t)=tv(1)+[v(t)-tv(1)]$ we have the direct sum decomposition $C_0=(t\mathbf{R}^n)\oplus Im\, L$. Thus L is a Fredholm map of index zero and we may set $Pu=u(1)$, $(Qv)(t)=tv(1)$, $J\bar{a}=ta$ and $\Phi=JP$.

We will assume $f:[0,1]\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous. Let $G:C_P\to C_0$ be defined by

(2.17) $$Gu(t)=\int_0^t f(s,u(s))ds.$$

It is well known that $G:C_P\to C_0$ is continuous and completely continuous.

Theorem 2.6. *Let $f:[0,1]\times\mathbf{R}^n\to\mathbf{R}^n$ be continuous. Suppose there is a constant $R>0$ with $|u|_0=\sup_{t\in[0,1]}|u(t)|<R$ for any solution $u\in C^1[0,1]$ to*

$(2.18)_\lambda$ $$\begin{cases} y'=\lambda f(t,y) & \text{for} \quad t\in[0,1] \\ y(0)=y(1) \end{cases}$$

for $0<\lambda<1$. Also assume

(2.19) $$\text{for all } a\in\mathbf{R}^n \text{ with } |a|=R \text{ we have } \int_0^1 f(s,a)ds\neq 0$$

and

(2.20) $$\text{for all } a\in\mathbf{R}^n \text{ with } |a|=R \text{ we have } \langle a, \int_0^1 f(s,a)ds\rangle\leq 0$$

hold. Then there exists a solution $u\in C^1[0,1]$ to (2.16) with $|u|_0\leq R$.

Proof. Let

$$U=\{u\in C_P:|u|_0<R\},\ X=C_P,\ E=C_0$$

and $G:C_P\to C_0$ be as in (2.17). Notice (see the definition of $\ker L$) we have $\partial U\cap\ker L=\{\bar{a}:\bar{a}\in\partial U\}$. If (2.9) fails then $QG(\bar{a})=0$ for some $\bar{a}\in\partial U\cap X_1$ i.e. $|\bar{a}|_0=R$ and

$$QG(\bar{a})=t\int_0^1 f(s,a)ds=0 \text{ for all } t\in[0,1].$$

Consequently $\int_0^1 f(s,a)ds=0$ for some $a\in\mathbf{R}^n$ with $|a|=R$. This contradicts (2.19), so (2.9) holds. If (2.15) fails then $\langle QG(\bar{a}), J(\bar{a})\rangle>0$ for some $\bar{a}\in\partial U\cap X_1$ i.e. $|\bar{a}|_0=R$ and

$$t\left\langle\int_0^1 f(s,a)ds, a\right\rangle>0 \text{ for all } t\in[0,1].$$

Consequently $\langle\int_0^1 f(s,a)ds,\ a\rangle>0$ for some $a\in\mathbf{R}^n$ with $|a|=R$. This contradicts

(2.20), so (2.15) holds. In addition since $|u|_0 < R$ for any solution u to $(2.18)_\lambda$ we have that (2.10) holds. Our result now follows from Theorem 2.4 with Remark 2.4. □

Remark 2.5. Notice (2.20) can be replaced by any condition that will guarantee for us that $QG \in M_{\partial U}(\bar{U}, E; L, JP)$ is an essential map.

Remark 2.6. We let $M^*_{\partial U}(\bar{U}, E; L, \Phi)$ denote the set of all continuous maps $F: \bar{U} \to E$ which satisfy the Mönch-Precup* condition (i.e. if $C \subseteq \bar{U}$ is countable, $W \subseteq ker\, L$ is compact and $C \subseteq \overline{co}(\{0\} \cup (L+\Phi)^{-1}F(C)) + W$ then $\bar{C}$ is compact) and with $(L-F)(x) \neq 0$ for $x \in \partial U \cap dom\, L$. There are obvious analogues of Theorem 2.1, Theorem 2.3 and Theorem 2.4 in this case (we leave the details to the reader). It is of interest to note that "$(L+\Phi)^{-1}\Phi(\bar{U})$ is a bounded set in $ker\, L$" is not needed in the analogue of Theorem 2.4 (however it is needed in the analogue of Theorem 2.3).

§3. Set Valued Maps

Let X and E be Fréchet spaces, U an open subset of X, $0 \in U$ and $\Phi \in H_L(X, E_1)$ is fixed (here L and E_1 are as described in Section 1).

Definition 3.1. A multivalued map $F: \bar{U} \to 2^E$ (here 2^E denotes the family of nonempty subsets of E) is said to be (L, Φ) upper semicontinuous if $(L+\Phi)^{-1}F: \bar{U} \to CK(X)$ is an upper semicontinuous (u.s.c.) map; here $CK(X)$ denotes the family of nonempty, compact, convex subsets of X.

Remark 3.1. It is possible to take $(L+\Phi)^{-1}F: \bar{U} \to AC(X)$ instead of $(L+\Phi)^{-1}F: \bar{U} \to CK(X)$ in the above definition and throughout this section (for example in Theorem 3.1); here $AC(X)$ denotes the family of nonempty, compact, acyclic subsets of X. Recall a subset A of X is acyclic if $H^m(A) = \delta_{0m}\mathbf{Z}$, where $\{H^m\}_{m \in \mathbf{N}}$ denotes the Čech cohomology functor with integer coefficients.

Definition 3.2. A multivalued map $F: \bar{U} \to 2^E$ is said to be (L, Φ) k-set contractive if $(L+\Phi)^{-1}F: \bar{U} \to CK(X)$ is a k-set contractive map.

Definition 3.3. We let $MM(\bar{U}, E; L, \Phi)$ denote the set of (L, Φ) u.s.c., k-set contractive (here $0 \leq k < 1$) maps $F: \bar{U} \to 2^E$ with $(L+\Phi)^{-1}F(\bar{U})$ a bounded set in X.

Remark 3.2. It is possible to discuss (L, Φ) condensing maps instead of (L, Φ) k-set contractive maps in Definition 3.3 (and throughout this section).

Remark 3.3. One could also discuss in this section (for example in Theorem 3.1) (L,Φ) admissible maps [6], (L,Φ) closed maps [7], and (L,Φ) approximable maps [7, 8].

Definition 3.4. We let $MM_{\partial U}(\bar{U},E;L,\Phi)$ denote the maps $F\in MM(\bar{U},E;L,\Phi)$ with $Lx\notin F(x)$ for $x\in\partial U\cap dom\, L$.

Definition 3.5. A map $F\in MM_{\partial U}(\bar{U},E;L,\Phi)$ is essential if for every $G\in MM_{\partial U}(\bar{U},E;L,\Phi)$ with $G|_{\partial U}=F|_{\partial U}$ we have that there exists $x\in\bar{U}\cap dom\, L$ with $Lx\in G(x)$.

Theorem 3.1. *Let X and E be Fréchet spaces, U an open subset of X, $0\in U$ and $\Phi\in H_L(X,E_1)$. Suppose $F\in MM_{\partial U}(\bar{U},E;L,\Phi)$ is an essential map and $H:\bar{U}\times[0,1]\to Cc(E)$ (here $Cc(E)$ denotes the family of nonempty, closed subsets of E) is a (L,Φ) u.s.c. map (i.e. $(L+\Phi)^{-1}H:\bar{U}\times[0,1]\to Cc(E)$ is u.s.c) with the following properties:*

$$H(x,0)=F(x) \quad \text{for} \quad x\in\bar{U} \tag{3.1}$$

$$Lx\notin H_t(x) \text{ for any } x\in\partial U\cap dom\, L \text{ and } t\in(0,1] \ (\text{here } H_t(x)=H(x,t)) \tag{3.2}$$

and

$$\begin{cases} \text{for any continuous } \mu:\bar{U}\to[0,1] \text{ with } \mu(\partial U)=0 \text{ the map} \\ R_\mu:\bar{U}\to Cc(E) \text{ defined by } R_\mu(x)=H(x,\mu(x)) \text{ is in } MM(\bar{U},E;L,\Phi). \end{cases} \tag{3.3}$$

Then there exists $x\in U\cap dom\, L$ with $Lx\in H_1(x)$.

Proof. Let

$$\begin{aligned} B&=\{x\in\bar{U}\cap dom\, L: Lx\in H_t(x) \text{ for some } t\in[0,1]\}\\ &=\{x\in\bar{U}: x\in(L+\Phi)^{-1}(H_t+\Phi)(x) \text{ for some } t\in[0,1]\}. \end{aligned}$$

As in Theorem 2.1, $B\neq\emptyset$. Moreover the continuity of Φ, $(L+\Phi)^{-1}$ and the (L,Φ) upper semicontinuity of H guarantees [1] that B is closed. Then there exists a continuous $\mu:\bar{U}\to[0,1]$ with $\mu(\partial U)=0$ and $\mu(B)=1$. Define a map $R:\bar{U}\to Cc(E)$ by

$$R(x)=H(x,\mu(x)).$$

By (3.3) we have $R\in MM(\bar{U},E;L,\Phi)$. In addition for $x\in\partial U\cap dom\, L$,

$$R(x)=H_0(x)=F(x)$$

and so $R\in MM_{\partial U}(\bar{U},E;L,\Phi)$. Also since $R|_{\partial U}=H_0|_{\partial U}=F|_{\partial U}$ and $F\in MM_{\partial U}(\bar{U},E;L,\Phi)$ is essential there exists $x\in\bar{U}\cap domL$ with $Lx\in R(x)$ (i.e. $Lx\in H_{\mu(x)}(x)$). Thus $x\in B$ and so $\mu(x)=1$. Consequently $Lx\in H_1(x)$. □

Next we recall a result [3] from the literature.

Theorem 3.2. *Let X be a Fréchet space and let D be a nonempty, closed, convex subset of X. Suppose $J_0:D\to CK(D)$ is a u.s.c., k-set contractive (here $0\le k<1$) map with $J_0(D)$ a bounded set in D. Then J_0 has a fixed point in D.*

Our next theorem extends a result of Volkmann [11 pp. 240].

Theorem 3.3. *Let X and E be Fréchet spaces, U an open subset of X, $0\in U\cap dom\, L$ and $\Phi\in H_L(X,E_1)$. Suppose $G:\bar{U}\to 2^E$ is a (L,Φ) u.s.c., k-set contractive (here $0\le k<1$) map with $(L+\Phi)^{-1}G(\bar{U})$ a bounded set in ker L. In addition assume*

$$(L+\Phi)^{-1}\Phi(\bar{U}) \text{ is a bounded set in } X \tag{3.4}$$

and

$$Lx\notin tG(x)+(1-t)(-\Phi(x)) \text{ for } x\in\partial U\cap dom\, L \text{ and } t\in(0,1) \tag{3.5}$$

are satisfied . Then there exists $x\in\bar{U}\cap dom\, L$ with $Lx\in G(x)$.

Proof. We assume $Lx\notin G(x)$ for $x\in\partial U\cap dom\, L$ (otherwise we are finished). Then

$$Lx\notin tG(x)+(1-t)(-\Phi(x)) \text{ for } x\in\partial U\cap dom\, L \text{ and } t\in[0,1]. \tag{3.6}$$

Let $H(x,t)=tG(x)+(1-t)(-\Phi(x))$ for $(x,t)\in\bar{U}\times[0,1]$ and $F(x)=-\Phi(x)$ for $x\in\bar{U}$. It is clear that $H:\bar{U}\times[0,1]\to Cc(E)$ is a (L,Φ) u.s.c. map. Also (3.1) and (3.2) hold. To see that (3.3) is true notice if $W\subseteq\bar{U}$ then

$$(L+\Phi)^{-1}R_\mu(W)\subseteq\overline{co}((L+\Phi)^{-1}G(W)\cup(L+\Phi)^{-1}(-\Phi(W))).$$

We now have immediately that $R_\mu\in MM(\bar{U},E;L,\Phi)$ and so (3.3) holds. We can apply Theorem 3.1 if we show $F\in MM_{\partial U}(\bar{U},E;L,\Phi)$ is essential. Notice it is immediate from (3.4) that $F\in MM_{\partial U}(\bar{U},E;L,\Phi)$ [Note if $Lx_0\in F(x_0)$ for some $x_0\in\partial U\cap dom\, L$ then $x_0=0$, a contradiction]. To show F is essential let $\theta\in MM_{\partial U}(\bar{U},E;L,\Phi)$ with $\theta|_{\partial U}=F|_{\partial U}=-\Phi|_{\partial U}$. Let $D=\overline{co}((L+\Phi)^{-1}(\theta+\Phi)(\bar{U}))$

and let $J_0 : D \to CK(D)$ be defined by

$$J_0(x) = \begin{cases} (L+\Phi)^{-1}(\theta+\Phi)(x), x \in \bar{U} \\ \{0\}, x \notin \bar{U}. \end{cases}$$

Note $0 \in D$ and it is easy to see that J_0 is a u.s.c., k-set contractive map (note $(L+\Phi)^{-1}\Phi(\Omega)$ is relatively compact for any bounded subset Ω of X) with $J_0(D)$ a bounded set in D. Theorem 3.2 implies that there exists $x \in D$ with $x \in J_0(x)$. Now if $x \notin U$, we have $x \in J_0(x) = \{0\}$, a contradiction. Thus $x \in U$ so $x \in J_0(x) = (L+\Phi)^{-1}(\theta+\Phi)(x)$ i.e. $x \in U \cap dom\, L$ and $Lx \in \theta(x)$. Hence F is essential and we may apply Theorem 3.1 to deduce the result. □

Remark 3.4. There is an obvious analogue of Theorem 2.4 in this setting. We leave the details to the reader. Also it is possible to apply our results to differential inclusions following the ideas in Section 2.

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